Mass quantization and minimax solutions for Neri's mean field equation in 2D-turbulence

T. Ricciardi*and G. Zecca

Dipartimento di Matematica e Applicazioni "R. Caccioppoli"

Università di Napoli Federico II, Via Cintia - 80126 Napoli - Italy

E-mail: tonricci@unina.it - g.zecca@unina.it

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Abstract

We study the mean field equation derived by Neri in the context of the statistical mechanics description of 2D-turbulence, under a "stochastic" assumption on the vortex circulations. The corresponding mathematical problem is a nonlocal semilinear elliptic equation with exponential type nonlinearity, containing a probability measure $\mathcal{P} \in \mathcal{M}([-1,1])$ which describes the distribution of the vortex circulations. Unlike the more investigated "deterministic" version, we prove that Neri's equation may be viewed as a perturbation of the widely analyzed standard mean field equation, obtained by taking $\mathcal{P} = \delta_1$. In particular, in the physically relevant case where $\mathcal{P}$ is non-negatively supported and $\mathcal{P}(\{1\}) > 0$, we prove the mass quantization for blow-up sequences. We apply this result to construct minimax type solutions on bounded domains in $\mathbb{R}^2$ and on compact 2-manifolds without boundary.

Key words and phrases: Mean field equation, exponential nonlinearity, mass quantization, mountain pass solution.

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1 Introduction and statement of the main results.

We are interested in the mean field equation derived by Neri [21] in the context of the statistical mechanics description of two-dimensional turbulence. Such an approach was introduced in 1949 by Onsager in the pioneering article [24], with the aim of explaining the formation of stable large-scale vortices, and is still of central interest in fluid mechanics, see [4, 8].

Neri's mean field equation [21] is derived under the "stochastic" assumption that the point vortex circulations are independent identically distributed random variables, with

*Corresponding author

probability distribution $\mathcal{P}$. On a bounded domain $\Omega \subset \mathbb{R}^2$, Neri's equation takes the form:

$$\begin{cases} -\Delta u = & \lambda \displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha u}\,\mathcal{P}(d\alpha)}{\iint_{[-1,1]\times\Omega} e^{\alpha u}\,\mathcal{P}(d\alpha)dx} \quad \text{in } \Omega \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, u denotes the stream function, $\lambda > 0$ is a constant related to the inverse temperature, dx is the volume element on Ω and $\mathcal{P}$ is a Borel probability measure defined on $[-1, 1]$ denoting the distribution of the circulations. We note that when $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$, equation (1.1) reduces to the standard mean field equation

$$\begin{cases} -\Delta u = & \lambda \dfrac{e^u}{\int_\Omega e^u\,dx} \quad \text{in } \Omega \\ \quad u = & 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) has been extensively analyzed in the context of turbulence in [5, 16]. It is also relevant in many other contexts, including the Nirenberg problem in differential geometry and the desciption of chemotaxis in Biology. See, e.g., [17, 33] and the references therein.

On the other hand, a "deterministic" assumption on the distribution of the vortex circulations yields the following similar equation

$$\begin{cases} -\Delta u = & \lambda \displaystyle\int_{-1}^{1} \frac{\alpha e^{\alpha u}\,\mathcal{P}(d\alpha)}{\int_\Omega e^{\alpha u}\,dx} \quad \text{in } \Omega \\ \quad u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

see [29]. An unpublished informal version of (1.3) was actually obtained by Onsager himself, see [12]. Equation (1.3) also includes the standard mean field equation (1.2) as a special case.

Thus, it is natural to ask for which probability measures $\mathcal{P}$ the results known for (1.2) may be extended to equations (1.1) and (1.3), and whether or not the equations (1.1) and (1.3) share similar properties. We note that, up to a rescaling with respect to α, we may assume without loss of generality that

$$\mathrm{supp}\mathcal{P} \cap \{-1, 1\} \neq \emptyset. \tag{1.4}$$

The "deterministic" equation (1.3) has been considered in [22, 25] from the point of view of the blow-up of solution sequences, and the optimal Moser-Trudinger constant. Liouville systems corresponding to discrete versions of (1.3) have been widely considered, see, e.g., [7, 10, 23] and the references therein. In these articles, it appears that equation (1.3) behaves quite differently from (1.2), particularly from the point of view of the corresponding optimal Moser-Trudinger constant, whose rather complicated expression depending on $\mathcal{P}$ was recently determined in [25].

On the other hand, fewer mathematical results are available for (1.1). In [33] it is conjectured as an open problem that the optimal Moser-Trudinger constant for (1.1) could depend on $\mathcal{P}$. However, in [26] we proved that this is not the case. More precisely, we showed that, assuming (1.4), the optimal Moser-Trudinger constant for the corresponding variational functional J_λ, given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \lambda \log\left(\iint_{[-1,1]\times\Omega} e^{\alpha u}\,\mathcal{P}(d\alpha)dx\right) \tag{1.5}$$

coincides with the optimal Moser-Trudinger constant for the "standard" case $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$. In other words, J_λ is bounded below if and only if $\lambda \leqslant 8\pi$. In this article we further confirm the significant differences between (1.1) and (1.3), which could in principle provide a criterion to identify the more suitable model among [21] and [29] to describe turbulent flows with variable intensities. More precisely, we prove that, under the additional assumption

$$\mathrm{supp}\mathcal{P} \subseteq [0,1], \qquad\qquad \mathcal{P}(\{1\}) > 0, \tag{1.6}$$

corresponding to the case where the vortices have the same orientation, as well as a non-zero probability of unit circulation, the blow-up masses have quantized values $8\pi m$, $m \in \mathbb{N}$. To this end, we follow the elegant complex analysis approach in [34]. We note that the sinh-Poisson case $\mathcal{P} = \tau\delta_1 + (1-\tau)\delta_{-1}$, $\tau \in [0,1]$ was studied in [27].

Concerning the existence problem for equation (1.1), we note that Neri himself derived an existence result in the subcritical case $\lambda < 8\pi$ by minimizing the functional (1.5). We shall here apply our blow-up results to prove the existence of saddle-type solutions in the supercritical case $\lambda > 8\pi$, following some ideas in [11, 32]. Such approaches employ the "Struwe monotonicity trick" [31] and an improved Moser-Trudinger inequality in the sense of Aubin [1].

We now state our main results. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let g be a metric on Ω. We consider solution sequences to Neri's equation in the following "local" form:

$$-\Delta_g u_n = \lambda_n \int_{[0,1]} \frac{\alpha e^{\alpha u_n}\,\mathcal{P}(d\alpha)}{\iint_{[0,1]\times\Omega} e^{\alpha u_n}\,\mathcal{P}(d\alpha) dv_g} + c_n \qquad \text{in } \Omega \tag{1.7}$$

where $c_n \in \mathbb{R}$, dv_g denotes the volume element and Δ_g denotes the Laplace-Beltrami operator. As usual, for every solution sequence u_n we define the blow-up set

$$\mathcal{S} = \{x \in \Omega \ s.t.\ \exists x_n \to x : \ u_n(x_n) \to +\infty\}.$$

Theorem 1.1 (Mass quantization)**.** *Assume* (1.6)*. Let u_n be a solution sequence to* (1.7) *with $\lambda_n \to \lambda_0$ and $c_n \to c_0$. Then there exists a subsequence, still denoted u_n, such that exactly one of the following holds:*

(i) u_n converges locally uniformly to a smooth solution u_0 for (1.7)*;*

(ii) $u_n \to -\infty$ locally uniformly in Ω;

(iii) The blow-up set $\mathcal{S}$ is finite and non-empty. Denoting $\mathcal{S} = \{p_1, \ldots, p_m\}$, there holds

$$\lambda_n \int_{[0,1]} \frac{\alpha e^{\alpha u_n}\,\mathcal{P}(d\alpha)}{\iint_{[0,1]\times\Omega} e^{\alpha u_n}\,\mathcal{P}(d\alpha) dv_g}\, dv_g \overset{*}{\rightharpoonup} \sum_{1=1}^{m} n_i \delta_{p_i} + r(x)\, dv_g$$

weakly in the sense of measures, for some $n_i \geqslant 4\pi$, $i = 1, \ldots, m$, and $r \in L^1(\Omega)$. In $\Omega \setminus \mathcal{S}$ either u_n is locally bounded, or $u_n \to -\infty$ locally uniformly.

If u_n is locally bounded in $\Omega \setminus \mathcal{S}$, then

$$\iint_{[0,1]\times\Omega} e^{u_n}\, dv_g \to +\infty,$$

$r \equiv 0$, $n_i = 8\pi$ for all $i = 1, \ldots, m$ and there exists $u_0 \in W^{1,q}_{loc}(\Omega)$ for any $q \in [1, 2)$ such that $u_n \to u_0$ in $W^{1,q}_{loc}(\Omega)$ and locally uniformly in $\Omega \setminus \mathcal{S}$. The function u_0 is of the form

$$u_0(x) = \sum_{j=1}^{m} \frac{1}{4} \log \frac{1}{d_g(x, p_i)} + b(x),$$

where b is locally bounded in Ω.

As an application of Theorem 1.1, we derive the existence of minimax type solutions in the supercritical range $\lambda > 8\pi$. Our first existence result is derived in the spirit of [11].

Theorem 1.2 (Existence of a minimax solution on annulus-type domains)**.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain whose complement contains a bounded region and assume* (1.4)*. Then, problem* (1.1) *admits a solution $u \in H^{1,2}_0(\Omega)$ for almost every $\lambda \in (8\pi, 16\pi)$. Furthermore, if $\mathcal{P}$ satisfies* (1.6)*, then* (1.1) *admits a solution $u \in H^{1,2}_0(\Omega)$ for all $\lambda \in (8\pi, 16\pi)$.*

We also consider solutions to Neri's equation on a compact orientable Riemannian surface without boundary M. On the manifold M, the corresponding problem is given by

$$\begin{cases} -\Delta_g v = \lambda \displaystyle\int_{[-1,1]} \frac{\alpha(e^{\alpha v} - \frac{1}{|M|}\int_M e^{\alpha v}\, dv_g)}{\iint_{[-1,1]\times M} e^{\alpha v}\, \mathcal{P}(d\alpha)\, dv_g} \mathcal{P}(d\alpha) \quad \text{in } M \\ \displaystyle\int_M v\, dv_g = 0, \end{cases} \tag{1.8}$$

Here, g denotes the Riemannian metric on M, dv_g denotes the volume element and Δ_g denotes the Laplace-Beltrami operator. We note that the proof of Theorem 1.2 may be adapted to problem (1.8) provided M has genus greater than or equal to one. However, in this case it is not clear in general whether or not the solution obtained is distinct from the trivial solution $u \equiv 0$. See [6] for some results and conjectures in this direction. On the other hand, a nontrivial solution in the supercritical range of λ for general manifolds may be obtained by the argument introduced in [32]. In order to state our second existence result, we denote by $\mu_1(M)$ the first non-zero eigenvalue of Δ_g, namely

$$\mu_1(M) := \inf_{\phi \neq 0,\ \phi \in \mathcal{E}} \frac{\int_M |\nabla \phi|^2}{\int_M \phi^2}, \tag{1.9}$$

where $\mathcal{E} = \{v \in H^1(M) :\ \int_M v\, dv_g = 0\}$. We prove:

Theorem 1.3 (Mountain-pass solution on manifolds)**.** *Let $\mathcal{P}$ satisfy* (1.4) *and let M be such that $\frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)} > 8\pi$. Then, for almost every $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)}\right)$ there exists a non-trivial solution to problem* (1.8)*. Furthermore, if $\mathcal{P}$ satisfies* (1.6) *and if M is such that $\frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)} \in (8\pi, 16\pi)$, then problem (1.8) admits a non-trivial solution for every $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)}\right)$.*

We organize this article as follows. In Section 2 we prove the mass quantization for blow-up sequences, as stated in Theorem 1.1. In Section 3 we establish an improved

Moser-Trudinger inequality for the Neri functional (1.5), on the line of Aubin [1]. In Section 4 we derive the Struwe's Monotonicity trick, originally introduced in [31] to construct bounded Palais-Smale sequences, in a form suitable for application to both Theorem 1.2 and Theorem 1.3. Our version of the monotonicity trick is therefore somewhat more general than the versions in [11, 32]. It should be mentioned that the monotonicity trick itself has attracted a considerable interest, and very general versions have been recently derived in [14, 15, 30]. Here, we choose to derive Struwe's argument in a specific form best suited to our applications, which also allows us to explicitly exhibit the corresponding deformations. Applying these results, in Section 5 we prove Theorem 1.2 and in Section 6 we prove Theorem 1.3, suitably adapting the ideas in [11] and [32] respectively, in order to take into account of the probability measure $\mathcal{P}$.

Notation Here and below, $\Omega \subset \mathbb{R}^2$ always denotes a smooth bounded domain and M always denotes a compact Riemannian 2-manifold without boundary. All integrals are taken with respect to the standard Lebesgue measure. When the integration measure is clear from the context, we may omit it for the sake of clarity. We denote by $C > 0$ a general constant whose actual value may vary from line to line. For every real number t we set $t^+ = \max\{0, t\}$.

2 Mass quantization and proof of Theorem 1.1

In this section we analyze the blow-up behavior of solution sequences for (1.7). Unlike the approaches in [20, 22, 26], where the cases of Dirichlet boundary conditions and of compact manifolds without boundary are considered, we establish our blow-up results in a more flexible local form, in the spirit of [3]. We prove the mass quantization extending the complex analysis approach introduced in [2, 34].

In order to prove Theorem 1.1, we begin by establishing a Brezis-Merle type alternative for equations with probability measures. More precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We consider solution sequences to the equation

$$-\Delta u_n = \int_{[0,1]} V_{\alpha,n}(x)\, e^{\alpha u_n}\, \mathcal{P}(d\alpha) + \varphi_n \qquad \text{in } \Omega. \tag{2.1}$$

where $\varphi_n \in L^\infty(\Omega)$. We begin by proving the existence of a "minimal mass" necessary for blow-up to occur.

Proposition 2.1 (Brezis-Merle alternative). *Assume $\mathcal{P}$ satisfies* (1.6) *and suppose the following bounds hold:*

(1) $0 \leqslant V_{\alpha,n} \leqslant C$, $\|\varphi_n\|_{L^\infty(\Omega)} \leqslant C$;

(2) $\iint_{[0,1]\times\Omega} V_{\alpha,n}(x)\, e^{\alpha u_n}\, \mathcal{P}(d\alpha)dx \leqslant C$

(3) $\|u_n^+\|_{L^1(\Omega)} \leqslant C$.

Then, exactly one of the following alternatives holds true:

(i) u_n converges locally uniformly in Ω to a bounded function u_0;

(ii) $u_n \to -\infty$ *locally uniformly in* Ω;

(iii) *There exists a finite set* $\mathcal{S} = \{p_1, \ldots, p_m\} \subset \Omega$ *such that*

$$\int_{[0,1]} V_{\alpha,n}(x)\, e^{\alpha u_n}\, \mathcal{P}(d\alpha)\, dx \overset{*}{\rightharpoonup} \sum_{i=1}^{m} n_i \delta_{p_i} + r(x)\, dx,$$

weakly in the sense of measures, with $n_i \geqslant 4\pi$, $i = 1, \ldots, m$ *and* $r \in L^1(\Omega)$. *Moreover,* u_n^+ *is locally uniformly bounded in* $\Omega \setminus \mathcal{S}$.

If u_n *is also locally uniformly bounded from below in* $\Omega \setminus \mathcal{S}$, *then* $\int_{B_\rho(p_j)} e^{u_n}\, dx \to +\infty$ *for any ball* $B_\rho(p_j) \subset \Omega$. *In particular, we have*

$$\iint_{[0,1]\times\Omega} e^{\alpha u_n}\, \mathcal{P}(d\alpha) dx \to +\infty.$$

Once Proposition 2.1 is established, setting

$$V_{\alpha,n}(x) = \alpha^{-1} \iint_{[0,1]\times\Omega} e^{\alpha u_n}\, \mathcal{P}(d\alpha) dx,$$

we readily derive alternatives (i)–(ii) and the first part of alternative (iii) in Theorem 1.1. In order to complete the proof of alternative (iii) in Theorem 1.1, we need to show that if $\mathcal{S} \neq \emptyset$, then $r \equiv 0$ and $n_i = 8\pi$ for all $i = 1, \ldots, m$. To this end, we prove that along a blow-up sequence (1.7) is equivalent to a nonlinear equation to which the complex analysis argument in [34] may be applied. More precisely, we show:

Proposition 2.2. *Let* (λ_n, u_n) *be a solution sequence for* (1.7) *with* $\lambda_n \to \lambda_0$. *Assume that* $\mathcal{S} \neq \emptyset$ *and* $u_n \geqslant -C$ *for some* $C > 0$. *Then* u_n *satisfies the equation*

$$-\Delta_g u_n = \kappa_n f(u_n) + c_n \qquad \text{in } \Omega, \tag{2.2}$$

for some $f(t) = e^t + o(e^t)$ *as* $t \to +\infty$, *and for some* $\kappa_n \to 0$. *Passing to a subsequence, we have* $u_n \to u_0$ *in* $W^{1,q}_{loc}(\Omega)$ *for all* $q \in [1, 2)$, *where*

$$u_0(x) = \frac{1}{4} \sum_{i=1}^{n} \log \frac{1}{d_g(x, p_i)} + b(x)$$

for some $b \in L^\infty_{loc}(\Omega)$. *At every blow-up point* p_i, $i = 1, \ldots, m$, *we have*

$$\nabla \left(b(p_i) + \frac{1}{4} \sum_{j \neq i} \log \frac{1}{d_g(p_i, p_j)} \right) = -\nabla \xi(p_i) \tag{2.3}$$

where ξ *is the conformal factor defined by* $g = e^{\xi(x)}(dx_1^2 + dx_2^2)$. *The blow-up masses satisfy* $n_{p_i} = 8\pi$, $i = 1, \ldots, m$ *and*

$$\kappa_n f(u_n) \overset{*}{\rightharpoonup} 8\pi \sum_{i=1}^{n} \delta_{p_i},$$

weakly in the sense of measures.

Theorem 1.1 will follow as a direct consequence of Proposition 2.1 and Proposition 2.2. We proceed towards the proof of Proposition 2.1. We recall the following well-known basic estimate.

Lemma 2.3 ([3]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, with $\|f\|_{L^1(\Omega)} < +\infty$. Then, for any $\eta \in (0,1)$ we have*

$$\int_\Omega \exp\left\{\frac{4\pi(1-\eta)}{\|f\|_{L_1(\Omega)}}|u|\right\} dx \leqslant \frac{\pi}{\eta}(\operatorname{diam}\Omega)^2.$$

Using Lemma 2.3 we can show the existence of a minimal mass for blow-up for equations containing a probability measure. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let u_n be a solution to (2.1) and let

$$\nu_n = \iint_{[0,1]\times\Omega} V_{\alpha,n}(x)e^{\alpha u_n}\,\mathcal{P}(d\alpha)dx.$$

In view of assumption (2) in Proposition 2.1, passing to a subsequence there exists $\nu_0 \in \mathcal{M}(D)$ such that $\nu_n \overset{*}{\rightharpoonup} \nu_0$ weakly in the sense of measures. The next lemma states that a minimal mass 4π is necessary for blow-up to occur.

Lemma 2.4 (Minimal mass for blow-up). *Let u_n be a solution to (2.1). Suppose $\|V_{\alpha,n}\|_{L^\infty(\Omega)} \leqslant C$, $\|u_n^+\|_{L^1(\Omega)} \leqslant C$, $\iint_{[0,1]\times\Omega} V_{\alpha,n}(x)e^{\alpha u_n}\,\mathcal{P}(d\alpha)dx \leqslant C$, $\|\varphi_n\|_{L^\infty(\Omega)} \leqslant C$ and suppose $x_0 \in \Omega$ is such that $\nu_0(\{x_0\}) < 4\pi$. Then, there exists $\rho_0 > 0$ such that $\|u_n^+\|_{L^\infty(B_{\rho_0}(x_0))} \leqslant C$.*

Proof. Let $\varepsilon_0, \rho_0 > 0$ be such that $\iint_{[0,1]\times B_{\rho_0}(x_0)} |V_{\alpha,n}(x)|e^{\alpha u_n}\,\mathcal{P}(d\alpha)dx \leqslant 4\pi(1-2\varepsilon_0)$ and $\|\varphi_n\|_{L^\infty(B_{\rho_0}(x_0))}|B_{\rho_0}(x_0)| \leqslant 4\pi\varepsilon_0$. Let w_n be defined by

$$\begin{cases} -\Delta w_n = \displaystyle\int_{[0,1]} V_{\alpha,n}(x)e^{\alpha u_n}\,\mathcal{P}(d\alpha) + \varphi_n & \text{in } B_{\rho_0}(x_0) \\ w_n = 0 & \text{on } \partial B_{\rho_0}(x_0). \end{cases} \tag{2.4}$$

Setting $\psi_n = \int_{[0,1]} V_{\alpha,n}(x)e^{\alpha u_n}\,\mathcal{P}(d\alpha) + \varphi_n$, we have $\|\psi_n\|_{L^1(B_{\rho_0}(x_0))} \leqslant 4\pi(1-\varepsilon_0)$. By elliptic estimates, $\|w_n\|_{L^1(B_{\rho_0}(x_0))} \leqslant C$. In view of Lemma 2.3 we derive for every $\eta \in (0,1)$ that

$$\int_{B_{\rho_0}(x_0)} \exp\left\{\frac{1-\eta}{1-\varepsilon_0}|w_n|\right\} \leqslant \frac{4\pi\rho_0^2}{\eta}.$$

Choosing $\eta < \varepsilon_0^2$, we find $\|\psi_n\|_{L^{1+\varepsilon_0}(B_{\rho_0}(x_0))} \leqslant C$.

On the other hand, the function $h_n := u_n - w_n$ is harmonic in $B_{\rho_0}(x_0)$ and

$$\|h_n^+\|_{L^1(B_{\rho_0}(x_0))} \leqslant \|u_n^+\|_{L^1(\Omega)} + \|w_n\|_{L^1(B_{\rho_0}(x_0))} \leqslant C.$$

Hence, the mean value theorem implies that $\|h_n^+\|_{L^\infty(B_{\rho_0/2}(x_0))} \leqslant C$. Inserting into (2.4), we find $\|w_n\|_{L^\infty(B_{\rho_0}(x_0))} \leqslant C\|\psi_n\|_{L^{1+\varepsilon_0}(B_{\rho_0}(x_0))} \leqslant C$. Finally, we have

$$\|u_n^+\|_{L^\infty(B_{\rho_0/2}(x_0))} \leqslant \|h_n^+\|_{L^\infty(B_{\rho_0/2}(x_0))} + \|w_n\|_{L^\infty(B_{\rho_0}(x_0))} \leqslant C.$$

Since ρ_0 is arbitrary, the asserted local uniform boundedness of u_n is established. □

Proof of Proposition 2.1. In view of Harnack's inequality, if $\mathcal{S}=\emptyset$ then (i) or (ii) hold. Therefore, we assume $\mathcal{S}\neq\emptyset$. By Harnack's inequality, either u_n is locally bounded from below in $\Omega\setminus\mathcal{S}$, or $u_n\to-\infty$ locally uniformly in $\Omega\setminus\mathcal{S}$. Let $p_i\in\mathcal{S}$ and let $\rho>0$ be such that $\overline{B_\rho(p_i)}\cap\mathcal{S}=\{p_i\}$. We assume that $u_n\geqslant -C$ on $\partial B_\rho(p_i)$. Similarly as in [3], we define

$$\begin{cases}-\Delta z_n=\displaystyle\int_{[0,1]}V_{\alpha,n}(x)\,e^{\alpha u_n}\,\mathcal{P}(d\alpha)+\varphi_n & \text{in } B_\rho(p_i)\\ \qquad z_n=-\,C & \text{on } \partial B_\rho(p_i).\end{cases}$$

Then, $u_n\geqslant z_n$ in $\overline{B_\rho(p_i)}$. On the other hand, $z_n\to z$ in $W^{1,q}(B_\rho(p_i))$ for all $q\in[1,2)$, with $z\geqslant\log|x-p_i|^{-2}-C$. By Fatou's lemma, we conclude that $\int_{B_\rho(p_i)}e^{u_n}\,dx\to+\infty$. In view of assumption (1.6) we derive in turn that

$$\iint_{[0,1]\times\Omega}e^{\alpha u_n}\,\mathcal{P}(d\alpha)dx\geqslant\mathcal{P}(\{1\})\int_\Omega e^{u_n}\,dx\to+\infty. \tag{2.5}$$

$\square$

Proof of Propostion 2.2. Since g is given in isothermal coordinates, namely $g=e^{\xi(x_1,x_2)}(dx_1^2+dx_2^2)$, then (1.7) takes the form

$$-\Delta u_n=\lambda_n e^\xi\int_{[0,1]}\frac{\alpha e^{\alpha u_n}}{\iint_{[0,1]\times\Omega}e^{\alpha u_n}}\,\mathcal{P}(d\alpha)+e^\xi c_n.$$

We apply Proposition 2.1–(iii) with $V_{\alpha,n}=e^\xi\lambda_n\alpha(\iint_{[0,1]\times\Omega}e^{\alpha u_n}\,\mathcal{P}(d\alpha)\,dv_g)^{-1}$ and $\varphi_n=e^\xi c_n$. Since $u_n\geqslant -C$, we conclude that

$$\iint_{[0,1]\times\Omega}e^{\alpha u_n}\,\mathcal{P}(d\alpha)dv_g\geqslant C^{-1}>0.$$

Moreover, (2.5) holds. We define

$$f(t):=(\mathcal{P}(\{1\}))^{-1}\int_{[0,1]}\alpha e^{\alpha t}\,\mathcal{P}(d\alpha)$$

and

$$\kappa_n:=\frac{\lambda_n\mathcal{P}(\{1\})}{\iint_{[0,1]\times\Omega}e^{\alpha u_n}\,\mathcal{P}(d\alpha)dx}.$$

With such definitions, u_n satisfies (2.2). In view of (2.5), we have $\kappa_n\to0$. Consequently, $r\equiv0$ and furthermore

$$\kappa_n f(u_n)\overset{*}{\rightharpoonup}\sum_{i=1}^n n_i\delta_{p_i}$$

weakly in the sense of measures. We are left to establish that $n_i=8\pi$, for all $i=1,\dots,m$ and that the blow-up points satisfy condition (2.3).

To this end, we adapt some ideas of [34]. We set

$$F(t)=(\mathcal{P}(\{1\}))^{-1}\int_{[0,1]}e^{\alpha t}\,\mathcal{P}(d\alpha).$$

Then, $F'(t) = f(t)$ and furthermore we have the following.

Claim A. As $t \to +\infty$, we have:

$$f(t) = e^t + o(e^t) \qquad\qquad F(t) = e^t + o(e^t). \tag{2.6}$$

Proof of Claim A. Let

$$p(t) = \int_{[0,1)} \alpha e^{\alpha t}\,\mathcal{P}(d\alpha), \qquad\qquad P(t) = \int_{[0,1)} e^{\alpha t}\,\mathcal{P}(d\alpha). \tag{2.7}$$

Then, $f(t) = e^t + \tau^{-1}p(t)$, $F(t) = e^t + \tau^{-1}P(t)$.

For any given $\varepsilon > 0$, we fix $0 < \delta_\varepsilon \ll 1$ such that

$$\int_{[1-\delta_\varepsilon,1)} e^{-(1-\alpha)t}\mathcal{P}(d\alpha) \leqslant \mathcal{P}([1-\delta_\varepsilon,1)) < \frac{\varepsilon}{2}.$$

Correspondingly, we take $t_\varepsilon \gg 1$ such that

$$\int_{[0,1-\delta_\varepsilon)} e^{-(1-\alpha)t}\mathcal{P}(d\alpha) \leqslant e^{-\delta_\varepsilon t} < \frac{\varepsilon}{2} \qquad \forall t \geqslant t_\varepsilon.$$

It follows that $e^{-t}P(t) = \int_{[0,1)} e^{-(1-\alpha)t}\mathcal{P}(d\alpha) < \varepsilon$ whenever $t \geqslant t_\varepsilon$. That is, $P(t) = o(e^t)$, and the second part of (2.6) is established. The first part of (2.6) follows by observing that $0 \leqslant p(t) \leqslant P(t)$. Hence Claim A is established.

Claim B. We have

$$\kappa_n F(u_n) \stackrel{*}{\rightharpoonup} \sum_{i=1}^n n_i \delta_{p_i}. \tag{2.8}$$

Proof of Claim B. Let $p(t)$ be the function defined in (2.7). For $p_i \in \mathcal{S}$, let $B_\rho(p_i)$ be such that $\overline{B_\rho(p_i)} \cap \mathcal{S} = \{p_i\}$. Let $\varphi \in C_c(B_\rho(p_i))$. Let $\varepsilon > 0$ and $t'_\varepsilon \gg 1$ be such that $e^{-t}p(t) < \varepsilon/2$ whenever $t \geqslant t'_\varepsilon$. We have

$$\begin{aligned}\left|\kappa_n \int_\Omega p(u_n)\varphi\right| \leqslant& \kappa_n \int_{u_n \geqslant t'_\varepsilon} p(u_n)|\varphi| + \kappa_n \int_{u_n < t'_\varepsilon} p(u_n)|\varphi| \\ \leqslant& \frac{\varepsilon}{2}\int_\Omega \kappa_n e^{u_n}|\varphi| + \kappa_n \max_{[0,t'_\varepsilon]} p \int_\Omega |\varphi| < c\varepsilon\end{aligned}$$

for sufficiently large n. Since ε and φ are arbitrary, we conclude that $\kappa_n p(u_n) \stackrel{*}{\rightharpoonup} 0$ weakly in the sense of measures. By the same argument, we conclude that $\kappa_n P(u_n) \stackrel{*}{\rightharpoonup} 0$ weakly in the sense of measures. Therefore,

$$\kappa_n \int_\Omega e^{u_n}\varphi = \kappa_n \int_\Omega (f(u_n) - p(u_n))\varphi \to n_i\varphi(p_i)$$

and

$$\kappa_n \int_\Omega F(u_n)\varphi = \kappa_n \int_\Omega (e^{u_n} + P(u_n))\varphi \to n_i\varphi(p_i).$$

Hence, (2.8) is established.

Claim C: There holds

$$n_i = 8\pi \tag{2.9}$$

for all $i = 1, \dots, m$.

Proof of Claim C. We adapt the complex analysis argument in [34]. For the sake of simplicity, throughout this proof we omit the index n. We fix a blow-up point $p \in \mathcal{S}$ and without loss of generality we assume that $p = 0$ and $\xi(0) = 0$. We define

$$W(t) = \kappa F(t) + c\, t$$

and we consider the Newtonian potential $N = (4\pi)^{-1} \log(z\bar{z})$ so that $\Delta N = \delta_0$. We define

$$H = \frac{u_z^2}{2}, \qquad\qquad K = N_z * \{e^{\xi} \chi_{B_\rho} [W(u)]_z\}.$$

It is readily checked that the function $S = H + K$ satisfies $\partial_{\bar{z}} S = 0$ in B_ρ. It follows that S converges uniformly to a holomorphic function S_0. On the other hand, we have $u_n \to u_0$ in $W^{1,q}(B_\rho)$, $q \in [1, 2)$, where

$$u_0(x) = \frac{n_p}{4\pi} \log(z\bar{z}) + \omega,$$

where ω is smooth in B_ρ. Taking limits for H we thus find that $H \to H_0$, where

$$H_0 = \frac{n_p^2}{32\pi^2 z^2} - \frac{n_p}{4\pi z} \omega_z + \frac{1}{2} \omega_z^2.$$

On the other hand, we may write

$$K = N_{zz} * \{e^{\xi} \chi_{B_\rho} W(u)\} - N_z * \{[e^{\xi} \chi_{B_\rho}]_z W(u)\}.$$

In view of Claim B we have $W(u) \overset{*}{\rightharpoonup} n_p \delta_0 + c_0 u_0$. Recalling that $N_z = (4\pi z)^{-1}$, $N_{zz} - (4\pi z^2)^{-1}$, we thus compute

$$N_{zz} * \{e^{\xi} \chi_{B_\rho} W(u)\} \to -\frac{n_p}{4\pi z^2} - N_z * \{e^{\xi} \chi_{B_\rho} c_0 u_{0,z}\}$$

and

$$N_z * \{[e^{\xi} \chi_{B_\rho}]_z W(u)\} \to \frac{n_p}{4\pi z} \xi_z(p) + c_0 N_z * \{e^{\xi} \chi_{B_\rho} u_0\},$$

pointwise in $B_\rho \setminus \{0\}$. Therefore, $K \to K_0$ where

$$K_0 = -\frac{n_p}{4\pi z^2} - N_z * \{e^{\xi} \chi_{B_\rho} c_0 u_{0,z}\} - \frac{n_p}{4\pi z} \xi_z(p) - c_0 N_z * \{e^{\xi} \chi_{B_\rho} u_0\}.$$

Since $S_0 = H_0 + K_0$ is holomorphic, by balancing singularities we derive

$$\frac{n_p^2}{32\pi^2} = \frac{n_p}{4\pi}, \qquad\qquad \frac{n_p}{4\pi} \omega_z(p) = -\frac{n_p}{4\pi} \xi_z(p).$$

Hence, (2.9) holds and Claim C is established. Moreover, we have

$$\omega_z(p) = -\xi_z(p).$$

Finally, observing that

$$\omega(x) = b(p) + \frac{1}{4} \sum_{p' \neq p} \log \frac{1}{d_g(p', p)},$$

we derive (2.3) $\square$

For later application, we now explicitely state the mass quantization for Neri's equation on a domain with Dirichlet boundary conditions and on a compact Riemannian 2-manifold without boundary. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let G_Ω be the Green's function defined by

$$\begin{cases} -\Delta_x G_\Omega(x,y) = \delta_y & \text{in } \Omega \\ G_\Omega(\cdot,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that

$$G_\Omega(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + h(x,y),$$

where $h(x,y)$ is the regular part of G_Ω. Assuming that $\mathcal{P}$ satisfies (1.6), problem (1.1) takes the form:

$$\begin{cases} -\Delta u = \lambda \displaystyle\int_{[0,1]} \frac{\alpha e^{\alpha u}\,\mathcal{P}(d\alpha)}{\iint_{[0,1]\times\Omega} e^{\alpha u}\,\mathcal{P}(d\alpha)dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

By the maximum principle, we have $u > 0$ in Ω. In the next lemma, we exclude the existence of blow-up on $\partial\Omega$.

Lemma 2.5. *Let (λ_n, u_n) be a solution sequence to (2.10) with $\lambda \to \lambda_0$. There exists a tubular neighborhood Ω_δ of $\partial\Omega$ and a constant $C > 0$ such that $\|u_n\|_{L^\infty(\Omega_\delta)} \leqslant C$.*

Proof. By a result in [13], p. 223, it is known that there exists a tubular neighborhood Ω_δ of $\partial\Omega$, depending on the geometry of Ω only, such that any solution to a problem of the form $-\Delta u = f(u)$ satisfying $u = 0$ on $\partial\Omega$, where $f(t) \geqslant 0$ is Lipschitz continuous, has no stationary points in Ω_δ. We may assume that $\partial\Omega_\delta \cap \Omega \cap \mathcal{S} = \emptyset$. Let $x_n \in \bar{\Omega}_\delta$ be such that $u_n(x_n) = \max_{\bar{\Omega}_\delta} u_n$. Arguing by contradiction, suppose that $u_n(x_n) \to +\infty$. Since $u_n = 0$ on $\partial\Omega$, and since u_n is uniformly bounded on $\partial\Omega_\delta \cap \Omega$, then, for n sufficiently large, $x_n \in \Omega_\delta$ and $\nabla u_n(x_n) = 0$, a contradiction. □

At this point, the following result readily follows.

Proposition 2.6 (Mass quantization for the Dirichlet problem)**.** *Assume (1.6). Let (λ_n, u_n) be a solution sequence to the problem (2.10) with $\lambda = \lambda_n \to \lambda_0$. Then, up to subsequences, exactly one of the following alternatives holds:*

(i) There exists a solution u_0 to equation (2.10) with $\lambda = \lambda_0$ such that $u_n \to u_0$;

(ii) There exists a finite set $\mathcal{S} = \{p_1, \ldots, p_m\} \subset \Omega$ such that $u_n \to u_0$ in $W_0^{1,q}(\Omega)$ for all $q \in [1,2)$, where

$$u_0(x) = 8\pi \sum_{i=1}^m G_\Omega(x, p_i).$$

Moreover, the points p_i satisfy the condition $\nabla R_i(p_i) = 0$, where

$$R_i(x) = h(x,p_i) + \sum_{j\neq i} G_\Omega(x,p_j)$$

and

$$\lambda_n \int_{[0,1]} \frac{\alpha e^{\alpha u_n}\,\mathcal{P}(d\alpha)}{\iint_{[0,1]\times\Omega} e^{\alpha u_n}\,\mathcal{P}(d\alpha)dx}\,dx \stackrel{*}{\rightharpoonup} 8\pi \sum_{j=1}^m \delta_{p_j}(dx),$$

weakly in the sense of measures.

We note that Proposition 2.6 is consistent with Theorem 1 in [18].

Proof of Proposition 2.6. The proof is a direct consequence of Proposition 2.1 and Proposition 2.2. In view of Lemma 2.5, blow-up does not occur on the boundary $\partial\Omega$. Since $u > 0$, alternative (ii) in Proposition 2.1 cannot occur. Moreover, at a given blow-up point $p_i \in \mathcal{S}$, we have

$$b(p_i) + \frac{1}{4}\sum_{j\neq i} \log \frac{1}{d_g(p_i, p_j)} = h(x, p_i) + \sum_{j\neq i} G_\Omega(x, p_j)$$

and since g is Euclidean, $\xi \equiv 0$. □

Similarly, let (M, g) be a compact orientable Riemannian surface without boundary. Let G_M be the Green's function defined by

$$\begin{cases} -\Delta_g G_M(x, y) = \delta_y - \dfrac{1}{|M|} \\ \displaystyle\int_M G_M(x, y) dv_g = 0. \end{cases}$$

Then,

$$G_M(x, y) = \frac{1}{2\pi} \log \frac{1}{d_g(x, y)} + h(x, y),$$

where h is the regular part of G_M, see [1]. Assuming (1.6), Neri's equation on a manifold (1.8) takes the form

$$\begin{cases} -\Delta_g v = \lambda \displaystyle\int_{[0,1]} \frac{\alpha(e^{\alpha v} - \frac{1}{|M|}\int_M e^{\alpha v}\, dv_g)}{\iint_{[0,1]\times M} e^{\alpha v}\, \mathcal{P}(d\alpha)\, dv_g} \mathcal{P}(d\alpha) \quad \text{in } M \\ \displaystyle\int_M v\, dv_g = 0, \end{cases} \tag{2.11}$$

The following holds.

Proposition 2.7 (Mass quantization for the problem on M)**.** *Assume* (1.6)*. Let* (λ_n, v_n) *be a solution sequence to the problem* (2.11) *with* $\lambda = \lambda_n \to \lambda_0$*. Then, up to subsequences, exactly one of the following alternatives holds:*

(i) There exists a solution v_0 *to equation* (2.11) *with* $\lambda = \lambda_0$ *such that* $v_n \to v_0$*;*

(ii) There exist a finite number of points $p_1, \ldots, p_m \in M$ *such that* $v_n \to v_0$ *in* $W^{1,q}(M)$ *for all* $q \in [1, 2)$*, where*

$$v_0(x) = 8\pi \sum_{j=1}^{m} G_M(x, p_j).$$

Moreover, the points p_i*,* $i = 1, \ldots, m$ *satisfy the condition* $\nabla R_i(p_i) = -\nabla\xi(p_i)$*, where* $g = e^{\xi(x)}(dx_1^2 + dx_2^2)$ *and*

$$R_i(x) = h(x, p_i) + \sum_{j\neq i} G_M(x, p_j).$$

Furthermore,

$$\lambda_n \int_{[0,1]} \frac{\alpha e^{\alpha v_n}\,\mathcal{P}(d\alpha)}{\iint_{[0,1]\times M} e^{\alpha v_n}\,\mathcal{P}(d\alpha)dv_g}\,dx \overset{*}{\rightharpoonup} 8\pi\sum_{j=1}^{m}\delta_{p_j},$$

weakly in the sense of measures.

Proof. The proof is analogous to the proof of Proposition 2.6. □

3 An improved Moser-Trudinger inequality

We derive an improved Moser-Trudinger inequality for the functional (1.5) defined on a bounded domain $\Omega \subset \mathbb{R}^2$ which will be needed in the proof of Theorem 1.2.

We recall that the classical Moser-Trudinger sharp inequality [19] states that

$$C_{MT} := \sup\left\{\int_\Omega e^{4\pi u^2}:\ u\in H_0^1(\Omega),\ \|\nabla u\|_2 \leqslant 1\right\} < +\infty, \tag{3.1}$$

where the constant 4π is best possible. Moreover, the embeddings $u \in H_0^1(\Omega) \to e^u \in L^1(\Omega)$ and $v \in H^1(M) \to e^v \in L^1(M)$ are compact. For a proof, see, e.g., Theorem 2.46 pag. 63 in [1].

In view of the elementary inequality

$$|u| \leqslant \frac{\|\nabla u\|_2^2}{16\pi} + 4\pi\frac{u^2}{\|\nabla u\|_2^2}$$

we derive using (3.1) that

$$\int_\Omega e^{|u|}\,dx \leqslant C_{MT}e^{\frac{1}{16\pi}\|\nabla u\|_2^2}, \qquad \forall u\in H_0^1(\Omega).$$

In particular, the standard Moser-Trudinger functional

$$I_\lambda(u) = \frac{1}{2}\|\nabla u\|_2^2 - \lambda\log\int_\Omega e^u\,dx$$

is bounded below for all $\lambda \leqslant 8\pi$ and

$$\inf_{u\in H_0^1(\Omega)} I_\lambda(u) = -\infty$$

whenever $\lambda > 8\pi$. From the arguments above it can be shown that if $\operatorname{supp}\mathcal{P}\cap\{-1,1\}\neq\emptyset$, then Neri's functional (1.5) is also bounded below on $H_0^1(\Omega)$ if and only if $\lambda \leqslant 8\pi$, and that

$$\inf_{u\in H_0^1(\Omega)} J_\lambda(u) = -\infty$$

whenever $\lambda > 8\pi$ and $\operatorname{supp}\mathcal{P}\cap\{-1,1\}\neq\emptyset$. More precisely, we have

Proposition 3.1. *Let* $\operatorname{supp}\mathcal{P}\cap\{-1,1\}\neq\emptyset$*. Then, the functional* $J_\lambda(u)$ *is bounded from below on* $H_0^1(\Omega)$*, if and only if* $\lambda \leqslant 8\pi$*.*

Proposition 3.1 was established for functions $u \in H^1(M)$ satisfying $\int_M u = 0$, where M is a two-dimensional Riemannian manifold, in [26]. The proof for $u \in H_0^1(\Omega)$ is similar. For the sake of completeness, we outline it below. In the improved Moser-Trudinger inequality we show that the best constant in Proposition 3.1 may be lowered if the "mass" of u is suitably distributed. Namely, following ideas of [1, 9], we prove:

Proposition 3.2 (Improved Moser-Trudinger inequality)**.** *Let $d_0 > 0$ and $a_0 \in (0, 1/2)$. Then, for any $\varepsilon > 0$, there exists a constant $K = K(\varepsilon, d_0, a_0) > 0$ such that if $u \in H_0^1(\Omega)$ satisfies*

$$\frac{\iint_{I\times\Omega_i} e^{\alpha u}\mathcal{P}(d\alpha)dx}{\iint_{I\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx} \geqslant a_0, \qquad i = 1, 2 \tag{3.2}$$

where $\Omega_1, \Omega_2 \subset \Omega$ are two measurable sets verifying dist$(\Omega_1, \Omega_2) \geqslant d_0$, then it holds

$$\iint_{I\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx \leqslant K \exp\left\{\left(\frac{1}{32\pi} + \varepsilon\right) \|\nabla u\|_2^2\right\}. \tag{3.3}$$

We begin by outlining the proof of Proposition 3.1.

Proof of Proposition 3.1. The "if" part is immediate and was already used in [21]. Indeed, since

$$\iint_{I\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx \leqslant \int_\Omega e^{|u|}dx \leqslant C_{MT} e^{\frac{1}{16\pi}\|\nabla u\|_2^2},$$

for all $u \in H_0^1(\Omega)$. Therefore J_λ is bounded below if $\lambda \leqslant 8\pi$. On the other hand the value 8π is also optimal, provided that supp$\mathcal{P} \cap \{-1, 1\} \neq \emptyset$. Indeed, the following holds: We need only prove that

$$\inf_{u\in H_0^1(\Omega),} J_\lambda(u) = -\infty, \qquad \forall\lambda > 8\pi. \tag{3.4}$$

Assume that $1 \in$ supp$\mathcal{P}$ (the case $-1 \in$ supp$\mathcal{P}$ is similar). Since the functional $I_\lambda(u)$ is unbounded below for $\lambda > 8\pi$, then also the functional

$$I_\lambda(u)_{|u\geqslant 0} = \frac{1}{2}\|\nabla u\|_2^2 - \lambda \log \int_\Omega e^u\, dx, \qquad u \in H_0^1(\Omega),\ u \geqslant 0$$

is unbounded below for $\lambda > 8\pi$. At this point we observe that for every $0 < \delta < 1$ and $u \geqslant 0$, $u \in H_0^1(\Omega)$, we have:

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \lambda \log \iint_{I\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx \leqslant \frac{1}{2}\|\nabla u\|_2^2 - \lambda \log \iint_{[1-\delta,1]\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx \\
&\leqslant \frac{1}{2}\|\nabla u\|_2^2 - \lambda \log \int_\Omega e^{(1-\delta)u}dx - \lambda\log(\mathcal{P}([1-\delta, 1])) \\
&= \frac{1}{(1-\delta)^2}\left[\frac{1}{2}\|(1-\delta)\nabla u\|_2^2 - \lambda(1-\delta)^2 \log\left(\int_\Omega e^{(1-\delta)u}dx\right)\right] - \lambda\log(\mathcal{P}([1-\delta, 1])) \\
&= \frac{1}{(1-\delta)^2} I_{\lambda(1-\delta)^2}\left((1-\delta)u\right) - \lambda\log(\mathcal{P}([1-\delta, 1])).
\end{aligned} \tag{3.5}$$

Hence, for $\lambda(1-\delta)^2 > 8\pi$, the right hand side of last inequality is unbounded from below and so

$$\inf_{u\in H_0^1(\Omega)} J_\lambda(u) = -\infty \qquad \text{for any } \lambda > \frac{8\pi}{(1-\delta)^2}.$$

Since $\delta \in (0, 1)$ is arbitrary, (3.4) follows. ∎

In order to prove Proposition 3.2, We adapt some ideas contained in [9], Proposition 1.

Proof. Let g_1 and g_2 be smooth functions defined on Ω such that $0 \leqslant g_i \leqslant 1$, $i = 1, 2$; $g_i \equiv 1$ on Ω_i, $i = 1, 2$; $\mathrm{supp} g_1 \cap \mathrm{supp} g_2 = \emptyset$; $|\nabla g_i| \leqslant c(d_0)$, $i = 1, 2$. We may assume that $\|g_1 \nabla u\|_{L^2(\Omega)} \leqslant \|g_2 \nabla u\|_{L^2(\Omega)}$ (otherwise it is sufficient to switch the functions g_1 and g_2). Denote, for every real number t, $t^+ = \max\{0, t\}$ and let $a > 0$. In view of the elementary inequality

$$g_1(|u| - a)^+ \leqslant \frac{1}{16\pi} \|\nabla \left[g_1(|u| - a)^+ \right] \|^2_{L^2(\Omega)} + \frac{4\pi (g_1(|u| - a)^+)^2}{\|\nabla \left[g_1(|u| - a)^+ \right] \|^2_{L^2(\Omega)}},$$

we derive from (3.1) that

$$\int_\Omega e^{g_1(|u|-a)^+} \leqslant C_{MT} \exp\left\{ \frac{1}{16\pi} \|\nabla \left[g_1(|u| - a)^+ \right] \|^2_{L^2(\Omega)} \right\}. \tag{3.6}$$

Hence, using (3.2) and (3.6), and using the elementary inequality $(A + B)^2 \leqslant (1 + \tau)A^2 + c(\tau)B^2$ for any $\tau > 0$, we have

$$\begin{aligned}
\iint_{I\times\Omega} e^{\alpha u} \mathcal{P}(d\alpha) dx &\leqslant \frac{e^a}{a_0} \iint_{I\times\Omega_1} e^{(\alpha u - a)^+} \mathcal{P}(d\alpha) dx \\
&\leqslant \frac{e^a}{a_0} \iint_{I\times\Omega} e^{g_1(\alpha u - a)^+} \mathcal{P}(d\alpha) dx \leqslant \frac{e^a}{a_0} \int_\Omega e^{g_1(|u|-a)^+} dx \\
&\leqslant \frac{C}{a_0} \exp\left\{ \frac{1}{16\pi} \|\nabla \left[g_1(|u| - a)^+ \right] \|^2_{L^2(\Omega)} + a \right\} \\
&\leqslant \frac{C}{a_0} \exp\left\{ \frac{1}{16\pi} \left[(1 + \tau) \|g_1 \nabla u\|^2_{L^2(\Omega)} \right.\right. \\
&\left.\left. + c(\tau) \|(|u| - a)^+ \nabla g_1\|^2_{L^2(\Omega)} \right] + a \right\} \\
&\leqslant \frac{C}{a_0} \exp\left\{ \frac{1}{32\pi} \left[(1 + \tau) \|(g_1 + g_2) \nabla u\|^2_{L^2(\Omega)} + c(\tau, d_0) \|(|u| - a)^+\|^2_{L^2(\Omega)} \right] + a \right\} \\
&\leqslant \frac{C}{a_0} \exp\left\{ \frac{1}{32\pi} \left[(1 + \tau) \|\nabla u\|^2_{L^2(\Omega)} + c(\tau, d_0) \|(|u| - a)^+\|^2_{L^2(\Omega)} \right] + a \right\}
\end{aligned}$$

for some small $\tau > 0$, where $C = C(\Omega)$. For a given real number $\eta \in (0, |\Omega|)$, let

$$a = a(\eta, u) = \sup \left\{ c \geqslant 0 : \text{ meas } \{ x \in \Omega : |u(x)| \geqslant c \} > \eta \right\}.$$

We have

$$\|(|u| - a)^+\|^2_{L^2(\Omega)} = \int_{\{x\in\Omega:|u|>a\}} (|u| - a)^2 \leqslant \eta^{\frac{1}{2}} C \|\nabla u\|^2_2$$

Using the Schwarz and Poincaré inequalities, we finally derive

$$a\eta \leqslant \int_{\{|u|\geqslant a\}} |u| \leqslant |\Omega|^{\frac{1}{2}} \left(\int_\Omega |u|^2 \right)^{\frac{1}{2}} \leqslant C \|\nabla u\|_2,$$

and therefore, for any small $\delta > 0$,

$$a \leqslant \frac{\delta}{2} \|\nabla u\|^2_2 + \frac{C^2}{2\delta\eta^2}.$$

The asserted improved Moser-Trudinger inequality (3.3) is completely established. □

4 Struwe's Monotonicity Trick: a unified form

The aim of this section is to establish Struwe's Monotonicity Trick in a unified form convenient for application to both Theorem 1.2 and Theorem 1.3. Let $\Lambda \subset \mathbb{R}_+$ be a bounded interval and let $\mathcal{H}$ be a Hilbert space. In this section, for $\lambda \in \Lambda$, we consider functionals of the form

$$\mathcal{J}_\lambda(w) = \frac{1}{2}\|w\|^2 - \lambda \mathcal{G}(w)$$

defined for every $w \in \mathcal{H}$, where $\mathcal{G} \in \mathcal{C}^2(\mathcal{H}; \mathbb{R})$ satisfies:

$$\mathcal{G}' \text{ is compact}, \qquad \langle \mathcal{G}''(w)\varphi, \varphi\rangle \geqslant 0 \text{ for every } w, \varphi \in \mathcal{H}. \tag{4.1}$$

We do not make any sign assumption on $\mathcal{G}$. Let $V \subset \mathbb{R}^m$ be a bounded domain. We consider the family

$$\mathcal{F}_\lambda := \{f \in \mathcal{C}(V; \mathcal{H}) : f \text{ satisfies } \mathscr{P}(\partial V)\}$$

where $\mathscr{P}(\partial V)$ is a set of properties defined on ∂V, including a property of the form:

$$\limsup_{\theta \to \partial V} \mathcal{J}_\lambda(f(\theta)) \leqslant A, \tag{4.2}$$

for some $A \in [-\infty, +\infty)$. We assume that for every $\lambda, \lambda' \in \Lambda$ it holds that $\mathcal{F}_\lambda \neq \emptyset$ and

$$\lambda' < \lambda \Longrightarrow \mathcal{F}_{\lambda'} \subseteq \mathcal{F}_\lambda.$$

Under these assumptions, we define the minimax value:

$$c_\lambda = \inf_{f \in \mathcal{F}_\lambda} \sup_{\theta \in V} \mathcal{J}_\lambda(f(\theta)).$$

and we assume that c_λ is finite for every λ. Since for every fixed $w \in \mathcal{H}$ the function $\lambda^{-1}\mathcal{J}_\lambda$ is non-increasing with respect to λ, $\lambda^{-1}c_\lambda$ is non-increasing as well. Therefore, writing $c_\lambda = \lambda(\lambda^{-1}c_\lambda)$ we see that

$$c'_\lambda = \left.\frac{dc_{\lambda+\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0}$$

is well-defined and finite for almost every $\lambda \in \Lambda$. We shall use the monotonicity trick in the following form.

Proposition 4.1 (Struwe's Monotonicity Trick)**.** *Suppose that $\mathcal{G}$ satisfies assumptions* (4.1) *and let $\lambda \in \Lambda$ be such that c'_λ exists. If*

$$c_\lambda > A, \tag{4.3}$$

then c_λ is a critical value for $\mathcal{J}_\lambda$. That is, there exists $w \in \mathcal{H}$ such that $\mathcal{J}_\lambda(w) = c_\lambda$ and $\mathcal{J}'_\lambda(w) = 0$.

In order to prove Proposition 4.1 we need some lemmas.

Lemma 4.1. *Let $\lambda \in \Lambda$ be such that c'_λ exists. Then, there exist two constants $K = K(c'_\lambda) > 0$ and $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and for all $w \in \mathcal{H}$ satisfying*

$$\mathcal{J}_\lambda(w) \geqslant c_\lambda - \varepsilon \qquad \text{and} \qquad \mathcal{J}_{\lambda-\varepsilon}(w) \leqslant c_{\lambda-\varepsilon} + \varepsilon \tag{4.4}$$

it holds that

(i) $\mathcal{G}(w) \leqslant K$*;*

(ii) $\|w\| \leqslant \sqrt{2(c_\lambda + \lambda K + 1)}$ *;*

(iii) $|\mathcal{J}_\lambda(w) - c_\lambda| \leqslant (\frac{c_\lambda}{\lambda} + |c'_\lambda| + 2)\varepsilon$.

Proof. Proof of (i). By (4.4) it follows that

$$\mathcal{G}(w) = \frac{\mathcal{J}_{\lambda-\varepsilon}(w) - \mathcal{J}_\lambda(w)}{\varepsilon} \leqslant \frac{1}{\varepsilon}(c_{\lambda-\varepsilon} - c_\lambda) + 2 = 2 - c'_\lambda + o(1)$$

for ε sufficiently small. Hence, (i) follows with $K = 3 - c'_\lambda$.

Proof of (ii). By the monotonicity property of $\lambda^{-1}\mathcal{J}_\lambda$, we have $\mathcal{J}_\lambda(w) \leqslant \frac{\lambda}{\lambda-\varepsilon}\mathcal{J}_{\lambda-\varepsilon}(w)$ for all $w \in \mathcal{H}$. Consequently, in view of (4.4) and (i) we have

$$\begin{aligned}\frac{1}{2}\|w\|^2 = \mathcal{J}_\lambda(w) + \lambda\mathcal{G}(w) &\leqslant \frac{\lambda}{\lambda-\varepsilon}(c_{\lambda-\varepsilon} + \varepsilon) + \lambda K \\ &\leqslant \left(1 + \frac{\varepsilon}{\lambda-\varepsilon}\right)(c_\lambda - \varepsilon c'_\lambda + o(\varepsilon) + \varepsilon) + \lambda K \leqslant c_\lambda + \lambda K + o(1)\end{aligned}$$

for ε sufficiently small, so that the (ii) follows.

Proof of (iii). Similarly,

$$\begin{aligned}c_\lambda - \varepsilon \leqslant \mathcal{J}_\lambda(w) \leqslant \frac{\lambda}{\lambda-\varepsilon}(c_{\lambda-\varepsilon} + \varepsilon) &= \left(1 + \frac{\varepsilon}{\lambda-\varepsilon}\right)(c_\lambda - \varepsilon c'_\lambda + o(\varepsilon) + \varepsilon) \\ &= c_\lambda + \varepsilon\left(\frac{c_\lambda}{\lambda-\varepsilon} - c'_\lambda + 1 + o(1)\right).\end{aligned}$$

for ε sufficiently small, and (iii) follows. □

In the next lemma we show the existence of bounded Palais-Smale sequences for $\mathcal{J}_\lambda$ at the level c_λ.

Lemma 4.2. *Under the assumptions of Proposition 4.1, for every $\varepsilon > 0$ sufficiently small there exists $w_\varepsilon \in \mathcal{H}$ such that*

(i) $|\mathcal{J}_\lambda(w_\varepsilon) - c_\lambda| \leqslant C\varepsilon$

(ii) $\|w_\varepsilon\| \leqslant C$,

(iii) $\|\mathcal{J}'_\lambda(w_\varepsilon)\|_{\mathcal{H}'} \to 0$ *as* $\varepsilon \to 0$.

Proof. Proof of (i)–(ii). Let $\varepsilon > 0$. By definition of $c_{\lambda-\varepsilon}$ and by the infimum property, there exists $f_\varepsilon \in \mathcal{F}_{\lambda-\varepsilon}$ such that

$$\sup_{\theta\in V} \mathcal{J}_{\lambda-\varepsilon}(f_\varepsilon(\theta)) \leqslant c_{\lambda-\varepsilon} + \varepsilon.$$

Moreover, since $f_\varepsilon \in \mathcal{F}_{\lambda-\varepsilon} \subseteq \mathcal{F}_\lambda$, by definition of c_λ and by the supremum property, there exists $\theta_\varepsilon \in V$ such that

$$\mathcal{J}_\lambda(f_\varepsilon(\theta_\varepsilon)) \geqslant c_\lambda - \varepsilon.$$

We conclude that $w_\varepsilon := f_\varepsilon(\theta_\varepsilon) \in \mathcal{H}$ satisfies (4.4). Consequently, Lemma 4.1 implies that, for ε sufficiently small, $\|w_\varepsilon\| \leqslant C$ and $|\mathcal{J}_\lambda(w_\varepsilon) - c_\lambda| \leqslant \varepsilon C$, for some $C > 0$ independent of ε.

Proof of (iii). For $\delta > 0$ we set

$$N_\delta := \{w \in \mathcal{H} : \|w\| \leqslant C, |\mathcal{J}_\lambda(w) - c_\lambda| < \delta\},$$

and we note that in view of the already established properties (i)–(ii) we have $N_\delta \neq \emptyset$.

Suppose that the claim is false. We shall derive a contradiction by constructing a suitable deformation. To this end, we assume that there exists $\delta > 0$ such that $\|\mathcal{J}'_\lambda(w)\| \geqslant \delta$ for every $w \in N_\delta$. Let $\xi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ be a cut-off function such that $0 \leqslant \xi \leqslant 1$, $\xi(t) = 0$ if and only if $t \leqslant -2$, $\xi(t) = 1$ if and only if $t \geqslant -1$. We set

$$\xi_\varepsilon(\theta) := \xi\left(\frac{\mathcal{J}_\lambda(f_\varepsilon(\theta)) - c_\lambda}{\varepsilon}\right), \qquad \forall \theta \in V.$$

For all $\theta \in V$ we define the deformation

$$\tilde{f}_\varepsilon(\theta) = f_\varepsilon(\theta) - \sqrt{\varepsilon}\ \xi_\varepsilon(\theta)\frac{\mathcal{J}'_\lambda(f_\varepsilon(\theta))}{\|\mathcal{J}'_\lambda(f_\varepsilon(\theta))\|}.$$

In view of (4.2) and (4.3), we have $\xi_\varepsilon(\theta) = 0$ when ε is sufficiently small and θ is near ∂V. In particular, $\tilde{f}_\varepsilon$ satisfies $\mathscr{P}(\partial V)$ and therefore $\tilde{f}_\varepsilon \in \mathcal{F}_{\lambda-\varepsilon}$. By Taylor expansion of $\mathcal{J}_\lambda$ and in view of assumption (4.1) we have for all $w, \varphi \in \mathcal{H}$

$$\mathcal{J}_\lambda(w + \varphi) \leqslant \mathcal{J}_\lambda(w) + \mathcal{J}'_\lambda(w)\varphi + \frac{1}{2}\|\varphi\|^2.$$

Consequently,

$$\mathcal{J}_\lambda(\tilde{f}_\varepsilon(\theta)) \leqslant \mathcal{J}_\lambda(f_\varepsilon(\theta)) - \sqrt{\varepsilon}\ \xi_\varepsilon(\theta)\frac{\|\mathcal{J}'_\lambda(f_\varepsilon(\theta))\|^2}{\|\mathcal{J}'_\lambda(f_\varepsilon(\theta))\|} + \frac{\varepsilon}{2}\xi_\varepsilon^2(\theta). \tag{4.5}$$

Now we claim that

$$\sup_{\theta \in V} \mathcal{J}_\lambda(\tilde{f}_\varepsilon(\theta)) \leqslant c_\lambda - \frac{\varepsilon}{2} \tag{4.6}$$

for all sufficiently small $\varepsilon > 0$. We prove (4.6) by considering two cases.

Case I: $\xi_\varepsilon(\theta) < 1$. In this case $\mathcal{J}_\lambda(f_\varepsilon(\theta)) < c_\lambda - \varepsilon$ and hence, using (4.5) we estimate

$$\mathcal{J}_\lambda(\tilde{f}_\varepsilon(\theta)) \leqslant c_\lambda - \frac{\varepsilon}{2}.$$

Case II: $\xi_\varepsilon(\theta) = 1$. In this case, we have that $f_\varepsilon(\theta)$ satisfies the assumptions of Lemma 4.1 and consequently $f_\varepsilon(\theta) \in N_\delta$ for all $\varepsilon > 0$ sufficiently small. By the contradiction assumption we then have $\|\mathcal{J}'_\lambda(f_\varepsilon(\theta))\| \geqslant \delta$. It follows that, using again (4.5) and Lemma 4.1,

$$\mathcal{J}_\lambda(\tilde{f}_\varepsilon(\theta)) \leqslant \mathcal{J}_\lambda(f_\varepsilon(\theta)) - \sqrt{\varepsilon}\delta + \frac{\varepsilon}{2} \leqslant c_\lambda + \varepsilon\left(\frac{c_\lambda}{\lambda} + |c'_\lambda| + \frac{5}{2}\right) - \sqrt{\varepsilon}\delta \leqslant c_\lambda - \frac{\sqrt{\varepsilon}\delta}{2}$$

for all $\varepsilon > 0$ sufficiently small. Taking $\varepsilon \leqslant \delta^2$ we conclude that $\mathcal{J}_\lambda(\tilde{f}_\varepsilon(\theta)) \leqslant c_\lambda - \frac{\varepsilon}{2}$ and claim (4.6) is established. But then we derive

$$c_\lambda = \inf_{f \in \mathcal{F}_\lambda} \sup_{\theta \in V} \mathcal{J}_\lambda(f(\theta)) \leqslant c_\lambda - \frac{\varepsilon}{2},$$

a contradiction. $\square$

At this point the proof of Proposition 4.1 follows by standard compactness arguments.

Proof of Proposition 4.1. Let $w_{\varepsilon_n} \in \mathcal{H}$ be a bounded Palais-Smale sequence as obtained in Lemma 4.2. Then there exists $w_0 \in \mathcal{H}$ such that $w_{\varepsilon_n} \rightharpoonup w_0$ weakly in $\mathcal{H}$. On the other hand, we have

$$\begin{aligned}\langle \mathcal{J}_\lambda'(w_{\varepsilon_n}), w_{\varepsilon_n} - w_0\rangle =& \|w_{\varepsilon_n} - w_0\|^2 + \langle w_0, w_{\varepsilon_n} - w_0\rangle \\ &- \lambda\langle \mathcal{G}'(w_{\varepsilon_n}) - \mathcal{G}'(w_0), w_{\varepsilon_n} - w_0\rangle - \lambda\langle \mathcal{G}'(w_0), w_{\varepsilon_n} - w_0\rangle.\end{aligned}$$

Hence, by the compactness assumption (4.1) on $\mathcal{G}'$ and using the fact that $\|\mathcal{J}_\lambda'(w_{\varepsilon_n})\| \to 0$, we obtain

$$o(1)\|w_{\varepsilon_n} - w_0\| = \langle \mathcal{J}_\lambda'(w_{\varepsilon_n}), w_{\varepsilon_n} - w_0\rangle = \|w_{\varepsilon_n} - w_0\|^2 + o(1)$$

so that $w_{\varepsilon_n} \to w_0$ strongly in $\mathcal{H}$. This implies, by the continuity of $\mathcal{J}_\lambda$ on $\mathcal{H}$, $\mathcal{J}_\lambda(w_{\varepsilon_n}) \to \mathcal{J}_\lambda(w_0) = c_\lambda$. Moreover, since $\mathcal{G}'(w_{\varepsilon_n})\varphi \to \mathcal{G}'(w_0)\varphi$ for all $\varphi \in \mathcal{H}$, we conclude that $o(1) = \mathcal{J}'{}_\lambda(w_{\varepsilon_n})\varphi = \langle w_{\varepsilon_n}, \varphi\rangle - \lambda\mathcal{G}'(w_{\varepsilon_n})\varphi \to \mathcal{J}'{}_\lambda(w_0)\varphi$ and therefore $\mathcal{J}'{}_\lambda(w_0) = 0$. $\square$

5 Proof of Theorem 1.2

We begin by establishing the existence almost everywhere of solutions. We note that assumption (1.6) is not necessary for this part of Theorem 1.2. We shall use the monotonicity trick, as established in Proposition 4.1, to construct the minimax critical values for Neri's functional (1.5) defined on a non-simply connected domain $\Omega \subset \mathbb{R}^2$. Namely, we consider the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \log\left(\iint_{I\times\Omega} e^{\alpha u}\,\mathcal{P}(d\alpha)dx\right),$$

for all $u \in H_0^1(\Omega)$. Our aim in this subsection is to show the following.

Proposition 5.1. *For almost every $\lambda \in (8\pi, 16\pi)$, there exists a saddle-type critical value $c_\lambda > -\infty$ for J_λ.*

The construction of c_λ relies on an idea originally introduced by [11] for the standard mean field equation (1.2). Such an idea was also exploited in [6] for the Toda system. Here, we extend the argument to the case of mean field equations including a probability measure.

For every $u \in H_0^1(\Omega)$ we consider the measure:

$$\mu_u = \frac{\int_I e^{\alpha u}\mathcal{P}(d\alpha)}{\iint_{I\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx}\, dx \in \mathcal{M}(\Omega)$$

and the corresponding "baricenter" of Ω:

$$m(u) = \frac{\iint_{I\times\Omega} x e^{\alpha u}\mathcal{P}(d\alpha)dx}{\iint_{I\times\Omega} e^{\alpha u}\mathcal{P}(d\alpha)dx} = \int_\Omega x\, d\mu_u \in \mathbb{R}^2.$$

We note that $\mu_u(\Omega) = 1$ and $|m(u)| \leqslant \sup\{|x| :\ x \in \Omega\}$. In the following lemma we show that, as a consequence of Proposition 3.2, if the functional J_λ given by (1.5) is unbounded below along a sequence $u_n \in H_0^1(\Omega)$, and if $\lambda \in (8\pi, 16\pi)$, then u_n blows up at exactly one point $x_0 \in \overline{\Omega}$.

Lemma 5.1. *Let $\lambda \in (8\pi, 16\pi)$ and $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that $J_\lambda(u_n) \to -\infty$. Then, there exists $x_0 \in \overline{\Omega}$ such that*

$$\mu_{u_n} \rightharpoonup \delta_{x_0} \quad \textit{weakly* in } \mathcal{C}(\bar{\Omega})' \qquad \textit{and} \qquad m(u_n) \to x_0.$$

Proof. Throughout this proof, for simplicity, we denote $\mu_n = \mu_{u_n}$. For every fixed $r > 0$ we denote by $\mathcal{Q}_n(r)$ the concentration function of μ_n, namely,

$$\mathcal{Q}_n(r) = \sup_{x\in\Omega} \int_{B(x,r)\cap\Omega} \mu_n, \qquad (r > 0).$$

For every n we take $\tilde{x}_n \in \overline{\Omega}$ such that $\mathcal{Q}_n(r/2) = \int_{B(\tilde{x}_n,r/2)\cap\Omega} \mu_n$. Upon taking a subsequence, we may assume that $\tilde{x}_n \to x_0 \in \overline{\Omega}$. We set $\Omega_1^n = B(\tilde{x}_n, r/2) \cap \Omega$ and $\Omega_2^n = \Omega \setminus B(\tilde{x}_n, r)$ and we note that $\operatorname{dist}(\Omega_1^n, \Omega_2^n) \geqslant r/2$. Since $J_\lambda(u_n) \to -\infty$ and $\lambda < 16\pi$, in view of Proposition 3.2 we conclude that $\min\{\mu_n(\Omega_1^n), \mu_n(\Omega_2^n)\} \to 0$. In particular, $\min\{\mathcal{Q}_n(r/2), 1 - \mathcal{Q}_n(r)\} \leqslant \min\left(\mu_n(\Omega_1^n), \mu_n(\Omega_2^n)\right) \to 0$. On the other hand, for every fixed $r > 0$ let $k_r \in \mathbb{N}$ be such that Ω is covered by k_r balls of radius $r/2$. Then, $1 = \mu_n(\Omega) \leqslant k_r Q_n(r/2)$, so that $Q_n(r/2) \geqslant k_r^{-1}$ for every n. We conclude that necessarily $\mathcal{Q}_n(r) \to 1$ as $n \to \infty$. Since $r > 0$ is arbitrary, we derive in turn that $1 - \mathcal{Q}_n(r/2) = \mu_n(\Omega \setminus B(\tilde{x}_n, r/2)) \to 0$ as $n \to \infty$. That is, $\mu_{u_n} \rightharpoonup \delta_{x_0}$. It follows that $m(u_n) = \int_\Omega x \, d\mu_n \to x_0$. □

Let $\Gamma_1 \subset \Omega$ be a non-contractible curve which exists in view of the non-simply connectedness assumption on Ω. Let $\mathbb{D} = \{(r, \theta) : 0 \leqslant r < 1, \ 0 \leqslant \theta < 2\pi\}$ be the unit disc. We now define the sets of functions which will be used in the minimax argument by setting

$$\mathcal{D}_\lambda = \left\{ \begin{array}{ll} h \in \mathcal{C}(\mathbb{D}, H_0^{1,2}(\Omega)) \ s.t. : & \text{(i)} \quad \lim\limits_{r\to 1} \sup\limits_{\theta\in[0,2\pi)} J_\lambda(h(r,\theta)) = -\infty, \\ & \text{(ii)} \quad m(h(r,\theta)) \text{ can be extended continuously to } \bar{\mathbb{D}}, \\ & \text{(iii)} \quad m(h(1,\cdot)) : \partial\mathbb{D} \to \Gamma_1 \text{ has degree } 1 \end{array} \right\}.$$

Lemma 5.2. *For any $\lambda \in (8\pi, 16\pi)$ the set $\mathcal{D}_\lambda$ is non-empty.*

Proof. We assume that $1 \in \operatorname{supp} \mathcal{P}$ (the case $-1 \in \operatorname{supp} \mathcal{P}$ can be treated in the same way). Let $\gamma_1(\theta) : [0, 2\pi) \to \Gamma_1$ be a parametrization of Γ_1 and let $\varepsilon_0 > 0$ be sufficiently small so that $B(\gamma_1(\theta), \varepsilon_0) \subset \Omega$. Let $\varphi_\theta(x) = \varepsilon_0^{-1}(x - \gamma_1(\theta))$ so that $\varphi_\theta(B(\gamma_1(\theta), \varepsilon_0)) = B(0, 1)$. We define "truncated Green's function":

$$V_r(X) = \begin{cases} 4 \log \dfrac{1}{1-r} & \text{for } X \in B(0, 1-r) \\ 4 \log \dfrac{1}{|X|} & \text{for } X \in B(0,1) \setminus B(0, 1-r) \end{cases}$$

and

$$v_{r,\theta}(x) = \begin{cases} 0 & \text{for } x \in \Omega \setminus B(\gamma_1(\theta), \varepsilon_0)) \\ V_r(\varphi_\theta(x)) & \text{for } x \in B(\gamma_1(\theta), \varepsilon_0). \end{cases}$$

We set

$$h(r, \theta)(x) = v_{r,\theta}(x), x \in \Omega. \tag{5.1}$$

Claim: The function h defined in (5.1) satisfies $h \in \mathcal{D}_\lambda$.

We check (i). As in (3.5), we note that for any $\delta \in (0,1)$ and for any (r,θ) it holds

$$J_\lambda(h(r,\theta)) \leqslant \frac{1}{2}\|\nabla h(r,\theta)\|_2^2 - \lambda \log \int_\Omega e^{(1-\delta)h(r,\theta)} - \lambda \log(\mathcal{P}([1-\delta,1]))$$

We have $\|\nabla h(r,\theta)\|_2^2 = -32\pi \log(1-r)$ and

$$\begin{aligned}\int_\Omega e^{(1-\delta)h(r,\theta)}\,dx \quad &= \varepsilon_0^2 \int_{B(0,1)} e^{(1-\delta)V_r(X)} dX + |\Omega| - \pi\varepsilon_0^2 \\ &= 2\pi\varepsilon_0^2 \left(\int_0^{1-r} e^{(1-\delta)4\log\frac{1}{1-r}} \rho d\rho + \int_{1-r}^1 e^{(1-\delta)4\log\frac{1}{\rho}} \rho d\rho \right) + O(1) \\ &= \pi\varepsilon_0^2 \left[\left(\frac{1}{1-r}\right)^{2-4\delta} + \frac{1}{1-2\delta}\left(\left(\frac{1}{1-r}\right)^{2-4\delta} - 1 \right) \right] + O(1),\end{aligned}$$

where $O(1)$ is bounded independently of (r,θ). It follows that if $0 < \delta \leqslant 1/4$, then

$$\log \int_\Omega e^{(1-\delta)h} = (2-4\delta)\log\frac{1}{1-r} + O(1).$$

We conclude that

$$J_\lambda(h) \leqslant 2\left(8\pi - \lambda(1-2\delta)\right)\log\frac{1}{1-r} - \lambda \log \mathcal{P}([1-\delta,1]) + O(1).$$

Since $\lambda > 8\pi$, by choosing $\delta > 0$ sufficiently small we conclude that $h(r,\theta)$ defined by (5.1) satisfies property (i).

We check (ii)–(iii). To this aim it is sufficient to prove that $\lim_{r\to 1} m(h(r,\theta)) = \gamma_1(\theta)$ uniformly with respect to $\theta \in [0,2\pi)$. We consider again the measures

$$\mu_{r,\theta} = \frac{\int_I e^{\alpha h(r,\theta)} \mathcal{P}(d\alpha)}{\iint_{I\times\Omega} e^{\alpha h(r,\theta)} \mathcal{P}(d\alpha)}.$$

We claim that for every $\varepsilon \in (0,\varepsilon_0)$, we have $\lim_{r\to 1}\int_{B(\gamma_1(\theta),\varepsilon)} \mu_{r,\theta} = 1$ uniformly with respect to $\theta \in [0,2\pi)$. Indeed, fix $\varepsilon \in (0,\varepsilon_0)$, and let $\delta = \delta(\varepsilon) \in (0,1)$ such that $B(0,\delta) \subset \varphi_\theta(B(\gamma_1(\theta),\varepsilon))$. We write

$$\int_{B(\gamma_1(\theta),\varepsilon)} \mu_{r,\theta} = \frac{I + II}{I + III} \tag{5.2}$$

where

$$I = \iint_{I\times\varphi_\theta^{-1}(B(0,\delta))} e^{\alpha v_{r,\theta}(x)} dx$$

and

$$II = \iint_{I\times(B(\gamma_1(\theta),\varepsilon)\setminus\varphi_\theta^{-1}(B(0,\delta)))} e^{\alpha v_{r,\theta}(x)}\,dx, \qquad III = \iint_{I\times(\Omega\setminus\varphi_\theta^{-1}(B(0,\delta)))} e^{\alpha v_{r,\theta}(x)}\,dx.$$

Since $1 \in \mathrm{supp}\mathcal{P}$, for every $\bar{\alpha} \in (0,1]$, $\mathrm{supp}\mathcal{P} \cap [\bar{\alpha}, 1] \neq \emptyset$. Then, for any $r > 1-\delta$ we have

$$\begin{aligned}
I &\geqslant \int_{\bar{\alpha}}^{1} \int_{\varphi_\theta^{-1}(B(0,\delta))} e^{\alpha v_{r,\theta}(x)}\, dx \\
&\geqslant \varepsilon_0^2 \mathcal{P}([\bar{\alpha}, 1]) \int_{B(0,\delta)} e^{\bar{\alpha} V_r(X)} dX \\
&= \varepsilon_0^2 \mathcal{P}([\bar{\alpha}, 1]) \left[\int_{B(0,1-r)} \frac{1}{(1-r)^{4\bar{\alpha}}} dX + \int_{B(0,\delta)\setminus B(0,1-r)} \frac{1}{|X|^{4\bar{\alpha}}} dX \right] \\
&= \pi \varepsilon_0^2 \mathcal{P}([\bar{\alpha}, 1]) \left[(1-r)^{2-4\bar{\alpha}} + \frac{1}{2\bar{\alpha}-1} \left((1-r)^{2-4\bar{\alpha}} - \delta^{2-4\bar{\alpha}} \right) \right]
\end{aligned}$$

Hence, choosing $\bar{\alpha} > 1/2$, we derive $I \to +\infty$ as $r \to 1$ uniformly with respect to θ. Moreover,

$$0 \leqslant II \leqslant III \leqslant \pi \varepsilon_0^2 \left(\frac{1}{\delta^2} - 1 \right) + |\Omega|,$$

Letting $r \to 1$ in (5.2) we conclude that $\int_{B(\gamma_1(\theta),\varepsilon)} \mu_{r,\theta} \to 1$ for any $\varepsilon > 0$, uniformly with respect to θ, and consequently $\mu_{h(r,\theta)} \overset{*}{\rightharpoonup} \delta_{\gamma(\theta)}$. In turn, we derive $\lim_{r\to 1} m(h(r,\theta)) = \gamma_1(\theta)$, and therefore h satisfies properties (ii)–(iii) in the definition of $\mathcal{D}_\lambda$. We conclude that $h \in \mathcal{D}_\lambda$. ☐

We define the minimax value:

$$c_\lambda = \inf_{h \in \mathcal{D}_\lambda} \sup_{(r,\theta)\in \mathbb{D}} J_\lambda(h(r,\theta)).$$

In view of Lemma 5.2, we have $c_\lambda < +\infty$. The following property relies on the nontrivial topology of Ω in an essential way.

Lemma 5.3. *For any $\lambda \in (8\pi, 16\pi)$, $c_\lambda > -\infty$.*

Proof. Denote by B a bounded component of $\mathbb{R}^2 \setminus \Omega$ such that Γ_1 encloses B. By the continuity and degree properties defining $\mathcal{D}_\lambda$, we have $m(h(\mathbb{D})) \supset B$ for all $h \in \mathcal{D}_\lambda$. Arguing by contradiction, we assume $c_\lambda = -\infty$. Then, there exists a sequence $\{h_n\} \subset \mathcal{D}_\lambda$ such that $\sup_{(r,\theta)\in\mathbb{D}} J_\lambda(h_n(r,\theta)) \to -\infty$. Let x_0 be an interior point of B. For every n we take $(r_n, \theta_n) \in \mathbb{D}$ such that $m(h_n(r_n,\theta_n)) = x_0$. In view of Lemma 5.1, there exists $\tilde{x}_0 \in \overline{\Omega}$ such that $m(h_n(r_n,\theta_n)) \to \tilde{x}_0$. But then $x_0 = \tilde{x}_0 \in \overset{o}{B} \cap \overline{\Omega} = \emptyset$, a contradiction. ☐

Lemma 5.4. *For $8\pi < \lambda_1 \leqslant \lambda_2 < 16\pi$, we have $\mathcal{D}_{\lambda_1} \subseteq \mathcal{D}_{\lambda_2}$.*

Proof. It is sufficient to note that whenever $J(u) \leqslant 0$ it is $\log \iint_{I\times\Omega} e^{\alpha u} \geqslant 0$, which implies that

$$J_{\lambda_1}(u) \geqslant J_{\lambda_2}(u) \qquad \text{for} \qquad 8\pi < \lambda_1 < \lambda_2 < 16\pi.$$

Hence, $\mathcal{D}_{\lambda_1} \subseteq \mathcal{D}_{\lambda_2}$ for every $8\pi < \lambda_1 < \lambda_2 < 16\pi$. ☐

We set

$$G(u) = \log \left(\iint_{I\times\Omega} e^{\alpha u} \mathcal{P}(d\alpha) \right) \tag{5.3}$$

so that Neri's functional (1.5) takes the form

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda G(u).$$

Lemma 5.5. *The function $G : H_0^1(\Omega) \to \mathbb{R}$ defined by* (5.3) *satisfies assumptions* (4.1).

Proof. For any $u, \phi, \psi \in H_0^1(\Omega)$ we have:

$$G'(u)\phi = \iint_{I\times\Omega} \frac{\alpha\phi e^{\alpha u}}{\iint_{I\times\Omega} e^{\alpha u}},$$

and therefore the compactness of G' follows by compactness of the Moser-Trudinger embedding as stated in Section 3. Moreover

$$\langle G''(u)\phi, \psi\rangle = \frac{\left(\iint_{I\times\Omega} \alpha^2\phi\psi e^{\alpha u}\right)\left(\iint_{I\times\Omega} e^{\alpha u}\right) - \left(\iint_{I\times\Omega} \alpha\phi e^{\alpha u}\right)\left(\iint_{I\times\Omega} \alpha\psi e^{\alpha u}\right)}{\left(\iint_{I\times\Omega} e^{\alpha u}\right)^2},$$

so that we obtain $\langle G''(u)\phi, \phi\rangle \geqslant 0$ using the Schwarz inequality. □

Now we are able to prove Proposition 5.1.

Proof of Proposition 5.1. In view of Lemma 5.5, Lemma 5.2, Lemma 5.3 and Lemma 5.4, we may apply Proposition 4.1 with $\mathcal{H} = H_0^1(\Omega)$, $\mathcal{G}(u) = G(u)$, $V = \mathbb{D}$, $A = -\infty$ and $\mathcal{F}_\lambda = \mathcal{D}_\lambda$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 5.1, for almost every $\lambda \in (8\pi, 16\pi)$, the value c_λ is a critical value for J_λ, which is achieved by a critical point $u \in H_0^1(\Omega)$. Hence, for almost every $\lambda \in (8\pi, 16\pi)$ we obtain a solution to (1.1). Now we assume that $\mathcal{P}$ satisfies (1.6). Let $\lambda_0 \in (8\pi, 16\pi)$. Using the first part of Theorem 1.2, there exists a solution sequence (λ_n, u_n) to (1.1) such that $\lambda_n \in (8\pi, 16\pi)$ and $\lambda_n \to \lambda_0$. In view of Proposition 2.6, blow-up cannot occur for $\lambda_0 \in (8\pi, 16\pi)$. Therefore, u_n converges uniformly to a solution u_0 to (1.1) with $\lambda = \lambda_0$. □

6 Proof of Theorem 1.3

Let $\mathcal{E} = \{v \in H^1(M) : \int_M v\, dv_g = 0\}$. The variational functional for Neri's equation (1.8) defined on a manifold M is given by

$$J_\lambda(v) = \frac{1}{2}\int_M |\nabla_g v|^2\, dv_g - \lambda \log\left(\frac{1}{|M|}\iint_{I\times M} e^{\alpha v}\, \mathcal{P}(d\alpha) dv_g\right),$$

where $v \in \mathcal{E}$. We begin by establishing the following.

Proposition 6.1. *For almost every $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)}\right)$, there exists a mountain pass critical value $c_\lambda > 0$ for J_λ.*

It is convenient to set

$$G(v) = \log\left(\frac{1}{|M|}\iint_{I\times M} e^{\alpha v}\mathcal{P}(d\alpha)dv_g\right)$$

so that $J_\lambda(v) = \frac{1}{2}\int_M |\nabla v|^2 - \lambda G(v)$. Henceforth, throughout this subsection, for simplicity we denote $\nabla = \nabla_g$, $\Delta = \Delta_g$, $dx = dv_g$, and we omit the integration measure when it is clear from the context. Then,

$$G'(v)\phi = \iint_{I\times M} \frac{\alpha e^{\alpha v}\phi}{\iint_{I\times M} e^{\alpha v}}$$

and

$$\langle G''(v)\phi, \psi\rangle = \frac{\left(\iint_{I\times M}\alpha^2 e^{\alpha v}\phi\psi\right)\left(\iint_{I\times M} e^{\alpha v}\right) - \left(\iint_{I\times M}\alpha e^{\alpha v}\phi\right)\left(\iint_{I\times M}\alpha e^{\alpha v}\psi\right)}{\left(\iint_{I\times M} e^{\alpha v}\right)^2}.$$

In particular, $G(0) = 0$; in view of Jensen's inequality we have $G(v) \geqslant 0$ for every $v \in \mathcal{E}$; $G'(0) = 0$ and G' is compact in view of Section 3. Furthermore, the Schwarz inequality implies that $\langle G''(v)\varphi, \varphi\rangle \geqslant 0$ for all $\varphi \in \mathcal{E}$, and we compute

$$\|G''(0)\| := \inf_{\phi\in\mathcal{E}\setminus\{0\}} \frac{\langle G''(0)\phi, \phi\rangle}{\|\nabla\phi\|_2^2} = \frac{\int_I \alpha^2\,\mathcal{P}(d\alpha)}{\mu_1(M)|M|}, \tag{6.1}$$

where $\mu_1(M)$ is the first nonzero eigenvualue defined in (1.9).

Lemma 6.1. *Let $\mathcal{P}$ satisfy assumption* (1.4) *and suppose $8\pi < \frac{\mu_1(M)|M|}{\int_I \alpha^2\mathcal{P}(d\alpha)}$. If $\lambda < \frac{\mu_1(M)|M|}{\int_I \alpha^2\mathcal{P}(d\alpha)}$, then $v \equiv 0$ is a strict local minimum for J_λ. Moreover, if $\lambda > 8\pi$, then there exists $v_1 \in \mathcal{E}$ such that $\|v_1\| \geqslant 1$ and $J_\lambda(v_1) < 0$. In particular, J_λ has a mountain-pass geometry for each $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2\mathcal{P}(d\alpha)}\right)$.*

Proof. By Taylor expansion and by properties of G, we have

$$J_\lambda(\phi) \geqslant \frac{1}{2}\left(1 - \lambda\|G''(0)\|\right)\|\phi\|^2 + o(\|\phi\|^2) \tag{6.2}$$

Therefore, in view of (6.1), $v \equiv 0$ is a strict local minimum for J_λ whenever $\lambda < \|G''(0)\|^{-1} = \frac{\mu_1(M)|M|}{\int_I \alpha^2\mathcal{P}(d\alpha)}$. We recall from [26] that $\int_M e^{\alpha v}$ is increasing with respect to α for all $v \in \mathcal{E}$. Indeed,

$$\frac{d}{d\alpha}\int_M e^{\alpha v} = \int_M v e^{\alpha v} = \int_{v\geqslant 0} v e^{\alpha v} - \int_{v<0} |v| e^{\alpha v} \geqslant 0.$$

Consequently, similarly as in (3.5), we compute

$$J_\lambda(v) \leqslant \frac{1}{(1-\delta)^2}\left[\frac{1}{2}\int_M |\nabla(1-\delta)v|^2 - (1-\delta)^2\lambda\log\int_M e^{(1-\delta)v}\right] - \lambda\log\mathcal{P}([1-\delta, 1]).$$

We fix $p_0 \in M$ and $r_0 > 0$ a constant smaller than the injectivity radius of M at p_0. For every $\varepsilon > 0$, we define

$$v_\epsilon(p) = \begin{cases} \log \frac{\epsilon^2}{(\epsilon^2 + d_g(p,p_0)^2)^2} & \text{in } \mathcal{B}_{r_0} \\ \log \frac{\epsilon^2}{(\epsilon^2 + r_0^2)^2} & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_{r_0} = \{p \in M : d_g(p,p_0) < r_0\}$ denotes the geodesic ball of radius r_0 centered at p_0. We set

$$\tilde{v}_\epsilon(x) := v_\epsilon(x) - \frac{1}{|M|} \int_M v_\epsilon dv_g.$$

so that $\tilde{v}_\varepsilon \in \mathcal{E}$. A standard computation yields

$$\int_M |\nabla_g \tilde{v}_\varepsilon|^2 dv_g = 32\pi \log \frac{1}{\varepsilon} + O(1),$$

and

$$\log \int_M e^{\tilde{v}_\varepsilon} = 2 \log \frac{1}{\varepsilon} + O(1)$$

where $O(1)$ is independent of ε. See, e.g., [28] for the details. Then,

$$J_\lambda \left(\frac{\tilde{v}_\varepsilon}{1-\delta} \right) \leqslant \frac{2}{(1-\delta)^2} \left[8\pi - (1-\delta)^2 \lambda \right] \log \frac{1}{\varepsilon} + O(1).$$

It follows for any $\lambda > 8\pi$ there exists $0 < \delta \ll 1$ such that $J_\lambda(\tilde{v}_\varepsilon/(1-\delta)) \to -\infty$ as $\varepsilon \to 0$. Since $\|\tilde{v}_\varepsilon\| \to +\infty$ as $\varepsilon \to 0$, taking $v_1 = v_\varepsilon$ with ε sufficiently small, we conclude the proof. □

We note that if λ belongs to a compact subset of $\left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)}\right)$, the function v_1 may be chosen independently of λ. We denote by Γ the set of paths

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], \mathcal{E}) : \gamma(0) = 0, \gamma(1) = v_1\}.$$

We set

$$c_\lambda = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\lambda(\gamma(t)).$$

Note that the value c_λ is finite and non negative. In particular, in the following lemma we prove that for $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)}\right)$, c_λ is strictly positive.

Lemma 6.1. *Let $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2 \mathcal{P}(d\alpha)}\right)$. For every $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ (independent on λ), such that*

$$c_\lambda \geqslant \frac{\rho_\varepsilon^2}{2} \left(1 - \lambda \frac{\int_I \alpha^2 \mathcal{P}(d\alpha)}{\mu_1(M)\,|M|} - \varepsilon \right).$$

Proof. In view of the Taylor expansion (6.2), $\varepsilon > 0$ there exists a constant $\rho_\varepsilon \in (0,1)$ independent of λ, such that for any $v \in \mathcal{E}$ satisfying $\|v\| \leqslant \rho_\varepsilon$ we have

$$J_\lambda(v) \geqslant \frac{1}{2} \left(1 - \lambda \frac{\int_I \alpha^2 \mathcal{P}(d\alpha)}{\mu_1(M)\,|M|} - \varepsilon \right) \|v\|^2.$$

In particular, for any $v \in \mathcal{E}$ such that $\|v\| = \rho_\varepsilon$ we have

$$J_\lambda(v) \geqslant \frac{\rho_\varepsilon^2}{2}\,(1 - \lambda\|G''(0)\| - \varepsilon)\,.$$

By continuity of $\gamma \in \Gamma$, we derive in turn that

$$\sup_{t\in[0,1]} J_\lambda(\gamma(t)) \geqslant \frac{\rho_\varepsilon^2}{2}\left(1 - \lambda\,\frac{\int_I \alpha^2\mathcal{P}(d\alpha)}{\mu_1(M)\,|M|} - \varepsilon\right),$$

and asserted strict positivity of c_λ follows. □

Proof of Proposition 6.1. We use the Struwe's Monotonicity Trick as stated in Proposition 4.1 with $\mathcal{H} = \mathcal{E}$, $V = [0,1]$, $\mathcal{F}_\lambda = \Gamma$ and $A = 0$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2\mathcal{P}(d\alpha)}\right)$ be such that c'_λ exists. By Proposition 6.1, we have that c_λ is a critical value for J_λ, which is achieved by a critical point $v \in \mathcal{E}$. Such a v is a solution to problem (1.8). This proves the first part of Theorem 1.3. To prove the second part of Theorem 1.3, assume that $\mathcal{P}$ satisfies (1.6). Let $\lambda_0 \in \left(8\pi, \frac{\mu_1(M)|M|}{\int_I \alpha^2\mathcal{P}(d\alpha)}\right)$. By the first part of Theorem 1.3 there exists a solution sequence (λ_n, v_n) to (1.8) with $\lambda = \lambda_n$, such that $\lambda_n \to \lambda_0$. In view of the mass quantization, as stated in Proposition 2.7, blow-up cannot occur. Therefore, v_n converges uniformly to a solution v_0 for problem (1.8) with $\lambda = \lambda_0$. □

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