SOME FORMULAE ARISING IN PROJECTIVE-DIFFERENTIAL GEOMETRY (*)

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Sunto. - Si stabilisce una terminologia di base che sembra essere particolarmente adatta a trattare alcune particolari questioni di geometria proiettivo-differenziale. Nell'ambito di questa si dimostrano alcuni risultati generali, tra cui delle formule 'di inversione' riguardanti le deformazioni di famiglie di sottoschemi non ridotti. Come applicazione, si propongono nuove definizioni di alcuni concetti classici, tra cui quello di fuochi di ordine qualsiasi; infine si dà una nuova dimostrazione del teorema di struttura delle fibre della prima mappa di Gauss.

Abstract. - We set up a framework in which some projective-differential intuitive concepts can be very easily formalized. Then we prove some general formulas. Among them, two 'inversion formulas' about deformations of families of nonreduced subschemes are particularly remarkable. As applications, we give convenient definitions about some classic topics, such as higher order foci; finally we propose a new proof of the classic structure theorem about the first Gauss map.

Introduction.

The study of varieties with a particular behaviour of the tangent spaces is a fundamental topic in projective-differential geometry. It is natural to consider similar questions about higher order osculating spaces, and in fact it is possible to find in the literature a certain amount of papers dealing with this subject. In this theory some basic operations such as spans, or intersections and unions of families of subspaces, are repeatedly used. It is also evident that the study of deformations of such families is crucial.

The classic authors treated deformations by the intuitive language of infinitely near objects. In some old papers one can find rigorous analytical explanations of the intuitive concepts involved (see for instance [T], [Seg]). In some other old papers, for instance in [C], there is lack of a rigorous foundation of some intuitive statements (fixing this problems

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was the starting point of our studies in this field). Certainly a rigorous treatment can be given on the same line of [T] and [Seg], but it is evident that modern techniques, such as the introduction of nonreduced schemes, allow handling this questions in a more concise and clear fashion. For instance, the osculating spaces can be simply (and completely rigorously) defined as the subspaces spanned by the infinitesimal neighborhoods (which are nonreduced schemes). At a very general level one can find a modern treatment of projective-differential geometry in [GH]. Also [Pe] contains many general results, which are useful in this context. However, the particular nature of the questions we are dealing with suggests to set up some additional simple definitions and notation, by which many concepts can be handled in a way that is very intuitive and very rigorous at the same time. It is the case, for instance, of Proposition 4.3 and its corollary (cf. also the end of Remark 5.4).

Meanwhile some definitions are completely natural, some others require to be modified with respect to what one could naïvely think, in a way which is also natural but not obvious at all. Once set the right definitions, basic properties which are extensions of the set-theoretic ones, are proved in detail: straightforwardly for the majority of them, some others with a little effort.

But a very natural equality between particular nonreduced schemes, which arises in this framework, seems to be rather deep. In fact the right hypotheses in which the two opposite inclusions hold, are far from being evident. Moreover, in the related proofs we need some advanced tools from deformation theory (such as 0-smoothness and infinitesimal lifting). This equality looks like a sort of inversion formula, similar to the exchange of summation indexes (or to Dirichlet inversion formula on double integrals).

In order to illustrate applications, we give simple definitions about classic concepts as higher order characteristic spaces and higher order foci. Finally, we give a proof of the classic theorem on the structure of the fibres for the first Gauss map, which is new as far as we know. These applications certainly provide a strong motivation for the results we discussed above (although we believe they have both an evident intrinsic interest).

Since we had to set up a convenient, but a little bit unusual terminology, we included detailed proofs, even of some very easy facts, which make the paper somewhat self-contained. In this way we could clarify differences with other approaches and avoid overlaps with them. We tried to organize the material in such a way that one can distinguish trivial or straightforward facts from deeper ones (but clearly it was only partially possible). In Section 1 we collect standard facts that are repeatedly used in the paper. The only unusual fact is the following. Exactly as a group (or other algebraic structures) is defined as a set equipped with an operation, although it could be formally described by the operation only, in this paper a family of subschemes, which is completely determined by a morphism f of schemes, is defined emphasizing the notation on the set of the closed fibres. We proceed similarly for families of subspaces (using a little bit more flexible definition). In Section 2 we define some basic operations, which in some cases are not obvious; the proofs are quite straightforward. In Section 3 we show as the framework of Section 2 allows to give concise definitions of some classic concepts such as (any order) foci, and particular types of varieties. In Section 4 there are the main results, roughly disposed by increasing order of difficulty. Finally in Section 5 there is our proof of the structure theorem for the first Gauss map, with a little discussion about possible extensions to higher order osculating spaces.

1. - Preliminaries.

— For the basic terminology we refer to [H], with only a few exceptions, explained through the paper.
— We fix once for all an algebraically closed field k and a k-vector space V, of finite dimension $n+1$.
— All the schemes will be defined over k. The fibred products, unless otherwise indicated, are supposed over Spec k. Tensor products without indication of the base ring are over k.
— If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of quasi-coherent sheaves, we denote by $\check{\varphi} : \check{\mathcal{G}} \to \check{\mathcal{F}}$ its transposed morphism, even if $\mathcal{F}$ and $\mathcal{G}$ are not locally free of finite rank.
— By abuse of notation, we shall often identify objects which are naturally isomorphic.
— We identify any k-vector space with its associated sheaf over Spec k; so $\mathbf{P}(\check{V})$ (cf. [H, II, 7]) is a scheme over k, denoted by $\mathbf{P}^n$.
— A *variety* is defined as in [H, II, 4], hence a variety in $\mathbf{P}^n$ is an irreducible, reduced subscheme of $\mathbf{P}^n$.
— A *subspace* of $\mathbf{P}^n$ is a variety of degree 1 in $\mathbf{P}^n$.
— Let X be a scheme over k and let W be a k-vector space. We denote by $W \otimes \mathcal{O}_X$ the sheaf f^*W, where f is the structural morphism $X \to$ Spec k.
— Let X be a scheme. A scheme Y will be *a (locally closed) subscheme* of X if is given an immersion $Y \to X$ which can be obtained composing a closed immersion with an open immersion. We shall say that a locally closed subscheme Y of $\mathbf{P}^n$ *is contained* in a locally closed subscheme X if the immersion of Y factors through that one of X. The subscheme $Y \times_X Y'$ of X will be the intersection $Y \cap Y'$ of two subschemes Y and Y' of X.
— The sheaf $\mathcal{O}_X(1)$ on a locally closed subscheme X of $\mathbf{P}^n$ will be the inverse image of $\mathcal{O}_{\mathbf{P}^n}(1)$. Its direct image on $\mathbf{P}^n$ will be often denoted again by $\mathcal{O}_X(1)$ (it is an usual abuse of notation; cf. [H]).
— If Y is a closed subscheme of X and i is a nonnegative integer, we call *i-th infinitesimal neighborhood of Y in X* the subscheme Y^i_X of X defined by the $(i+1)$-th power of the ideal sheaf of Y in X (notice that in [H] this is the $(i+1)$-th neighborhood).

Since we are interested in quasi-projective varieties we shall restrict our propositions and definitions to quasi-projective schemes (but in many cases will be fundamental that they may be nonreduced, since we often deal with infinitesimal neighborhoods). We shall use the following terminology about families of schemes.

Definition 1.1. A *family of quasi-projective schemes* is given by the set of the closed fibres of a morphism $\pi : X \to S$ of quasi-projective schemes. We denote the family by $\{X_P\}_{`P\in S'}$. We shall also say that *the structure of the family $\{X_P\}_{`P\in S'}$ is defined by* π. When $\{X_P\}_{`P\in S'}$ and $\{X'_P\}_{`P\in S'}$ are families of schemes defined respectively by π and π' such thata X is a subscheme of X' and π is the restriction of π', we write

$$X_P \subseteq X'_P \quad `\forall P \in S\,'.$$

If X is a nonempty open subscheme of X', we shall say that $\{X_P\}_{`P\in S'}$ is a nonempty open subfamily of $\{X'_P\}_{`P\in S'}$. If T is a subscheme of S then the family $\{X_P\}_{`P\in T'}$ will have the structure obtained by base change.

Clearly the same set of schemes may have different structures, but throughout the paper, when we deal with a family of schemes, it will be always supposed to be defined by a fixed (sometimes understood) structure. We consider now those families of schemes which are subschemes of a fixed quasi-projective scheme Y. This is a particular case of the situation described above, and precisely when the second family is constant.

Definition 1.2. Let Y and S be quasi-projective schemes, let X be a subscheme of $S \times Y$, and let π be the restriction on X of the first projection $S \times Y \to S$. We say that the family $\{X_P\}_{`P\in S'}$ defined by $\pi$ is a *family (over $S$) of (locally closed) subschemes of* $Y$ and that *the structure of the family* $\{X_P\}_{`P\in S'}$ *is defined by* $X \subseteq S \times Y$. The notation is slightly different with respect to [H], where the parameter space is the second factor.

Now we introduce a convenient definition about families of subspaces.

Definition 1.3. Let $\lambda : \check{V} \otimes \mathcal{O}_S \to \mathcal{L}$ be a morphism of quasi-coherent $\mathcal{O}_S$-modules over a quasi-projective scheme S, let $P \in S$ and $i_P : \mathrm{Spec}\ k(P) \to S$ be its immersion in S. The morphism $H^0(i_P^*\lambda)$ is (up to natural isomorphisms) a homomorphism $\check{V} \otimes k(P) \to i_P^*\mathcal{L}$ of $k(P)$-vector spaces. If the point P is closed, applying $\mathbf{P}(\cdot)$ to the morphism we obtain a rational map (linear projection) $\mathbf{P}(i_P^*\mathcal{L}) \to \mathbf{P}^n$, whose image will be denoted by L_P. Note that the linear projection can be degenerate and the space $\mathbf{P}(i_P^*\mathcal{L})$ can be infinite-dimensional. We call the set of L_P a *family (over S) of subspaces of* $\mathbf{P}^n$, or sometimes simply a *family of subspaces.* We denote it by $\{L_P\}_{`P\in S'}$, and we shall say that *the structure of the family is defined by* $\lambda$. If $T$ is a subscheme of $S$ the family $\{L_P\}_{`P\in T'}$ will be defined by the restriction (i.e. the inverse image) of φ on T .

Some families of subspaces can be also naturally regarded as a family of subschemes, as follows.

Definition 1.4. Let $\{L_P\}_{`P\in S'}$ be a family of subspaces defined by a morphism $\lambda : \check{V} \otimes \mathcal{O}_S \to \mathcal{L}$. Suppose that the sheaves Coker $\lambda$ and Im $\lambda$ are flat over $S$. We shall call $\{L_P\}_{`P\in S'}$ a *flat family of subspaces.*

Remark 1.5. In the situation of the above definition, consider the morphism $\lambda' : \check{V} \otimes \mathcal{O}_S \to \mathrm{Im}\ \lambda$, obtained by restriction of λ on its image. This morphism defines another structure on the same set $\{L_P\}_{`P\in S'}$: the only difference is the fact that the linear projections (whose images are the spaces $L_P$) are injective. Notice that $\{L_P\}_{`P\in S'}$ has also a natural structure of family of subschemes of $\mathbf{P}^n$, which is flat in the ordinary sense, given by $\mathbf{P}(\lambda')$, which is a closed immersion of $\mathbf{P}(\mathrm{Im}\ \lambda)$ in $\mathbf{P}(\check{V} \otimes \mathcal{O}_S) \cong S \times \mathbf{P}^n$ (cf. Definition 1.2). If S is connected then all the spaces L_P have the same dimension, say d, so in this case we shall say that $\{L_P\}_{`P\in S'}$ is a flat family of subspaces of dimension d.

The flat families above are in a natural correspondence with morphisms of S into a grassmannian. This fact is explained by the universal property of the grassmannian, as follows.

PROPOSITION 1.6. *There is a locally free rank $d+1$ quotient $q : \check{V} \otimes \mathcal{O}_{\mathbf{G}(V,d+1)} \to \mathcal{Q}$ on the grassmannian $\mathbf{G}(V, d+1)$ such that for every surjective morphism $\lambda : \check{V} \otimes \mathcal{O}_S \to \mathcal{L}$ over a noetherian scheme S over k, with $\mathcal{L}$ locally free of rank $d+1$, there is a unique morphism $l : S \to \mathbf{G}(V, d+1)$, such that $l^*q = \lambda$ (up to isomorphisms).*

PROOF. See [Ser].□

REMARK 1.7. The universal property of the grassmannian implies that the set of closed points of $\mathbf{G}(V, d+1)$ can be naturally identified with the set of the $(d+1)$-subspaces of V, or equivalently with the linear subspaces of dimension d of $\mathbf{P}^n$. In fact, if W is a subspace of V, the homomorphism $\varphi : \check{V} \to \check{W}$ obtained transposing the immersion of W can be regarded as a morphism of sheaves on Spec k. Hence, by the above universal property, it gives rise to an immersion Spec $k \to \mathbf{G}(V, d+1)$, which is the immersion of the required closed point representing W. The corresponding linear subscheme L of $\mathbf{P}^n$ is the image of the immersion $\mathbf{P}(\varphi) : \mathbf{P}(\check{W}) \to \mathbf{P}(\check{V}) = \mathbf{P}^n$. In our settings we can express this fact by saying that φ is a structure of family of subspaces on the set $\{L\}$, and the universal quotient is a structure of family of subspaces on the set of all d-dimensional subspaces of $\mathbf{P}^n$.

Proposition 1.6 deals only with surjective morphisms. This is not a restriction since, as pointed out in Remark 1.5, if we replace a morphism λ defining a flat family with its restriction (on the codomain) λ', we obtain another structure on the same set, obviously fulfilling again the 'flat' condition. However, in the next section we shall introduce some operations on families of subspaces, and in some cases we shall need to replace λ by λ' without affecting the result. It will be an immediate fact if the sheaves Coker λ and Im λ are locally free. For the image there are no problems since it is coherent, but the cokernel can be only quasi-coherent, hence 'locally free' is a condition stronger than 'flat'. This is the reason for the following definition.

DEFINITION 1.8. Throughout this paper, a family of subspaces $\{L_P\}_{`P \in S'}$, defined by a morphism $\lambda : \check{V} \otimes \mathcal{O}_S \to \mathcal{L}$, will be called a *good* family if the sheaves Coker λ and Im λ are locally free.

REMARK 1.9. Let us consider a family of subspaces $\{L_P\}_{`P \in S'}$, over a quasi-projective variety S, defined by $\lambda : \check{V} \otimes \mathcal{O}_S \to \mathcal{L}$. Since Im λ is a coherent sheaf and S is reduced (since it is a variety) over a nonempty open subscheme U of S it is locally free of finite rank, say $d+1$. If additionally Coker λ is locally free over a nonempty subscheme S' of U (if $\mathcal{L}$ is coherent there are no problems), then $\{L_P\}_{`P \in S''}$ is a good (hence flat) family of subspaces of dimension $d$. We say that the integer $d$ is the dimension of the generic space of $\{L_P\}_{`P \in S'}$. From now on, for every such family of subspaces $\{L_P\}_{`P \in S'}$, we shall always take S' as 'big' as possible. We shall also consider the associated morphism $S' \to \mathbf{G}(V, d+1)$.

2. - Some operations on families.

Families of spans.

Definition 2.1. Let X be a (locally closed) subscheme of $\mathbf{P}^n$. We shall denote by $\langle X\rangle$ the subspace spanned by X. More precisely, $\langle X\rangle$ is the minimum subspace of $\mathbf{P}^n$ which contains X.

Remark 2.2. We shall often deal with families of subspaces which are spanned by subschemes in a family, then we need an appropriate technical definition. If X is a subscheme of $\mathbf{P}^n$ then the homomorphism $\check{V} \cong H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(\mathcal{O}_X(1))$ defines a structure of a family of subspaces on the single space $\langle X\rangle$. This gives the idea of the required definition, but we have to carefully point out some details. We begin by reminding the following useful lemma.

Lemma 2.3. *Let A be a noetherian integral domain.*

(I) *If B is a finitely generated A-algebra and M is a finitely generated B-module, then there is an $f \in A - \{0\}$ such that M_f is free over A_f;*

(II) *if $\varphi : M \to N$ is a homomorphism of a finitely generated A-module M to a free (possibly infinite-rank) A-module N, then there is an $f \in A - \{0\}$ such that* Coker φ_f *is free over A_f.*

Proof. (I). See [Mu, lecture 8, Proposition pag. 57 and its proof].

(II). Take a (possibly infinite) base X of N and a finite set of generators $\{g_j\}$ of Im φ. Each g_j is a linear combination of elements of a finite subset $X_j \subseteq X$, hence we can consider the finitely generated free submodule N' of N generated by $\cup_j X_j$. We have $N = N' \oplus N"$, with $N"$ free and Im $\varphi \subseteq N'$. Since N' is finitely generated $F := N'/\mathrm{Im}\ \varphi$ is finitely generated. Hence there is an $f \in A - \{0\}$ such that F_f is free over A_f. Then Coker $\varphi_f \cong F_f \oplus N"_f$ is free over A_f.□

Proposition 2.4. *Let $\{X_P\}_{`P\in S'}$ be a family of subschemes of $\mathbf{P}^n$ defined by $X \subseteq S \times \mathbf{P}^n$, let π be the projection $S \times \mathbf{P}^n \to S$ and π^X its restriction to X. Consider the morphism $\varphi : \check{V} \otimes \mathcal{O}_S \cong \pi_*\mathcal{O}_{S\times\mathbf{P}^n}(1) \to \pi_*\mathcal{O}_X(1)$. If X is affine over S (i.e. the morphism $\pi^X : X \to S$ is affine) then*

(a) *for any morphism of quasi-projective schemes $f : S' \to S$, setting $X' := X \times_S S' \subseteq S' \times \mathbf{P}^n$ and denoting by π' the projection on S' of $S' \times \mathbf{P}^n$, the morphism $\varphi' : \check{V} \otimes \mathcal{O}_{S'} \cong \pi'_*\mathcal{O}_{S'\times\mathbf{P}^n}(1) \to \pi'_*\mathcal{O}_{X'}(1)$ is equal (up natural isomorphisms) to $f^*\varphi$;*

(b) *φ is a structure for the family $\{\langle X_P\rangle\}_{`P\in S'}$;*

(c) *if S is reduced then over a nonempty open $U \subseteq S$ the family $\{\langle X_P\rangle\}_{`P\in U'}$ is good.*

Proof. (a). It is enough to show that the natural morphism $f^*\pi^X_*\mathcal{O}_X(1) \to {\pi'}^{X'}_*\mathcal{O}_{X'}(1)$ is an isomorphism, where $\pi^{X'}_*$ is the restriction of π' to X'. The question is local on the base, hence we can suppose $S = \mathrm{Spec}\ A$, $S' = \mathrm{Spec}\ A'$ and f induced by $A \to A'$. Since π^X is affine we can also suppose $X = \mathrm{Spec}\ B$, π^X induced by $A \to B$, $\mathcal{O}_X(1) = \tilde{M}$, with M a finitely generated B-module (where the tilde indicate the associated sheaf), and $X' = \text{Spec } B \otimes_A A'$. Then, our natural map is $(M \otimes_A A')\tilde{} \to (M \otimes_B (B \otimes_A A'))\tilde{}$ (where the associated sheaves are taken on A'), which is clearly an isomorphism.

(b). It is an immediate consequence of (a), taking f equal to the immersion of P, for each closed point $P \in S$ (cf. Remark 2.2).

(c). Again the question is local on the base, hence we can keep the above notation. Moreover φ is induced by a homomorphism $h : \check{V} \otimes A \to M$ of A-modules. Since we are looking for a nonempty open subscheme, we can suppose S integral. Then A is an integral domain, hence we can apply lemma 2.3. By (I) we get an $f \in A - \{0\}$ such that M_f is free over A_f. By (II) we get an $f' \in A - \{0\}$ such that Coker $h_{ff'}$ is free over $A_{ff'}$. Hence on the nonempty open $D(ff')$, φ has a locally free (actually, free) cokernel. Since Im h is finitely generated we can get a nonempty open U over which Im φ is locally free too. Then the family defined by φ is good over U.□

Corollary 2.5. *Let $\{X_P\}$ ‘$_{P \in S}$’ be a family of locally closed subschemes of $\mathbf{P}^n$, defined by $X \subseteq S \times \mathbf{P}^n$, and let $\{X_i\}$ be a finite covering of X, such that each X_i is open and affine over S (such a covering always exists). Let φ_i be the morphisms $\check{V} \otimes \mathcal{O}_S \cong \pi_* \mathcal{O}_{S \times \mathbf{P}^n}(1) \to \pi_* \mathcal{O}_{X_i}(1)$. Finally, let $\varphi : \check{V} \otimes \mathcal{O}_S \to \oplus_i \pi_* \mathcal{O}_{X_i}(1)$ be the morphism induced by the morphisms φ_i. Then φ defines a structure of family of subspaces on the set $\{\langle X_P \rangle\}$ ‘$_{P \in S}$’. If S is reduced, over a nonempty open subscheme U of S the family $\{\langle X_P \rangle\}$ ‘$_{P \in U}$’ is good.*

Proof. It follow easily from Proposition 2.4 and lemma 2.3 again, taking into account that if a set of morphisms $\{\varphi_i : \check{V} \to M_i\}$ define a set of subspaces L_i, then the induced morphism $\check{V} \to \oplus_i M_i$ define the minimum subspace containing each L_i.□

Definition 2.6. Let $\{X_P\}$‘$_{P \in S}$’ be a family of subschemes of $\mathbf{P}^n$. We shall call an *associated structure* on $\{\langle X_P \rangle\}$‘$_{P \in S}$’ any structure defined as in the statement of corollary 2.5. The choice of an associated structure depends on the choice of the open covering. In the following, when we deal with a family $\{X_P\}$‘$_{P \in S}$’ of subschemes of $\mathbf{P}^n$, we shall always suppose the family $\{\langle X_P \rangle\}$‘$_{P \in S}$’ defined by one of these structures. If the scheme $X \subseteq S \times \mathbf{P}^n$ is affine over S, we shall always choice the (associated) structure corresponding to the trivial covering of X (i.e. $\{X\}$; in other words, it is the structure defined in the statement of Proposition 2.4).

Remark 2.7. Once fixed an associated structure on $\{\langle X_P \rangle\}$‘$_{P \in S}$’, with respect to a covering $\{X_i\}$, if $f : T \to S$ is a morphism of quasi-projective schemes, the family $\{X_P\}$‘$_{P \in S}$’ induces (by base change) a family of subschemes $\{X_Q\}$‘$_{Q \in T}$’ over T. Consider the set $\{\langle X_Q \rangle\}$‘$_{Q \in T}$’. It is easy to see that on this set, up to natural isomorphisms, the structure obtained by base change from the structure of $\{\langle X_P \rangle\}$‘$_{P \in S}$’ coincides (up to natural isomorphisms) with the associated structure to $\{X_Q\}$‘$_{Q \in T}$’, with respect to the covering $\{f^{-1}(X_i)\}$.

Union of schemes in a family.

Definition 2.8. Let $\{X_P\}_{‘P\in S’}$ be a family of subschemes of a quasi-projective scheme Y, defined by $X \subseteq S \times Y$. The scheme-theoretic image of X via the second projection $S \times Y \to Y$ will be denoted by $\overline{\bigcup_{‘P\in S’} X_P}$, or sometimes simply by $\bigcup_{‘P\in S’} X_P$.

Remark 2.9. Notice that in the situation of Definition 2.8, we have also a natural definition for $\overline{\bigcup_{‘P\in T’} X_P}$ for any subscheme T of S, in fact the family $\{X_P\}_{‘P\in T’}$ has the natural structure obtained by base change, as already pointed out in Definition 1.1.

Span of families.

Definition 2.10. Let $\{L_P\}_{‘P\in S’}$ be a family of subspaces defined by $\lambda : \check{V}\otimes\mathcal{O}_S \to \mathcal{L}$. Consider the morphism φ, obtained by the composition $\check{V} \to H^0(\check{V}\otimes\mathcal{O}_S) \to H^0(\mathcal{L})$, where the first map is the natural one and the second map is $H^0(\lambda)$. In other words, φ is the morphism of sheaves on Spec k corresponding to λ via the structural map $s : S \to \text{Spec } k$, by the adjointness of s^* and s_*. As a morphism of sheaves on Spec k, φ defines a family made of a single subspace L (by Definition 1.3, L is the image of the linear projection $\mathbf{P}(\varphi) : \mathbf{P}(H^0(\mathcal{L})) \to \mathbf{P}^n$). We shall denote the subspace L by $\langle\bigcup_{‘P\in S’} L_P\rangle$.

The above definition agrees with the naïve notion of $\langle\bigcup_{P\in S} L_P\rangle$ when S is a variety (hence reduced) and the family is good. When the base scheme S is the i-th neighborhood of a point P in some variety, we have a good definition of ‘the space spanned by the i-th order infinitely near spaces to L_P’.

Remark 2.11. Let $\{S_P\}_{‘P\in T’}$ be a family of subschemes, defined by an affine morphism $f : S \to T$ of quasi-projective schemes, and let $\{L_Q\}_{‘Q\in S’}$ be a family of subspaces defined by a morphism $\lambda : \check{V}\otimes\mathcal{O}_S \to \mathcal{L}$. The family $\{\langle\bigcup_{‘Q\in S_P’} L_Q\rangle\}_{‘P\in T’}$ has a natural structure given by the morphism of $\mathcal{O}_T$-modules $\check{V}\otimes\mathcal{O}_T \to f_*\mathcal{L}$, canonically obtained from λ by the adjointness of f^* and f_*. If we want to remove the affineness hypothesis on f, in order to get a structure for $\{\langle\bigcup_{‘Q\in S_P’} L_Q\rangle\}_{‘P\in T’}$, we have to proceed as in the statement of corollary 2.5.

Remark 2.12. For a flat family of subspaces $\{L_P\}_{‘P\in S’}$, which can be also regarded as a family of subschemes (cf. Remark 1.5), Definition 2.10 of $\langle\bigcup_{‘P\in S’} L_P\rangle$ agrees with simultaneous application of Definitions 1.2 and 2.1 (which gives rise to a subspace which, according to these definitions, could be denoted in the same way). In fact, let $\lambda : \check{V}\otimes\mathcal{O}_S \to \mathcal{L}$ be the morphism defining $\{L_P\}_{‘P\in S’}$, and let $\lambda' : \check{V}\otimes\mathcal{O}_S \to \mathcal{L}' := \text{Im } \lambda$ its restriction on the image (which is another structure for the same family). We can replace the structure λ with the structure λ', since the space $L = \langle\bigcup_{‘P\in S’} L_P\rangle$ of Definition 2.10 is the same in both cases. In fact, in the sequence

$$\check{V} \to H^0(\check{V}\otimes\mathcal{O}_S) \xrightarrow{\lambda'} H^0(\mathcal{L}') \to H^0(\mathcal{L}),$$

the last morphism is injective. Let X be the subscheme of $\mathbf{P}(\check{V}\otimes\mathcal{O}_S) \cong S \times \mathbf{P}^n$ defining $\{L_P\}_{‘P\in S’}$ as a family of subschemes. Since the immersion of X in $S \times \mathbf{P}^n$ is $\mathbf{P}(\lambda')$ (up to natural isomorphisms), then λ' is equal, up to natural isomorphisms, to the morphism $\mathcal{O}_{S\times\mathbf{P}^n}(1) \to \mathcal{O}_X(1)$. Now, by Definition 1.2, $Y = \bigcup_{`P\in S'} L_P$ is the (scheme-theoretic) image of $X$ in $\mathbf{P}^n$, via the projection $\pi : S \times \mathbf{P}^n \to \mathbf{P}^n$. By Remark 2.2, our space $\langle\bigcup_{`P\in S'} L_P\rangle = \langle Y\rangle$ can be recovered by the morphism $\check{V} \cong H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(\mathcal{O}_Y(1))$. But composing with the morphism $H^0(\pi^\sharp(1)) : H^0(\mathcal{O}_Y(1)) \to H^0(\mathcal{O}_X(1))$ we get exactly the morphism defining L. Since the map $X \to Y$ is dominant, $H^0(\pi^\sharp(1))$ is injective, hence $\langle Y\rangle = L$, as required.

On the same line we can prove the following two propositions, which will be useful in Section 4.

Proposition 2.13. *If $\{X_P\}_{`P\in S'}$ is a family of subschemes of $\mathbf{P}^n$, choosing an associated structure for $\{\langle X_P\rangle\}_{`P\in S'}$ we have*

$$\langle \bigcup_{`P\in S'} \langle X_P\rangle\rangle = \langle \bigcup_{`P\in S'} X_P\rangle.$$

In particular $\langle\bigcup_{`P\in S'}\langle X_P\rangle\rangle$ does not depend on the choice of the associated structure for $\{\langle X_P\rangle\}_{`P\in S'}$ (cf. Definition 2.6).

Proof. Let $X \subseteq S \times \mathbf{P}^n$ be the subscheme defining the family $\{X_P\}_{`P\in S'}$, let $\pi^S$ and $\pi^{\mathbf{P}^n}$ be the projections of $S \times \mathbf{P}^n$ and let $Y = \overline{\bigcup_{`P\in S'} X_P} \subseteq \mathbf{P}^n$. In order to define an associated structure λ for $\{\langle X_P\rangle\}_{`P\in S'}$, according to Definition 2.6, we choose a finite open affine covering $\{X_i\}$ of X and define $\lambda : \check{V}\otimes\mathcal{O}_S \cong \pi^S_*\mathcal{O}_{S\times\mathbf{P}^n}(1) \to \mathcal{L} := \oplus_i\pi^S_*\mathcal{O}_{X_i}(1)$.

By Definition 2.10, the first side of the equality in the statement is given by the following morphism φ:

$$\check{V} \longrightarrow H^0(\check{V}\otimes\mathcal{O}_S) \stackrel{H^0(\lambda)}{\longrightarrow} H^0(\mathcal{L}).$$

The second side of our equality is $\langle Y\rangle$. By Remark 2.2, it is given by the morphism $\nu : \check{V} \cong H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(\mathcal{O}_Y(1))$.

The equality will follow if we find an injective morphism $i : H^0(\mathcal{O}_Y(1)) \to H^0(\mathcal{L})$, such that $\varphi = i\circ\nu$. Since Y is the image of X via the projection $\pi^{\mathbf{P}^n}$, we have morphisms $X_i \to Y$, hence morphisms $\mathcal{O}_Y(1) \to \pi_*\mathcal{O}_{X_i}(1)$ of sheaves over $\mathbf{P}^n$ which patch together to give a morphism $\iota : \mathcal{O}_Y(1) \to \oplus_i\pi^{\mathbf{P}^n}_*\mathcal{O}_{X_i}(1)$. This morphism is injective, since the map $X \to Y$ is dominant, and $\{X_i\}$ is an open covering of X. It is easy to check that we can take $i = H^0(\iota)$.

Finally, the first side of the equality is independent of the choice of the structure for the family of subspaces $\{\langle X_P\rangle\}_{`P\in S'}$, because clearly the second side is.□

Proposition 2.14. *Let X be a subscheme of $\mathbf{P}^n$ and let $\{L_P\}_{`P\in X'}$ be a flat family of subspaces of $\mathbf{P}^n$ such that (considering it as a family of subschemes) $P \in L_P$ '$\forall P \in X$'. Fix an integer $i > 0$ and give the family $\{P^i_{L_P}\}_{`P\in X'}$ its obviuos structure. Then*

$$\langle \bigcup_{`P\in X'} P^1_{L_P}\rangle = \langle \bigcup_{`P\in X'} L_P\rangle.$$

PROOF. By Proposition 2.13, it is enough to show that

$$(*) \qquad \langle \bigcup_{`P\in X'} \langle P^1_{L_P}\rangle\rangle = \langle \bigcup_{`P\in X'} L_P\rangle$$

(clearly $\{\langle P^i_{L_P}\rangle\}_{`P\in X'}$, as a set of subspaces, is nothing else than $\{L_P\}_{`P\in X'}$, but we have to be careful with the structures). We can suppose that the structure $\varphi : \check{V}\otimes\mathcal{O}_X \to \mathcal{L}$ is surjective (arguing as in Remark 2.12). Let $L \cong \mathbf{P}(\mathcal{L}) \subseteq X \times \mathbf{P}^n$ be the scheme definining $\{L_P\}_{`P\in X'}$ as a family of subschemes and let $\Delta \subseteq X \times X \subseteq X \times \mathbf{P}^n$ be the diagonal. By our hypotheses $\Delta \subseteq L$, and the structure of $\{P^i_{L_P}\}_{`P\in X'}$ is Δ^i_L which is finite over X, hence affine over X. Then the natural associated structure α for $\{\langle P^i_{L_P}\rangle\}_{`P\in X'}$ is $\check{V}\otimes\mathcal{O}_X \cong \pi_*\mathcal{O}_{X\times P_n}(1) \to \pi_*\mathcal{O}_{\Delta^i_L}(1)$, where π is the projection $X\times\mathbf{P}^n \to X$. Hence the first side of $(*)$ is the space defined by

$$\check{V} \longrightarrow H^0(\check{V}\otimes\mathcal{O}_X) \overset{H^0(\alpha)}{\longrightarrow} H^0(\mathcal{O}_{\Delta^i_L}(1)).$$

The second side of $(*)$ is the space defined by

$$\check{V} \longrightarrow H^0(\check{V}\otimes\mathcal{O}_X) \cong H^0(\mathcal{O}_{X\times P_n}(1)) \overset{H^0(\varphi)}{\longrightarrow} H^0(\mathcal{L}) \cong H^0(\mathcal{O}_L(1))$$

(cf. Remark 2.12). It is enough to show that the natural morphism $\nu : \mathcal{O}_L(1) \to \mathcal{O}_{\Delta^i_L}(1)$ is injective (in fact the morphism defining the first side is the composition of $H^0(\nu)$ with the morphism defining the second side). The question is local on X, hence we can replace X by Spec A, where A is a noetherian *local* ring. Then we can suppose $L = \text{Proj } A[x_0,\dots,x_d]$ (where d is the dimensions of the spaces L_P), and Δ defined by an ideal I in $A[x_0,\dots,x_d]$, which is the kernel of a suitable graded homomorphism $h : A[x_0,\dots,x_d] \to A[x]$ of A-algebras. As such, h is completely determined in degree 1, i.e. by the homomorphism h_1 of the free A-modules generated by $x_0,\dots,x_d$ and x respectively. Since A is local, the domain of h_1 splits into a direct sum of I_1 (the homogeneous part of degree 1 of I) and a rank 1 submodule generated by a linear form with at least one invertible coefficient. Then, up to an invertible change of variables, we can suppose I generated by $x_1,\dots,x_d$. But the kernel of ν is given by the degree 1 homogeneous part of the saturation of I^i, which clearly vanishes.□

Intersection.

DEFINITION 2.15. Let $\{L_P\}_{`P\in S'}$ be a family of subspaces defined by $\lambda : \check{V}\otimes\mathcal{O}_S \to \mathcal{L}$. Let $\check{\lambda} : \check{\mathcal{L}} \to V\otimes\mathcal{O}_S$ be the transposed of λ, and let $\omega : V \to H^0(V\otimes\mathcal{O}_S)$ be the natural morphism. Starting from $H^0(\check{\lambda})$ and ω construct the pull back diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{p_2} & H^0(\check{\mathcal{L}}) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle H^0(\check{\lambda})} \\ V & \xrightarrow{\omega} & H^0(V\otimes\mathcal{O}_S) \end{array} .$$

The subspace of $\mathbf{P}^n$ defined by $\check{p}_1$, i.e. the image of $\mathbf{P}(\check{p}_1) : \mathbf{P}(\check{\mathcal{G}}) \to \mathbf{P}^n$, will be denoted by $\bigcap`_{P\in S}' L_P$.

The above definition agrees with the naïve notion of $\langle \bigcap_{P\in S} L_P \rangle$ when S is a variety (hence reduced) and the family is good.

3. - Some classic definitions revisited.

Definition 3.1. Let X be a subscheme of $\mathbf{P}^n$, let P be a closed point of X, let i be a nonnegative integer, and let P^i_X be the i-th infinitesimal neighborhood of P in X. The space $\langle P^i_X \rangle$ is called *the i th order osculating space to X at P*, and it is denoted by $T^i(X,P)$, or simply by T^i_P. The family $\{P^i_X\}`_{P\in X}'$ has a natural structure of a family of subschemes of $\mathbf{P}^n$, given by $\Delta^i_{X\times X} \subseteq X \times X \subseteq X \times \mathbf{P}^n$, where Δ is the diagonal subscheme of $X \times X$, and $\Delta^i_{X\times X}$ is its i-th infinitesimal neighborhood. By Definition 2.6 $\{T^i_P\}`_{P\in X}'$ is in a natural way a family of subspaces. We denote its structure by $\tau^i_X : \check{V} \otimes \mathcal{O}_X \to \mathcal{T}^i_X$. The associated morphism of the appropriate open subscheme of X into the appropriate grassmannian (cf. Remark 1.7) will be called *i-th Gauss map*, and denoted by $t^i : X^{(i)} \to \mathbf{G}$.

Remark 3.2. The sheaf $\mathcal{T}^i_X$ introduced above is the sheaf $P^i(\mathcal{O}_X(1))$ of i-th order principal parts (or i-jets) of $\mathcal{O}_X(1)$ (over Spec k; cf. [Pe, Appendix A, Definition 1.2]). In fact, since any infinitesimal neighborhood of the diagonal Δ is finite over X, hence affine over X, then the structure τ^i_X is the morphism

$$\check{V} \otimes \mathcal{O}_S \cong (\pi_X)_* \mathcal{O}_{X\times \mathbf{P}^n}(1) \to (\pi_X)_* \mathcal{O}_{\Delta^i_{X\times X}}(1).$$

Thus $\mathcal{T}^i_X = (\pi_X)_* \mathcal{O}_{\Delta^i_{X\times X}}(1)$. Following the notation of [Pe, Appendix A, Section A1], where we put $S = \text{Spec } k$, the first projection $\pi_1 : X \times X \to X$ is the restriction of $\pi_X : X \times \mathbf{P}^n \to X$, hence p is the restriction of π_X to the subscheme $\Delta^i_{X\times X}$ (which in [Pe] is denoted by $\Delta_{(i)}$). Moreover, looking at the diagram

$$\begin{array}{ccccc} \Delta^i_{X\times X} & \xrightarrow{\iota} & X \times X & \xrightarrow{\pi_2} & X \\ & & \downarrow & & \downarrow \\ & & X \times \mathbf{P}^n & \longrightarrow & \mathbf{P}^n \end{array},$$

$\mathcal{O}_{\Delta^i_{X\times X}}(1)$ (as a sheaf on $\Delta^i_{X\times X}$) is the inverse image of $\mathcal{O}_{\mathbf{P}^n}(1)$ on $\Delta^i_{X\times X}$ and $\mathcal{O}_X(1)$ is the inverse image of $\mathcal{O}_{\mathbf{P}^n}(1)$ on X, then $\mathcal{O}_{\Delta^i_{X\times X}}(1) = q^*\mathcal{O}_X(1)$, where $q = \pi_2 \circ \iota$. Hence (taking in account the abuse of notation $\mathcal{O}_X(1) = \iota_* \mathcal{O}_X(1)$) we have $\mathcal{T}^i_X = (\pi_1)_* \mathcal{O}_{\Delta^i_{X\times X}}(1) = (\pi_1)_* \iota_* \mathcal{O}_{\Delta^i_{X\times X}}(1) = p_* q^* \mathcal{O}_X(1) = P^i(\mathcal{O}_X(1))$ as required.

Definition 2.15 allows us to give in a rigorous form some classic definitions about i-th order characteristic space (i.e. the intersection of the spaces of the family which are 'i-th order infinitely near' to a given one) and i-th order foci.

DEFINITION 3.3. Let $\{L_P\}_{`P\in S'}$ be a family of subspaces. If $P \in S$ is a closed point, and $P^i$ is its $i$-th infinitesimal neighborhood, the space $\bigcap_{`Q\in P^i'} L_Q$ is called the *i-th order characteristic space* in L_P of the family. The *i-th order foci* in L_P of the family are the closed points $Q \in L_P$ such that there exists a curve $C \subseteq S$ passing smoothly through P and $Q \in \bigcap_{`R\in P^i_C'} L_R$. In other words, Q is an i-th order focus if it lies on the i-th order characteristic space in L_P of a one-dimensional subfamily which contains L_P 'regularly'.

REMARK 3.4. Our definition of 'foci', like that one of 'Gauss map', is a natural extension of the classic ones (cf. [CS]). However, modern terminology about them is not standard at all, since these notion can be extended in many ways.

EXAMPLE 3.5. Consider the nondegenerate conic $C : x_0x_1 - x_2^2$ in $\mathbf{P}^2$, over a field k of characteristic $\neq 2$. We can assume $\check{V} = \langle x_0, x_1, x_2\rangle_k$. The family of the tangents $\{T^1_P\}_{`P\in C'}$ is given by the morphism $\varphi : \check{V} \otimes \mathcal{O}_C \to \mathcal{T}^1_C$. Let us calculate the first order characteristic space in T^1_O where $O = [1, 0, 0]$. Parametrizing an affine open U of C by Spec $k[t]$, with

$$x_0 \mapsto 1, \qquad x_1 \mapsto t^2, \qquad x_2 \mapsto t,$$

over U we have that $\mathcal{T}^1_U$ is the sheaf associated to the $k[t_1]$-module $\frac{k[t_1,t_2]}{(t_1-t_2)^2}$ and $\varphi|_U$: $\check{V} \otimes \mathcal{O}_U \to \mathcal{T}^1_U$ corresponds to the homomorphism $\check{V} \otimes k[t_1] \to \frac{k[t_1,t_2]}{(t_1-t_2)^2}$ given by

$$x_0 \mapsto 1, \qquad x_1 \mapsto t_2^2, \qquad x_2 \mapsto t_2.$$

Pulling-back over $O^1_C \cong$ Spec $k[\epsilon]$, with $\epsilon^2 = 0$, we get the homomorphism of $k[\epsilon]$-modules

$$\check{V} \otimes k[\epsilon] \to \frac{k[\epsilon, t_2]}{(\epsilon - t_2)^2} = k[\epsilon] \oplus (k[\epsilon] \cdot \bar{t_2})$$

given by

$$x_0 \mapsto 1, \qquad x_1 \mapsto 2\epsilon\bar{t_2} (= \bar{t_2}^2), \qquad x_2 \mapsto \bar{t_2}.$$

Now, according to Definition 2.15, dualizing over $k[\epsilon]$ construct the diagram

$$\begin{array}{ccc} \mathcal{G} & \to & k[\epsilon] \oplus k[\epsilon] \\ \Big\downarrow p & & \Big\downarrow \alpha \\ V & \to & V \otimes k[\epsilon] \end{array} .$$

The matrix of the homomorphism α (of $k[\epsilon]$-modules) in the above diagram, with respect to the dual bases of $(1, \bar{t_2})$ and (x_0, x_1, x_2) respectively is

$$\begin{pmatrix} 1 & 0 \\ 0 & 2\epsilon \\ 0 & 1 \end{pmatrix},$$

hence $\mathcal{G} = \alpha^{-1}(V) = k$ and p is defined by $c \mapsto c \cdot (1,0,0)$. Thus $\check{p}$ has kernel $\langle x_1, x_2 \rangle$, then the characteristic space is just the point O, as one could intuitively expect.

Definition 3.6. Let $\{L_P\}_{`P\in C'}$ be a flat family of subspaces of dimension $d$, where $C$ is a curve and such that the generic space of the family $\{\langle \bigcup_{`Q\in P^1_C'} L_Q \rangle\}_{`P\in C'}$ (cf. Remark 2.11) has dimension $d+1$. Thinking $\{L_P\}_{`P\in C'}$ as a family of subschemes, consider $\overline{\bigcup_{`P\in C'} L_P}$. Any nonempty open subvariety of $\overline{\bigcup_{`P\in C'} L_P}$ is called an *ordinary developable* variety.

Definition 3.7. Let $\{L_P\}_{`P\in S'}$ be a flat family of lines, where $S$ is a surface, such that the generic space of $\{\langle \bigcup_{`Q\in P^1_S'} L_Q \rangle\}_{`P\in S'}$ has dimension 3. In this situation, the family $\{L_P\}_{`P\in S'}$ is called a *Laplace congruency*; we shall also call Laplace congruency any nonempty open subvariety of $\overline{\bigcup_{`P\in S'} L_P}$.

Remark 3.8. In the above definition we used the term 'Laplace congruency' following [C, pag. 349]. As pointed out there, the term was introduced first for a more restricted class of congruencies, arising in the theory of Laplace equations (hence the name).

4. - Main results.

Proposition 4.1. *Let $\{X_P\}_{`P\in S'}$ be a family of subschemes of $\mathbf{P}^n$ defined by a closed subscheme $X \subseteq S \times \mathbf{P}^n$ and let Y be a subscheme of $\mathbf{P}^n$. If the family is flat (i.e. X is flat over S) and S is a variety then*

$$X_P \subseteq Y \ \forall P \text{ closed point of } S \Rightarrow X_P \subseteq Y \ `\forall P \in S\,'.$$

Proof. We have to show that $X \subseteq S \times Y$. We can suppose Y closed. In fact, if $\bar{Y}$ is the closure of Y and $X \subseteq S \times \bar{Y}$, if X were not in the open subscheme $S \times Y$ of $S \times \bar{Y}$, then there would be a closed point of X lying out of $S \times Y$, and then some X_P would not be contained in Y.

Let $X' = X \cap (S \times Y)$, so that X' is a closed subscheme of X. Let $\mathcal{I}$ be the ideal sheaf of X' in X, let P be a closed point of X and let Q be its projection in S. Consider the exact sequence

$$0 \to \mathcal{I}_P \to \mathcal{O}_{X,P} \to \mathcal{O}_{X',P} \to 0.$$

The closed fibres of X and X' over S are the same and since X is flat, their Hilbert polynomial is the same. Hence X' is flat over S, and in particular $\mathcal{O}_{X',P}$ is a flat $\mathcal{O}_{S,Q}$-module. Then, if k_Q is the residue field of Q, tensorizing the above sequence by k_Q over $\mathcal{O}_{S,Q}$ we get the exact sequence

$$0 \to \mathcal{I}_P \otimes_{\mathcal{O}_{S,Q}} k_Q \to \mathcal{O}_{X_Q,P} \to \mathcal{O}_{X'_Q,P} \to 0.$$

Since $X_Q = X'_Q$, we have $\mathcal{I}_P \otimes_{\mathcal{O}_{S,Q}} k_Q = 0$. By Nakayama's lemma $\mathcal{I}_P = 0$. This holds for any closed point P of X, hence $\mathcal{I} = 0$. Thus $X = X' = X \cap (S \times Y)$, then $X \subseteq S \times Y$.$\square$

When $\{L_P\}_{`P \in S'}$ is a flat family of subspaces, and $l : S \to \mathbf{G}$ is the associated morphism into the grassmannian, two closed points $P, Q \in S$ lies on the same fibre of l if and only if $L_P = L_Q$. Unfortunately this condition cannot be used when P and Q are 'infinitely near'. More precisely, we shall need a criterion to know when a nonreduced subscheme of S lies on a fibre of l. A good idea is to write the condition $L_P = L_Q$ in the equivalent form $\langle L_P \cup L_Q \rangle = L_P$ (notice in fact that L_P and L_Q have the same dimension) or even $L_P \cap L_Q = L_P$. In fact we state the following proposition.

Proposition 4.2. *Let $\{L_P\}_{`P \in S'}$ be a good family of subspaces, defined by $\lambda : \check{V} \otimes \mathcal{O}_S \to \mathcal{L}$, let $l : S \to \mathbf{G}$ be the associated morphism into the suitable grassmannian, and let L be a space of the family. Then the following conditions are equivalent:*

(a) $\langle \bigcup_{`P \in S'} L_P \rangle = L$.

(b) $\bigcap_{`P \in S'} L_P = L$.

(c) *l is a constant map, on the point corresponding to L.*

Proof. Up to a restriction on the image, we can suppose λ surjective. Let $i : \{L\} \cong \operatorname{Spec} k \to \mathbf{G}$ be the immersion of the point of $\mathbf{G}$ corresponding to L and let $\pi : S \to \{L\}$ the constant morphism (i.e. the structural morphism of S over k). Let W be the vector subspace of V corresponding to L (cf. Remark 1.7), and let $\varphi : \check{V} \to \check{W}$ be the transposed of the immersion of W in V. We show that (a), (b), (c) are all equivalent to the following fact:

(d) $\pi^*\varphi = \lambda$, up to an isomorphism $\pi^*\check{W} \cong \mathcal{L}$.

(a)$\Rightarrow$(d). Let $\lambda_\pi : \check{V} \to \pi_*\mathcal{L}$ be the morphism of vector spaces corresponding to λ via π. Condition (a), by Definition 2.10, means that $\mathbf{P}(\lambda_\pi)$ has image L, therefore it factors through the immersion of L in $\mathbf{P}^n$, which is $\mathbf{P}(\varphi)$; then λ_π factors through φ. Pulling back on S and composing with $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ we get $\lambda = \psi \circ \pi^*\varphi$. Now $\psi : \pi^*W \to \mathcal{L}$ must be surjective since λ is surjective, and π^*W and $\mathcal{L}$ have the same rank, hence ψ is an isomorphism.

(d)$\Rightarrow$(a). The morphism of vector spaces on $\{L\} = \operatorname{Spec} k$ corresponding to $\pi^*\varphi$ is the composition $\check{V} \to \pi_*\pi^*\check{V} \to \pi_*\pi^*\check{W}$, which is equal to $\check{V} \xrightarrow{\varphi} \check{W} \to \pi_*\pi^*\check{W}$. Hence the linear projection defining $\langle \bigcup_{`P \in S'} L_P \rangle$ factors through the immersion of $L$, and since $\check{W} \to \pi_*\pi^*\check{W}$ is injective, its image must be equal to $L$, hence $\langle \bigcup_{`P \in S'} L_P \rangle = L$ as required.

(b)$\Rightarrow$(d). Looking at Definition 2.15, let us construct the pull-back diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{p_2} & H^0(\check{\mathcal{L}}) \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle H^0(\check{\lambda})} \\ V & \xrightarrow{\omega} & \pi_*(V \otimes \mathcal{O}_S) \end{array} .$$

The space $\bigcap_{`P \in S'} L_P$ is the image of $\mathbf{P}(\check{p}_1)$, so (b) implies that this map factors through the immersion of L, via a surjective linear projection $\mathbf{P}(\check{\mathcal{M}}) \to L$; we can get then a factorization of p_1 through the immersion $\check{\varphi}$ of W in V, via a surjective homomorphism $\mathcal{M} \to W$. Choosing a right inverse $\psi : W \to \mathcal{M}$ of this homomorphism, we have a factorization $\check{\varphi} = p_1 \circ \psi$. Composing with the preceding pull-back diagram, we have the following commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\check{\varphi}"] & H^0(\check{\mathcal{L}}) \arrow[d, "H^0(\check{\lambda})"] \\ V \arrow[r, "\omega"] & \pi_*(V \otimes \mathcal{O}_S) \end{tikzcd} .$$

Applying π^* and taking in account the adjointness of π^* and π_*, we get a factorization of $\pi^*\check{\varphi}$ through $\check{\lambda}$, and then a factorization of $\pi^*\varphi$ through λ. Now we can conclude like at the end of the proof of (a)$\Rightarrow$(d).

(d)$\Rightarrow$(b). It is enough to notice that the natural diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\check{\varphi}"] & W \otimes \mathcal{O}_S \arrow[d, "\check{\varphi}\otimes \mathrm{id}_{\mathcal{O}_S}"] \\ V \arrow[r] & V \otimes \mathcal{O}_S \end{tikzcd}$$

is exactly the pull-back diagram required by Definition 2.15, so $\bigcap_{`P\in S'} L_P$ is defined by $\mathbf{P}(\varphi)$ (as a single subspace), which is nothing else the immersion of L.

(c) $\Longleftrightarrow$ (d). Let q be the universal quotient over $\mathbf{G}$. By definition of l and i, $\lambda = l^*q$ and $\varphi = i^*q$. So (d) means that $l^*q = \pi^*i^*q$, which by the universal property of $\mathbf{G}$ (Proposition 1.6) is equivalent to $l = i \circ \pi$, i.e. (c).$\square$

Proposition 4.3. *Let X be a quasi-projective variety, let P be a closed point of X, and let i, j be nonnegative integers. Then we have*

$$\overline{\bigcup_{`Q\in P^i\,'} Q^j} \subseteq P^{i+j} \quad .$$

Proof. Restricting to an affine open neighborhood of P, we can assume $X = \mathrm{Spec}\ A$, and let $\mathfrak{m}$ be the ideal of P. The family $\{Q^j_X\}_{`Q\in X'}$ is defined by the j-th infinitesimal neighborhood of the diagonal subscheme $D \subseteq X \times X = \mathrm{Spec}\ (A \otimes A)$ and recall that the ideal $\mathfrak{p}$ of D is the kernel of the multiplication map $A \otimes A \to A$, and it is generated by the elements of the form $1 \otimes a - a \otimes 1$. The scheme $\overline{\bigcup_{`Q\in P^i_X\,'} Q^j_X}$ is obtained restricting the family $\{Q^j_X\}_{`Q\in X'}$ over P^i (just changing base over the first factor X by the immersion of P^i) and then taking the image via the second projection on X. Hence we first take the tensor product

$$B = \frac{A}{\mathfrak{m}^{i+1}} \otimes_A \frac{A \otimes A}{\mathfrak{p}^{j+1}},$$

where the structure of A-algebra on $\frac{A\otimes A}{\mathfrak{p}^{j+1}}$ is given by the first factor, then we take the kernel of the map $A \to B$ defined starting from the second immersion $A \to A \otimes A$: it is enough to prove that $\mathfrak{m}^{i+j+1}$ is contained in this kernel. Notice that B is the quotient of $A \otimes A$ over the ideal $\mathfrak{q}$, generated by $\mathfrak{p}^{j+1}$ and by the elements which are products of $i+1$ elements of the form $m \otimes 1$, with $m \in \mathfrak{m}$. Now it is enough to prove that the elements which are products of $i+j+1$ elements of the form $1 \otimes m$, with $m \in \mathfrak{m}$, are in $\mathfrak{q}$. Write $1 \otimes m = m \otimes 1 + (1 \otimes m - m \otimes 1)$ and observe that every 'monomial' of a product of $i+j+1$ elements of this form must contain either a product of $j+1$ elements of the form $(1 \otimes m - m \otimes 1)$ or a product of $i+1$ elements of the form $m \otimes 1$, hence belongs to $\mathfrak{q}$, as required.□

Corollary 4.4. *Let X be a quasi-projective variety, and let i, j be nonnegative integers. The following inclusion holds:*

$$\langle \bigcup_{\text{`}Q \in P^i\text{'}} T^j \rangle \subseteq T^{i+j}.$$

Proof. By Proposition 4.3, we have $\overline{\bigcup_{\text{`}Q \in P^i_X\text{'}} Q^j_X} \subseteq P^{i+j}_X$. Now it is enough to take the subspaces spanned by each members of the inclusion, and apply Proposition 2.13.□

An 'inversion formula'.

Let $\{X_P\}_{\text{`}P\in S\text{'}}$ be a family of subschemes of a scheme $Y$ such that $Y = \overline{\bigcup_{\text{`}P\in S\text{'}} X_P}$ and let P be a closed point of S. Consider the schemes

$$\overline{\bigcup_{\text{`}Q \in X_P\text{'}} Q^i_Y} \qquad \text{and} \qquad \overline{\bigcup_{\text{`}Q \in P^i_S\text{'}} X_Q}.$$

It is natural to conjecture that they are equal. Moreover, one could think that they are nothing else than the i-th neighborhood of X_P in Y. This is true in many cases, but sometimes this neighborhood is 'thicker' than the two schemes above, as shown by the following example.

Example 4.5. Let $S = Y = \mathbf{P}^1$, and $X_P = P^1_Y$ '$\forall P \in Y$'. Obviously $Y = \overline{\bigcup_{\text{`}P\in S\text{'}} X_P}$. It is easy to see that for any closed point $P \in Y$, the two schemes $\overline{\bigcup_{\text{`}Q\in X_P\text{'}} Q^1_Y}$ and $\overline{\bigcup_{\text{`}Q\in P^1_S\text{'}} X_Q}$ are equal to P^2_Y (cf. also Proposition 4.3), while the first neighborhood of X_P in Y is P^3_Y.

Now we give some examples which show that there are exceptions to the equality of the schemes $\overline{\bigcup_{\text{`}Q\in X_P\text{'}} Q^i_Y}$ and $\overline{\bigcup_{\text{`}Q\in P^i_S\text{'}} X_Q}$.

Example 4.6. (a) Let $S = \mathbf{P}^1$, let Y be a (reduced, irreducible) quadric cone and let $\{X_P\}_{\text{`}P\in S\text{'}}$ the family of its lines. If $P$ is any closed point of $S$, then $\overline{\bigcup_{\text{`}Q\in X_P\text{'}} Q^1_Y} \supseteq \overline{\bigcup_{\text{`}Q\in P^1_S\text{'}} X_Q}$, but they are not equal, since the first one has an embedded component at the vertex of Y. However, notice that they are equal apart from the vertex.

(b) Let $X \subset \mathbf{A}^1 \times \mathbf{A}^1$ be the graph of the morphism $F : \mathbf{A}^1 \to \mathbf{A}^1$ defined by $y = x^2$ (i.e. X is the subscheme defined by the map $\mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ which gives the identity on the first factor and F on the second). If $S = \mathbf{A}^1$ is the first factor, X defines a family $\{X_P\}_{`P\in S'}$ of subschemes of $\mathbf{A}^1$. If $P$ is the origin ($x = 0$) of $S = \mathbf{A}^1$ then $\overline{\bigcup_{`Q\in X_P'} Q^1_Y} = X^1_P$ while $\overline{\bigcup_{`Q\in P^1_S'} X_Q} = X_P$ (and $X_P = F(P) = P$). Again the first scheme properly contains the second one. Notice that if the characteristic of k is 2 (so that F is the Frobenius morphism) this holds for any closed point $P \in S$.

Example 4.7. Exchange the factors in Example 4.6b (i.e. now X is defined by the map $\mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ which gives F on the first factor and the identity on the second) and let S be again the first factor. If P is the origin of $S = \mathbf{A}^1$ then $\overline{\bigcup_{`Q\in X_P'} Q^1_Y} = P^2$ while $\overline{\bigcup_{`Q\in P^1_S'} X_Q} = P^3$ (and $X_P = P^1$). In this case the first scheme is properly contained in the second one.

The examples above allow us to conjecture that the equality

$$\overline{\bigcup_{`Q\in X_P'} Q^i_Y} = \overline{\bigcup_{`Q\in P^i_S'} X_Q}$$

holds over an open subfamily of $\{X_P\}_{`P\in S'}$, which may be empty only in special cases in positive characteristic. We prove now two useful proposition about the two inclusions. The first one requires the hypothesis of regularity of $X_P$, and holds even if $Y$ is not equal to $\overline{\bigcup_{`P\in S'} X_P}$.

Proposition 4.8. *Let $\{X_P\}_{`P\in S'}$ be a family of subschemes of a quasi-projective scheme Y, let $P \in S$ be such that X_P is smooth and let i be a nonnegative integer. Then*

$$\overline{\bigcup_{`Q\in X_P'} Q^i_Y} \supseteq \overline{\bigcup_{`Q\in P^i_S'} X_Q}.$$

Proof. We can replace S by the affine scheme $P^i_S \cong \operatorname{Spec} A_S$, where $A_S = \frac{\mathcal{O}_{P,S}}{\mathfrak{m}^{i+1}_{P,S}}$. Restricting to an open set of Y in which X_P is closed, and considering an open affine cover of it, we reduce to the case Y affine, $\cong \operatorname{Spec} A_Y$ and X (the scheme defining the family) closed in $S \times Y$. Hence X is affine, say $\cong \operatorname{Spec} A_X$. Let $\mathfrak{m}$ be the ideal of X_P in A_X, which is the extension in A_X of the maximal ideal of A_S, hence $\mathfrak{m}^{i+1} = (0)$. The ideal of $\overline{\bigcup_{`Q\in P^i_S'} X_Q}$ is the kernel $\mathfrak{k}$ of the homomorphism $A_Y \to A_X$ and the ideal of $X_P$ in $Y$ is the contraction $\mathfrak{m}_Y$ of $\mathfrak{m}$ via the same homomorphism. Now we have to prove that $\mathfrak{k}$ contains the ideal $\mathfrak{h}$ of $\overline{\bigcup_{`Q\in X_P'} Q^i_Y}$, and this one is obtained in the following way. Let $\mathfrak{p}$ be the extension in $A_Y/\mathfrak{m}_Y \otimes A_Y$ of the kernel of the multiplication map $A_Y \otimes A_Y \to A_Y$. Then $\mathfrak{h}$ is the contraction of $\mathfrak{p}^{i+1}$ via the second factor immersion of A_Y in $A_Y/\mathfrak{m}_Y \otimes A_Y$. Let us look at the commutative diagram

$$\begin{array}{ccc} A_Y & \longrightarrow & A_X \\ \big\downarrow & & \big\downarrow \\ \frac{A_Y}{\mathfrak{m}_Y} \otimes A_Y & \longrightarrow & \frac{A_X}{\mathfrak{m}} \otimes A_X \end{array}$$

(where the vertical arrows are immersions of the second factor), and consider the ideal $\mathfrak{q}$ of $A_X/\mathfrak{m} \otimes A_X$, extension of the kernel of the multiplication map $A_X \otimes A_X \to A_X$. The contraction of $\mathfrak{q}$ in $A_Y/\mathfrak{m}_Y \otimes A_Y$ is $\mathfrak{p}$. In fact, $A_X/\mathfrak{m} \cong A_Y/\mathfrak{m}_Y$ (since they are both the coordinate ring of X_P), and notice that the two ideals can be obtained as the contraction of the kernel of the multiplication map $A_X/\mathfrak{m} \otimes A_X/\mathfrak{m} \to A_X/\mathfrak{m}$. Hence the contraction of $\mathfrak{q}^{i+1}$ contains $\mathfrak{p}^{i+1}$. Then $\mathfrak{h}$ is contained in the contraction of $\mathfrak{q}^{i+1}$ in A_Y via the above diagram. Now we prove that this one is $\mathfrak{k}$: clearly it is enough to show that the contraction of $\mathfrak{q}^{i+1}$ in A_X is (0). Starting from the identity $A_X/\mathfrak{m} \to A_X/\mathfrak{m}$, since X_P is smooth, we can apply the infinitesimal lifting property [H, II, ex. 8.6] in order to get a homomorphism $A_X/\mathfrak{m} \to A_X/\mathfrak{m}^2$. Iterating, we get a homomorphism $\varphi : A_X/\mathfrak{m} \to A_X/\mathfrak{m}^{i+1} \cong A_X$ such that $A_X/\mathfrak{m} \xrightarrow{\varphi} A_X \to A_X/\mathfrak{m}$ is the identity. From φ and the identity on A_X, we get a homomorphism $\psi : A_X/\mathfrak{m} \otimes A_X \to A_X$ such that the contraction of $\mathfrak{m}$ is $\mathfrak{q}$. Since $\mathfrak{m}^{i+1} = (0)$, the ideal $\mathfrak{q}^{i+1}$ is contained in the kernel of ψ. But the composition $A_X \longrightarrow A_X/\mathfrak{m} \otimes A_X \xrightarrow{\psi} A_X$ is the identity, hence the contraction of $\mathfrak{q}^{i+1}$ in A_X is (0), as required.□

Let us now consider the reverse inclusion. Example 4.6 indicates that it should be true on an open subfamily of the assigned one. Unfortunately this subfamily may be empty in some special situations in positive characteristic or in some nonreduced cases. The inclusion $\overline{\bigcup`_{Q \in X_P}' Q_Y^i} \subseteq \overline{\bigcup`_{Q \in P_S^i}' X_Q}$ is useful in nonreduced situation: we shall see an example in our proof of theorem 5.3 in the next section. Hence we have to find some good conditions under which the subfamily is nonempty. The Proposition 4.11 below will be enough in the proofs of the next section. We need to state a preliminary proposition (4.10), which is interesting in its own.

Remark 4.9. In the next proposition we shall adopt the definition of separability of [Ma, Section 26, pag. 198], which is a generalization of the usual one, and allows to speak about separability of any (not necessarily algebraic) extension of a field. Then we can say that a dominant morphism $X \to Y$ of quasi-projective varieties is *separable* if $K(Y)$ is separable over $K(X)$. By [Ma, theorem 26.1], separability is automatic in characteristic 0.

Proposition 4.10. *Let f be a dominant morphism from a quasi-projective irreducible scheme X to a quasi-projective variety Y, let $\mathcal{I}$ be the ideal sheaf of X_{red} in X (i.e. the nilradical of $\mathcal{O}_X$) and let i be a nonnegative integer. If $f|_{X_{\mathrm{red}}}$ is separable and $\mathcal{I}^2 = 0$ then there is a nonempty open subscheme U of X such that for any subscheme Z of U such that $f|_Z$ is an immersion, the scheme-theoretic image of $\overline{\bigcup`_{P \in Z}' P_X^i}$ in $Y$ is $\overline{\bigcup`_{P \in Z}' P_Y^i}$ (where we consider Z also as a subscheme of Y, via $f|_Z$).*

Proof. Let L be the local ring at the generic point of X and K be the function field of Y. Since $f|_{X_{\mathrm{red}}}$ is separable, L_{red} is a separable extension of K, hence is 0-smooth over K by [Ma, Section 26, theorem 26.9]. Thus, we can lift the identity of L_{red} to a morphism $L_{\mathrm{red}} \to L$, which gives rise to a morphism ψ of a nonempty open subscheme U of X to U_{red} such that the composition with $f|_{U_{\mathrm{red}}}$ is $f|_U$. Shrinking U if necessary, by [Mu, lecture 8, Proposition pag. 57] we can suppose that $\mathcal{I}$ is flat over U_{red} via the morphism $\psi : U \to U_{\mathrm{red}}$. Moreover, since $\mathcal{I}^2 = 0$, we can suppose (shrinking again if necessary) that

U is a product $U|_{\mathrm{red}} \times T$, with T a 0-dimensional scheme (in general nonreduced). Then $f|_U$ is a composition $U \to U_{\mathrm{red}} \to Y$, hence we can reduce ourselves to prove the statement in the following two cases:

(a) $X = Y \times T$ (and f is the first projection).

(b) X is reduced. Let $G \subseteq X \times Y$ be the graph of f and $\Delta_X \subseteq X_1 \times X_2$, $\Delta_Y \subseteq Y_1 \times Y_2$ be the diagonals, where we set $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$ in order to distinguish the factors. It is enough to show that the scheme-theoretic image of $\Delta^i_X \times_{X_1} Z$ in $G^i \times_X Z$ is the whole $G^i \times_X Z$: in fact, $\overline{\bigcup_{`P\in Z'} P^i_X}$ is the scheme-theoretic image of $\Delta^i_X \times_{X_1} Z$ in $X_2$ and $\overline{\bigcup_{`P\in Z'} P^i_Y}$ is the scheme-theoretic image of $\Delta^i_Y \times_{Y_1} Z$ in Y_2, which is equal to the scheme-theoretic image of $G^i \times_X Z$ in Y.

Let us prove the statement in the case (a). If $X = Y \times T$, then $X \times Y \cong Y \times Y \times T$ and $X_1 \times X_2 \cong Y_1 \times Y_2 \times T \times T$, hence the diagonal morphism $T \to T \times T$ induces a morphism $X \times Y \to X \times X$ such that the composition $X \times Y \to X \times X \to X \times Y$ is the identity of $X \times Y$. Then we get a composition $G^i \to \Delta^i_X \to G^i$ which gives the identity, hence the scheme-theoretic image of $\Delta^i_X \times_{X_1} Z$ in $G^i \times_X Z$ is the whole $G^i \times_X Z$, as required.

Let us prove the statement in the case (b). Let $\mathcal{G}$ and $\mathcal{D}$ be the direct images on X of the sheaves $\mathcal{O}_{G^i}$ and $\mathcal{O}_{\Delta^i_X}$. We have to find U in such a way that the restriction on Z of the morphism $\varphi : \mathcal{G} \to \mathcal{D}$ is injective.

At the generic point η of X, the morphism φ has a left inverse. In fact, if K and L are as above, looking at the map $(K \otimes L)/I^{i+1} \to L$, where I is the kernel of $K \otimes L \to L$, and using again the 0-smoothness of L over K, we can lift the identity of L to a homomorphism $L \to (K \otimes L)/I^{i+1}$ which, together with $L \to K \otimes L \to (K \otimes L)/I^{i+1}$ induces a homomorphism $(L \otimes L) \to (K \otimes L)/I^{i+1}$. This homomorphism sends the kernel J of $L \otimes L \to L$ to I/I^{i+1}. Thus we get a homomorphism $(L \otimes L)/J^{i+1} \to (K \otimes L)/I^{i+1}$, which is (up to natural isomorphisms) a left inverse of φ_η. Now it is enough to take U as a nonempty open subscheme over which this left inverse is defined.□

Proposition 4.11. *Let $Y \subseteq \mathbf{P}^n$ be a (quasi-projective) variety, let i be a nonnegative integer and let $\{X'_P\}_{`P\in S'}$ be a family of subschemes of $Y$ defined by an irreducible subscheme $X' \subseteq S \times Y$, such that $\overline{\bigcup_{`P\in S'} X'_P} = Y$. If the restriction of the morphism $X' \to Y$ to X'_{red} is separable (in particular if char $k = 0$) and the square of the ideal sheaf of X'_{red} in X' is 0, then there is a nonempty open subfamily $\{X_P\}_{`P\in S'}$ of $\{X'_P\}_{`P\in S'}$ such that for any closed point $P \in S$*

$$\overline{\bigcup_{`Q\in X_P'} Q^i_Y} \subseteq \overline{\bigcup_{`Q\in P^i_S'} X_Q}.$$

Proof. It is enough to take the family $\{X_P\}_{`P\in S'}$ be defined by a nonempty open subscheme $X$ of the scheme $X'$ as in the statement of Proposition 4.10. In fact, $\overline{\bigcup_{`Q\in P^i_S'} X_Q}$ is the scheme-theoretic image in Y of the i-th infinitesimal neighborhood of X_P, which contains $\overline{\bigcup_{`Q\in X_P'} Q^i_X}$, whose scheme-theoretic image in $Y$ is $\overline{\bigcup_{`Q\in X_P'} Q^i_Y}$, by Proposition 4.10.□

5. - Some classic results revisited

We begin this section giving a new proof of a classic result, using the framework developed in the previous sections. We state before a simple proposition and a technical lemma.

PROPOSITION 5.1. *Let P be a closed point of a variety X in $\mathbf{P}^n$. Then the first neighborhood P^1_X of P in X is equal to the first neighborhood $P^1_{T^1_P}$ of P in the tangent space T^1_P.*

PROOF. It is enough to notice that P^1_X is contained in T^1_P, hence is contained in $P^1_{T^1_P}$, but P^1_X and $P^1_{T^1_P}$ have clearly coordinate rings of equal length.□

LEMMA 5.2. *Let X be a variety of dimension d in $\mathbf{P}^n$, over a field k of characteristic 0, let $t^1 : X^{(1)} \to \mathbf{G} := \mathbf{G}(V, d+1)$ be the first Gauss map. For any closed point $P \in X^{(1)}$ let F_P be the fibre of t^1 containing P and let $L_P = \langle F_P \rangle$. Then there is a nonempty open subset U of X, such that*

(a) $\{F_P\}_{P\in U}$ *is a family of smooth subschemes of* $\mathbf{P}^n$;

(b) *there is an associated structure of good family of subspaces for* $\{L_P\}_{P\in U}$;

(c) *for any closed point* $P \in U$, $\langle \bigcup_{Q\in P^1_{L_P}} Q^1_X \rangle \subseteq \langle \bigcup_{Q\in P^1_X} Q^1_{L_Q} \rangle$.

PROOF. There is no loss of generality if we suppose $X = X^{(1)}$. Let Y be the scheme-theoretic image of X in $\mathbf{G}$ via t^1, which is a variety since X is a variety. The restriction of t^1 on its image Y defines a family $\{F_y\}_{y\in Y}$ (whose members are fibres of t^1). It is obviously a family of subschemes of $\mathbf{P}^n$, since X can be embedded in $Y \times X$ as the graph of $X \to Y$, hence in $Y \times \mathbf{P}^n$. Then we can apply Definition 2.6 and get a structure for the family $\{\langle F_y \rangle\}_{y\in Y}$. By corollary 2.5 there is a nonempty open subset U_1 of Y over which the family $\{\langle F_y \rangle\}_{y\in U_1}$ is good. By generic smoothness, there is a nonempty open subset U_2 such that F_y is smooth for any closed $y \in U_2$. Let U' be the preimage of $U_1 \cap U_2$ and define the families $\{F_P\}_{P\in U'}$ and $\{L_P\}_{P\in U'}$ as the pull-backs on U of $\{F_y\}_{y\in Y}$ and $\{\langle F_y \rangle\}_{y\in Y}$ respectively. The family $\{L_P\}_{P\in U'}$ is good since the pull-back of a good family is good, and its structure is (up to natural isomorphisms) associated to $\{F_P\}_{P\in U'}$, as already pointed out at the end of Definition 2.6. Let $L \subseteq U' \times \mathbf{P}^n$ be the scheme defining $\{L_P\}_{P\in U'}$ as a family of subschemes and let $\Delta \subseteq U' \times U' \subseteq U' \times \mathbf{P}^n$ be the diagonal. By Proposition 4.1, $\Delta \subseteq L$ and the family $\{P^1_{L_P}\}_{P\in U'}$ is defined by Δ^1_L. It is a flat family since all closed fibres have Hilbert polynomial $l+1$, where l is the dimension of the subspaces L_P. Moreover, for any closed point $P \in U'$, $P^1_{L_P} \subseteq U'$. In fact, $L_P = \langle F_P \rangle \subseteq T^1_P$, hence $P^1_{L_P} \subseteq P^1_{T^1_P} = P^1_X = P^1_{U'} \subseteq U'$ by Proposition 5.1. Hence by Proposition 4.1 $\{P^1_{L_P}\}_{P\in U'}$ is a family of subschemes of U' and $\bigcup_{P\in U'} P^1_{L_P} = U'$ since U' is a variety and any closed point of U' lies on $\bigcup_{P\in U'} P^1_{L_P}$. But U' is open in X, hence it is also a family of subschemes of X and, as such, $\bigcup_{P\in U'} P^1_{L_P} = U'$. Since the scheme defining the family is Δ^1_L and $\Delta \cong U'$ is reduced, the square of the ideal sheaf of Δ^1_L is zero. Hence the hypotheses of Proposition 4.11 are fulfilled. Then, for $i = 1$, we get a nonempty open subscheme D of Δ_L^1 such that for any closed point P in U',

$$(*) \qquad \overline{\bigcup_{`Q\in D_P'} Q_X^1} \subseteq \overline{\bigcup_{`Q\in P_X^1'} D_Q}.$$

(where we wrote P_X^1 instead of $P_{U'}^1$, since they are clearly the same). Now notice that the map $\Delta_L^1 \to U'$ is an homeomorphism (since $\Delta \to U'$ is an isomorphism), hence D is the preimage of a nonempty open subset U of U', which means $\{D_P\}_{`P\in U'} = \{P_{L_P}^1\}_{`P\in U'}$. Then if P is a closed point of U, $(*)$ become

$$\overline{\bigcup_{`Q\in P_{L_P}^1'} Q_X^1} \subseteq \overline{\bigcup_{`Q\in P_X^1'} Q_{L_Q}^1}.$$

Taking the span, we have $\langle \bigcup_{`Q\in P_{L_P}^1'} Q_X^1\rangle \subseteq \langle \bigcup_{`Q\in P_X^1'} Q_{L_Q}^1\rangle$, as required.$\square$

THEOREM 5.3. *Let X be a variety of dimension d in $\mathbf{P}^n$, over a field k of characteristic 0. If the first Gauss map $t^1 : X^{(1)} \to \mathbf{G} := \mathbf{G}(V, d+1)$ is not birational on the image, then its generic closed fibre is an open subset of a linear space of positive dimension in $\mathbf{P}^n$.*

PROOF. For any closed point $P \in X$ let F_P be the fibre of t^1 containing P and let $L_P = \langle F_P\rangle$. By lemma 5.2, there is a nonempty open subset U of X, such that $\{F_P\}_{`P\in U'}$ is a family of smooth subschemes of $\mathbf{P}^n$, and there is an associated structure of good family of subspaces for $\{L_P\}_{`P\in U'}$. By the hypotheses, t^1 is not birational on the image, hence the dimension l of the subspaces of $\{L_P\}_{`P\in U'}$ is positive. We show now that F_P contains the first infinitesimal neighborhood $P_{L_P}^1$ of P in L_P. According to Proposition 4.2, it is enough to show that

$$\langle \bigcup_{`Q\in P_{L_P}^1'} T_Q^1\rangle = T_P^1.$$

One inclusion is obvious, and the other follows from

$$\langle \bigcup_{`Q\in P_{L_P}^1'} T_Q^1\rangle \overset{2.13}{=} \langle \bigcup_{`Q\in P_{L_P}^1'} Q_X^1\rangle \overset{5.2c}{\subseteq} \langle \bigcup_{`Q\in P_X^1'} Q_{L_Q}^1\rangle \overset{2.14}{=} \langle \bigcup_{`Q\in P_X^1'} L_Q\rangle \overset{2.13}{=}$$

$$\overset{2.13}{=} \langle \bigcup_{`Q\in P_X^1'} F_Q\rangle \overset{4.8}{\subseteq} \langle \bigcup_{`Q\in F_P'} Q_X^1\rangle \overset{2.13}{=} \langle \bigcup_{`Q\in F_P'} T_Q^1\rangle \overset{4.2}{=} T_P^1 \quad .$$

Then, since P is smooth for F_P (because of smoothness of F_P) the fact that F_P contains the first infinitesimal neighborhood of P in L_P means that $F_P(\subseteq L_P)$ has at least dimension $l = \dim L_P$, hence it is an open subvariety of L_P.$\square$

REMARK 5.4. In the proof above we needed Proposition 5.1 (since it is used in the proof of Lemma 5.2). Clearly this proposition does not extend to osculating spaces of order > 1, so we can expect that theorem 5.3 does not extend to osculating spaces of order > 1.

In [T] is claimed that this fact is proved in [C]. Actually this paper gives a classification of surfaces with second Gauss map t^2 not birational on the image, and one class is made by surfaces for which t^2 has finite generic fibre. In this way the author gets a counterexample. But her proof that this class is nonempty is incomplete. In fact, it is not obvious that any variety contained in a Laplace congruency and meeting each fibre in a finite number ($s > 1$) of points, is in this class: it would follow just if the 2-th osculating spaces have the expected dimension 5. This fact is not obvious at all, since the surface is not a general one, and in [C] there are no proofs on it. In [C] there are also some unexplained statements and a proof in local coordinates. As one can see, using the settings of the present paper, we can get coordinate-free proofs and statements which are rigorous and close to the intuitive form of [C]. For instance, our corollary 4.4 provides us with a rigorous proof (and with a generalization) of a statement in [C, first (complete) paragraph at pag. 349].

A complete treatment of this interesting subject goes beyond of the limits of a single paper. As further examples in our settings we only give a quick proof of two well-known results, which show that varieties as ordinary developable or Laplace congruencies (and their generalizations) play certainly an important role in this kind questions (one can also have an idea of this looking at the cited result of [C]).

Corollary 5.5. *Let S be a smooth surface in $\mathbf{P}^n$, with char $k = 0$. If the first Gauss map t^1 is not birational on the image then S is either a developable surface or an open subset of a plane.*

Proof. If the generic fibre has dimension 2, then t^1 is constant. It follows that T^1_P is the same for any $P \in S$, then $S \subseteq T^1_P$ and it is an open subset of a plane. If the generic fibre has dimension 1, the image C of t^1 is a curve in the grassmannian. Since the set of the fibres $\{F_P\}_{`P\in C'}$ is a family of open subsets of the lines $L_P = \langle F_P \rangle$, we have $\overline{S} = \overline{\bigcup_{`P\in C'} F_P} = \overline{\bigcup_{`P\in C'} L_P}$ and then $S$ is an open subset of $\overline{\bigcup_{`P\in C'} L_P}$. Since the fibres F_P are smooth, Proposition 4.8 applies, and up to a shrinking $\{F_P\}_{`P\in C'}$ to a nonempty open subfamily, by Proposition 4.11 the reverse inclusion holds too. Hence for the generic closed $P \in C$ we have

$$\langle \bigcup_{`Q\in P^1_C'} L_Q \rangle = \langle \bigcup_{`Q\in P^1_C'} F_Q \rangle = \langle \bigcup_{`Q\in F_P'} Q^1_S \rangle = \langle \bigcup_{`Q\in F_P'} T^1_Q \rangle = T^1_P,$$

and dim $T^1_P = 2$, hence (cf. Definition 3.6) S is an ordinary developable surface.□

Corollary 5.6. *Let X be a smooth variety of dimension 3 in $\mathbf{P}^n$, with char $k = 0$. If the first Gauss map t^1 is not birational on the image then X is either a Laplace congruency, or an ordinary developable 3-fold, or an open subset of a linear space.*

Proof. It is enough to mimic the proof of corollary 5.5, in the three cases where the generic fibre has dimension one, two or three, taking in account Definitions 3.6 and 3.7.□

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