Testing hypothesis in VARs with I(2) and near-I(2) latent stochastic trends

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Abstract We review the I(2) model in an empirical perspective, and report the results of some Monte Carlo simulations on the small sample performance of asymptotic tests on the long-run coefficients in I(2) and near-I(2) systems. The results show that, although moderate size bias tend to be present in both cases, the I(2) model may provide an suitable tool for analysing near-I(2) systems

Key words: VAR, I(2), latent trend

1 Introduction

Vector Autoregressions (VAR), introduced by Sims (1980), are one of the most powerful and popular ways of modelling the interactions between economic variables. Assume we are interested in a set of $I(d)$ variables such that $\Delta^d X_{it} = \varepsilon_{it}$, $i = 1,\ldots,p$, $t = 1,\ldots,T$ and $\varepsilon_{it} \sim IID(0,\sigma_i^2)$. When $d = 1$ and the variables are linked by some long-run equilibrium (*cointegration*) relationship we know that the Vector Error Correction Mechanism (VECM) $\Delta\mathbf{X}_t = \Pi\mathbf{X}_{t-1} + \sum_{j=1}^{k}\Gamma_j\Delta\mathbf{X}_{t-j} + \varepsilon_t$ holds, where the matrix Π satisfies the reduced rank restriction $\Pi = \alpha\beta'$ with α and β $p \times r$ matrices (Johansen, 1988). Of particular interest here is that in this case the variables of the system are driven by a small number of latent stochastic trends. From the VECM representation is easy to derive the VMA form; setting the initial

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values to zero and denoting by $\alpha_\perp, \beta_\perp$ the orthogonal complements of α and β, this is $\mathbf{X}_t = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' \sum_{s=1}^{t} \varepsilon_s = \widetilde{\beta}_\perp \left(\alpha_\perp' \sum_{s=1}^{t} \varepsilon_s\right)$ (see, e.g., Juselius, 2006). This representation shows clearly that the pushing forces of the system are the latent stochastic trends $\alpha_\perp' \sum_{s=1}^{t} \varepsilon_s$, linear combinations of the cumulated shocks to each equation of the VAR, which are loaded onto the $X's$ through the loadings $\widetilde{\beta}_\perp$. Both the coefficients of the linear combinations and the loadings can be recovered from the estimates of the VAR obtained under the reduced rank restriction.

The cointegrated $I(1)$ VAR has proved to be immensely popular and to be able to shed considerable light on the long-run dynamics of economic systems. However, some recent contributions (*e.g.*, Johansen, Juselius Frydman and Goldberg, 2010) pointed out that in empirically important cases (for instance, when modelling exchange rates) the $I(1)$ model is not adequate, as the variables of interest may be $I(2)$, or nearly so. That this could be the case is empirically suggested by estimated $I(1)$ cointegrated VARs having unrestricted characteristic roots very close to 1, so that very long swings in the data are not accounted for. The statistical analysis of $I(2)$ systems, started by Johansen (1992, 1997), is now two decades old and fairly well developed. However, much still remains to be done: for instance, Johansen (2006) showed that likelihood ratio tests of many hypothesis on the long-run structure are asymptotically χ^2, but nothing is known on the validity of the approximation (i) in small samples, an empirically crucial point, and (ii), in near-$I(2)$ systems. Our aim is thus (i) to review the $I(2)$ model in an empirical perspective, and (ii), to report the results of some Monte Carlo simulations on the small sample performance of tests on the long-run coefficients.

2 The I(2) and near-I(2) VAR

Consider for simplicity a VAR(2)

$$\Delta^2\mathbf{X}_t = \alpha\beta'\mathbf{X}_{t-1} + \Gamma\Delta\mathbf{X}_{t-1} + \varepsilon_t \tag{1}$$

where $\varepsilon_t = (\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_3)'$ and $\Pi = \alpha\beta'$. If further $\alpha_\perp'\Gamma\beta_\perp = \xi\eta'$ for some matrices ξ, η of dimensions $(p-r)\times s$, $s < p-r$, the $X's$ are $I(2)$. The common latent stochastic trends representation is then

$$\mathbf{X}_t = C_2 \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i + C_1 \sum_{i=1}^{t} \varepsilon_i + A \tag{2}$$

where $C_2 = \beta_{\perp 2}(\alpha_{\perp 2}'\theta\beta_{\perp 2})^{-1}\alpha_{\perp 2}'$, $\beta_{\perp 2} = \beta_\perp\eta_\perp$ and $\alpha_{\perp 2} = \alpha_\perp\xi_\perp$. For convenience of notation the deterministic and stationary parts are both included in the term A . Thus, analogously what seen above, the I(2) common trends $\alpha_{\perp 2}' \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_i$ are linear combinations of the twice cumulated shocks of the various equations of the VAR that load onto the variables through the loadings $\beta_{\perp 2}(\alpha_{\perp 2}'\theta\beta_{\perp 2})^{-1}$. Paruolo and Mosconi (2011) recently showed that the following reparametrisation of (1) holds:

$$\Delta^2\mathbf{X}_t = \alpha(\beta'\mathbf{X}_{t-1} + \upsilon'\Delta\mathbf{X}_{t-1}) + (\xi\gamma' + \varsigma\beta')\Delta\mathbf{X}_{t-1} + \varepsilon_t \tag{3}$$

where the matrices υ and γ are respectively $p \times r$ and $p \times s$. The terms $(\beta'\mathbf{X}_{t-1} + \upsilon'\Delta\mathbf{X}_{t-1})$ and $(\gamma' + \beta')\Delta\mathbf{X}_{t-1}$ are both stationary; the former is known as multicointegration relation, or integral control term, the latter as mediun run relation, or proportional control term. As we will see, for testing purposes it is convenient to rewrite (3) as

$$\Delta^2\mathbf{X}_t = \alpha(\beta' : \upsilon') \begin{bmatrix} \mathbf{X}_{t-1} \\ \Delta\mathbf{X}_{t-1} \end{bmatrix} + (\xi : \varsigma) \begin{bmatrix} \gamma' \\ \beta' \end{bmatrix} \Delta\mathbf{X}_{t-1} + \varepsilon_t \tag{4}$$

$$= (\alpha : \xi : \varsigma) \begin{bmatrix} \beta' & \upsilon' \\ \mathbf{0} & \gamma' \\ \mathbf{0} & \beta' \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t-1} \\ \Delta\mathbf{X}_{t-1} \end{bmatrix} + \varepsilon_t \tag{5}$$

$$= \eta\zeta'(\mathbf{X}'_{t-1} : \Delta\mathbf{X}'_{t-1}) + \varepsilon_t \tag{6}$$

Estimation of (4) can be carried out by a two-step switching algorithm similar to that proposed by Johansen (1997), see Paruolo and Mosconi (2011).

The matrix $\zeta = (\zeta^1 : \zeta^2)$, which collects the cointegration parameters, is divided in two $2r \times p$ blocks, with the second block a linear combination of the first. Hence, any hypothesis on β and υ can be expressed as an hypothesis on ζ^1. More precisely, a convenient formulation is $H_0 : vec(\zeta^1) = B\Phi$, with $\Phi = (\phi^\beta : \phi^\upsilon)'$ the $(m^\beta + m^\upsilon) \times 1$ vector collecting the unconstrained coefficients of the both vectors and B a $2pr \times (m^\beta + m^\upsilon)$ block matrix given by

$$B = \begin{bmatrix} B^\beta & \mathbf{0} \\ B^{\beta\upsilon} & B^\upsilon \end{bmatrix}.$$

Here B^β is the $pr \times m^\beta$ matrix projecting the m^β unconstrained elements of β onto the full pr vector, B^υ fulfills the same purpose for υ, and $B^{\beta\upsilon}$ defines possible cross-restrictions between the two vectors. Although this formulation allows joint hypothesis on β and υ as well hypothesis on each of these two vectors, Johansen (2006) showed that in the case of hypothesis on all elements of the multicointegration relation inference is not LAMN. Hence, in our experiments we will concentrate on hypothesis on β, expressed as $H_0 : vec(\zeta^1) = B\Phi$ with $B^{\beta\upsilon} = \mathbf{0}$, $m^\upsilon = pr$ and $B^\upsilon = I_{pr}$.

3 Simulation experiments

Our experiments will be based on a Data Generating Process (DGP) derived by Johansen (1992) from Hendry and Von Urgen-Stenberg's (1981) famous specification of the consumption function, namely:

$$\Delta c_t = \nu \Delta y_{t-1} + a_{11}(y_{t-1} - c_{t-1}) + a_{12}(z_{t-1} - l_{t-1}) + \varepsilon_{1t} \tag{7a}$$

$$\Delta l_t = a_{21}(y_{t-1} - c_{t-1}) + \varepsilon_{2t} \tag{7b}$$

$$\Delta y_t = \rho \Delta y_t + \varepsilon_{3t} \tag{7c}$$

where c is consumption, l liquid assetts, and y disposable income. This DGP is particularly convenient for our purposes, as setting $\rho = 1$ we have an $I(2)$ system, whereas setting $\rho = 1 - \varepsilon$ for some arbitrarily small ε we have a near-$I(2)$ one. Further, since $r = 2$ and $s = 0$ the proportional control term is simply $\beta' \Delta \mathbf{X}_{t-1}$, and

$$\alpha' = \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & 0 & 0 \end{bmatrix}, \beta' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Defining $\mathbf{X}_t = (c_t \ l_t \ y_t)'$, $\varepsilon_t = (\varepsilon_{1t} \ \varepsilon_{2t} \ \varepsilon_3)'$, and assuming $\rho = 1$ the DGP can be compactly written as in equation (4) as:

$$\Delta^2 \mathbf{X}_t = (\alpha : \varsigma) \begin{bmatrix} \beta' & \mathbf{0} \\ \upsilon' & \beta' \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t-1} \\ \Delta \mathbf{X}_{t-1} \end{bmatrix} + \varepsilon_t \tag{8}$$

$$= \eta \zeta' (\mathbf{X}'_{t-1} : \Delta \mathbf{X}'_{t-1}) + \varepsilon_t \tag{9}$$

To evaluate Type I errors of the likelihood ratio tests we shall let

$$B^\beta = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}, \phi^\beta = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

To evaluate the performances of the test in both correctly and misspecified VARs we considered $\rho = 1, 0.90$, while the loadings a_{ij} have been chosen so to ensure control of the roots of the VAR(2) in levels which can be derived from (7a)-(7c). The following combinations, covering a wide range of adjustment speeds, have been chosen:

1. $a_{11} = k0.20, a_{21} = -k0.20, a_{12} = k0.20, k = 1, 2, 3$; in this case all loadings have the same absolute value;
2. $a_{11} = h0.20, a_{21} = -k0.20, a_{12} = k0.20, h = 2, 3, k = 1, 2$; in this case the loading of $(y_{t-1} - c_{t-1})$, a_{11}, is larger in absolute value than the other two loadings, a_{12} and a_{21};

The lagged effect of income growth on consumption growth, ν, is always fixed at 0.5, and the noises $\varepsilon_{it}, i = 1, 2, 3$, are $NID(0, 1)$, while sample size is $T = 100, 200$t and the number of Monte Carlo replications has been fixed to 1000. The results are reported in Tables 1 and 2 respectively for each case. As it can be immediately appreciated for most parameter combinations the test has some size bias. This generally, but not always, falls with sample size. For the empirical point of view an incovenient feature is the presence of both positive and negative bias, which makes the interpretation of results very tricky. In near-I(2) systems the tendency to under-reject seems to prevail, but the bias is generall small even for $T = 100$. Summing up, the message is that in both I(2) and near-I(2) systems asymptotic tests might be applied, but the interpretation of results requires some care. Ongoing research is exploring the performances delivered by the bootstrap, already successfully applied to hypothesis testing in I(1) VARs by Fachin (2000), Omtzigt and Fachin (2006).

Table 1

Size of Likelihood Ratio tests on long-run coefficients

		I(2)				*near I(2)*			
		$T = 100$		$T = 200$		$T = 100$		$T = 200$	
a_{11}	α	5.0	10.0	5.0	10.0	5.0	10.0	5.0	10.0
0.20		6.5	9.1	2.6	4.1	11.4	14.1	6.8	8.6
0.40		2.9	4.6	0.6	2.6	6.1	8.3	2.5	3.9
0.60		14.5	15.6	10.5	11.6	6.8	8.8	3.9	5.2
0.80		19.0	20.2	12.9	14.3	8.3	9.7	5.3	6.5
median		*10.5*	*12.4*	*6.6*	*7.9*	*7.6*	*9.3*	*4.6*	*5.9*
mean		*10.7*	*12.4*	*6.7*	*8.2*	*8.2*	*10.2*	*4.6*	*6.1*

$a_{21} = -a_{11}, a_{12} = a_{11}; I(2)$: $\rho = 1$; near-$I(2)$: $\rho = 0.90$.

Table B

Size of Likelihood Ratio tests on long-run coefficients

				I(2)				*near-I(2)*			
				$T = 100$		$T = 200$		$T = 100$		$T = 200$	
a_{11}	a_{21}	a_{12}	α	5.0	10.0	5.0	10.0	5.0	10.0	5.0	10.0
0.40	-0.20	0.20		7.2	10.3	4.1	5.9	8.8	12.5	7.1	8.7
0.60				11.1	15.6	8.8	10.8	7.8	11.5	6.8	9.0
0.80				15.2	20.1	12.7	14.4	9.8	13.8	8.0	9.4
0.60	-0.40	0.40		12.6	14.5	9.2	10.2	6.6	8.0	4.5	6.6
0.80				14.6	16.6	8.3	9.6	12.1	14.0	12.1	13.3
0.80	-0.60	0.60		11.3	12.9	9.8	10.9	5.5	7.2	2.6	4.1
median				*12.0*	*15.6*	*9.5*	*10.6*	*8.3*	*10.9*	*6.3*	*8.0*
mean				*13.4*	*16.0*	*10.0*	*11.3*	*8.5*	*10.7*	*6.8*	*8.4*

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