ON CERTAIN CLASSES OF CURVE SINGULARITIES WITH REDUCED TANGENT CONE

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Abstract. We study a class of rational curves with an ordinary singular point, which was introduced in [GO]. We find some conditions under which the tangent cone is reduced and we show that the tangent cone is not always reduced. We construct another class of rational curves with an ordinary singular point satisfying the condition required in [GO] and whose tangent cone is always reduced.

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Introduction.

In the paper [GO] cones made by s lines through the origin of the affine $n+1$ space $\mathbf{A}^{n+1}$ are studied. In Section 5 it is asked if such a cone always arise as the tangent cone of a suitable irreducible curve. The answer given there is positive in general, namely if the lines are in generic position. More precisely,

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let $P_1, \ldots, P_s$ be a set of distinct points of the projective space $\mathbf{P}^n$. In [GO, Example 13] a rational curve $C_{P_1,\ldots,P_s}$ in $\mathbf{A}^{n+1}$ having at the origin an s-fold ordinary singularity whose projectivized tangent cone is exactly $\{P_1, \ldots, P_s\}$ is constructed. Then the tangent cone is exactly the cone over $P_1, \ldots, P_s$ if and only if it is reduced. By [O1, Theorem 3.3], if $P_1, \ldots, P_s$ are in generic position then the tangent cone is reduced. The question was still open for points in special position, and here we give a complete answer.

In Section 1 we set up the basic definitions, and introduce the reader to the subject by a few simple remarks. In particular, we point out that the curves constructed in [GO, Example 13] satisfy a natural necessary condition for having a reduced tangent cone. This fact could induce to believe that the answer to the question which remained open at the end of [GO, Section 5] might be positive. But in Section 3 we find a curve in $\mathbf{A}^3$ of type $C_{P_1,\ldots,P_6}$ with non-reduced tangent cone, solving the question in the negative. In Section 2 we find some classes of sets of points not in generic position, which give rise to a curve with reduced tangent cone. In particular we show that if we have a group of less than 6 points in $\mathbf{A}^3$ then the tangent cone of the corresponding curve is reduced; so the example found in Section 3 is the simplest one.

Finally, in Section 4 we give a construction similar to the one of [GO], with the property that the tangent cone is exactly the (reduced) cone over the assigned points, for any choice of them.

1. Basic constructions and remarks.

We recall the construction of [GO, Example 13].

Remark 1.1. Let $(a_1, \ldots, a_s), (b_1, \ldots, b_s)$ be s-tuples of elements of the algebraically closed field k such that $a_i \neq a_j$ for $i \neq j$. Then there is a unique polynomial $f \in k[x_1, \ldots, x_s]$ of degree $\leq s-1$ such that $f(a_i) = b_i$, for all $i \in \{1, \ldots, s\}$. In fact the conditions $f(a_i) = b_i$ form a square linear system in the coefficients of f, whose determinant is the Vandermonde determinant in $a_1, \ldots, a_s$.

Definition 1.2. Let $\mathbf{P}^n$ be the projectivized tangent space at the origin of $\mathbf{A}^{n+1}$, and let $P_1, \ldots, P_s$ be a set of distinct points in $\mathbf{P}^n$. Up to a change of coordinates we can suppose that for any $i \in \{1, \ldots, s\}$ $P_i = [1, a_{i1}, \ldots, a_{in}]$ and $a_{i1} \neq a_{j1}$ for $i \neq j$. Now let

$$g(t) = \prod_{i=1}^{s} (t - a_{i1}) \in k[t]$$

and for $j \in \{2, \ldots, n\}$ let $f_j(t)$ be the unique polynomial of degree $s-1$ such that $f_j(a_{i1}) = a_{ij}$ for any $i \in \{1, \ldots, s\}$ (cf. Remark 1.1). We define $C_{P_1,\ldots,P_s}$ as the curve in $\mathbf{A}^{n+1}$ given parametrically by

$$\begin{cases} x_0 = g(t) \\ x_1 = tg(t) \\ x_2 = f_2(t)g(t) \\ \vdots \\ x_n = f_n(t)g(t) \end{cases}.$$

Remark 1.3. In the above situation, let $B = k[g, tg, f_2g, \ldots, f_ng]$ and N be the ideal of B generated by $g, tg, \ldots, f_ng$. Clearly $C_{P_1,\ldots,P_s} \cong \operatorname{Spec} B$ and the local ring (A, η) at the origin of $C_{P_1,\ldots,P_s}$ is the localization of B with respect to N. The tangent cone (resp. the projectivized tangent cone) at the origin is the Spec (resp. the Proj) of the associated graded ring $G_\eta(A) \cong G_N(B)$. From [O2, Section 4] we get that Proj $G_\eta(A) = \{P_1, \ldots, P_s\}$, hence Spec $G_\eta(A)_{red}$ is the cone over $\{P_1, \ldots, P_s\}$. From [O1, Theorem 3.3] we have that if $\{P_1, \ldots, P_s\}$ are in generic position (i.e. their Hilbert function is $H(d) = \min\{\binom{d+n}{n}, s\}$), then $G_\eta(A)$ is reduced, so the tangent cone is exactly the cone over $\{P_1, \ldots, P_s\}$.

We want to know whether the tangent cone Spec $G_\eta(A)$ is always the cone over $\{P_1, \ldots, P_s\}$, i.e. if it is always reduced. We recall the following useful characterization.

Proposition 1.4. *Let A, η, B and N be as in Remark 1.3, and let H be the Hilbert function of (A, η), i.e. the Hilbert function of $G_\eta(A)$: $H(i) := \dim_k(G_\eta(A))_i = \dim_k(\eta^i/\eta^{i+1})$. Let $S(\{P_1, \ldots, P_s\})$ be the graded ring associated to $\{P_1, \ldots, P_s\}$, and let H' be its Hilbert function, i.e. $H'(i) = \dim_k(S(\{P_1, \ldots, P_s\}))_i$. Then $G_\eta(A) \cong G_N(B)$ is reduced if and only if $H(i) = H'(i)$ for $i < s-1$.*

Proof. The result comes from [O2, Theorem 2.15], as $S(\{P_1, \ldots, P_s\}) \cong (G_\eta(A))_{red}$ (cf. Remark 1.3), and its Hilbert function is the same of that of the local ring obtained by localizing at its irrelevant maximal ideal. □

Remark 1.5. The curve $C_{P_1,\ldots,P_s}$ satisfies "the first step" of the above condition, i.e. if H and H' are as in the statement of Proposition 1.4, then $H(1) = H'(1)$. In fact, the ring $S(\{P_1, \ldots, P_s\}) = G_\eta(A)_{red}$ is the quotient of $G_\eta(A) \cong G_N(B)$ over its nilradical, so $H' \leq H$. In order to prove the opposite inequality, it suffices to show that for any linear homogeneous polynomial $\lambda_0 x_0 + \cdots + \lambda_n x_n$ vanishing on $P_1, \ldots, P_s$, we have $\lambda_0 \overline{g} + \lambda_1 \overline{tg} +$

$\lambda_2\overline{f_2 g}+\cdots+\lambda_n\overline{f_n g}=0$ in $G_N(B)$, i.e. $\lambda_0 g+\lambda_1 tg+\lambda_2 f_2 g+\cdots+\lambda_n f_n g\in N^2$. In geometrical terms this condition can be expressed in the following way. Let O_2 be the first order infinitesimal neighborhood of $C_{P_1,\ldots,P_s}$ at the origin, which in [H, II, Example 3.2.5] is called the 2th infinitesimal neighborhood. We require that the linear space of $\mathbf{A}^{n+1}$ whose projectivized tangent space at the origin is spanned by $P_1,\ldots,P_s$, contains O_2. We show more, namely that this linear space contains the whole $C_{P_1,\ldots,P_s}$, i.e. that any linear homogeneous polynomial $l(x_0,\ldots,x_n)=\lambda_0 x_0+\cdots+\lambda_n x_n$ vanishing on $P_1,\ldots,P_s$ is such that $\lambda_0 g+\lambda_1 tg+\lambda_2 f_2 g+\cdots+\lambda_n f_n g=0$ in B. In fact $\lambda_0 g+\lambda_1 tg+\lambda_2 f_2 g+\cdots+\lambda_n f_n g=gh$, where $h(t)=\lambda_0+\lambda_1 t+\lambda_2 f_2(t)+\cdots+\lambda_n f_n(t)$. Then $h(a_{i1})=l(P_i)=0$ for any $i\in 1,\ldots,s$, hence h is divisible by g in $k[t]$. But deg $h\leq s-1<s=$ deg g, hence $h=0$. Since the most common examples of curves with non-reduced tangent cone do not satisfy "the first step", this fact could induce to believe that the tangent cone to $C_{P_1,\ldots,P_s}$ is always reduced. At least, it indicates that a counterexample must be searched looking at the further steps of the Hilbert functions: this will be done in Section 3.

Remark 1.6. From the discussion in the above proof we can immediately deduce that $G_\eta(A)$ is reduced when the points $P_1,\ldots P_s$ are in a very special position, namely when they lie on a line. In fact in this case we have that $C_{P_1,\ldots,P_s}$ is a plane curve, then the tangent cone is reduced (since it is Cohen-Macaulay by [S, II, Proposition 3.4] and Proj $G_\eta(A)$ is the reduced scheme $\{P_1,\ldots,P_s\}$). In the next section we shall prove directly this fact as a particular case of a more general discussion about points in special position which give rise to a reduced tangent cone.

2. Some classes of points in special position in $\mathbf{P}^2$.

In this section we restrict ourselves to curves in $\mathbf{A}^3$ and points in $\mathbf{P}^2$. Similar results for higher dimension can be easily obtained from the same line of proof.

Remark 2.1. A first obvious result is that if $s\leq 3$ then the tangent cone is reduced. In fact in this case the points are in generic position or on a line (cf. Remarks 1.3 and 1.6; see also [O2, Section 4, discussion after Example 1.a)]).

Proposition 2.2. *If $n=2$ and the polynomial f_2 of Definition 1.2 has degree $d\leq 2+\sqrt{s+2}$ then the tangent cone of $C_{P_1,\ldots,P_s}$ is reduced.*

Proof. By Proposition 1.4, it is enough to show that the two Hilbert functions H and H' defined in its statement are the same. Let $f(x_0,x_1,x_2)$

be a homogeneous polynomial of degree i vanishing on $P_1, \ldots, P_s$. We have to show that $f(g, tg, f_2 g) \in N^{i+1}$.

Since f is homogeneous of degree i, $f(g, tg, f_2 g) = g^i h$, where $h(t) = f(1, t, f_2)$. But $h(a_{j1}) = f(P_j) = 0$ for any $j \in \{1, \ldots, s\}$, so h is divisible by g in $k[t]$. Let $h' = h/g \in k[t]$. We have $f(g, tg, f_2 g) = g^{i+1} h'$ and notice that h' has degree $\leq \max\{di - s, i - s\}$.

If there exists a homogeneous polynomial $p(x_0, x_1, x_2)$ of degree $i + 1$ such that $p(1, t, f_2) = h'$ then $f(g, tg, f_2 g) = g^{i+1} h' = g^{i+1} p(1, t, f_2) = p(g, tg, f_2 g) \in N^{i+1}$. Hence it is enough to show that the k-subspace of $k[t]$ spanned by the polynomials of the form $p(1, t, f_2)$, with $p(x_0, x_1, x_2)$ homogeneous of degree $i + 1$ contains the space of polynomials of degree $\leq \max\{di - s, i - s\}$. If $d \leq 1$ it is obvious, so we can suppose $d \geq 2$. Now it is enough to show that for any j such that $0 \leq j \leq di - s$ there is a monomial $m(x_0, x_1, x_2)$ of degree $i + 1$ such that $m(1, t, f_2)$ has degree (exactly) j.

Write $j = ad + b$ with $0 \leq b \leq d - 1$. We claim that

$$(*) \qquad a + b \leq i + 1.$$

Since $b \leq d - 1$, if $a \leq (i + 1) - (d - 1) = i - d + 2$ then $(*)$ is verified, and it is also verified if $a = i - d + 3$ and $b \leq d - 2$. Hence $(*)$ is true if $j \leq (i - d + 3)d + d - 2 = -d^2 + (i + 4)d - 2$. But we know that $j \leq id - s$ and then the claim will follow if we show that $id - s \leq -d^2 + (i + 4)d - 2$, which is equivalent to $2 - \sqrt{s + 2} \leq d \leq 2 + \sqrt{s + 2}$. But $2 - \sqrt{s + 2} \leq d$ as we are supposing $d \geq 2$, and $d \leq 2 + \sqrt{s + 2}$ by hypotheses, so the claim is true.

Now, by $(*)$, we can consider the monomial $x_0^{i+1-a-b} x_1^b x_2^a$: it has degree $i + 1$ and $1^{i+1-a-b} t^b f_2^a$ has degree $b + ad = j$, as required.□

Corollary 2.3. *If $s \leq 5$ then the tangent cone to $C_{P_1, \ldots, P_s}$ is reduced.*

Proof. The polynomial f_2 has degree $\leq s - 1$, and if $s \leq 5$ then $s - 1 \leq 2 + \sqrt{s + 2}$, hence the result follows from Proposition 2.2.□

Corollary 2.4. *If $P_1, \ldots, P_s$ lie on a line, then the tangent cone to $C_{P_1, \ldots, P_s}$ is reduced.*

Proof. If $P_1, \ldots, P_s$ lie on a line, then the polynomial f_2 has degree ≤ 1, hence the result follows from Proposition 2.2.□

Remark 2.5. By Remark 1.1 the polynomial f_2 depends only on the points $P_1, \ldots, P_s$ and the choice of coordinates. Notice also that we can always find a set of points with an assigned polynomial f_2 of degree $\leq s - 1$ (it follows from Remark 1.1 too). So Proposition 2.2 really gives a class of sets of points which give rise to a curve with a reduced tangent cone. These sets in most of the cases are not in generic position. In fact the points lie on the curve $x_2 - f_2(x_1)$, which has degree $d = \max\{\deg f_2, 1\}$, and if $\binom{d+2}{2} \leq s$, s points in generic position can not lie on such a curve. For example, if $s \geq 66$ all the sets of points satisfying the hypotheses of Proposition 2.2 are not in generic position.

3. A curve $C_{P_1,\ldots,P_6}$ with non-reduced tangent cone.

In the previous section we have seen that the tangent cone to $C_{P_1,\ldots,P_s}$ is reduced if the points are either in generic position or in a very special position. Now we show that in the "middle range" the tangent cone can be non-reduced, giving a negative answer to a question left open at the end of [GO, Example 13]. In the following example $n = 2$ and $s = 6$ so the bound of Corollary 2.3 is sharp.

Example 3.1. Let α_i, $i \in \{1, \ldots, 5\}$, the fifth roots of the unity in a field k of characteristic $\neq 5$. Let $P_i = [1, \alpha_i, 0]$ for $i \in \{1, \ldots, 5\}$ and $P_6 = [1, 0, -1]$. Applying Definition 1.2 we have $f_2 = t^5 - 1$ and $g = t(t^5 - 1) = tf_2$. We shall show that the tangent cone is not reduced by finding a homogeneous polynomial p of degree 2 vanishing on the points $P_1, \ldots, P_6$ and such that $p(g, tg, f_2 g) \notin N^3$ (where N is as in Remark 1.3). Consider $p = x_2^2 + x_0 x_2$. We have $p(P_i) = 0$ for all $i \in \{1, \ldots, 6\}$ and $p(g, tg, f_2 g) = t^4 g^3$. Suppose that $t^4 g^3 \in N^3$. Then there is a homogeneous polynomial $q(x_0, x_1, x_2)$ of degree 3 and a polynomial $h(t)$ such that $t^4 g^3 = q(g, tg, f_2 g) + g^4 h = g^3(q(1, t, f_2) + gh)$. Hence $t^4 = q(1, t, f_2) + t f_2 h$. Now write $q(x_0, x_1, x_2) = \bar{q}(x_0, x_1) + x_2 q'(x_0, x_1, x_2)$ with $\bar{q}$ homogeneous of degree 3 in the variables x_0, x_1. We have $t^4 - \bar{q}(1, t) = f_2 \cdot (q'(1, t, f_2) + th)$. But this is impossible since $t^4 - \bar{q}(1, t)$ has degree (exactly) 4 and so it can not be divisible by f_2 which has degree 5.

4. Curves with assigned reduced tangent cone.

We have seen in the previous section that the curve $C_{P_1,\ldots,P_s}$ may not have a reduced tangent cone. In this section, given $P_1, \ldots, P_s$, we construct a curve whose tangent cone is the (reduced) cone over $P_1, \ldots, P_s$.

Remark 4.1. In this section we shall use the fact that any polynomial $f \in k[t]$ can be expressed in a unique way as a linear combination of powers of $t - a$, $a \in k$. In characteristic 0 or if the degree is less than the characteristic, the coefficients may be calculated by the Taylor formula. We start from the following geometric idea: imposing conditions on the derivatives of the polynomials $f_1, \ldots, f_s$, in order to have a curve such that a sufficiently thick neighborhood of the origin is the same as that one of the cone over $P_1, \ldots, P_s$. In order to make things working in any characteristic, we shall impose conditions on the coefficients mentioned above.

Lemma 4.2. *Let $(a_1, \ldots, a_s)$, $(b_1, \ldots, b_s)$ be s-tuples of elements of k such that $a_i \neq a_j$ for $i \neq j$. For any natural number m there is a polynomial $f(t)$ of degree $\leq sm + s - 1$ such that writing f as a linear combination of powers of $t - a_i$ we have*

$$f(t) = b_i + \text{linear combination of powers of order } > m \text{ of } t - a_i$$

for any $i \in \{1, \ldots, s\}$.

Proof. The proof is by induction on m. For $m = 0$ this is just Remark 1.1. We can suppose the statement true for $m - 1$, i.e. that there is a polynomial f' of degree $\leq s(m-1) + s - 1$ such that for any $i \in \{1, \ldots, s\}$, $f'(t) = b_i + c_i(t - a_i)^m + \cdots$. By Remark 1.1 there is a polynomial $h(t)$ of degree $\leq s - 1$ such that for any $i \in \{1, \ldots, s\}$ $h(a_i) = c_i/(\prod_{j \neq i}(a_i - a_j))^m$. Now it is sufficient to take $f = f' - hg^m$, where $g = \prod_{i=1}^{s}(t - a_i)$.□

Definition 4.3 Let $\mathbf{P}^n$ be the projectivized tangent space at the origin of $\mathbf{A}^{n+1}$, and let $P_1, \ldots, P_s$ be a set of distinct points in $\mathbf{P}^n$. Up to a change of coordinates we can suppose that for any $i \in \{1, \ldots, s\}$ $P_i = [1, a_{i1}, \ldots, a_{in}]$ and $a_{i1} \neq a_{j1}$ for $i \neq j$. Let

$$g(t) = \prod_{i=1}^{s}(t - a_{i1}) \in k[t].$$

By Lemma 4.2 (with $m = s - 1$), for any $j \in \{1, \ldots, n\}$ there is a polynomial $f_j(t)$ of degree $\leq s^2 - 1$ such that

$$f_j = a_{ij} + c_{ij}(t - a_{i1})^s + d_{ij}(t - a_{i1})^{s+1} + \cdots$$

for any $i \in \{1, \ldots, s\}$. We define $C'_{P_1,\ldots,P_s}$ to be the curve in $\mathbf{A}^{n+1}$ given parametrically by

$$\begin{cases} x_0 = g(t) \\ x_1 = f_1(t)g(t) \\ x_2 = f_2(t)g(t) \\ \vdots \\ x_n = f_n(t)g(t) \end{cases}.$$

Proposition 4.4. *For any set of distinct points $\{P_1, \ldots, P_s\}$ of $\mathbf{P}^n$, the tangent cone at the origin of $C'_{P_1,\ldots,P_s}$ is the (reduced) cone over $P_1, \ldots, P_s$.*

Proof. Let A be the local ring at the origin of $C'_{P_1,\dots,P_s}$ and η its maximal ideal. Let $B = k[g, f_1g, \dots, f_ng]$ and N the ideal of B generated by $g, f_1g, \dots, f_ng$. By [O2, Section 4] Proj $G_\eta(A) = \{P_1, \dots, P_s\}$, so it is enough to show that $G_\eta(A) \cong G_N(B)$ is reduced. By Proposition 1.4 we reduce ourselves to proving that for any i, if $p_i(x_0, \dots, x_n)$ is a homogeneous polynomial of degree i which vanishes on $P_1, \dots, P_s$, then $p_i(g, f_1g, \dots, f_ng) \in N^{i+1}$. Looking at the isomorphism $N^i/N^{i+1} \cong \eta^i/\eta^{i+1}$ the result will follow if we prove that the image of $p_i(g, f_1g, \dots, f_ng)$ in A (which is the localization of B with respect to N) is contained in η^{i+1}.

First notice that $p_i(g, f_1g, \dots, f_ng) = g^ip_i(1, f_1, \dots, f_n)$. Fix $j \in \{1, \dots, s\}$ and write for any $h \in \{1, \dots, n\}$

$$f_h = a_{jh} + c_{jh}(t - a_{j1})^s + d_{jh}(t - a_{j1})^{s+1} + \cdots.$$

Then $p_i(1, f_1, \dots, f_n)$ is $p_i(1, a_{j1}, \dots, a_{jn})$ plus a multiple (in k[t]) of $(t - a_{j1})^s$. But $p_i(1, a_{j1}, \dots, a_{jn}) = p_i(P_j) = 0$, so $p_i(1, f_1, \dots, f_n)$ is divisible (in k[t]) by $(t - a_{j1})^s$. This holds for any $j \in \{1, \dots, s\}$, hence $p_i(1, f_1, \dots, f_n)$ is divisible in $k[t]$ by g^s, and then $p_i(g, f_1g, \dots, f_ng)$ is divisible by g^{i+s}. Now notice that the normalization $\bar{A}$ of A is the localization of $k[t]$ with respect to the multiplicative system $B - N$, and its Jacobson radical J is generated by the image of g. Hence the image of $p_i(g, f_1g, \dots, f_ng)$ belongs to J^{i+s}. But $J^{i+s} = \eta^{i+s}$ by [O2, Theorem 2.13, (1)⇒(3)], and $i + 1 \le i + s$. Hence the image of $p_i(g, f_1g, \dots, f_ng)$ belongs to η^{i+1}, as required.□

Remark 4.5. In every proof involving Proposition 1.4, we have shown that the two Hilbert functions are the same everywhere, not only for $i < s - 1$. Notice that Proposition 1.4 without the assumption $i < s - 1$ is very easy to prove.

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