# On the blowup of solutions to Liouville type equations 

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#### Abstract

We estimate some complex structures related to perturbed Liouville equations defined on a compact Riemannian 2-manifold. As a byproduct, we obtain a quick proof of the mass quantization and we locate the blow-up points.


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## 1 Introduction and main results

In the article [6] the authors considered the following Liouville type problem:

$$
\left\{\begin{array}{cl}
-\Delta u=\rho f(u) &  \tag{1.1}\\
\text { in } \Omega \\
u=0 & \\
\text { on } \partial \Omega
\end{array}\right.
$$

where $\Omega \subset \mathbb{R}^{2}$ is a smooth bounded domain, $\rho>0$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that

$$
\begin{equation*}
f(t)=e^{t}+\varphi(t) \text { with } \varphi(t)=o\left(e^{t}\right) \text { as } t \rightarrow+\infty \tag{1.2}
\end{equation*}
$$

[^0]Equations of the form (1.1) are of actual interest in several contexts, including turbulent Euler flows, chemotaxis, the Nirenberg problem in geometry. See, e.g., [5] and the references therein. A recent example is given by the mean field equation for turbulent flows with variable intensities derived in [7]:

$$
\left\{\begin{align*}
-\Delta u & =\lambda \int_{[-1,1]} \frac{\alpha e^{\alpha u} \mathcal{P}(d \alpha)}{\iint_{[-1,1] \times \Omega} e^{\alpha u} \mathcal{P}(d \alpha) d x} & & \text { in } \Omega  \tag{1.3}\\
u & =0 & & \text { on } \partial \Omega
\end{align*}\right.
$$

where $\mathcal{P} \in \mathcal{M}([-1,1])$ is a probability measure related to the vortex intensity distribution. In this case, setting

$$
f(t)=\int_{[-1,1]} \alpha e^{\alpha t} \mathcal{P}(d \alpha), \quad \rho=\lambda\left(\iint_{[-1,1] \times \Omega} e^{\alpha u} \mathcal{P}(d \alpha) d x\right)^{-1}
$$

it is readily seen that if $\mathcal{P}(\{1\})>0$, then along a blow-up sequence problem (1.3) is of the form (1.1). See $[10,11,12,13]$ for details, where the existence of solutions by variational arguments and blowup analysis are also considered. Blowup solution sequences for (1.3) have also recently been constructed in [9] following the approach introduced in [4].

In [6] the authors derived a concentration-compactness principle for (1.1), mass quantization and location of blowup points, under some additional technical assumptions for $f$. More precisely, they assumed:

$$
\begin{align*}
& \left|\varphi(t)-\varphi^{\prime}(t)\right| \leqslant \mathcal{G}(t) \text { for some } \mathcal{G} \in C^{1}(\mathbb{R}, \mathbb{R}) \\
& \text { satisfying } \mathcal{G}(t)+\left|\mathcal{G}^{\prime}(t)\right| \leqslant C e^{\gamma t} \text { with } \gamma<1 / 4 \tag{1.4}
\end{align*}
$$

and

$$
\begin{equation*}
f(t) \geqslant 0 \quad \forall t \geqslant 0 \tag{1.5}
\end{equation*}
$$

By a complex analysis approach, they established the following result.
Theorem 1.1 ([6]). Let $f$ satisfy assumptions (1.2)-(1.4)-(1.5). Let $u_{n}$ be a solution sequence to (1.1) with $\rho=\rho_{n} \rightarrow 0$. Suppose $u_{n}$ converges to some nontrivial function $u_{0}$. Then,

$$
\begin{equation*}
u_{0}(x)=8 \pi \sum_{j=1}^{m} G_{\Omega}\left(x, p_{j}\right) \tag{1.6}
\end{equation*}
$$

for some $p_{1}, \ldots, p_{m} \in \Omega, m \in \mathbb{N}$, where $G_{\Omega}$ denotes the Green's function for the Dirichlet problem on $\Omega$. Furthermore, at each blow-up point $p_{j}, j=1, \ldots, m$, it holds that

$$
\left.\nabla\left[G_{\Omega}\left(x, p_{j}\right)+\frac{1}{2 \pi} \log \left|x-p_{j}\right|\right]\right|_{x=p_{j}}+\nabla \sum_{i \neq j} G_{\Omega}\left(p_{j}, p_{i}\right)=0
$$

The original estimates in [6] are involved and require the technical assumption $\gamma \in(0,1 / 4)$. It should be mentioned that this assumption was later weakened to the natural assumption $\gamma \in(0,1)$ in $[14]$, by taking a different viewpoint on the line of [1].

Here, we are interested in revisiting the complex analysis framework introduced in [6]. In particular, we study the effect of the lower-order terms which naturally appear when the equation is considered on a compact Riemannian 2-manifold. We observe that, although the very elaborate key $L^{\infty}$-estimate obtained in [6], namely Proposition 1.1 below, may be extended in a straightforward manner to the case of manifolds (see the Appendix for the details), the lower-order terms are naturally estimated only in $L^{1}$. Therefore, we are led to consider an $L^{1}$-framework, which turns out to be significantly simpler and which holds under the weaker assumption $\gamma \in(0,1 / 2)$. As a byproduct, we obtain a quick proof of mass quantization and blowup point location for the case $\gamma \in(0,1 / 2)$.

In order to state our results, for a function $u \in C^{2}(\Omega)$ we define the quantity

$$
\begin{equation*}
S(u)=u_{z}^{2} / 2-u_{z z} \tag{1.7}
\end{equation*}
$$

where $\partial_{z}=\left(\partial_{x}-i \partial_{y}\right) / 2, \partial_{\bar{z}}=\left(\partial_{x}+i \partial_{y}\right) / 2$. Then, if $u$ is a solution to (1.1), we have

$$
\begin{equation*}
\partial_{\bar{z}}[S(u)]=-\frac{\rho}{4} u_{z}\left[f(u)-f^{\prime}(u)\right]=\frac{\rho}{4} u_{z}\left[\varphi(u)-\varphi^{\prime}(u)\right] . \tag{1.8}
\end{equation*}
$$

In particular, in the Liouville case $f(u)=e^{u}$, the function $S(u)$ is holomorphic. Therefore, the complex derivative $\partial_{\bar{z}}[S(u)]$ may be viewed as an estimate of the "distance" between the equation in (1.1) and the standard Liouville equation.

We recall that the main technical estimate in [6] is given by the following.
Proposition 1.1 ([6]). Let $u_{\rho}$ be a blow-up sequence for (1.1). Assume (1.2)-(1.4)-(1.5). Then,

$$
\left\|\partial_{\bar{z}} S(u)\right\|_{L^{\infty}(\Omega)}=\frac{\rho}{4}\left\|\nabla u_{\rho}\left(f^{\prime}\left(u_{\rho}\right)-f\left(u_{\rho}\right)\right)\right\|_{L^{\infty}(\Omega)} \rightarrow 0
$$

It is natural to expect that corresponding results should hold on a compact Riemannian 2-manifold $(M, g)$ without boundary. We show that, in fact, the $L^{\infty}$ convergence as stated in Proposition 1.1 still holds true on $M$ (see Proposition 3.1 in the Appendix). However, a modified point of view is needed in order to suitably locally define a function $S$ corresponding to (1.7), such that the lower-order terms may be controlled, as well as to prove its convergence to a holomorphic function in some suitable norm, so that the mass quantization and the location of the blowup points may be derived. As we shall see, our point of view holds under the weaker assumption $\gamma \in(0,1 / 2)$ and is significantly simpler than the original $L^{\infty}$-framework.

More precisely, on a compact Riemannian 2-manifold without boundary ( $M, g$ ), we consider the problem

$$
\left\{\begin{array}{l}
-\Delta_{g} u=\rho f(u)-c_{\rho} \quad \text { in } M  \tag{1.9}\\
\int_{M} u d x=0
\end{array}\right.
$$

where $c_{\rho}=\rho|M|^{-1} \int_{M} f(u) d x \in \mathbb{R}, d x$ denotes the volume element on $M$ and $\Delta_{g}$ denotes the Laplace-Beltrami operator. We assume that $f(t)=e^{t}+\varphi(t)$ satisfies condition (1.2) and moreover that

$$
\begin{align*}
& \left|\varphi(t)-\varphi^{\prime}(t)\right| \leqslant \mathcal{G}(t) \text { for some } \mathcal{G} \in C^{1}(\mathbb{R}, \mathbb{R}) \\
& \text { satisfying } \mathcal{G}(t)+\left|\mathcal{G}^{\prime}(t)\right| \leqslant C e^{\gamma t} \text { with } \gamma<1 / 2 \tag{1.10}
\end{align*}
$$

and

$$
\begin{equation*}
f(t) \geqslant-C \quad \forall t \geqslant 0 \tag{1.11}
\end{equation*}
$$

In the spirit of [3], we assume that along a blowup sequence we have

$$
\begin{equation*}
\rho \int_{M} f(u) d x \leq C \tag{1.12}
\end{equation*}
$$

In particular, without loss of generality we may assume that

$$
\begin{equation*}
c_{\rho} \rightarrow c_{0} \text { as } \rho \rightarrow 0^{+} \tag{1.13}
\end{equation*}
$$

We note that assumption (1.11) implies $u \geqslant-C$. We now define the modified quantity corresponding to $S(u)$. Let $\mathcal{S}=\left\{p_{1}, \ldots, p_{m}\right\}$ denote the blow-up set. Let $p \in \mathcal{S}$ and denote $X=\left(x_{1}, x_{2}\right)$. We consider a local iso-thermal chart $(\Psi, \mathcal{U})$ such that $\mathcal{B}_{\varepsilon}(p) \subset \mathcal{U}, \Psi(p)=0, \mathcal{B}_{\varepsilon}(p) \cap \mathcal{S}=\emptyset, g(X)=e^{\xi(X)}\left(d x_{1}^{2}+d x_{2}^{2}\right), \xi(0)=0$. For the sake of simplicity, we identify here funtions on $M$ with their pullback functions to $B=B(0, r)=\Psi\left(\mathcal{B}_{\varepsilon}(p)\right)$. We denote by $G_{B}(X, Y)$ the Green's function of $\Delta_{X}=\partial_{x_{1}}^{2}+\partial_{x_{2}}^{2}$ on $B$. We set

$$
\begin{equation*}
K(X)=-\int_{B} G_{B}(X, Y) e^{\xi(Y)} d Y+c_{1} z \tag{1.14}
\end{equation*}
$$

with $c_{1} \in \mathbb{C}$ defined by

$$
\begin{equation*}
\left.\partial_{z}\left[\xi(z, \bar{z})+c_{0} K(z, \bar{z})\right]\right|_{z=0}=0 \tag{1.15}
\end{equation*}
$$

where $c_{0}$ is defined in (1.13). Let $u$ denote a solution sequence to (1.9). We define $w(z)=u-c_{\rho} K$, so that $-\Delta w=e^{\xi} \rho f(u)$ in $B$. Finally, consider $S(w)$, where $S$ is defined in (1.7). Our main estimate is the following.

Theorem 1.2. Assume that $f(t)=e^{t}+\varphi(t)$ satisfies assumptions (1.2)-(1.10)(1.11). Let $u_{\rho}$ be a blow-up solution sequence for (1.9). Then,
(i) For every $1 \leqslant s<(\gamma+1 / 2)^{-1}$,

$$
\begin{equation*}
\rho\left\|\nabla u_{\rho}\left(f^{\prime}\left(u_{\rho}\right)-f\left(u_{\rho}\right)\right)\right\|_{L^{s}(M)} \rightarrow 0, \quad \text { as } \rho \rightarrow 0^{+} \tag{1.16}
\end{equation*}
$$

(ii) For every blow-up point $p \in \mathcal{S}$, the function $S(w) \rightarrow S_{0}$ in $L^{1}(B)$ as $\rho \rightarrow 0^{+}$, where $S_{0}$ is holomorphic in $B$.

Consequently, we derive

Corollary 1.1. Assume that $f(t)=e^{t}+\varphi(t)$ satisfies (1.2)-(1.10)-(1.11). Suppose $u_{n}$ converges to some nontrivial function $u_{0}$. Then,

$$
\begin{equation*}
u_{0}(x)=8 \pi \sum_{j=1}^{m} G_{M}\left(x, p_{j}\right) \tag{1.17}
\end{equation*}
$$

Moreover, the following relation holds for all $p \in \mathcal{S}$ :
$\left[\nabla_{X}\left(\sum_{q \in \mathcal{S} \backslash\{p\}} G_{M}\left(\Psi^{-1}(X), q\right)+G_{M}\left(\Psi^{-1}(X), p\right)+\frac{1}{2 \pi} \log |X|+\frac{1}{8 \pi} \xi(X)\right)\right]_{X=0}=0$.

We provide the proofs of Theorem 1.2 and of Corollary 1.1 in Section 2. For the sake of completeness, and in order to readily allow a comparison with the $L^{\infty}$ framework employed in [6], in the Appendix we extend Proposition 1.1 to the case of Riemannian 2-manifolds without boundary.

Throughout this note we denote by $C>0$ a constant whose actual value may vary from line to line.

## 2 Proof of Theorem 1.2

We begin by establishing the following result.
Lemma 2.1. Let $u$ be a solution to (1.9). For every $r>0$ we have

$$
\begin{equation*}
r \int_{M} e^{-r u}|\nabla u|^{2} d x \leqslant C \tag{2.19}
\end{equation*}
$$

where $C=C\left(r, M, \varphi, c_{0}\right)$.
Proof. We multiply by $e^{-r u}$ the equation $-\Delta_{g} u=\rho f(u)-c_{\rho}$. Integrating we have:

$$
\begin{aligned}
r \int_{M} e^{-r u}|\nabla u|^{2} d x & =\int_{M} e^{-r u} \Delta_{g} u d x=-\rho \int_{M} e^{-r u} f(u) d x+c_{\rho} \int_{M} e^{-r u} d x \\
& \leqslant \rho \int_{M} e^{-r u}|\varphi(u)| d x+c_{\rho} \int_{M} e^{r C} d x \\
& \leqslant \rho \int_{M} e^{-r u}|\varphi(u)| d x+c_{\rho} e^{r C}|M|
\end{aligned}
$$

since $u \geqslant-C$. Using the assumptions on $\varphi$, there exists $t_{0}>0$ such that $|g(u)|<e^{u}$ for $u>t_{0}$, so that

$$
\begin{aligned}
r \int_{M} e^{-r u}|\nabla u|^{2} \mathrm{~d} x & \leqslant C+\rho\left(\int_{\left\{u>t_{0}\right\}} e^{(1-r) u} d x+\int_{\left\{u \leqslant t_{0}\right\}} e^{-r u}|\varphi(u)| d x\right) \\
& \leqslant C+\rho\left(\int_{M} e^{u} d x+\int_{\left\{u \leqslant t_{0}\right\}} e^{-r u}|\varphi(u)| d x\right)
\end{aligned}
$$

and the claim follows using again the fact that $u \geqslant-C$.
The following Proposition 2.1 proves the (i)-part of Theorem 1.2.
Proposition 2.1. Let $u$ be a solution to (1.9). Then, for every $1 \leqslant s<(\gamma+1 / 2)^{-1}$ and for every $\varepsilon>0$,

$$
\begin{equation*}
\left\|\nabla u\left(f^{\prime}(u)-f(u)\right)\right\|_{L^{s}(M)} \leqslant C \rho^{-\gamma-\varepsilon} \tag{2.20}
\end{equation*}
$$

for $0<\rho<1$.
Proof. In view of assumption (1.10) we have

$$
0 \leqslant\left|f(u)-f^{\prime}(u)\right| \leqslant C e^{\gamma u}
$$

Hence,

$$
\begin{equation*}
\left\|\left(f(u)-f^{\prime}(u)\right) \nabla u\right\|_{L^{s}} \leqslant C\left\|e^{\gamma u} \nabla u\right\|_{L^{s}} \tag{2.21}
\end{equation*}
$$

Moreover, assumption (1.12) implies that

$$
\int_{M} e^{u} d x \leqslant c \rho^{-1}
$$

Then, for every $1 \leqslant q<\gamma^{-1}$, using the Hölder inequality we have

$$
\begin{equation*}
\left\|e^{\gamma u}\right\|_{L^{q}(M)} \leqslant C|M|^{1 / q-\gamma} \rho^{-\gamma} \tag{2.22}
\end{equation*}
$$

Let $0<r<1-s\left(\gamma+\frac{1}{2}\right)$. By Lemma 2.1, using the Hölder inequality again we have, for $q=\frac{s+\frac{r}{\gamma}}{1-\frac{s}{2}}<\frac{1}{\gamma}$,

$$
\begin{align*}
\left\|e^{\gamma u} \nabla u\right\|_{L^{s}(M)}^{s} & =\int_{M} e^{(s \gamma+r) u}\left(e^{-r u}|\nabla u|^{s}\right) \mathrm{d} x \\
& \leqslant\left(\int_{M} e^{\gamma u q} d x\right)^{1-\frac{s}{2}}\left(\int_{M} e^{-2 r u}|\nabla u|^{2} d x\right)^{\frac{s}{2}} \\
& \leqslant C\left\|e^{\gamma u}\right\|_{L^{q}(M)}^{s+\frac{r}{\gamma}} \tag{2.23}
\end{align*}
$$

Then, by (2.22) and (2.23) we have

$$
\begin{equation*}
\left\|e^{\gamma u} \nabla u\right\|_{L^{s}(M)} \leqslant C \rho^{-\gamma-\frac{r}{s}} . \tag{2.24}
\end{equation*}
$$

Combining (2.21) and (2.24) the claim is proved.
Let $p \in \mathcal{S}$. We denote by $(\Psi, \mathcal{U})$ an isothermal chart satisfying

$$
\begin{align*}
& \overline{\mathcal{U}} \cap \mathcal{S}=\{p\}, \quad \Psi(\mathcal{U})=\mathcal{O} \subset \mathbb{R}^{2} \\
& \Psi(p)=0, \quad g(X)=e^{\xi(X)}\left(d x_{1}^{2}+d x_{2}^{2}\right), \quad \xi(0)=0 \tag{2.25}
\end{align*}
$$

where $X=\left(x_{1}, x_{2}\right)$ denotes a coordinate system on $\mathcal{O}$. We consider $\varepsilon>0$ sufficiently small so that $\mathcal{B}(p, \varepsilon) \Subset \mathcal{U}$ and let $B=B(0, r)=\Psi(\mathcal{B}(p, \varepsilon))$. The Laplace-Beltrami operator $\Delta_{g}$ is then mapped to the operator $e^{-\xi(X)} \Delta_{X}$ on $\mathcal{O}$, where $\Delta_{X}=\partial_{x_{1}^{2}}^{2}+\partial_{x_{2}^{2}}^{2}$. By $G_{B}(X, Y)$ we denote the Green's function of $\Delta_{X}$ on $B$, namely

$$
\begin{cases}-\Delta_{X} G_{B}(X, Y)=\delta_{Y} & \text { in } B \\ G_{B}(X, Y)=0 & \text { on } \partial B\end{cases}
$$

We recall from (1.14) that

$$
K(X)=-\int_{B} G_{B}(X, Y) e^{\xi(Y)} d Y+c_{1} z
$$

with $c_{1}$ the constant defined by (1.15), namely

$$
\left.\partial_{z}\left[\xi(z, \bar{z})+c_{0} K(z, \bar{z})\right]\right|_{z=0}=0
$$

where $c_{0}=\lim _{\rho \rightarrow 0} c_{\rho}$. Then, $K \in C^{\infty}(B)$ and

$$
\begin{equation*}
\Delta_{X} K=e^{\xi} \quad \text { in } \bar{B} \tag{2.26}
\end{equation*}
$$

Let $u_{\rho}$ be a blow-up solution sequence for (1.9). As $\rho \rightarrow 0, u \rightarrow u_{0}$ in $C_{l o c}^{\infty}(M \backslash \mathcal{S})$, $u-u_{0} \in W^{1, q}(M)$ for $1 \leqslant q<2$, and $f(u) \rightarrow f\left(u_{0}\right)$ in $C_{l o c}^{\infty}(M \backslash \mathcal{S})$ and $\Delta_{g} u \rightarrow \Delta_{g} u_{0}$ in $C_{l o c}^{\infty}(M \backslash \mathcal{S})$ so that

$$
\begin{equation*}
\Delta_{g} u_{0}=c_{0} \quad \text { in } M \backslash \mathcal{S} \tag{2.27}
\end{equation*}
$$

We consider the following functions defined in $B$

$$
\begin{align*}
& \tilde{u}=u \circ \Psi^{-1}, \quad \tilde{u}_{0}=u_{0} \circ \Psi^{-1} \\
& w(z)=\tilde{u}-c_{\rho} K \\
& w_{0}(z)=\tilde{u}_{0}-c_{0} K \\
& S(w)=w_{z z}-\frac{1}{2} w_{z}^{2}  \tag{2.28}\\
& S_{0}=w_{0 z z}-\frac{1}{2} w_{0 z}^{2} .
\end{align*}
$$

The following Proposition 2.2 proves the (ii)-part of Theorem 1.2.
Proposition 2.2. The complex function $S_{0}$ defined in (2.28) is holomorphic in $B$ and $S \rightarrow S_{0}$ in $L^{1}(B)$.
Proof. By (2.28) we have

$$
-\Delta_{X} w=\rho f(\tilde{u}) e^{\xi} \quad \text { and } \quad w_{z}=\tilde{u}_{z}-c_{\rho} K_{z}
$$

Then, using $\Delta_{X}=4 \partial_{z \bar{z}}$ we compute

$$
\begin{align*}
\partial_{\bar{z}}[S(w)] & =\frac{1}{4}\left(\partial_{z} \Delta_{X} w-w_{z} \Delta_{X} w\right) \\
& =-\frac{\rho}{4} e^{\xi}\left[f(\tilde{u}) \xi_{z}+\tilde{u}_{z} f^{\prime}(\tilde{u})\right]+\frac{\rho}{4} e^{\xi} f(\tilde{u})\left[\tilde{u}_{z}-c_{\rho} K_{z}\right] \\
& =\frac{\rho}{4} e^{\xi}\left(f(\tilde{u})-f^{\prime}(\tilde{u})\right) \tilde{u}_{z}-\frac{\rho}{4} e^{\xi} f(\tilde{u})\left[\xi_{z}+c_{0} K_{z}\right]+\left(c_{0}-c_{\rho}\right) \frac{\rho}{4} e^{\xi} f(\tilde{u}) K_{z} \tag{2.29}
\end{align*}
$$

Using (2.29) we derive that

$$
\begin{equation*}
\partial_{\bar{z}} S \rightarrow 0 \quad \text { in } L^{1}(B) \tag{2.30}
\end{equation*}
$$

Indeed, this follows by Proposition 2.1, (1.15) and by the fact that $|\rho f(\tilde{u})| \xrightarrow{*} a \delta_{0}(d x)$ for some $a>0$. On the other hand, by (2.28), since $u \rightarrow u_{0}$ in $C_{l o c}^{\infty}(M \backslash \mathcal{S})$, we have

$$
w \rightarrow w_{0} \quad \text { in } C_{l o c}^{\infty}(\bar{B} \backslash\{0\})
$$

and then

$$
\begin{equation*}
S \rightarrow S_{0} \quad \text { in } C_{l o c}^{\infty}(\bar{B} \backslash\{0\}) \tag{2.31}
\end{equation*}
$$

At this point we set $\Xi=\left(\xi_{1}, \xi_{2}\right)$ and $\zeta=\xi_{1}+i \xi_{2}$ and we observe that by the Cauchy integral formula we may write:

$$
\begin{equation*}
[S(w)](\zeta)=\frac{1}{\pi} \int_{B} \frac{\partial_{\bar{z}} S(z)}{\zeta-z} d X+\frac{i}{2 \pi} \int_{+\partial B} \frac{[S(w)](z)}{\zeta-z} d z=g(\zeta)+h(\zeta) \tag{2.32}
\end{equation*}
$$

We have

$$
\begin{equation*}
h(\zeta) \rightarrow h_{0}(\zeta)=\frac{i}{2 \pi} \int_{+\partial B} \frac{S_{0}(z)}{\zeta-z} d z \quad \text { in } C_{l o c}^{0}(B) \tag{2.33}
\end{equation*}
$$

and $h_{0}$ is holomorphic in $B$. On the other hand, we have

$$
\begin{equation*}
g \rightarrow 0 \quad \text { in } L^{1}(B) \tag{2.34}
\end{equation*}
$$

To prove (2.34) it is sufficient to observe that for every $z \in B=B(0, r)$ we have $B \subset B(z, 2 r)$ and then

$$
\begin{aligned}
\|g\|_{L^{1}(B)} & \leqslant \iint_{B \times B}\left|\partial_{\bar{z}} S(z)\right| \frac{1}{|\zeta-z|} d X d \Xi \\
& \leqslant \int_{B}\left|\partial_{\bar{z}} S(z)\right|\left(\int_{B(z, 2 r)} \frac{1}{|\zeta-z|} d \Xi\right) d X \\
& =4 \pi r \int_{B}\left|\partial_{\bar{z}} S(z)\right| d X
\end{aligned}
$$

which tends to zero by (2.30). Combining (2.32), (2.33) and (2.34) we have

$$
S \rightarrow h_{0} \quad \text { in } L^{1}(B), \text { as } \rho \rightarrow 0
$$

and hence, up to subsequences,

$$
S \rightarrow h_{0} \quad \text { a.e. in } B, \text { as } \rho \rightarrow 0
$$

so that by (2.31)

$$
S_{0}(\zeta)=h_{0}(\zeta) \quad \forall \zeta \in B \backslash\{0\}
$$

This completes our proof.
Finally, we use the following result from [2].

Proposition 2.3 ([2]). For $B=B(0,1) \subset \mathbb{R}^{n}(n \geqslant 2)$ the conditions

$$
v \in W^{1, p}(B)(1<p<\infty) \quad \text { and } \quad \Delta v=0 \quad \text { in } B \backslash\{0\}
$$

imply that $H=v-\ell E$ is harmonic in $B$ where $\ell$ is some constant and

$$
E(x)= \begin{cases}|x|^{2-n}, & \text { if } n>2 \\ \log |x|, & \text { if } n=2\end{cases}
$$

Now we are ready to prove Corollary 1.1. By $G_{M}$ we denote the Green's function on the manifold $M$, defined by

$$
\left\{\begin{array}{l}
-\Delta_{g} G_{M}(x, y)=\delta_{y}-\frac{1}{|M|} \\
\int_{M} G_{M}(x, y) d x=0
\end{array}\right.
$$

Proof of Corollary 1.1. Assume $p \in \mathcal{S}$. Let us start by observing that $w_{0}$ in (2.28) is harmonic in $B \backslash\{0\}$ by definition, and that $w_{0} \in W^{1, q}(B) \forall 1<q<2$. Hence, also by using Proposition 2.3,

$$
\begin{equation*}
w_{0}(z)=\ell \log \frac{1}{|z|}+H(z) \tag{2.35}
\end{equation*}
$$

where $H$ is harmonic in $B$ and $\ell \neq 0$. Then, using the fact that $\partial_{z} \log |z|=$ $\frac{1}{2} \partial_{z} \log (z \bar{z})=(2 z)^{-1}$, we compute

$$
\begin{aligned}
& w_{0 z}=-\frac{\ell}{2 z}+H_{z} \\
& w_{0 z z}=\frac{\ell}{2 z^{2}}+H_{z z}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_{0} & =w_{0 z z}-\frac{1}{2} w_{0 z}^{2} \\
& =\frac{\ell}{2 z^{2}}+H_{z z}-\frac{1}{2}\left(\frac{\ell}{2 z}-H_{z}\right)^{2} \\
& =\frac{\ell(4-\ell)}{8 z^{2}}+\frac{\ell}{2 z} H_{z}+H_{z z}-\frac{1}{2} H_{z}^{2}
\end{aligned}
$$

By Proposition 2.2 we know that $S_{0}$ is holomorphic. Hence we can conclude that $\ell=4$ and $H_{z}(0)=0$. Since

$$
\begin{equation*}
H=w_{0}-4 \log \frac{1}{|z|} \text { is harmonic in } B \tag{2.36}
\end{equation*}
$$

we have

$$
\Delta_{X}\left(\tilde{u}_{0}-4 \log \frac{1}{|z|}\right)=c_{0} e^{\xi} \quad \text { in } B(0, r)
$$

and therefore

$$
\Delta_{g}\left(u_{0}(x)-8 \pi G_{M}(x, p)\right)=c_{0}-\frac{8 \pi}{|M|}+h_{p} \quad \text { in } \mathcal{B}(p, \varepsilon)
$$

for some harmonic function $h_{p}$. Arguing similarly for each $p \in \mathcal{S}=\left\{p_{1}, p_{2}, \ldots, p_{m}\right\}$, we conclude that

$$
\Delta_{g}\left(u_{0}(x)-8 \pi \sum_{j=1}^{m} G_{M}\left(x, p_{j}\right)\right)=c_{0}-\frac{8 \pi m}{|M|} \quad \text { in } M
$$

In particular we obtain

$$
u_{0}(x)-8 \pi \sum_{j=1}^{m} G_{M}\left(x, p_{j}\right)=\mathrm{cost} \quad \text { in } M
$$

Observing that $\int_{M} u_{0}=0$, this completes the proof of (1.17). To obtain (1.18) it is sufficient to observe that, in view of (2.36) and (1.15),

$$
\begin{aligned}
0 & =\frac{1}{8 \pi} \partial_{z} H(X)_{\mid X=0} \\
& =\partial_{z}\left[\sum_{q \in \mathcal{S}} G_{M}\left(\Psi^{-1}(X), q\right)+\frac{1}{2 \pi} \log |X|\right]_{X=0}-\left[\frac{m}{|M|} \partial_{z} K(X)\right]_{X=0} \\
& =\partial_{z}\left[\sum_{q \in \mathcal{S}} G_{M}\left(\Psi^{-1}(X), q\right)+\frac{1}{2 \pi} \log |X|-\frac{1}{8 \pi} \xi(X)\right]_{X=0} .
\end{aligned}
$$

Now, Corollary 1.1 is completely established.

## 3 Appendix: the $L^{\infty}$-estimate on $M$

In this Appendix, for the sake of completeness and in order to outline the original arguments in [6], so that the simplification of our $L^{1}$-approach may be seen, we check that Proposition 1.1 may be actually extended to problem (1.9) on a compact Riemannian 2-manifold without boundary $(M, g)$ with minor modifications. We consider a solution sequence for problem (1.9). We assume that $f$ satisfies (1.2)-(1.4)-(1.5). Moreover, we assume (1.12) so that $c_{\rho} \rightarrow c_{0}$ as $\rho \rightarrow 0^{+}$. We show the following.

Proposition 3.1. Let $u$ be a solution to (1.9). Then,

$$
\rho\left\|\nabla u\left(f^{\prime}(u)-f(u)\right)\right\|_{L^{\infty}(M)} \rightarrow 0, \quad \text { as } \rho \rightarrow 0
$$

The proof relies on the following relation, due to M. Obata.

Lemma 3.1 ([8]). Let $w=w(x)>0$ be a solution to

$$
\begin{equation*}
\Delta w=\frac{|\nabla w|^{2}}{w}+F(w) \quad \text { on } M \tag{3.37}
\end{equation*}
$$

where $F$ is a $C^{1}$-function. Then, it holds the identity

$$
\begin{equation*}
\operatorname{div} V=J+\frac{1}{2}|\nabla w|^{2} w^{-2}\left(F(w)+w F^{\prime}(w)\right) \tag{3.38}
\end{equation*}
$$

where, in local coordinates,

$$
\begin{align*}
& V_{j}=w^{-1}\left\{\nabla\left(\frac{\partial w}{\partial x_{i}}\right) \cdot \nabla w-\frac{1}{2} \frac{\partial w}{\partial x_{i}} \Delta w\right\}, \quad j=1,2 \\
& J=w^{-1}\left\{\sum_{i, j=1}^{2}\left(\frac{\partial^{2} w}{\partial x_{i} \partial x_{j}}\right)^{2}-\frac{1}{2}(\Delta w)^{2}\right\} \geqslant 0 . \tag{3.39}
\end{align*}
$$

Lemma 3.2. Let $u$ be a solution to (1.9). Then, for every $r>0$ it holds

$$
\begin{equation*}
\rho \int_{M} e^{-r u}|\nabla u|^{2}\left(2 r f(u)-f^{\prime}(u)\right) \leqslant 2 r c_{\rho} \int_{M} e^{-r u}|\nabla u|^{2} . \tag{3.40}
\end{equation*}
$$

Proof. Let u be a solution to (1.9). Denoting $w=e^{-r u}$, it is easy to see that Obata's assumption (3.37) is satisfied by the function $w$ with

$$
F(w)=r e^{-r u}\left(\rho f(u)-c_{\rho}\right)
$$

On the other hand we have,

$$
F(w)+w F^{\prime}(w)=\rho e^{-r u}\left(2 r f(u)-f^{\prime}(u)\right)-2 r e^{-r u} c_{\rho}
$$

In view of Obata's identity (3.38), we conclude that

$$
\int_{M} \frac{|\nabla w|^{2}}{w^{2}}(F(w)+w F(w)) \leqslant 2 \int_{M} \operatorname{div} V=0
$$

In particular, since $\frac{\nabla w}{w}=-r \nabla u$, by the last inequality we obtain

$$
\begin{aligned}
& \int_{M} r^{2}|\nabla u|^{2}\left(F(w)+w F^{\prime}(w)\right) \\
& =r^{2} \rho \int_{M} e^{-r u}|\nabla u|^{2}\left(2 r f(u)-f^{\prime}(u)\right)-2 r^{3} c_{\rho} \int_{M} e^{-r u}|\nabla u|^{2} \leqslant 0
\end{aligned}
$$

We note that combining (3.40) and (2.19) we obtain, for $\frac{1}{2}<r<1$,

$$
\begin{equation*}
\rho \int_{M} e^{-r u}|\nabla u|^{2} f(u) \leqslant C\left(1+\rho \int_{M} e^{-(r-\gamma) u}|\nabla u|^{2}\right) \tag{3.41}
\end{equation*}
$$

Since $\gamma<\frac{1}{4}$, combining (2.19) and (3.41) we obtain

$$
\rho \int_{M} e^{-r u}|\nabla u|^{2} f(u) \mathrm{d} x \leqslant C
$$

and then, since $u \geqslant-C$, using (2.19) again we have

$$
\begin{equation*}
\rho \int_{M} e^{-r u}|\nabla u|^{2}|f(u)| d x \leqslant C \quad \text { if } \frac{1}{2}<r<1 \tag{3.42}
\end{equation*}
$$

We define, for $r>0$,

$$
G_{r}(t)=\int_{0}^{t} e^{-\frac{r}{2} s} \sqrt{|f(s)|} d s
$$

Then, the estimate (3.42) may be written in the form

$$
\begin{equation*}
\left\|\nabla G_{r}(u)\right\|_{L^{2}(M)} \leqslant \frac{C}{\sqrt{\rho}} \tag{3.43}
\end{equation*}
$$

Lemma 3.3. It holds true that

$$
\begin{equation*}
\left\|G_{r}(u)\right\|_{L^{1}(M)} \leqslant \frac{C}{\sqrt{\rho}} \tag{3.44}
\end{equation*}
$$

Proof. The proof can be easily obtained as in Lemma 2.1. Let us observe that in our assumption we have, for every $\frac{1}{2}<r<1$,

$$
\begin{align*}
\int_{\{x \in M: u(x) \geqslant 0\}} G_{r}(u) d x & \leqslant \frac{2}{r} \int_{\{u \geqslant 0\}} \sqrt{|f(u)|} d x \\
& \leqslant C\left(\int_{M}|f(u)| d x\right)^{\frac{1}{2}} \leqslant \frac{C}{\sqrt{\rho}} \tag{3.45}
\end{align*}
$$

On the other hand, since $-u \leqslant C$,

$$
\begin{equation*}
\int_{\{x \in M: u(x) \leqslant 0\}}\left|G_{r}(u)\right| d x \leqslant C \int_{\{u \leqslant 0\}} d x \int_{u}^{0} e^{\frac{C_{r}}{2}} \leqslant C e^{\frac{C_{r}}{2}}|M| \leqslant C . \tag{3.46}
\end{equation*}
$$

Combining (3.45) and (3.46) we conclude the proof of (3.44).
Reducing (3.43) to

$$
\begin{equation*}
\left\|\nabla G_{r}(u)\right\|_{L^{p}(M)} \leqslant \frac{C}{\sqrt{\rho}} \quad \text { for } 1<p<2 \tag{3.47}
\end{equation*}
$$

and using (3.44) and the Sobolev imbedding we obtain

$$
\begin{equation*}
\left\|G_{r}(u)\right\|_{L^{p^{*}}(M)} \leqslant \frac{C}{\sqrt{\rho}}, \quad \frac{1}{p^{*}}=\frac{1}{p}-\frac{1}{2} \tag{3.48}
\end{equation*}
$$

Moreover we have

$$
\begin{equation*}
|f(t)|^{\frac{1}{2 \sigma}} \leqslant C\left(\left|G_{r}(t)\right|+1\right) \tag{3.49}
\end{equation*}
$$

for $\sigma=\frac{1}{1-r}(>2)$. We choose $\frac{1}{2}<r<1$ such that

$$
\begin{equation*}
\left(\gamma+\frac{1}{2}\right) \sigma<\frac{3}{2} \tag{3.50}
\end{equation*}
$$

Arguing as in [6] we obtain, for every $\varepsilon>0$,

$$
\begin{equation*}
\|f(u)\|_{L^{p}(M)} \leqslant C \rho^{-\sigma+\frac{\sigma-1}{p}-\varepsilon} \quad(1<p<\infty) \tag{3.51}
\end{equation*}
$$

and for $q>2$,

$$
\begin{equation*}
\|\nabla u\|_{L^{q}(M)} \leqslant C \rho^{\left(-\frac{1}{2}+\frac{1}{q}\right)(\sigma-1)-\varepsilon} \tag{3.52}
\end{equation*}
$$

Now we conclude the proof of Proposition 3.1.
Proof of Proposition 3.1. It holds

$$
\begin{equation*}
\left\|\left(f^{\prime}(u)-f(u)\right) \nabla u\right\|_{L^{\infty}(M)} \leqslant C\left\|e^{\gamma u} \nabla u\right\|_{L^{\infty}(M)}=\frac{C}{\gamma}\left\|\nabla e^{\gamma u}\right\|_{L^{\infty}(M)} . \tag{3.53}
\end{equation*}
$$

Moreover, by (1.9),

$$
-\Delta_{g} e^{\gamma u}=-\gamma^{2} e^{\gamma u}|\nabla u|^{2}+\rho \gamma e^{\gamma u} f(u)-c_{\rho} \gamma e^{\gamma u} \quad \text { in } M
$$

Hence we have, for $p>2$,

$$
\begin{align*}
\left\|\nabla e^{\gamma u}\right\|_{L^{\infty}(M)} & \leqslant C\left(\left\|\Delta_{g} e^{\gamma u}\right\|_{L^{p}(M)}+\left\|e^{\gamma u}\right\|_{L^{1}(M)}\right) \\
& \leqslant C\left\{\left\|e^{\gamma u}|\nabla u|^{2}\right\|_{L^{p}(M)}+\rho\left\|e^{\gamma u} f(u)\right\|_{L^{p}(M)}+\left\|c_{\rho} e^{\gamma u}\right\|_{L^{p}(M)}\right\} \tag{3.54}
\end{align*}
$$

Now, observing that $e^{u} \leqslant C(f(u)+1)$ by (3.51) we obtain

$$
\begin{align*}
\rho\left\|e^{\gamma u} f(u)\right\|_{L^{p}(M)} & \leqslant C \rho\left\|e^{(\gamma+1) u}\right\|_{L^{p}(M)}  \tag{3.55}\\
& =C \rho\left\|e^{u}\right\|_{L^{p(\gamma+1)}(M)}^{\gamma+1} \leqslant C \rho^{\tau-\varepsilon}
\end{align*}
$$

for every $\varepsilon>0$ with

$$
\begin{equation*}
\tau=1+(\gamma+1)\left[\frac{\sigma-1}{p(\gamma+1)}-\sigma\right]=1+\frac{\sigma-1}{p}-\sigma(\gamma+1) \tag{3.56}
\end{equation*}
$$

Hence, as $p \downarrow 2$, we have

$$
\begin{equation*}
\tau \uparrow 1+\frac{1}{2}(\sigma-1)-\sigma(\gamma+1)>-1 \tag{3.57}
\end{equation*}
$$

by (3.50). On the other hand, by (2.22) for $1 \leqslant p<\frac{1}{\gamma}$,

$$
\begin{equation*}
\left\|c_{\rho} e^{\gamma u}\right\|_{L^{p}(M)} \leqslant C \rho^{-\gamma} \tag{3.58}
\end{equation*}
$$

Moreover, if $q>\frac{1}{2 \gamma}(>2)$, then

$$
\left\|e^{\gamma u}|\nabla u|^{2}\right\|_{L^{p}(M)} \leqslant\left\|e^{\gamma u}\right\|_{L^{p q}(M)} \cdot\|\nabla u \mid\|_{L^{2 p q^{\prime}}(M)}^{2}
$$

where $q q^{\prime}=q+q^{\prime}$. By (3.52), for every $\varepsilon>0$, since $2 p q^{\prime}>2$ we have

$$
\|\nabla u\|_{L^{2 p q^{\prime}}(M)}^{2} \leqslant C \rho^{\left(-1+\frac{1}{p q^{\prime}}\right)(\sigma-1)-\varepsilon}
$$

Using again (3.51), for every $\varepsilon>0$ we have

$$
\left\|e^{\gamma u}\right\|_{L^{p q}(M)} \leqslant C\left\|e^{u}\right\|_{L^{p q \gamma}(M)}^{\gamma} \leqslant C \rho^{-\gamma \sigma+\frac{\sigma-1}{p q}-\varepsilon}
$$

Then, for every $\varepsilon>0$,

$$
\begin{equation*}
\left\|e^{\gamma u}|\nabla u|^{2}\right\|_{L^{p}(M)} \leqslant C \rho^{\tau-\varepsilon} \tag{3.59}
\end{equation*}
$$

with $\tau$ defined by (3.56). Combining (3.53), (3.54), (3.55), (3.57), (3.58) and (3.59) we complete the proof.

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