On the injectivity of the nodal map

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Abstract. In this paper we study (in characteristic 0) the nodal map, which is here defined as the application $p_{d,g} : \Sigma_{d,g} \to Sym^{(\delta)}(\mathbf{P}^2)$, where $\Sigma_{d,g}$ is the open set (subvariety) of the Severi variety, which parametrizes all curves of degree d and geometric genus g in $\mathbf{P}^2$ having at most nodes as singularities, and where $p_{d,g}$ sends a curve to the set of its nodes. In particular, we study the injectivity of $p^r_{d,g}$, the restriction of $p_{d,g}$ to the subset $\Sigma^r_{d,g}$ given by those curves, which are projections of smooth, non degenerate, irreducible curves in $\mathbf{P}^r$. We classify completely the (few) values of d, r, g which make $p^r_{d,g}$ not injective (for $r > 5$ there are no such values, and for $3 \le r \le 5$ see table 0.1), and obtain a simple necessary and sufficient condition for the injectivity (corollary 2.9), involving the canonical and hyperplane divisor series on curves of $\mathrm{Hilb}_{r,d,g}$.

0. Introduction

The aim of the paper is to investigate some connections between the theory of algebraic curves in characteristic 0 and the study of groups of points in the plane. Starting from the idea of Brill and Noether (see [2]), of considering plane nodal models of curves, in order to obtain information on families and linear series, we deal with a “nodal map”. Stictly speaking, we can define this map as the function, with domain in the set of plane curves of degree d and genus g, having only nodes as singularities, which sends a curve to the set of its nodes; but this map becomes really interesting when we insert it in a more general context, and this can be done both from the viewpoint of moduli and from the viewpoint of Hilbert schemes.

Let $\mathcal{M}_g$ be the moduli space of curves of genus g. Locally over $\mathcal{M}_g$, it is possible to construct varieties $\mathcal{W}^r_{d,g}$, together with morphisms $\mathcal{W}^r_{d,g} \to \mathcal{M}_g$, in such a way that for every curve C, as a point of $\mathcal{M}_g$, the fibre of the morphism parametrizes complete linear series of degree d and dimension $\ge r$. Then, we can consider the variety $\mathcal{C}^{2,r}_{d,g}$, with the morphism $\mathcal{C}^{2,r}_{d,g} \to \mathcal{W}^r_{d,g}$, such that for every point of $\mathcal{W}^r_{d,g}$, which represents a linear series on a curve C, the fibre parametrizes the subseries of dimension two and their bases. More precisely, we are interested only in those subseries which send C birationally on a plane curve having only nodes as singularities, and then we can relate $\mathcal{C}^{2,r}_{d,g}$ with $\mathrm{Sym}^\delta(\mathbf{P}^2)$ (with appropriate δ), via the nodal map.

For example, recall that the Brill-Noether conjecture, proved by several authors in the last ten years, says the generic fibre of the morphism $\mathcal{W}^r_{d,g} \to \mathcal{M}_g$ has dimension $\rho(d,g,r) = g-(r+1)(g+r-d)$ (Brill-Noether number), if it is ≥ 0, otherwise the morphism is not dominant. One immediately checks that, once proved the dominancy, and after some easy calculation on the morphism $\mathcal{C}^{2,r}_{d,g} \to \mathcal{W}^r_{d,g}$, we can reduce the problem to the determination of the image and of the generic fibre of some restrictions of the nodal map (see also [5]).

As an other example, note that in the paper [1], of Arbarello and Cornalba, they obtain results on Hurwitz spaces and Severi varieties, using considerations on the sets of nodes of plane curves.

From the viewpoint of Hilbert schemes, which we shall use in this paper, we can consider the open subvariety $V_{r,d,g} \subset \mathrm{Hilb}_{r,d,g}$, given by smooth, irreducible, non-degenerate curves of degree d and genus g in $\mathbf{P}^r$; if we fix a projection $\mathbf{P}^r \to \mathbf{P}^2$, this one sends the generic curve of $V_{r,d,g}$ on a nodal curve, and one easily checks that the set of nodal plane curves obtained in this way, say $\Sigma^r_{d,g}$, is independent of the projection. So, we can obtain information about $V_{r,d,g}$ by considering the nodal map $p^r_{d,g}\colon \Sigma^r_{d,g} \to \mathrm{Sym}^\delta(\mathbf{P}^2)$, and reducing problems on components of Hilbert schemes to the study of subvarieties of $\mathrm{Sym}^\delta(\mathbf{P}^2)$. This construction, in the open set which we are interested in, is essentially equivalent to the previous one.

Here we do not consider the interesting problem of the determination of the locus inside $\mathrm{Sym}^\delta(\mathbf{P}^2)$ formed by groups of points which are nodes of curves of fixed type (and of its equations); what we are going to do, is to establish the injectivity of the nodal maps $p^r_{d,g}$.

Treger, in [15], has improved some of the results of Arbarello and Cornalba, and has proved, in the expected range, the birationality of the nodal map, defined over all the set $\Sigma_{d,g}$ of plane nodal curves of given degree and genus. These varieties were intoduced by Severi in [14]; later, Zariski in [16] has proved they are open sets of these, just called Severi Varieties, which parametrize all plane (not necessarily nodal) curves of given degree and genus; of course, as $\Sigma^r_{d,g}$ is a closed, pratically always proper subset of $\Sigma_{d,g}$, we can not directly use Treger's results; indeed it may happen, and actually in many cases happens, that $\Sigma^r_{d,g}$ lies outside the open set where the nodal map is an isomorphism.

We show that $p^r_{d,g}$ is injective, except for a few numerical cases, which we completely classify (see table 0.1). Furthermore, we prove (corollary 2.9) that the locus in $V_{r,d,g}$, of curves projecting on elements of $\Sigma^r_{d,g}$ that makes the nodal map not injective, is defined by the simple equation $h^0(\mathcal{O}(2K-(d-6)H)) > 0$ over the Hilbert scheme; notice that this equation involves the canonical and hyperplane series, which are in many cases the generators of the Picard group of the universal curve (see [12], or, more generally, [7] and [13]). It is also interesting to note (see Remark 3.3), that in some cases, this equation separates components of Hilbert scheme, while sometimes defines proper subvarieties of components.

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Table 0.1
(All the values of r, d, g, such that $p^r_{d,g}$ is not injective)

r	d	g
3	3	0
	4	0, 1
	5	0, 1, 2
	6	1, 2, 3, 4
	7	3, 4, 5, 6
	8	5, 6, 7, 8, 9
	9	10, 12
	10	11, 12, 15, 16
	11	20
	12	19, 24, 25
	[13,∞]	$[\frac{1}{4}d^2 - d, \frac{1}{4}d^2 - d + 1]$
4	4	0
	5	0, 1
	6	1, 2
	7	3
	8	5
5	5	0
	6	1

1. Notations and recap

All the schemes will be supposed over a field k, algebraically closed of characteristic 0. If D is a divisor, we shall denote by $|D|$ the complete linear series of D, which will be often supposed with its natural projective structure.

In this paper, a *curve* will mean *projective reduced scheme, pure of dimension 1 over k* or *divisor of* $\mathbf{P}^2$, and will be clear from the context which meaning we are using; points will be always closed.

When we shall deal with a singular plane curve, we shall consider divisors as cut on the desingularization.

Div X = divisor's group of X.

$\mathbf{C}_d = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^2}(d)))$.

$V_{r,d,g}$ = the set of smooth, irreducible, non-degenerate curves of degree d and genus g in $\mathbf{P}^r$ (is an open set in the Hilbert scheme).

$\Sigma_{d,g}$ = the set of irreducible curves in $\mathbf{P}^2$, of degree d and geometric genus g, having only nodes as singularities (is an open set in the Severi variety; see [14], Anhang F, and [16]).

$\Sigma^r_{d,g}$ $(r > 2)$ = the set of irreducible curves of $\Sigma_{d,g}$, which can be obtained from a curve C of $V_{r,d,g}$, by birational projection from a point $P \in \mathbf{P}^r$, $P \notin C$.

$\delta_{d,g} = \frac{(d-1)(d-2)}{2} - g$ = number of nodes of any curve of $\Sigma_{d,g}$.

$p_{d,g} : \Sigma_{d,g} \to \mathrm{Sym}^{\delta_{d,g}}(\mathbf{P}^2)$ = function which sends a curve to the set of its nodes.

$p^r_{d,g}$ = restriction of $p_{d,g}$ to $\Sigma^r_{d,g}$.

$G(r,d) = \frac{M(M-1)}{2}(r-1) + Mq$, where M and q are the quotient and the remainder of the division of $d-1$ by $r-1$

Finally, it is necessary to recall two classical results, which we shall often use in the paper.

Proposition 1.1. : *If C is a smooth, irreducible, non- degenerate curve in $\mathbf{P}^r$, of degree d and genus g, then $g \leq G(r,d)$ (Castelnuovo bound).*

Proof. [3] and [4]. □

Proposition 1.2. *If C is a smooth, irreducible curve in $\mathbf{P}^3$ of degree d and genus g, not contained in any quadric surface, then $g \leq \frac{1}{6}d^2 - \frac{1}{2}d + 1$.*

Proof. [8]. □

2. Preliminary results

The problem of injectivity of $p^r_{d,g}$, in many cases can be approached, starting from the propositions of Section 1, by elementary considerations:

Proposition 2.1. *If $g < \frac{1}{4}d^2 - \frac{3}{2}d + 1$ then $p^r_{d,g}$ is injective.*

Proof. If there are $C_0, C'_0 \in \Sigma_{d,g}$, $C_0 \neq C'_0$, with $p_{d,g}(C_0) = p_{d,g}(C'_0)$, then, considering the intersection in $\mathbf{P}^2$, must be $4\delta_{d,g} \leq C_0.C'_0 = d^2$, hence, using the expression of $\delta_{d,g}$, we have $g \geq \frac{1}{4}d^2 - \frac{3}{2}d + 1$.

Hence, if $g < \frac{1}{4}d^2 - \frac{3}{2}d + 1$, $p_{d,g}$ must be injective, and so its restriction $p^r_{d,g}$. □

Corollary 2.2. *If $r \geq 6$, then $p^r_{d,g}$ is injective, for all values of d and g.*

Proof. It suffices to note that $r \geq 6$ implies $G(r,d) < \frac{1}{4}d^2 - \frac{3}{2}d + 1$; hence we have that $r \geq 6$ and $g \geq \frac{1}{4}d^2 - \frac{3}{2}d + 1$ imply $\Sigma^r_{d,g} = \emptyset$ (by Proposition 1.1), while if $g < \frac{1}{4}d^2 - \frac{3}{2}d + 1$, then $p^r_{d,g}$ is injective by Proposition 2.1. □

If we notice that $\delta_{d,g}$ fixed singular points impose $3\delta_{d,g}$ (not necessarily independent) linear conditions, we can easily find, given a curve $C_0 \in \Sigma_{d,g}$, a sufficient condition for the existence of another curve C'_0, having singular points in all the nodes of C_0; but we can not deduce from this fact that $p^r_{d,g}$ is not injective, as we need that C'_0 has no other singularities out of the nodes of C_0, and that the singularities in these points are exactly nodes. We should also have that even C'_0 comes, by birational projection from an external point, from a curve of $V_{r,d,g}$. We shall prove now some propositions, which allow us to solve these questions.

Proposition 2.3. *Let $C_0 \in \Sigma_{d,g}$. If there exists a curve $A \in \mathbf{C}_d$, $A \neq C_0$, having singularities in all the points of the set $p_{d,g}(C_0)$ of the nodes of C_0, then there exists a curve $C'_0 \in \Sigma_{d,g}$, $C'_0 \neq C_0$, with $p_{d,g}(C_0) = p_{d,g}(C'_0)$.*

Proof. Let $L \subset \mathbf{C}_d$ the pencil generated by C_0 and A. The condition $p_{d,g}(C_0) = p_{d,g}(C'_0)$ is equivalent to the following two open conditions:

a) C'_0 is smooth outside $p_{d,g}(C_0)$.

b) Singularities in C'_0 are at most nodes.

The subset of L given by those $C'_0 \in L$ satisfying a) and b) is hence open and nonempty (because it contains C_0), thus in this set there are infinitely many elements among which we can find a curve distinct from C_0. □

Lemma 2.4. *Let $C_0, C'_0 \in \Sigma_{d,g}$, with $p_{d,g}(C_0) = p_{d,g}(C'_0)$ and with respective desingularizations C and C', let R be a line of $\mathbf{P}^2$, which cuts both C_0 and C'_0 in d distinct points, and finally let $H \in \operatorname{Div} C$ and $H' \in \operatorname{Div} C'$ be the divisors cut by R. Then* $\dim|H| = \dim|H'|$.

Proof. Since C_0 and C'_0 have the same degree and geometric genus, it suffices to show that H and H' have the same index of speciality; so, let i be the index of H and i' that one of H', and let us prove that $i' \geq i$:

If $i = 0$ it is trivial; otherwise, we find i canonical groups containing H and, consequently, i adjoint curves of C_0, say $A_1, \cdots, A_i$, of degree $d-3$, which are linearly independent on $\mathbf{P}^2$ and contain $R \cap C_0$. Since $R \cap C_0$ has d points, the adjoints, which have degree $d-3$, must have R as a component, and then they contain $R \cap C'_0$; furthermore they are adjoint of C'_0 (because $p_{d,g}(C_0) = p_{d,g}(C'_0)$). Hence, $A_1, \cdots, A_i$ cut i canonical groups of C', which will be independent and contain H', and this means that $i' \geq i$. Exchanging the roles of C_0 and C'_0, we obtain $i \geq i'$ and then the equality. □

Lemma 2.5. *Let $C_0, C'_0 \in \Sigma_{d,g}$, with $p_{d,g}(C_0) = p_{d,g}(C'_0)$ and with respective desingularizations C and C'; let R be a line of $\mathbf{P}^2$, which cuts both C_0 and C'_0 in a fixed common node and in other $d-2$ distinct points; finally, let $N \in \mathrm{Div}\ C$, $N' \in \mathrm{Div}\ C'$ be the divisors which correspond to the node, and $H \in \mathrm{Div}\ C$, $H' \in \mathrm{Div}\ C'$ the divisors cut by R. Then* $\dim|H - N| = \dim|H' - N'|$.

Proof. Similar of that of Lemma 2.4, except for the fact that now $R \cap C_0$ is a set of $d-1$ distinct points. □

Proposition 2.6. *Let $C_0 \in \Sigma^r_{d,g}$ and $C'_0 \in \Sigma_{d,g}$ such that $p_{d,g}(C_0) = p_{d,g}(C'_0)$, then $C'_0 \in \Sigma^r_{d,g}$.*

Proof. Let C and C' the respective desingularizations of C_0 and C'_0. If $|H|$ is the hyperplane section on C, relative to C_0, we must have $\dim|H| \geq r$ and $|H|$ very ample. If $|H'|$ is the hyperplane section on C', Lemma 2.4 gives that $\dim|H| \geq r$, and implies, togheter with Lemma 2.5, that $|H'|$ separates the pairs of points of C', which correspond to each node of C'_0 (indeed $\dim|H'-N'| = \dim|H-N| = \dim|H|-2 = \dim|H'|-2$); hence $|H'|$ is very ample.

By the above discussion, we can think of C' as embedded in $\mathbf{P}^{r'}$, with $r' \geq r$, and we have a projection of C' on $C'_0 \subset \mathbf{P}^2$ from a subspace L of $\mathbf{P}^{r'}$, of dimension $r'-3$. The secant variety of C' only meet L in the points of the intersection of L with the secant lines which meet the nodes of C'_0, while the tangent variety does not meet L (because C'_0 has no cusps), hence it is possible to find a subspace L' of L, of dimension $r'-r-1$ and not passing thru those points of intersection. If we project C' from L' in a $\mathbf{P}^r \supset \mathbf{P}^2$ such that $L' \cap \mathbf{P}^r = \emptyset$, we have a smooth, non- degenerate, irreducible curve $C'' \subset \mathbf{P}^r$, and the projection of $\mathbf{P}^r$ from $L \cap \mathbf{P}^r$ send C'' on C_0. □

Proposition 2.7. *If $g > \frac{1}{3}d^2 - 2d + 1$ and $\Sigma^r_{d,g} \neq \emptyset$, then $p^r_{d,g}$ is not injective.*

Proof. By some easy calculations, $g > \frac{1}{3}d^2 - 2d + 1$ is equivalent to $3\delta_{d,g} < \frac{d(d+3)}{2}$; now, take a $C_0 \in \Sigma^r_{d,g}$: we have that the linear system of curves having singular points in all the nodes of C_0 has dimension $\geq \frac{d(d+3)}{2} - 3\delta_{d,g}$, which in turn (according to our hypotheses) is > 0; hence there exists a curve A, which satisfy the hypotheses of the Proposition 2.3, and then we can find a curve $C'_0 \in \Sigma_{d,g}$ such that $C'_0 \neq C_0$ and $p_{d,g}(C'_0) = p_{d,g}(C_0)$. Finally, by Proposition 2.6, C'_0 belong to $\Sigma^r_{d,g}$. □

The previous proposition is far from providing us with a complete answer to the problem of the injectivity of $p^r_{d,g}$, because its hypothesis is too strong; however, the propositions proved till now, suggest that this problem can be solved by studying the linear series; indeed, as we have seen, the injectivity of $p^r_{d,g}$ depends on the existence, once fixed a $C_0 \in \Sigma^r_{d,g}$, of curves A, which are singular in the nodes of C_0. Such curves are called *biadjoints* (see [6]), and we have that the biadjoints of degree $2(d-3)$ cut, outside the doubled adjoint divisor (i.e. the double of the divisor corresponding to the nodes), the double of the canonical series; starting from this fact, we can state the following proposition.

Proposition 2.8. *Let $C_0 \in \Sigma_{d,g}$, with desingularization C; then there is a curve $C_0' \in \Sigma_{d,g}$, distinct from C_0, with $p_{d,g}(C_0') = p_{d,g}(C_0)$, if and only if $|2K-(d-6)H|$ is effective on C, where $|K|$ is the canonical series and $|H|$ is the hyperplane section relative to C_0.*

Proof. From Proposition 2.3 we have that the existence of a C_0' which satisfies the imposed condition is equivalent to the existence of a biadjoint of C_0 of degree d. Hence, it suffices to show that the biadjoints of degree d cut, outside of $2N$, the complete series $|D| = |2K-(d-6)H|$ (here we write N for the adjoint divisor of C). This means that each biadjoint of degree d cuts a divisor linearly equivalent to $2N+D$, and for each divisor $D' \in |2K-(d-6)H|$ there is a biadjoint A which cuts $2N+D'$ on C. Now, it is clear that curves of degree d cut divisors linearly equivalent to $2N+D$; in order to prove the completeness, we observe that if we fix a divisor $D' \in |2K-(d-6)H|$, then $2N+D' = N+(N+D')$ will be cut by an adjoint A, of degree d, as we know the adjoints cut complete series outsides the nodes. But this one must be a biadjoint too: indeed, since C_0 has at most nodes as singularities, the fact that A cuts a divisor containing $2N$, implies that A has multiplicity at least 2 in each node of C_0. □

Corollary 2.9 *$p^r_{d,g}$ is not injective if and only if there is a smooth, irreducible, non-degenerate curve in $\mathbf{P}^r$, of degree d and genus g, on which $|2K-(d-6)H|$ is effective, where $|K|$ is the canonical series and $|H|$ is the hyperplane section.*

Proof. It follows immediately from Proposition 2.6, Proposition 2.8 and the fact that every smooth curve can be projected on a plane nodal curve. □

The previous assertion says that the locus of the curves in $V_{r,d,g}$ which can be projected on curves of $\Sigma^r_{d,g}$ which make $p^r_{d,g}$ not injective, is described by the "equation" $h^0(\mathcal{O}(2K-(d-6)H)) > 0$, defined on the Hilbert scheme which contain $V_{r,d,g}$; it is interesting to notice that this equation involves the series $|K|$ and $|H|$, which in many cases are "the unique" series on the universal curve over the Hilbert scheme (i.e. the Picard group is generated by these series; see [12], or, more generally [7] and [13]). Finally we remark that this locus is independent of the choosen projection (on the other hand, this fact comes from elementary considerations of projective geometry).

3. Curves in $\mathbf{P}^4$ and $\mathbf{P}^5$

In this section we solve our problem for $r = 4, 5$:

Proposition 3.1. *$p^5_{d,g}$ is not injective if and only if $(d,g) \in \{(5,0),(6,1)\}$; $p^4_{d,g}$ is not injective if and only if $(d,g) \in \{(4,0),(5,0),(5,1),(6,1),(6,2),(7,3),(8,5)\}$.*

Proof. By Propositions 1.1 and 2.1, $p^r_{d,g}$ can be not injective only if $\frac{1}{4}d^2 - \frac{3}{2}d + 1 \le g \le G(r,d)$. For $r = 5$ this condition holds only if $(d,g) \in \{(5,0),(6,1)\}$, and in this cases $p^r_{d,g}$ is actually not injective: indeed in the case $(d,g) = (5,0)$ it suffices to consider a plane nodal projection of the quintic rational normal curve in $\mathbf{P}^5$, and then apply Proposition 2.7, while in the case $(d,g)=(6,1)$ we can consider an elliptic smooth curve, embed it in $\mathbf{P}^5$ by a complete series $|H|$, of degree 6, and use Corollary 2.9, as in this case $|2K-(d-6)H|=0$.

For $r=4$, the unique pairs of values of d and g, which satisfy the condition $\frac{1}{4}d^2-\frac{3}{2}d+1\le g\le G(r,d)$, are those predicted, and in these cases $p^4_{d,g}$ is actually not injective, indeed:

$(d,g)=(4,0)$: $\Sigma^4_{4,0}\neq\emptyset$, as there is a plane nodal projection of a quartic rational normal curve in $\mathbf{P}^4$, then $p^4_{4,0}$ is not injective by Proposition 2.7.

$(d,g)=(5,0)$: $\Sigma^4_{5,0}\neq\emptyset$, as we can obtain, projecting the quintic rational normal curve, a smooth rational quintic in $\mathbf{P}^4$; then a plane nodal projection of the last one is clearly an element of $\Sigma^4_{5,0}$. Now it suffices to apply Proposition 2.7.

$(d,g)=(5,1)$: $\Sigma^4_{5,1}\neq\emptyset$, as we can embed a smooth elliptic curve in $\mathbf{P}^4$, by any complete series of degree 5, and then as above obtain an element of $\Sigma^4_{5,1}$; again, Proposition 2.7 implies $p^4_{5,1}$ is not injective.

$(d,g)=(6,1)$: It suffices to take a smooth projection in $\mathbf{P}^4$ of the elliptic sestic in $\mathbf{P}^5$ considered in the case $r=5$, and apply Corollary 2.9, as the condition of effectivity on $|2K-(d-6)H|$ is clearly preserved.

$(d,g)=(6,2)$: $\Sigma^4_{6,2}\neq\emptyset$, because we can take a plane nodal projection of any smooth curve in $\mathbf{P}^4$, obtained from a smooth curve of genus 2 embedded by any complete series of degree 6 (which is certainly very ample, see e.g. [10], IV, 3.2). Then, apply Proposition 2.7.

$(d,g)=(7,3)$: take a smooth curve of genus 3. A canonical divisor K has degree 4, then we can take any divisor H of degree 7 contained in $2K$, and embed by $|H|$ the curve in $\mathbf{P}^4$ (as in the above case, $|H|$ is certainly very ample). Hence $p^4_{7,3}$ is not injective, by Corollary 2.9, as the above construction assures $2K-H$ is effective.

$(d,g)=(8,5)$: $p^4_{8,5}$ is not injective by Corollary 2.9, applied to a canonical curve of genus 5 in $\mathbf{P}^4$. □

Remark 3.2. The same argument which we used in the case $(d,g)=(6,1)$, shows that if $p^r_{d,g}$ is not injective, with $r>3$, then $p^{r-1}_{d,g}$ is not injective, and we shall use this fact in the following. We should expect that a similar property holds for the genus; indeed, once fixed the degree, we expect that it is easier that the nodal map is not injective, as the number of the nodes drops. At the end of our study (see table 0.1), by a direct cheking, we shall see that if $p^r_{d,g}$ is not injective and $\Sigma^r_{d,g+1}$ is nonempty then $p^r_{d,g+1}$ is not injective. However there exists (only) a case, where $p^r_{d,g}$ is not injective, $p^r_{d,g'}$ is injective, but $g'>g$, precisely when $r=3$, $d=12$, $g=19$, $g'=21$ (this is not in contradiction with the previous assertion, as $(d,g)=(12,20)$ is an Halphen lacuna).

Remark 3.3. In the case $(d,g)=(7,3)$, as follows immediately from the construction, there are both curves on which $|2K-(d-6)H|$ is effective and curve on which it not happens; since the corresponding Hilbert scheme is irreducible, we have that the locus where the nodal map is not injective is a proper subvariety of a component. Later, in the case $(r, d, g) = (3, 9, 10)$, we shall see an example of an Hilbert scheme which this locus is a whole proper component of.

4. Curves on a quadric surface in $\mathbf{P}^3$

When $r = 3$, there are infinitely many values of d and g satisfying the condition $\frac{1}{4}d^2 - \frac{3}{2}d + 1 \leq g \leq G(r, d)$, of the Proposition 2.1; but, by Proposition 1.2, all of these values, except a finite number, are relative to curves which must lie on a quadric surface. So in this section we shall consider curves which satisfy the condition

$$(*) \qquad \frac{1}{6}d^2 - \frac{1}{2}d + 1 < g \leq G(3, d)$$

Remark 4.1. Notice that $G(3, d)$ is the integer part of $\frac{1}{4}d^2 - d + 1$ and that, $(*)$ implies $d \geq 7, g \geq 6$ (we clearly do not consider negative values); remember also that a divisor of type (a, b) on a smooth quadric surface (see [10], III, ex. 5.6), have degree $a + b$ and genus $(a - 1)(b - 1)$.

Proposition 4.2. *If $C \in V_{3,d,g}$, with (d,g) satisfying $(*)$, is a divisor of type (a, b) on a smooth quadric surface Q, and if $|K|$ is the canonical series and $|H|$ the plane section of C, then $|2K - (d - 6)H|$ is effective if and only if $|a - b| \leq 2$.*

Proof. Canonical divisors on C are cut on Q by divisors of type $(a - 2, b - 2)$, while divisors which are linearly equivalent to $|(d-6)H|$ are cut by divisors of type $(d-6, d-6)$; then divisors linearly equivalent to $\Delta = 2K - (d - 6)H$, are cut by divisors of type $(a - b + 2, b - a + 2)$. Now, consider the exact sequence

$$(**) \qquad 0 \to \mathcal{O}_Q(2 - b, 2 - a) \to \mathcal{O}_Q(a - b + 2, b - a + 2) \to \mathcal{O}_C(\Delta) \to 0.$$

First, we can assume $2 - a < 0$, $2 - b < 0$, in fact:

If $a = 2$ then $d = b + 2$, $g = b - 1$, hence $g = d - 3$, so, by easy calculations, $d - 3 > \frac{1}{6}d^2 - \frac{1}{2}d + 1$ if and only if $d^2 - 9d + 24 < 0$, which is impossible on the real numbers; then $(*)$ excludes $a = 2$. If $a = 1$ then $g = 0$, and again this is incompatible with $(*)$ (see Remark 4.1). Of course the case $a = 0$ can not occur.

So, let $2 - a < 0$, and similarly let $2 - b < 0$. We have $H^0(\mathcal{O}_Q(2 - b, 2 - a)) = H^1(\mathcal{O}_Q(2-b, 2-a)) = 0$ (see [10], III, ex. 5.6). Hence, from the cohomology exact sequence of $(**)$ follows $h^0(\mathcal{O}_C(\Delta)) = h^0(\mathcal{O}_Q(a - b + 2, b - a + 2))$, and this is zero if and only if $|a - b| \leq 2$. □

Proposition 4.3. *If $g \geq \frac{1}{4}d^2 - d$ and $(*)$ holds, then $p^3_{d,g}$ is not injective.*

Proof. We know that for $a > 0$, $b > 0$, there is always a smooth curve of type (a, b) on a smooth quadric surface (see [10], III, ex. 5.6), thus we can apply Proposition 4.2 and Corollary 2.9, indeed from the relations $a + b = d$ and $(a - 1)(b - 1) = g$ we have that $\frac{1}{4}d^2 - d \leq g \leq \frac{1}{4}d^2 - d + 1$ is equivalent to $|a - b| \leq 2$. □

Proposition 4.4 *If* $\frac{1}{6}d^2 - \frac{1}{2}d + 1 < g < \frac{1}{4}d^2 - d$ *then* $p^3_{d,g}$ *is injective.*

Proof. By Corollary 2.9, it is enough to prove that there exist no smooth, irreducible, non-degenerate curves in $\mathbf{P}^3$, of degree d and genus g, with $|2K - (d-6)H|$ effective, satisfying the condition $\frac{1}{6}d^2 - \frac{1}{2}d + 1 < g < \frac{1}{4}d^2 - d$. Suppose the converse, and let C be such a curve. By Proposition 1.2, C must lie on a quadric surface, which must be smooth, as curves of degree d on a quadric cone have genus equal to the integer part of $\frac{1}{4}d^2 - d + 1$, and g is less than this number. Hence we can apply Proposition 4.2 and find a contradiction, since, as above, $|a - b| \leq 2$ is equivalent to $\frac{1}{4}d^2 - d \leq g \leq \frac{1}{4}d^2 - d + 1$. □

Proposition 4.5. *Suppose* $d > 12$. $p^3_{d,g}$ *is not injective if and only if* $\frac{1}{4}d^2 - d \leq g \leq \frac{1}{4}d^2 - d + 1$.

Proof. If $d > 12$ then $\frac{1}{6}d^2 - \frac{1}{2}d + 1 < \frac{1}{4}d^2 - \frac{3}{2}d + 1 < \frac{1}{4}d^2 - d$.

Thus, if $g < \frac{1}{4}d^2 - d$ then $p^r_{d,g}$ is injective: indeed in the range $\frac{1}{6}d^2 - \frac{1}{2}d + 1 < g < \frac{1}{4}d^2 - d$ we can apply Proposition 4.4, otherwise we can apply Proposition 2.1. On the other hand, if $g > \frac{1}{4}d^2 - d + 1$, $\Sigma^3_{d,g}$ is empty.

Conversely, if $\frac{1}{4}d^2 - d \leq g \leq \frac{1}{4}d^2 - d + 1$ then $p^3_{d,g}$ is not injective by Proposition 4.3. □

5. Curves in $\mathbf{P}^3$ of low degree

In order to complete the table 0.1, we have to consider the cases with $r = 3$ and $d \leq 12$. As usual, it is enough to consider the values of d and g satisfying $\frac{1}{4}d^2 - \frac{3}{2}d + 1 \leq g \leq G(3, d)$. Thus $p^3_{d,g}$ can be not injective for the following values only:

$d = 3,\ \ g = 0$
$d = 4,\ \ 0 \leq g \leq 1$
$d = 5,\ \ 0 \leq g \leq 2$
$d = 6,\ \ 1 \leq g \leq 4$
$d = 7,\ \ 3 \leq g \leq 6$
$d = 8,\ \ 5 \leq g \leq 9$
$d = 9,\ \ 8 \leq g \leq 12$
$d = 10,\ \ 11 \leq g \leq 16$
$d = 11,\ \ 15 \leq g \leq 20$
$d = 12,\ \ 19 \leq g \leq 25$

First of all, Proposition 2.7 implies $p^3_{d,g}$ is not injective in the following cases (in which $\Sigma^3_{d,g} \neq \emptyset$; see [10], IV, 6.4.2; V, 4.13.1; V, ex. 4.14):

$d = 3,\ \ g = 0$
$d = 4,\ \ 0 \leq g \leq 1$
$d = 5,\ \ 0 \leq g \leq 2$
$d = 6,\ \ 2 \leq g \leq 4$
$d = 7,\ \ 4 \leq g \leq 6$
$d = 8,\ \ 7 \leq g \leq 9$
$d = 9,\ \ g = 12$

The Propositions of section 4 imply that if $(d, g) \in \{(10, 15), (10, 16), (11, 20), (12, 24), (12, 25)\}$, $p^3_{d,g}$ is not injective (see Proposition 4.3), while it is injective by Proposition 4.4 if $(d, g) \in \{(9, 11), (10, 13), (10, 14), (11, 16), (11, 17), (11, 18), (11, 19), (12, 20), (12, 21), (12, 22), (12, 23)\}$ (really in some of these cases $V_{3,d,g} = \emptyset$).

So, let us consider the remaining cases:

$(d, g) \in \{(6, 1), (7, 3), (8, 5)\}$: $p^3_{d,g}$ is not injective by Proposition 3.1 and Remark 3.2.

$(d, g) = (8, 6)$: $p^3_{8,6}$ is not injective. Indeed, take a smooth cubic surface X in $\mathbf{P}^3$; as we know, this one is isomorphic to the blowing up of $\mathbf{P}^2$ along six points in general position. So, let l be the divisor which corresponds to a line in $\mathbf{P}^2$ not passing thru any of these points, and let $e_1, \cdots, e_6$ be the exceptional divisors. Consider a smooth, irreducible curve on X, linearly equivalent to $8l - 4e_1 - 3e_2 - 3e_3 - 2e_4 - 2e_5 - 2e_6$ (certainly there exists such a curve, see [10], V, 4.13), and apply Corollary 2.9. So we are done, as $|2K - (d-6)H|$ is cut by divisors equivalent to $4l - 4e_1 - 2e_2 - 2e_3$, which are effective.

$(d, g) = (9, 8)$: $p^3_{9,8}$ is injective. Indeed, by Corollary 2.9, it is enough to show that there are no smooth, irreducible, non-degenerate curves in $\mathbf{P}^3$, of degree 9 and genus 8, with $|2K - 3H|$ effective.

Let C be such a curve. By Proposition 1.1, we have that $\dim|H| = 3$; furthermore letting $|L| = |K - H|$, we have $\dim|L| = 1$, by Riemann-Roch. An effective divisor equivalent to $2K - 3H$ is a point P, then $|H| = |2L - P|$ and $|K| = |3L - P|$; hence P is a base point of $|3L|$, as $\dim|3L| = \dim|K + P| = 7 = \dim|K| = \dim|3L - P|$, and then is a base point of $|L|$. Let $|L_0| = |L - P|$, so $|H| = |2L_0 + P|$. Since $|H|$ is very ample, we have that for every point Q, $\dim|H - P - Q| = 1$, $|H - P - Q| = |2L_0 - Q|$ and since $\dim|2L_0| = 2$, looking at the dimensions (see e.g. [10], IV, proof of Lemma 5.5), every effective divisor of $|2L_0|$ is a sum of a L' and a L'' both equivalent to L_0. It follows that every effective divisor of $|2L_0 - Q|$ contains a divisor of $|L_0 - Q|$; but $|L_0|$ has dimension 1, so the unique effective divisor of $|L_0 - Q|$ is formed by three base points of $|H - P - Q|$, so they must lie on the line thru P and Q.

Now, project C from P: the previous discussion says that every fiber of the projection is a group of fours points, hence the image of C is a conic. Thus C lies on a quadric cone, but this is impossible, as curves of degree 9 on a quadric cone have genus 12.

$(d, g) = (9, 9)$: $p^3_{9,9}$ is injective. Ideed, by Corollary 2.9, it is enogh to show that are no smooth, irreducible, non-degenerate curves in $\mathbf{P}^3$, of degree 9 and genus 9, with $|2K - 3H|$ effective.

Let C be such a curve, let $D \in |2K - 3H|$ ($\deg D = 5$) and $L \in |K - H|$ ($\deg L = 7$), so $|H| = |2L - D|$. We have that $\dim|H| = 3$, by Proposition 1.1, and that $\dim|L| = 2$ by

Riemann-Roch. First note that $|L-D|$ can not be effective, otherwise, taking an effective divisor $E \in |L-D|$ ($\deg E = 2$), $\dim|H-E| = \dim|L| = 2$, and this is impossible because $|H|$ is very ample of dimension 3 . It follws that D imposes exactly two conditions to $|2L|$, as $\dim|2L| = 14 - 9 + \dim|L-D| + 1 = 5$ while $\dim|2L-D| = \dim|H| = 3$.

Let $P \in D$ a point of $|L|$, not base for $|L|$, and let $Q \in D$, not base for $|L-P|$ (remember that $\dim|L| = 2$). Looking at the dimensions, P can not be a base point for $|L-Q|$. From the above construction and from the fact that D imposes two conditions to $|2L|$, we have that for all $D_1 \in |L-P|$ and for all $D_2 \in |L-Q|$, $D_1 + D_2 \in |2L-P-Q|$ must contain the (three) points of D out of P and Q, say P_1, P_2, P_3. These points will be base points for $|D_1|$ or $|D_2|$. Certainly P_1, P_2 and P_3 all can not be base points of $|D_1|$ (respectively of $|D_2|$), otherwise we can impose $|D_1|$ (respectively $|D_2|$) passes thru Q (respectively thru P) and obtain $|D| \subset |L|$ (and above we have excluded this fact). So, up to a change of indexes, we can suppose P_1 and P_2 are base points of $|D_1|$, and P_3 is a base point of $|D_2|$, and let $|H_1| = |D_1 - P_1 - P_2|$ and $|H_2| = |D_2 - P_3|$.

Now we have $|H| = |H_1 + H_2|$ and $\dim|H_1| = \dim|H_2| = 1$, hence C lies on a smooth quadric surface (since the immersion of C factor through an immersion of $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$), and this is impossible, as $a + b = 9$, $(a-1)(b-1) = 9$, are incompatibles on the integers (see Remark 4.1).

$(d, g) = (9, 10)$: $p^3_{9,10}$ is not injective. It is enough to consider (see [10], II, ex. 8.4) a smooth, irreducible, non-degenerate complete intersection of two cubic sufaces in $\mathbf{P}^3$, and apply Corollary 2.9, as $|K| = |2H|$.

Remark 5.1. Notice that $\mathrm{Hilb}_{3,9,10}$ has a component of curves which lie on a quadric surface too. On this component $p^3_{9,10}$ is injective, as we can easily mimic the proof of Proposition 4.2. So, in this case we have a caracterization of those curves which make $p^3_{9,10}$ not injective: they are the complete intersections.

$(d, g) = (10, 11)$: $p^3_{10,11}$ is not injective. Here (as usual by Corollary 2.9) it suffices to find a 2-subcanonical curve of degree 10 in $\mathbf{P}^3$. We can do this using the Serre correspondence (see [11]). Indeed if $\mathcal{E}$ is the bundle associated to an elliptic quintic curve, then it has Chern classes $c_1 = 4$ and $c_2 = 5$; so $\mathcal{E}(1)$ has $c_1 = 2$ and $c_2 = 10$, and it is generated by global sections, as $\mathcal{E}$ is. One easily see that a generic section of $\mathcal{E}(1)$ gives rise to a smooth, irreducible, 2-subcanonical curve of degree 10. □

$(d, g) = (10, 12)$: $p^3_{10,12}$ is not injective. We can mimic the proof of the case $(8, 6)$, with the divisor $10l - 4e_1 - 4e_2 - 3e_3 - 3e_4 - 3e_5 - 3e_6$ on the smooth cubic surface. □

$(d, g) = (11, 15)$: $p^3_{11,15}$ is injective. Indeed, by Corollary 2.9, it is enough to show that there are no smooth, irreducible, non-degenerate curves in $\mathbf{P}^3$, of degree 11 and genus 15, with $|2K - 5H|$ effective.

Let C be such a curve, let $|L| = |K - 2H|$ ($\deg L = 6$), and notice that the effective divisor of $|2K - 5K|$ is a point, say P. So $|H| = |2L - P|$ and $|K| = |5L - 2P|$.

Suppose that P is not a base point for $|L|$. By Riemann-Roch, $\dim|5L| = \dim|5L - 2P| + 1$, thus there are no divisors of $|5L|$ which contain P but not $2P$. Hence there are no divisors in $|2L|$ which contain P but not $2P$, and then $|H| = |2L - P|$ has P as a base point, but this is impossible as $|H|$ is very ample. Therefore P is a base point of $|L|$, so let

$|L_0| = |L - P|$ and notice that $|2L_0| = |H - P|$, which implies $\dim|2L_0| = 2$ ($\dim|H| = 3$, by Proposition 1.1).

Now C can not lie on a quadric surface, as in the smooth case the conditions $a+b = 11$, $(a-1)(b-1) = 15$ are impossible, while in a quadric cone curves of degree 11 have genus 20. So we have $\dim|2H| \geq \dim \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^2}(2))) = 9$, which implies, by Riemann- Roch, $\dim|L| \geq 1$, hence $\dim|L_0| \geq 1$. Thus $\dim|2L_0| = 2$ implies $\dim|L_0| = 1$, so projecting C from P, we can mimic the end of the proof of the case $(d, g) = (9, 8)$. □

$(d, g) = (12, 19)$: $p^3_{12,19}$ is not injective. Indeed, we cas use the same argument as in the case $(9, 10)$, considering now a complete intersection of a cubic and a quartic.

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