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Casimir energy in a spherical surface within surface impedance approach: The Drude model

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Summary. — The Casimir Energy of a spherical cavity whose surface is characterized by means of its surface impedance is calculated. The material properties of the boundary are described by means of the Drude model, so that a generalization of a previous result, based on plasma model, is obtained. The limits of the proposed approach are analyzed and a possible solution is suggested. The possibility of modulating the sign of the Casimir force from positive (repulsion) to negative (attraction) is studied.

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1. – Introduction

The Casimir force is one of the few macroscopic manifestations of quantum mechanics. Indeed the (generally attractive) force between two parallel conducting plates in vacuum is directly connected to the vacuum fluctuations of the electro-magnetic field within the plates [1]. Because the Casimir force becomes dominant at the nanometer scale it could constitute a strong limitation in the production of nanodevices [2]. For this reasons, during the last years people tried to understand under what conditions it is possible to change the sign of the force from attractive to repulsive [3].

It was Boyer [4] in 1974 the first one to obtain a repulsive Casimir force between two plates, one perfectly conducting and the other infinitely permeable (see also [5-8]). In [3]

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the authors showed that, under suitable conditions, the transition between attractive and repulsive regime only depends on the Surface Impedance (SI) of the material constituting the boundary. Since then a lot of effort in that direction has been done, see [9-12] and references there in. As far as we know the first ones to use the SI to compute Casimir energy were Mostepanenko and Trunov in [13].

In this paper we follow a twofold line of reasoning, from one side we want to extend a previous analysis [12] (to which we refer for details), based on a plasma model, to a more general case, based on the Drude model. At the same time, we would like to understand the formal limitation of that approach (if any), and to better analyze the structure of the divergences appearing, see conclusions in [12]. A careful study of the contributions from TE and TM modes allowed us to better understand this structure and to fix some errors present in [12], see appendix A.

The paper is organized as follows: in sect. **2** the general setup of the calculation is introduced, in sect. **3** it is applied to the case under consideration, and in sect. **4** we give our conclusions. In appendix A we discuss the structure of the divergencies with respect to the finding in [12], and in appendix B some useful formulae are given.

2. – Casimir energy and the zeta function regularization

The Casimir energy is the vacuum expectation value of the Hamiltonian operator on the ground state [14, 15]:

$$E_{Cas} = \langle 0\,|H|\,0\rangle = \sum_J \frac{1}{2} E_J, \tag{1}$$

where E_J are the energy eigenvalues labeled by some index J. In general the sum is divergent and a regularization is necessary. In the following we will use the zeta-function regularization [14]. In this approach a convenient exponent s is introduced in the sum eq. (1) so to make it convergent. The final result is recovered by a limiting procedure:

$$E_{Cas} = \lim_{s\to -1/2} \frac{\mu^{2s+1}}{2} \sum E_J^{(-2s)} =: \lim_{s\to -1/2} \mu^{2s+1} \zeta_H(s), \tag{2}$$

where $\zeta_H(s)$ is the Riemann zeta function relative to the operator H and μ is a dimensional parameter introduced so to have E_J adimensional. It will disappear on removing the regularization in the limit $s \to -1/2$.

The main problem is that no explicit expression for the eigenvalues exists. One way of overcoming this difficulty is by means of the Cauchy argument principle [16]:

If Δ is analytic in a region Ω, except for poles x and zeros y, and $g(z)$ is analytic in Ω then

$$\sum_l g(x_l) - \sum_m g(y_m) = \frac{1}{2\pi i} \oint_\gamma g(k) \log(\Delta(k)) \tag{3}$$

where γ is a closed contour in Ω that does not pass through any of the zeros and poles, and contains all the zeros x_l and poles y_l of the function Δ.

Thus, choosing the function $\Delta(k)$ such that its zeros coincide with the eigenvalues of our problem and has no poles inside γ we have

$$\zeta_H = \sum_J \frac{1}{2\pi i} \oint_\gamma g(k) \log(\Delta_J(k)). \tag{4}$$

For this reason Δ is called the generating function. The sum over J takes into account possible degeneracy of the eigenvalues.

In the following we will use the SI to obtain the generating function. The SI of a surface material (Σ) is defined through the equation [14, 17]

$$\mathbf{E}_t \mid_\Sigma = Z(\mathbf{H} \times n) \mid_\Sigma, \tag{5}$$

where n is the outward normal to the surface. It relates the tangential components of the fields outside the material whose properties are encoded in Z [18]. The big advantage is that, in all the cases in which the dielectric properties of the material $\epsilon,\ \mu$ are known, a direct relation exists between them and Z:

$$Z = \sqrt{\frac{\mu}{\varepsilon}}, \tag{6}$$

otherwise eq. (5) can be viewed as a functional definition for the SI (see [11] and references there in).

3. – The Casimir energy for a sphere

In this section, we will concentrate on the case of the electromagnetic field in a sphere of radius a. By means of eq. (5) we will obtain the generating function $\Delta_J(k)$ so to compute the ζ_H.

The electric and magnetic field within the sphere can be written as [14, 17, 19-21]

$$\begin{aligned}\mathbf{E} = \sum_{l=1}^{\infty} \sum_{l_z=-l}^{l} \frac{i}{ka} A^{TE} [i\hat{\mathbf{n}} j_\nu(kr) Y_{l,l_z}(\theta,\varphi) + (kr j_\nu(kr))' \hat{n} \times \mathbf{X}_{l,l_z}(\theta,\varphi)] \\ + A^{TM} j_\nu(kr) \mathbf{X}_{l,l_z}(\theta,\varphi)\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{H} = \sum_{l=1}^{\infty} \sum_{l_z=-l}^{l} \frac{k}{\omega\mu} \Big\{ A^{TE} j_\nu(kr) \mathbf{X}_{l,l_z}(\theta,\varphi) - i \frac{A^{TM}}{rk} [i\hat{\mathbf{n}} j_\nu(kr) Y_{l,l_z}(\theta,\varphi) \\ + (kr j_\nu(kr))' \hat{n} \times \mathbf{X}_{l,l_z}(\theta,\varphi)] \Big\}\end{aligned} \tag{8}$$

where Y_{l,l_z} and $\mathbf{X}_{l,l_z}$ are the scalar and the vectorial spherical harmonics, respectively [17], $k = \sqrt{\epsilon\mu}\omega$, $j_\nu(x) = \sqrt{\frac{\pi}{2x}} J_\nu(x)$ [22], and $\nu = l + \frac{1}{2}$.

In the spherical geometry the regularized Casimir energy is given by

$$E_{Cas} = \lim_{s \to -\frac{1}{2}} \mu^{2s+1} \zeta_H(s) \quad \text{with} \quad \zeta_H(s) = \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} (2l+1)(\omega_{n,l}^2 + m^2)^{-s},$$

where $\omega_{n,l}$ are the eingenmodes, and $\hbar = c = 1$ is assumed. The mass m is introduced for convenience. Indeed, the following representation eq. (10) of the zeta function is not defined for any value of s if $m = 0$. Thus we will do all the calculations with $m \neq 0$ and only in the end we let m go to zero. A different procedure can be used in which $m = 0$ but is more involved [23].

On imposing the boundary conditions eq. (5) the eigenfrequencies for the TE and TM modes are implicitly obtained [12]:

$$\begin{cases} \Delta_\nu^{TE}(ka) := [\frac{i}{ka}(kaj_\nu(ka))' - Zj_\nu(ka)]A^{TE} = 0, \\ \Delta_\nu^{TM}(ka) := [-Z\frac{i}{ka}(kaj_\nu(ka))' + j_\nu(ka)]A^{TM} = 0, \end{cases} \tag{9}$$

so that the generating function $\Delta_J(k)$ is given by: $\Delta_\nu(k) = \Delta_\nu^{TE}\Delta_\nu^{TM}$.

In conclusion, in our case, eq. (4) can be written [24]

$$\zeta_H(s) = \sum_{\nu=3/2}^{\infty} \frac{\nu}{\pi i} \oint_\gamma \mathrm{d}k(k^2+m^2)^{-s} \frac{\partial}{\partial k} log[(ka)^{-2\nu}\Delta_\nu(ka)], \tag{10}$$

where the factor ν before the integral takes into account for the degeneracy with respect to the index l_z.

Shifting the integration contour along the imaginary axis $k \to ik$, we get the following formulae valid in the strip $1/2 < \mathcal{R}(s) < 1$:

$$\begin{aligned} \zeta_H(s) &= \frac{\sin(s\pi)}{\pi} \sum_{\nu=3/2}^{\infty} \nu \int_m^\infty \mathrm{d}k(k^2-m^2)^{-s} \frac{\partial}{\partial k} \log\left[(ka)^{-2\nu}\tilde{\Delta}_\nu(ka)\right] \\ &= \frac{\sin(s\pi)}{\pi} \sum_{\nu=3/2}^{\infty} \nu \int_{\frac{ma}{\nu}}^\infty \mathrm{d}y \left[\left(\frac{y\nu}{a}\right)^2 - m^2\right]^{-s} \frac{\partial}{\partial y} \log\left[(y\nu)^{-2\nu}\tilde{\Delta}_\nu(y\nu)\right], \end{aligned} \tag{11}$$

where $\tilde{\Delta}_\nu = \tilde{\Delta}_\nu^{TE}\tilde{\Delta}_\nu^{TM}$ is the mode generating function written along the imaginary axis and the $\tilde{\Delta}_\nu^{TE(TM)}$ are given by

$$\begin{aligned} \tilde{\Delta}_\nu^{(TM)}(y) &= -Z\,(iy/a)\left[\frac{1}{2y}I_\nu(y) + \dot{I}_\nu(y)\right] + I_\nu(y), \\ \tilde{\Delta}_\nu^{(TE)}(y) &= \left[\frac{1}{2} - Z\,(iy/a)\,y\right] I_\nu(y) + y\,\dot{I}_\nu(y), \end{aligned}$$

$I_\nu(y) = \exp(-i\nu\frac{\pi}{2})J_\nu(iy)$ being the modified Bessel functions [22].

Unfortunately we need to compute $\zeta_H(s)$ to the left of the strip $1/2 < \mathcal{R}(s) < 1$ where it is not defined. The general technique [24] to overcome this problem is to add and subtract the asymptotic term

$$h_{as}(y,\nu) = \log\left[(y\nu)^{-2\nu}\tilde{\Delta}_\nu(y\nu)\right]_{\nu\to\infty} \tag{12}$$

to the integrand so to move the strip of convergence to the left. In this way an asymptotic zeta function, $\zeta_{as}(s)$, is defined. If we are able to compute analytically the asymptotic term and, at the same time, to treat, at least numerically, the remaining one we can find the result. In particular, let us define $\zeta_N(s)$ as

$$\zeta_H(s) - \zeta_{as}(s) + \zeta_{as}(s) =: \zeta_N(s) + \zeta_{as}(s). \tag{13}$$

We will compute $\zeta_{as}(s)$ analytically, while the remaining part, $\zeta_N(s)$, must be computed numerically but, because in general it is very small, we will ignore it.

A careful study of the various terms constituting $\tilde{\Delta}_\nu^{TE}$ ($\tilde{\Delta}_\nu^{TM}$) [25] showed that a lot of cancellations occur between the asymptotic contributions of $\tilde{\Delta}_\nu^{TE}$ and $\tilde{\Delta}_\nu^{TM}$. Thus, from a computational point of view, it is worth to use $\tilde{\Delta}_\nu$ instead of $\tilde{\Delta}_\nu^{TE}$, $\tilde{\Delta}_\nu^{TM}$ separately. This simplification allowed us to fix some errors present in [12] and to point out some problems connected with the emerging divergences (see appendix A and conclusions).

3˙1. *The Drude model.* – In this section we compute the Casimir energy for a spherical cavity whose surface is characterized by means of the surface impedance Z of a Drude model [25], *i.e.*

$$Z_{Drude}\left(i\frac{y\nu}{a}\right) = \sqrt{\frac{y(y+\sigma_\nu)}{y(y+\sigma_\nu)+\delta_\nu^2}}, \tag{14}$$

where ω_P is the plasma frequency of the material, γ is the relaxation parameter, $\delta_\nu = \frac{y_a}{\nu}$, $y_a = a\omega_P\sqrt{\varepsilon\mu}$, $\sigma_\nu = \frac{d_a}{\nu}$, $d_a = a\gamma\sqrt{\varepsilon\mu}$.

To obtain the asymptotic values of $\tilde{\Delta}_\nu$ for $\nu \to \infty$ for fixed $\frac{k}{\nu}$ we make use of the following uniform asympotic expansion of the Bessel functions [22]:

$$I_\nu(y\nu) = \frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta(y)}}{(1+y^2)^{\frac{1}{4}}}\left[1+\sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\right],$$
$$\dot{I}_\nu(y\nu) = \frac{1}{\sqrt{2\pi\nu}}e^{\nu\eta(y)}\frac{(1+y^2)^{\frac{1}{4}}}{y}\left[1+\sum_{k=1}^{\infty}\frac{v_k(t)}{\nu^k}\right],$$

with $t = \frac{1}{\sqrt{1+y^2}}$, $\eta = \sqrt{1+y^2} + \log(\frac{y}{1+\sqrt{1+y^2}})$ and $v_k(t)$ and $u_k(t)$ are the Debye's polynomials defined by the following recurrence relation:

$$\begin{cases} u_{k+1}(t) = \dfrac{t^2(1-t^2)}{2}u'_k(t) + \dfrac{1}{8}\displaystyle\int_0^t \mathrm{d}z(1-5z^2)u_k(z), \\ v_k(t) = u_k(t) - t(1-t^2)\left[\dfrac{1}{2}u_{k-1}(t) + tu'_{k-1}(t)\right]. \end{cases} \quad k = 0,1,2\dots, \tag{15}$$

Inserting this expansion into the expression eq. (11) and developing for $\nu \to \infty$ we obtain the asymptotic expression for the ζ-function:

$$\zeta_{as}(s) = \frac{\sin(s\pi)}{\pi}\sum_{l=3/2}^{\infty}\nu\int_{\frac{ma}{\nu}}^{\infty}\mathrm{d}y\left[\left(\frac{y\nu}{a}\right)^2 - m^2\right]^{-s} h_{as}(\nu,y), \tag{16}$$

where $h_{as}(\nu, y)$ can be written as

(17) $$h_{as}(y,\nu) = \sum_{i=-1}^{n_{max}} \frac{D_i(y)}{\nu^i}.$$

The coefficients $D_i(y)$ are given in appendix B up to $n_{max} = 5$.

4. – Results and conclusions

All the integrals in eq. (16) can be computed using eq. (A.4) given in the appendix. In this way, after developing around $m = 0$, summing over ν and developing around $s = 1/2$ [12], we obtain for the Casimir energy:

(18) $$E_{Cas} = \frac{1}{a}\left[-0.328 - 0.504{y_a}^4 - 0.441 d_a {y_a}^4\right] + E_{div},$$

where

(19) $$E_{div} = \frac{1}{a}\left[-0.111\log(a\mu) - \frac{0.055}{\left(s+\frac{1}{2}\right)} + {y_a}^4\left(0.318\log(a\mu) + \frac{0.035}{s+\frac{1}{2}}\right)\right. \\ \left. + d_a{y_a}^4\left(0.132\log(a\mu) + \frac{0.330}{s+\frac{1}{2}}\right) + \left(0.248{y_a}^4 - 0.528 d_a {y_a}^4\right)\log(\frac{m}{\mu})\right] \\ + \frac{1}{a}\sum_{j=1}^{4}\frac{c_j}{(am)^j}$$

with the various coefficients c_j explicitly given in the appendix B. We observe that the finite part of the Casimir energy is always negative and that the first corrections due to the material are at least of the fourth (y_a^4) and fifth ($d_a y_a^4$) order, respectively. The structure of the divergencies appear, in a sense, more conventional than the one obtained in [12]. Indeed both the terms $1/(s+1/2)^2$ and $\log^2(a\mu)$ disappeared. They originate from an inappropriate regularization procedure adopted in [12], see appendix A. This time all the (standard) divergencies are linear and can be eliminated by computing the principal part of the zeta function as usual in the zeta function regularization [26]. We underline that this way of doing deserve the use of the outer modes and, in a sense, this is unsatisfactory within this approach whose peculiarity is the use of the internal modes of the field only. Very interesting is the new term appearing in E_{div} : $\frac{1}{a}\sum_{j=1}^{4}\frac{c_j}{(am)^j}$. This term is completely absent in [12], as far we understand it is a consequence of the fact that the asymptotic expansions we used are non uniform with respect to $y_a(d_a) \in [0, \infty]$. This is suggested by the lack of these terms in the limit of ideal conducting surface. For this reason to obtain the limit for $y_a \simeq \infty$, which reproduces the case of ideal conducting surface, it is necessary to develop with respect $y_a \simeq \infty$ first, and then perform the asymptotic expansion with respect to ν. In this way we find:

$$
E_{Cas} = \frac{1}{a}\left\{0.084 + 0.008\log(a) + \frac{0.004}{\left(s+\frac{1}{2}\right)} + \frac{1}{y_a}\left[0.070 + d_a\left(-0.038 - 0.075\log\left(\frac{a}{m}\right) + \frac{0.001}{\left(s+\frac{1}{2}\right)}\right)\right]\right\} \tag{20}
$$

in agreement with [23, 27].

In conclusion in this paper we extended the results of [12], obtained for the plasma model, to the case of the Drude model. We developed the calculation up to $n_{max} = 5$. The simultaneous treatment of the TE and TM generating functions allowed for a great simplification of the calculations. In this way we could fix some errors present in [12]. However, to better understand the structure of divergencies it would be desirable to have a *uniform* asymptotic expansion over the whole interval $y_a(d_a) \in [0, \infty]$ [28].

The possibility of obtaining negative values for the (finite part) of the Casimir energy (attractive force) is confirmed. The approach seems to be very general but the appearance of divergencies of increasing order with respect to the negative power of m deserves further analysis. The possibility of regularizing the obtained energy remains still an open question and it is currently under investigation.

Appendix A

In the following $\xi = \sqrt{1+y^2}$.

In this appendix we give, by means of an example, a detailed treatment of the structure of the divergencies appearing in the formula (18).

Let us consider the TM and TE contributions to the coefficient $D_2(y)$ in eq. (17):

$$
\begin{aligned}
a_2^{TM} &= \frac{1}{8y^2\xi^9}\left[-9y^8 + 5y^6 + 12y^4 - 4\xi^8 {y_a}^2 - 2y^2 - \xi\left(9y^7 + 10y^5 + 4y^3\right)\right] \\
a_2^{TE} &= \frac{1}{8y^3\xi^9}\left[-9y^9 + 5y^7 + 12y^5 - 2y^3 + \xi\left(-9y^8 + 10y^6 + 4y^4\right)\right. \\
&\quad + \left(4y + 4y^9 + 16y^7 + 24y^5 + 16y^3\right. \\
&\quad \left.\left. + \xi\left(8y^8 + 32y^6 + 48y^4 + 32y^2 + 8\right)\right){y_a}^2\right].
\end{aligned} \tag{A.1}
$$

Once expanded, all the terms (there are $25 = 8 + 17$) can be integrated by means of the Euler beta function:

$$
\int_0^\infty y^a(1+y^2)^b \mathrm{d}y = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(-\frac{a}{2} - b - \frac{1}{2}\right)}{2\Gamma(-b)} =: \beta(a+1, -b-a-1) \tag{A.2}
$$

and its analytic properties. Incidentally we note that this was the procedure used in ref. [12]. On the contrary by summing and simplifying we get

$$
D_2(y) = \frac{{y_a}^2}{y^3} + \frac{\xi\left(-9y^5 + 10y^3 + 4y\right) - 2 - 9y^6 + 5y^4 + 12y^2}{4\xi^9}. \tag{A.3}
$$

All the terms except the first one can be handled by means of formula (A.2) while for the first one we must use the integral formula:

$$\text{(A.4)} \int_{\frac{ma}{\nu}}^{\infty} \frac{y^b}{(1+y^2)^c}\left(\frac{\nu^2 y^2}{a^2}-m^2\right)^{-s} \mathrm{d}y = \frac{\left(\frac{\nu^2}{a^2}\right)^{c-\frac{b+1}{2}}\Gamma(1-s)}{2\,(m^2)^{s+c-\frac{b+1}{2}}}\Gamma\left(c+s-\frac{b+1}{2}\right) \times {}_2F_1\left(c, c+s-\frac{b+1}{2}, c-\frac{1-b}{2}; -\frac{\nu^2}{a^2m^2}\right),$$

where ${}_2F_1$ is the hypergeometric function [22]. Let us stress that the divergent term of the form $\frac{1}{y^3}$ can be present in one (or both) of the two contributions (TE, TM) but only after a careful simplification we have been able to single it out. By expanding around $m=0$ we obtain, in this case, a pole $\sim 1/(am)$ in formula (19). Applying the same procedure to the others D_i terms one obtains poles of increasing order in $1/(am)$.

Appendix B

$$D_{-1}(y) = \frac{2}{y}(\xi-1); \;\; D_0(y) = -\frac{(\xi+y)^2}{\xi^2 y}; \qquad D_1(y) = \frac{1-y^2}{\xi^4} - \frac{5y^3}{4\xi^5},$$

$$D_2(y) = \frac{{y_a}^2}{y^3} + \frac{\xi\left(-9y^5+10y^3+4y\right)-2-9y^6+5y^4+12y^2}{4\xi^9},$$

$$D_3(y) = -\frac{3d_a{y_a}^2}{2y^4} + \frac{-401y^7+928y^5+112y^3-112y}{64\xi^{11}} + \frac{1-25y^6+70y^4-24y^2}{4\xi^{10}},$$

$$D_4(y) = \frac{2{d_a}^2{y_a}^2}{y^5} + \frac{36y-341y^9+1330y^7-376y^5-316y^3}{16\xi^{14}} + {y_a}^4\left(\frac{4y^2+3}{4\xi y^4} - \frac{1-y^2}{y^5}\right) + \frac{-1363y^8+5980y^6-4292y^4+512y^2-8}{64\xi^{13}},$$

$$D_5(y) = -\frac{5{d_a}^3{y_a}^2}{2y^6} + d_a{y_a}^4\left(-\frac{1}{y^2}+\frac{5-6y^2}{2y^6}+\frac{-2y^4-7y^2-4}{2\xi y^5}\right) + {y_a}^4\left(-\frac{3}{8y^4}+\frac{8y^4+13y^2-7}{8\xi^4y^2}+\frac{-2y^4-3y^2-2}{2\xi^3y^3}\right) + \frac{-43085y^{11}+252384y^9-210320y^7-56784y^5+33568y^3-1312y}{512\xi^{17}} + \frac{-1346y^{10}+8545y^8-10660y^6+3027y^4-151y^2+1}{16\xi^{16}}.$$

The c_j coefficients in eq. (19) are

$$c_1 = 0.818d_a{y_a}^4 + 0.234{y_a}^4 + 0.115{y_a}^2,$$
$$c_2 = 0.080{y_a}^4 - 0.073d_a{y_a}^2,$$
$$c_3 = 0.057{d_a}^2{y_a}^2 - 0.125d_a{y_a}^4 - 0.0286{y_a}^4,$$
$$c_4 = 0.0486d_a{y_a}^4 - 0.0486{d_a}^3{y_a}^2.$$

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