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Hints for the existence of hexaquark states in the baryon-antibaryon sector

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The discovery of some baryon-antibaryon resonances has led us to consider $3q\ 3\bar{q}$ systems as possible candidates. We predict their spectrum in the framework of a constituent model, where the chromo-magnetic interaction plays the main role. The relevant parameters are fixed by the present knowledge on tetraquarks. The emerging scenario complies well with experiment, besides the description of the baryon-antibaryon resonances, we find evidence for new tetraquark states, namely the $a_0(Y)$ in the hidden strangeness sector and, in the $cs\overline{cs}$ sector, the $Y(4140)$ and the $X(4350)$. A detailed account of the spectra and the decay channels is provided for future comparisons with data.

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I. INTRODUCTION

The presence in the hadron spectrum of mesons consisting of two q's and two $\bar{q}$'s [1–3] as well as of baryons consisting of $4q$ and a $\bar{q}$ has been considered since many years [4].

A long time ago Jaffe proposed that the lightest scalar states, f_0/σ, κ together with the rest of their nonet, should be interpreted as $qq\bar{q}\bar{q}$ states [1].

The simplifying assumption [5] of considering only $2q$ pairs transforming as a $(\bar{3}_c, 1_s, \bar{3}_F)$ representation of $SU(3)_c \times SU(2)_s \times SU(3)_F$, straitens the whole spectrum to the lightest scalar nonet, namely $f_0(600)$, $\kappa(800)$ and $f_0/a_0(980)$ as built with a pair of such a diquark and anti-diquark [6]. This interpretation was recently enforced by experiments confirming the presence of hidden strangeness in both the states $f_0(980)$ and $a_0(980)$ [7], promoting the tetraquarks to a more solid status.

Candidates with open or hidden charm come from the study of non-leptonic B decays at BABAR and BELLE, as anticipated in [8], and from BES.

In this paper we study the spectrum of the states consisting of three quarks and three antiquarks in S-wave, interacting via chromo-magnetism. Besides strangeness we include also charm and assume for chromo-magnetism its full content [9], treated along the lines of ref. [10].

It happens that this hypothesis can successfully interpret some observed baryon-antibaryon negative parity states in $p\overline{p}$ [11], $\Lambda_c\overline{p}$ [12] and $\Lambda_c\overline{\Lambda}_c$ [13], assuming for the parameters (constituent masses and effective couplings) those values obtained from the tetraquarks phenomenology. To study the case of broken flavor symmetry we had to resort to machine computation.

The paper is organized as follows: in section I we introduce the basics of chromo-magnetism with a formulation more suitable for algebraic computation. Section II deals with the formalism for the construction of the tetraquarks states and the study of the open door decays. In section III, IV and V we discuss the phenomenology of tetraquarks states and the parameter fixing of the model. Hexaquark states are introduced in sections VI along with the details entering the calculation. In section VII we present the results we found for the spectrum and compare them with the relevant experimental data. Section VIII contains our conclusions. Finally, the appendix A contains a table with the full spectrum of the baryon-antibaryon systems that were taken under consideration, while in appendix B the crossing matrices required for the study of the decays of tetraquarks are reported. The matrix elements of the chromo-magnetic operator are given, for all cases, in appendix C.

II. THE CHROMO-MAGNETIC INTERACTION

The hyperfine interaction arising from one gluon exchange between constituents leads to a simple Hamiltonian involving the color and spin degrees of freedom:

$$H_{CM} = \sum_i m_i - \sum_{i<j} C_{ij}\, O_{CM}^{(i,\,j)} \tag{1}$$

the index i (j) refers to the ith (jth) quark, m_i its mass and C_{ij} appropriate coupling constants. The kinetic energy is absorbed in the mass term, so it is not surprise that the quarks masses depend on the system under consideration. The C_{ij}'s depend not only on the m_i's (as $1/m_i m_j$) but also on the wave function at zero distance of the pair $(i,\ j)$, so depending on the system as well. Chromo-Magnetism (CM) is encoded in $O_{CM}^{(i,\,j)}$, the two particles chromo-magnetic operator, which is given by:

$$O_{CM}^{(i,\,j)} = \frac{1}{4} \sum_{a=1}^{8} \sum_{k=1}^{3} (\lambda_a \otimes \sigma_k)^{(i)}\, (\lambda_a \otimes \sigma_k)^{(j)} \tag{2}$$

where λ_a are the Gell-Mann matrices and σ_k the Pauli matrices. It is reminiscent of the well known exchange interaction and can be expressed in terms of permutation operators for color and spin $P_c^{(i,j)}$, $P_s^{(i,j)}$ respectively. The action on a (i,j) quark-quark (antiquark-antiquark) pair is given by

$$O_{CM}^{qq} = (P_c - 1/3) \otimes (P_s - 1/2) \quad (3)$$

where $P_c^{(i,j)}$ and $P_s^{(i,j)}$ exchange the colors and spins (acting independently), of the pair (i,j). Eigenvectors of 3 are the diquark states of definite symmetry in color and spin $(6,3)(SS)$, $(6,1)(SA)$, $(\overline{3},3)(AS)$, $(\overline{3},1)(AA)$ with eigenvalues $(-1/3, 1, 2/3, -2)$ respectively.

To express the result for a quark-antiquark pair it is useful to define a generic T_N for the group $SU(N)$ as the object: $T_N : \Psi_A\Xi^B \rightarrow 1/N\ \Psi_A\Xi^B - \delta_A^B\Psi_C\Xi^C$, with Ψ_A in the representation N and Ξ^B in the c.c. representation $\overline{N}$. Making the identification $N = 3$ for T_c and $N = 2$ for T_s we can write quite simply:

$$O_{CM}^{q\overline{q}} = -T_c \otimes T_s \quad (4)$$

The eigenvectors of T_N are the singlet representation $(\delta_A^B\Psi_c\Xi^C)$ with eigenvalue $(1/N - N)$ and the adjoint representation $(\Psi_A\Xi^B - 1/N\delta_A^B\Psi_c\Xi^C)$ with eigenvalue $1/N$. So eigenvectors and eigenvalues of the chromo-magnetic operator in the present case are: $(8,3), (8,1), (1,3), (1,1)$ with eigenvalues $(-1/6, 1/2, 4/3, -4)$ respectively.

By far the more bonded diquark is the $(\overline{3},1)(AA)$ whose $SU(3)_F$ flavor content, as dictated by the Pauli principle, is $\overline{3}_F$. This is the so called good diquark, it transforms as a scalar antiquark. If one assumes the hypothesis of Jaffe and Wilczek [5], the spectrum of the tetraquarks remains restricted to the scalar nonet suggested by Jaffe a long time ago. The vector, or bad diquark $(\overline{3},3)(AS)$, allows for higher spin states but, since it is a 6_F, it also introduces exotics, i.e. multiplets higher than $SU(3)_F$ nonets and are excluded from most models. The other two 6_c states, that Jaffe [3, 14] called sometimes “worse” are not in general taken into account neither.

In the present approach in searching for the eigenstates of the chromo-magnetic operator we do not truncate the space in any way, such that, in some sense, all four possible diquarks enter the game.

It is easy to see that we have the following spin-flavor multiplets: spin 0 has four nonets and two 27_F’s , spin 1 has two nonets, four octets, one 27_F, two decuplets and two antidecuplets, finally spin 2 has two nonets and one 27_F. Exotics, as $I = 2$ states, are not excluded a priori but we think that these states are much less stable and difficult to be observed.

Often, we have found a number of near threshold decays, usually attributed as molecular states, that are well described by chromo-magnetism. In particular the introduction of the $(6,3)$ diquark encompass the dichotomy between diquark and molecular models as clearly argued in [15].

They showed that the molecular state is not an independent state, but is a linear combination of $(\overline{3},1)(3,1)$ and $(6,3)(\overline{6},3)$, the later $(6,3)$, by the way, is the only other diquark with negative chromo-magnetic energy $(-1/3)$. Their observation indicates that a minimal diquark model should include both pairs, and interestingly enough, it would comprise all spin cases as S-wave tetraquarks lying in only $SU(3)_F$ nonets. From the point of view of $SU(6)_{cs}$ this means that a diquark should transform as the symmetric representation, 21 (so as $\overline{3}_F$).

A purely phenomenological motivation to include the $(6,3)$ diquark is that the mass of the $\overline{3}, S = 0, (ud)_{I=0}$ pair, say μ, is related to the mass of the Λ hyperon by the relation [1]: $\mu = m_\Lambda - m_s$, which for a state consisting of two of these objects which have no mutual chromo-magnetic interaction imply about twice the mass of the $f_0(600)$. Instead, by considering the vector space consisting of both the $(\overline{3},1)(3,1)$ and $(6,3)(\overline{6},3)$, $S = 0$ color singlet states, the lightest state has a binding energy about 2.7 times larger than the diagonal matrix element for $(\overline{3},1)(3,1)$ [10].

In the flavor symmetry limit, i.e. when the couplings C_{ij} are all equal to each other, it is well known that O_{CM} can be expressed as a combination of Casimirs. This fact has been extensively exploited in the pioneering works of Jaffe [3] and in many other works [4]. In the present paper we shall attack the more complicated issue of considering different masses and couplings, in most of such cases we have to rely on symbolic manipulations that we performed with FORM [16]. The expressions in Eqs. (3) and (4) result quite suitable for computer implementation.

III. “OPEN DOOR” CHANNELS FOR TETRAQUARKS

It has been observed for the first time by Jaffe [1] that $qq\overline{q}\overline{q}$ mesons may decay into two ordinary (i.e. color singlet) mesons PP, PV, VV (P stands for a pseudoscalar and V for a vector) by simply separating from each other, as long as it is kinematically allowed. He called these channels “open door” or “Ozi super-allowed” decays, since they can occur without gluon exchange or quark annihilation. In open door channels, S-wave states have to decay into S-wave mesons with zero relative angular momentum.

In general calculations are performed in the diquark-antidiquark basis, i.e. the tetraquark is represented as $q_1q_2\overline{q}_3\overline{q}_4$ denoted **[12,34]** in the following. Evidently the diquark and the antidiquark cannot separate from each other as they can never be color singlets. So, in order to access the open door channels it is convenient to pass to the meson-meson basis **[13,24]** and **[14,23]** which, obviously, coincide if antiquarks 3 and 4 have the same

[1] We are indebted to Prof. P. Minkowski for bringing this remark to our knowledge

flavor.

In order to have some uniformity in the conventions, we maintain those of [10]. We call the basis for spin 0: as ϕ in **[12,34]**, α in **[13,24]** and ϵ in **[14,23]**, in the same order one has ψ, β and χ for spin 1, while those of spin 2 are called ξ, γ and δ. To characterize each basis, we have only to specify the color-spin content of the first and second pairs in the brackets, which combine to form the color singlets i.e. the set of physical states.

Spin 0

$$
\begin{aligned}
(\phi)\mathbf{[12{,}34]} &: [(6,3)(\overline{6},3)];\ [(\overline{3},1)(3,1)];\\
&\quad [(6,1)(\overline{6},1)];\ [(\overline{3},3)(3,3)]\\
(\alpha)\mathbf{[13{,}24]} &: [(1,1)(1,1)];\ [(1,3)(1,3)];\\
&\quad [(8,1)(8,1)];\ [(8,3)(8,3)]\\
(\epsilon)\mathbf{[14{,}23]} &: \text{as } \mathbf{[13{,}24]}
\end{aligned}
\tag{5}
$$

For α and ϵ the first components are PP and the second VV. The last two are P_8P_8 and V_8V_8, where P_8 is a colored pseudoscalar and V_8 a colored vector

Spin 1

$$
\begin{aligned}
(\psi)\mathbf{[12{,}34]} &: [(6,3)(\overline{6},3)];\ [(\overline{3},3)(3,3)];\ [(\overline{3},1)(3,3)];\\
&\quad [(6,3)(\overline{6},1)];\ [(\overline{3},3)(3,1)];\ [(6,1)(\overline{6},3)]\\
(\beta)\mathbf{[13{,}24]} &: [(1,1)(1,3)];\ [(1,3)(1,1)];\ [(1,3)(1,3)];\\
&\quad [(8,1)(8,3)];\ [(8,3)(8,1)];\ [(8,3)(8,3)]\\
(\chi)\mathbf{[14{,}23]} &: \text{as } \mathbf{[13{,}24]}
\end{aligned}
\tag{6}
$$

So β_1, χ_1 (β_2 χ_2) are PV(VP) and β_3, χ_3 are VV.

Spin 2

$$
\begin{aligned}
(\xi)\mathbf{[12{,}34]} &: [(6,3)(\overline{6},3)];\ [(\overline{3},3)(3,3)]\\
(\gamma)\mathbf{[13{,}24]} &: [(1,3)(1,3)];\ [(8,3)(8,3)]\\
(\delta)\mathbf{[14{,}23]} &: \text{as } \mathbf{[13{,}24]}
\end{aligned}
\tag{7}
$$

The only open door channel for a tensor meson is, evidently, VV.

The relative probability for the particle decaying through a specific channel is given by the square of the corresponding component of the normalized eigenvector of the state multiplied by phase space (as is assumed all dynamical amplitudes to be the same). For convenience we call the square of the component along the channel the probability factor (PF) for that channel. In some cases we have also to consider the non open door channels, if for instance, the open door have negligible probabilities or are kinematically forbidden, and so violations of the OZI rule would enter the game. In particular the P_8P_8 or V_8V_8 channel can become relevant at order $O(\alpha_s)$, as the exchange of one gluon in the t-channel converts this object into an ordinary PP or VV pairs.

The so called crossing matrices operating the change of a basis into another, arise from well known Fierz identities for color and spin [3] and are available in many places, for definiteness we will refer to [10]. They are reproduced, together with a necessary completion, in Eqs. [B1-B5].

IV. TETRAQUARK STATES

It is immediate to realize that the overall chromomagnetic contribution in Eq. 1 (let us call it O_{CM} and assume thoroughly $C_{q\overline{q'}} = C_{qq'}$ for any (anti) quarks pair) greatly simplifies for 0^+ and 2^+ states made of at least three constituents with the same flavor, say of type $q\overline{q}q\overline{q}'$ (q is not necessarily a light quark and q and $q\prime$ can incidentally coincide), since the corresponding matrices depend exclusively on the combination $(C_{qq} + C_{qq'})$, which factorizes out. For 2^+ we have: $O_{CM} = -4/3(C_{qq} + C_{qq'})\, diag(1,1)$, while for 0^+:

$$
O_{CM} = -1/2(C_{qq} + C_{qq'}) \cdot
\begin{pmatrix}
8 & 0 & 0 & -4\sqrt{\frac{2}{3}} \\
0 & -\frac{8}{3} & -4\sqrt{\frac{2}{3}} & 0 \\
0 & -4\sqrt{\frac{2}{3}} & -1 & -\frac{5}{\sqrt{3}} \\
-4\sqrt{\frac{2}{3}} & 0 & -\frac{5}{\sqrt{3}} & \frac{19}{3}
\end{pmatrix}.
\tag{8}
$$

The eigenvalues of the above matrix are $\lambda_1 = 1/3(17 + \sqrt{241})$, $\lambda_2 = 1/3(\sqrt{241} - 1)$, $\lambda_3 = 1/3(17 - \sqrt{241})$, $\lambda_4 = -1/3(\sqrt{241} + 1)$, with corresponding eigenvectors (for briefness we give decimal approximations) $(-0.74, 0.04, 0.17, 0.65)$, $(0.64, 0.18, -0.41, 0.62)$, $(0.18, -0.64, 0.62, 0.41)$ and $(0.04, 0.74, 0.64, 0.17)$.

The spectrum is given by $M_a^{(0)} = 3m_q + m_{q'} - 1/2\ \lambda_a\ (C_{qq} + C_{qq'})$, $(a = 1, ..., 4)$ for 0^+ and by $M_b^{(2)} = 3m_q + m_{q'} + 4/3\ (C_{qq} + C_{qq'}), (b = 1, 2)$ for 2^+. These considerations apply also to the case of three light constituents within the approximation of exact isospin symmetry.It is worth to stress that this phenomenon does not happen for 1^+.

A simple consequence of the fact that the eigenvectors do not depend on the masses and couplings is that the scalar nonet presents an universal pattern of decays, the lowest state has about 55% probability to decay into PP (negligible in VV) and for the next states, in order of increasing mass: 41% in VV, 41% in PP and 55% in VV. Identifying the lowest state of the light nonet with the $\sigma/f_0(600)$ and the third one with the $f_0(1370)$ we get the mass of light quarks m_q and C_{qq}, we find $m_q \cong 351.65\ MeV$ and $C_{qq} \cong 74.4\ MeV$. Notice that the quark mass and the coupling can be expressed in terms of the masses of σ and f_0 by:

$$\begin{aligned} 4m_q &= m_\sigma + \left(1+\frac{17}{\sqrt{241}}\right)\frac{m_{f_0}-m_\sigma}{2} \\ C_{qq} &= \frac{3}{\sqrt{241}}\frac{m_{f_0}-m_\sigma}{2}. \end{aligned} \tag{9}$$

So it is immediate to realize that, if we would take for m_σ a lower value, around $450MeV$, as suggested by some authors, the change in m_q would be negligible but C_{qq} would rise to $89\,MeV$

A similar determination of the parameters concerning the s and c quarks is not feasible because presently we dispose only of one strange scalar as a possible candidate for a tetraquark ($\kappa(800)$) and none for charm. For the s quark we choose the parameters in order to reproduce the masses of the $\kappa(800)$ as a $(qq\overline{qs})$ state, the $a_0(980)$ as a $(qs\overline{qs})$ and the $f_1(1420)$ as a 1^+ $(qs\overline{qs})$ state, getting $m_s \cong 455.21\ MeV$, $C_{qs} \cong 58.04\ MeV$ and $C_{ss} \cong 43.2 MeV$.

It is quite unexpected the almost exact agreement with the parameters of our previous calculation for the pentaquarks [17], where we found: $m_q \cong 346.8\,MeV$, $C_{qq} \cong 74.\,MeV$, $m_s \cong 480\,MeV$ and for C_{qs} and C_{ss} we assumed the hyperfine prescription $\frac{C_{qs}}{C_{qq}} = \frac{C_{ss}}{C_{qs}} = \frac{m_q}{m_s}$ which, as a matter of fact, is also well satisfied by the tetraquark determinations.

The parameters related to charm have been obtained requiring agreement with the masses of the following states: $X(3872)$ as a 1^+ $(qc\overline{qc})$ state, the pair $D_s(2317)$ and $D_s(2573)$ as 0^+ $(qc\overline{qs})$ states and finally $D_s(2460)$ as a 1^+ $(qc\overline{qs})$ state. The values obtained for the parameters are: $m_c \cong 1631\,MeV$, $C_{qc} = 26\,MeV$, $C_{cc} = 18\,MeV$, $C_{sc} = 17.6\,MeV$. A direct determination from the J/ψ and η_c masses gives $m_c \simeq 1534\,MeV$, $C_{cc} = 21.6\,MeV$. Since we expect a bigger kinetic energy for the tetraquark together with a broader wave function, the discrepancy goes in the right direction [2]. On the other side if we determine C_{qc} from the $D^* - D$ mass splitting, we get $C_{qc} = 26.2MeV$, in excellent agreement with the determination via the tetraquaks spectrum.

Here it is interesting to notice that the system $Q\overline{q}$ should obey some general property as a consequence that the recoil of Q can be safely neglected. So it should not depend on the mass of Q, but only on the radial and orbital quantum numbers of $\overline{q}$. Since $\overline{q}$ is very light the system would have a spatial extension that falls in the region of dominance of the linear part of the confinement potential, (phenomenological analysis demonstrate that the $c\overline{c}$ system falls in the logarithmic dominated region) for which well known scaling laws [19] prescribe that the wave function at the origin does not depend on the Q mass, so we should expect the product $m_Q\,C_{qQ}$ to be constant. A law equivalent to the constancy of the product $m_Q\,C_{qQ}$ has been inferred some time ago in ref. [20] and verified for a great number of states involving charm or beauty.

J^P	$qq\overline{qq}$	$qq\overline{qs}$	$qq\overline{qc}$	$qs\overline{ss}$	$ss\overline{ss}$	*Decays*
0^+	$600.^{(*)}$ $I=0$	$792.3^{(*)}$ $I=1/2$	2141.7	$-$	$-$	$0.55(PP);\ 1.7 10^{-3}(VV)$
Exp	$f_0(600)$	$\kappa(800)$				
0^+	1046.4 $I=0,1,2$	1189.6 $I=1/2,3/2$	2442.9	1472.2	1611.6	$0.41(PP);\ 3.1 10^{-2}(VV)$
0^+	$1370.^{(*)}$ $I=0$	1477.6 $I=1/2$	2661.2	$-$	$-$	$3.1 10^{-2}(PP);\ 0.41(VV)$
Exp	$f_0(1370)$					
0^+	1816.4 $I=0,1,2$	1874.9 $I=1/2,3/2$	2962.4	1996.1	2058.9	$1.7 10^{-3}(PP);\ 0.55(VV)$
2^+	1605. twice $I=0$ and $I=1,2$	1686.7$I=1/2,3/2$	2819.8	1852.3	1936.1	$0.5(VV);$ 0.5 (light mesons)
Exp	$X(1600)$ $I=2$[21]				$f_2(2010)$?[22]	

TABLE I. 0^+ and 2^+ states with 3 light (strange) quarks calculated exactly according to section IV. Values of masses used in the fit are distinguished with a (*). Experimental results, when available, are displayed in the next row, numbers in square brackets give the reference to the experimental data. Pauli principle fixes the isospins of the various states,so $qq\overline{qc}$ have the same isospins as $qq\overline{qs}$ while $qs\overline{ss}$ have $I=1/2$ and $ss\overline{ss}$ $I=0$. The states forbidden by the Pauli principle are indicated by $(-)$. Masses are given in MeV.

[2] The hyperfine law $\simeq 1/m_i m_j$ does not apply to the charm sector, since the wave function, due to a much higher mass, is much peaked around the origin, partially compensating the mass powers in the denominator. Actually, recent data on the η_b suggest a mass splitting with the Υ of the same order of the $\eta_c - \psi$, and not a factor $(m_c/m_b)^2$ smaller [18].

In the case of a "neutral" state ($qq'\overline{qq}'$), as for hidden strangeness or charm, the 1^+ CM matrix in the β-basis is block diagonal, with a 2×2 block corresponding to $C=+$ and the other 4×4 block to $C=-$. So, independently of the parameters, we have two exact eigenvectors, one along the direction β_3 (pair of color singlet vectors) and the other along β_6 (pair of color octet vectors). On the other hand all scalars and tensors have the same charge conjugation, $C=+$.

It is immediate to calculate the masses of the two C-even states: the first has mass $2m_q+2m_{q'}+4/3(C_{qq}+C_{q'q'})$ and the second $2m_q+2m_{q'}-1/6(C_{qq}+18\ C_{qq'}+C_{q'q'})$. We can also calculate exactly the 2^+ sector getting for the mass $2m_q+2m_{q'}+4/3(C_{qq}+C_{q'q'})$, the corresponding eigenvector being along γ_1 (pair of color singlet vectors), the value of the other mass is $2m_q+2m_{q'}-1/6(C_{qq}-18\ C_{qq'}+C_{q'q'})$ corresponding to γ_2 (pair of color octet vectors). A general trend for this case is that the highest 1^{++} state is degenerate with the highest 2^+ state, both decaying exclusively into $V_{q\overline{q}}V_{q'\overline{q}'}$. The other 1^{++} is below the light tensor state and has dominant decay into $P_{q'\overline{q}}V_{\overline{q}'q}+P_{\overline{q}'q}V_{q'\overline{q}}$ while the light tensor decays into $V_{q'\overline{q}}V_{\overline{q}'q}$. The states 0^{++} and 1^{+-} have to be calculated numerically, with the exception of the case $q=q'$, when the spectrum of the 1^+ becomes highly degenerate. In such a case, the C-even state β_6 is paired with a C-odd state with eigenvector $\chi_6 = 2/3(-1,1,0,1/(2\sqrt{2}),-1/(2\sqrt{2}),0)$, the other C-even state β_3 becomes degenerate with the C-odd state with eigenvector $\chi_3 = 2/3(-1/(2\sqrt{2}),1/(2\sqrt{2}),0,-1,1,0)$. As can be seen from the table below, the mass region $1100-1950\,MeV$ could seem to be populated by some controversial peaks with no definite spin or C-parity, due to states overlapping.

C	$-$	$+$	$-$	$-$	$+$	$-$
$qq\overline{qq}$	1109. $I=0$	1158.6 $I=1$	1158.6 $I=1$	1406.6 $I=0,1,2$	1605. $I=1$	1605. $I=1$
$ss\overline{ss}$	$-$	$-$	$-$	1820.8	$-$	$-$
Decays	PV	PV	PV	PV	VV	VV

TABLE II. Axial states made of all light (in the limit of exact isospin) or strange (anti) quarks calculated exactly, according to section IV. They have definite charge conjugation. The states forbidden by the Pauli principle are indicated by ($-$). Masses are given in MeV.

Even if no candidates have been observed let us, for completeness, give the spectrum of strange and charmed axials:

$qc\overline{qq}$	2329.3	2515.6	2611.7	2727.8	2785.8	2877.73
Decays	$0.55(\pi,\eta)D^*$	$0.39(\pi,\eta)D^*$	$0.47(\omega,\rho)D$	$0.28(\omega,\rho)D$	$0.47(\omega,\rho)D^*$ $0.11(\omega,\rho)D$	$0.46(\omega,\rho)D^*$
$qq\overline{qs}$	1207.5	1302.7	1308.6	1513.4	1672.7	1703.
Decays	$0.56(\pi,\eta)K^*$	$0.17(\omega,\rho)K$ $0.28(\pi,\eta)K^*$	$0.52(\omega,\rho)K$	$0.21(\omega,\rho)K$ $0.12(\pi,\eta)K^*$	$0.50(\omega,\rho)K^*$	$0.50(\omega,\rho)K^*$
Isospin	$1/2$	$1/2,3/2$	$1/2$	$1/2,3/2$	$1/2$	$1/2,3/2$

TABLE III. Charmed and strange axial mesons, calculated numerically. Masses are in MeV. The non negligible decay channels are indicated in the row below.

When an object contains a pair of (anti) quarks, Pauli principle implies the absence of some states or, otherwise, if the pair is made of light quarks, restrictions on the isospin content, according to the correspondence $I=0\rightarrow 21_{cs}$ and $I=1\rightarrow 15_{cs}$. This has been taken into account in the elaboration of Tables I, II, III, where Pauli forbidden states are indicated by a hyphen. The very interesting cases of hidden strangeness/charm and tetraquarks with $C=\pm\ S=1$ were calculated numerically and are given in the Table V. The interest for the somewhat chimerical states with $C=-\ S=1$ and $C=2$ i.e. of kind ($cs\overline{qq}$) and ($cc\overline{qq}$), is justified by the fact that they provide a clear signature for tetraquarks. In the case of $I=0$ the first decays into D^+K^- and $D^0\overline{K}^0$ and the second into D^+D^0. Since in both cases the objects carrying strangeness or charm are necessarily a pair of quarks and obviously they cannot form by themselves color singlets, so the occurrence of such states is possible only if the pair of quarks combine with at least a pair of antiquarks.

V. DISCUSSION ON THE RESULTS FOR TETRAQUARKS

First of all, let us recall that the information we used in the fit involves only the mass spectrum, so the pattern of decays may be considered as “predictions”. Let us cite the observed dominance of $\pi\pi$ in the $f_0(600)$ decay and of $\rho\rho$ in that of $f_0(1370)$[23][24], the dominance of the πK channel for $\kappa(800)$ (unfortunately, by now, omitted from PDG).

For the axials we obtained the dominance of $\overline{K}K^* + cc$ ($KK\pi$ probably arising from a off-shell K^*) for the $f_1(1420)$ and, analogously, the dominance of $\overline{D}D^* + cc$ for the $X(3872)$.

J^P	$qs\overline{qs}$	$cs\overline{qq}$	$qc\overline{qs}$	$qc\overline{qc}$	$cc\overline{qq}$	$cs\overline{cs}$
0^+	$981.^{(*)}$	2326.7	$2315.^{(*)}$	3562.7	3643.1	3904.5
Exp	$a_0(980)$		$D_{s_0}^{*\pm}(2317)$			
0^+	1330.3	2592.3	$2574.^{(*)}$	3799.3	3870.8	4060.8
Exp	$a_0(Y)$[29]		$D_{s_1}^{\pm}(2573)$			
0^+	1586.1	2757.6	2773.7	3979.3	3898.6	4181.
0^+	1934.9	3028.	3028.4	4148.4	4144.5	4295.
Exp						$X(4350)$[28]
1^+	1327.6	2503.7	$2469.3^{(*)}$	3682.9	3795.3	4016.41
Exp			$D_{s_1}^{\pm}(2460)$			
1^+	$1420.^{(*)}$	2674.5	2634.6	$3871.9^{(*)}$	3847.8	4109.4
Exp	$f_1(1420)$			$X(3872)$		$Y(4140)$[26]
1^+	1461.	2692.	2736.4	3924.6	3927.8	4132.1
1^+	1618.9	2822.8	2823.	3980.5	3991.6	4172.5
1^+	1770.5	2857.8	2889.3	4057.5	3992.2	4225.9
1^+	1773.	2959.5	2951.	4088.5	4084.78	4254.
2^+	1768.2	2889.	2900.2	4027.9	4021.2	4215.
2^+	1770.5	2906.9	2912.2	4088.5	4061.6	4254.

TABLE IV. Spectrum of the tetraquarks calculated numerically. States used in the fit are marked with a (*). When experimental data are available they are displayed in the next row, reference to the source are given in square brackets. Masses are in Mev.

Since they are pure β_6 states these channels are exclusive. In particular, for $X(3872)$, the observed decays into $\rho(\omega)J/\psi$, can be explained by one gluon exchange in the t-channel, since those rates are comparable with the process being $O(\alpha_s)$. For $D_{s_0}^{*\pm}(2317)$ the only kinematically allowed open door channel is $\pi^0 D_s^{\pm}$, it is just below the DK threshold, at $2359\,MeV$. The relevant components are $\alpha_1 = 0.78$, $\epsilon_1 = 0.70$, so predicting strong dominance of the $\pi^0 D_s^{\pm}$ decay. In the case of $D_{s_2}^{*\pm}(2573)$ that we interpreted to be 0^+ (even if it is also consistent with a 2^+) the only observed decay is $D^0K^{\pm}$ while $D^{0*}(2007)K^{\pm}$ is not, so in agreement with PP prescription arising from scalar nature of the state. Nevertheless, also in this case the components are almost equal $\alpha_1 = 0.60$ ($\pi^0 D_s^{\pm}$), $\epsilon_1 = 0.68$ ($D^0K^{\pm}$) and so we could expect the $\pi^0 D_s^{\pm}$ to be relevant, as well. Experimental data neither confirm nor disprove this point. Finally the axial state $D_{s_1}^{\pm}(2460)$, that we put at $2469.3\,MeV$, has a large component along $\beta_1(0.87)$, which corresponds to the dominant $\pi^0 D_s^{\pm *}$ channel. The $\omega D_s^{\pm}$ decay (notice the state $D_{s_1}^{\pm}(2460)$ has $I = 0$) has a tiny component $\beta_2 = 0.024$ and is also kinematically inaccessible. It remains to explain the large branching fraction in $D_s^{\pm}\gamma$, suggesting that the state is very narrow, albeit the experimental upper bound is not much restrictive, $\Gamma \leq 3.5 MeV$.

Concerning the two degenerate states, the isoscalar f_0 and the isovector a_0, at $980 MeV$, they can only decay into $\eta\pi$ and $K\overline{K}$, other channels being too high. We predict $\alpha_1 = 0.75$, $\epsilon_1 = 0.74$, and if we take the corrections for the mixing $\eta_0 - \eta_8$ with a mixing angle $\theta = -16°$ (as obtained recently in $\gamma\gamma \to X$), we find for the ratio of $g^2_{a_0 K\overline{K}}/g^2_{a_0\eta\pi} \cong 2.48$, to be compared with the value recently obtained by the KLOE experiment [7] of $0.67 \pm 0.06 \pm 0.13$. This abnormally large coupling for $\eta\pi$ cannot be obtained by chromo-magnetism alone, it has been explained recently [25] by non perturbative effects induced by instantons. Analogously for the dominant decay $f_0 \to \pi\pi$, which violates OZI rule, we have to rely on the above solution, in association with $f_0(980) - \sigma$ mixing.

We predict a companion (which is a mixture of 8_F and 27_F), for the $a_0(980)$ at $1330.3 MeV$ coupled to $\eta\pi$, $\eta'\pi$ and $K\overline{K}$. It was recently observed [29] in $\gamma\gamma \to \eta\pi^0$ and named $a_0(Y)$ with an observed mass of $1316 \pm 25 MeV$.

In the hidden charm-strange sector ($cs\overline{cs}$)we have found two candidates for newly discovered states. The first is the pure β_6 1^{++} state at $4109.4 MeV$ which we propose to identify to the narrow state $Y(4140)$ found at CDF [26] in $B^+ \to XK^+, X \to J/\psi\phi$, with a mass $4143 \pm 2.9 \pm 1.2 MeV$ and a width of $11.7^{+8.3}_{-5.0} \pm 3.7 MeV$. As the $X(3872)$ the later has dominant decays into $\overline{D_s} D_s^* + cc$ (threshold at $4080 MeV$), but can also decay into $J/\psi\phi$ (threshold at $4116.4 MeV$). The choice of spin one[3] is strongly suggested by the fact that it was not observed in $\gamma\gamma \to X$ by BELLE[28]. The second state is a 0^{++} at $4295 MeV$, with predominant decays into $J/\psi\phi$ ($\alpha_2 \simeq 0.81$) and $D_s^* \overline{D_s^*}$ ($\beta_2 \simeq 0.69$), to be interpreted as the $X(4350)$, discovered by BELLE in the same experiment[28], with a mass $4350.6^{+4.6}_{-5.1} \pm 0.7 MeV$ and width $13.3^{+17.9}_{-9.1} \pm 4.1 MeV$. Taking into account phase space, we find the $J/\psi\phi$ channel to be twice more probable than the $D_s^* \overline{D_s^*}$ one.

Among the non well established states there is a 2^+ state $X(1600)$ (with I=2) [21] at $1600 \pm 100 MeV$ that, if interpreted as ($qq\overline{qq}$), is compatible with our predictions and, according to the previous section, it has to be degenerate with the highest 1^{++}, the later being possibly hidden by some ($L = 1$ $q\overline{q}$) state of the a_1 family.

It is not excluded that presently we have already seen some ($ss\overline{ss}$) states, one of these could be the $f_0(2010)$ found around $2011 \pm 70 MeV$ [22] that is identifiable to our 2^+ state at $1936\, MeV$. We predict a 1^+, ($qs\overline{qs}$) state, with a mass of $1327.6 MeV$ decaying predominantly into $\pi\phi$ ($\beta_1 \backsimeq 0.91$) and another one at $1773 MeV$ with important components along $\beta_1 \backsimeq 0.34$ ($\eta_s V$) and $\beta_2 \backsimeq 0.15$ ($\pi\phi$, $\eta\phi$),while χ_3 is also very large, the state is below threshold for $K^* \overline{K}^*$. The later could be, possibly identified to the $X(1835)$ found at BES [30] at $1834 \pm 6\, MeV$ and width $67.7 \pm 20.3 \pm 7.7\, MeV$, decaying into $\pi^+\pi^-\eta'$. The spin-parity of the $X(1835)$ is not known and it was, initially, supposed to be related to a $p\overline{p}$ threshold enhancement, due to the strong dominance of the channel $\pi^+\pi^-\eta'$.

We also predict a 0^+ $ss\overline{s}\overline{s}$ state at $2058.9 MeV$, is strongly coupled to $\phi\phi$, so it would arise as a $\phi\phi$ threshold enhancement.

VI. NEGATIVE PARITY STATES BUILT WITH THREE QUARKS AND THREE ANTIQUARKS

Today it seems to exist experimental evidence for the occurrence of baryon-antibaryon states. People could have the tendency to interpret them as molecular states, but as said before, there is no clear distinction between chromo-magnetism and the molecular point of view as long as we do not neglect some configurations of the diquarks. In obtaining the predictions of chromo-magnetism, since the number of candidates is not enough to completely determine the parameters, we will tentatively assume for the masses and chromo-magnetic couplings of the quarks in the baryon-antibaryon system the same as for tetraquarks. As mentioned before, masses could be larger due to the fact that they are defined including the kinetic energy. On the other hand, couplings could be smaller mainly because the wave function is more spread.

A complete calculation is very complex and probably not of immediate utility in view of the scarcity of these states. We treat two cases, the first is related to $p\overline{p}$ states and concerns ($qqq\overline{qqq}$) systems, the second deals with the production of a variety of states of the kind ($qqq\overline{qq}\overline{Q}$) or ($qqQ\overline{qq}\overline{Q}$), where Q denotes an s or c quark.

It is natural to work with what we call the baryon-antibaryon basis. In the first case, since we are interested in a $p\overline{p}$ pair, it is enough to take the sub block qqq in the 70 of $SU(6)_{cs}$ (and $\overline{qqq}$ in the $\overline{70}$). The decomposition of the 70, under $SU(3)_c \otimes SU(2)_s$ is given by: $70_{cs} = (8_c, 4_s) + (8_c, 2_s) + (10_c, 2_s) + (1_c, 2_s)$. We can construct 4 color singlets of spin 0 and 6 of spin 1, which are below:

Spin 0

$$|1\rangle = [(1_c, 2_s), (1_c, 2_s)];\ |2\rangle = [(8_c, 2_s), (8_c, 2_s)];$$
$$|3\rangle = [(8_c, 4_s), (8_c, 4_s)];\ |4\rangle = [(10_c, 2_s), (\overline{10}_c, 2_s)] \tag{10}$$

Spin 1

$$|1\rangle = [(1_c, 2_s), (1_c, 2_s)];\ |2\rangle = [(8_c, 2_s), (8_c, 2_s)];$$
$$|3\rangle = [(8_c, 4_s), (8_c, 4_s)];\ |4\rangle = [(10_c, 2_s), (\overline{10}_c, 2_s)];$$
$$|5\rangle = [(8_c, 2_s), (8_c, 4_s)];\ |6\rangle = [(8_c, 4_s), (8_c, 2_s)] \tag{11}$$

Evaluating the chromo-magnetic operator of Eq. [1] between these states we get the 2 matrices, describing

[3] The interpretation of the $Y(4140)$ as an axial was already contemplated in ref [27], albeit not excluding the 0^{++} alternative.

chromo-magnetism in the 2 sectors, given in Eqs. [C1,C2], where it was assumed the same ordering as above.

This has been done using a computer, but since we are in fact in the symmetry limit, it can also be calculated by purely group theoretical means. It furnishes a valuable check of the machine's symbolic calculation. It is straightforward to obtain the expression in terms of Casimir operators:

$$\begin{aligned} O_{CM} &= [C_6(R_{3q}) + C_6(R_{3\overline{q}}) - \frac{1}{2}C_3(R_{3q}) - \frac{1}{2}C_3(R_{3\overline{q}}) \\ &\quad -\frac{1}{3}S_{3\overline{q}}(S_{3\overline{q}}+1) - \frac{1}{3}S_{3\overline{q}}(S_{3\overline{q}}+1) - 12] \\ &\quad -[C_6(H) - C_6(R_{3q}) - C_6(R_{3\overline{q}}) + \frac{1}{2}C_3(R_{3q}) \\ &\quad +\frac{1}{2}C_3(R_{3\overline{q}}) - \frac{1}{3}S_H(S_H+1) \\ &\quad +\frac{1}{3}S_{3q}(S_{3q}+1) + \frac{1}{3}S_{3\overline{q}}(S_{3\overline{q}}+1)] \qquad (12) \end{aligned}$$

where H stands for the representation of the hexaquark in $SU(6)_{cs}$, with S_H being its spin (0 or 1 in the present case), R_{3q} and $R_{3\overline{q}}$ the representations of the 3 quarks and 3 antiquarks subsystems, respectively (of both groups , $SU(6)_{cs}$ and $SU(3)_c$), S_{3q} and $S_{3\overline{q}}$ being their spins. As before, C_6 and C_3 are the quadratic Casimir operators of $SU(6)_{cs}$ and $SU(3)_c$. In the first square brackets we have isolated the contribution of the quark-quark and antiquark-antiquark interactions, while in the second the contribution for quark-antiquark interactions. Here a severe complication arises: the Casimir operators in the second bracket are not diagonal. As the operator O_{CM} transforms as the 35 of $SU(6)_{cs}$, it does not leave the 70 and, thus the Casimir operators present in the first bracket are diagonal, while for the second one, representation mixing remains possible and in fact it occurs.

The hexaquark state $(qqq\overline{q}\overline{q}\overline{q})$, we have designated by H, transforms under $SU(6)_{cs}$ as one of irreducible representations (or mixings thereof) arising in the product below: $70\otimes\overline{70} = 1+35_1+35_2+189+280+\overline{280}+405+3675$. For 0^- we have to select the blocks that contain components transforming as $(1_c, 1_s)$, and for the 1^- as $(1_c, 3_s)$. It is indicated below the relevant representations and the number of components of the suitable color singlets contained in each one:

$$0^- : (1_c, 1_s) \subset 1;\ 189(1);\ 405(1);\ 3675(1)$$

$$1^- : (1_c, 3_s) \subset 35_1(1);\ 35_2(1);\ 280(1);\ \overline{280}(1); 3675(2).$$

The matrix elements were found through the determination of the appropriate Clebsch Gordan coefficients for the above decomposition.

Let us now consider states of the kind $(qqQ\overline{q}\overline{q}\overline{Q})$ (Q being an s or c quark), for which some experimental evidence is available. The Pauli principle implies that the pair of light (anti-)quarks in the (anti-)baryonic block qqQ ($\overline{q}\overline{q}\overline{Q}$) must transform under $SU(6)_{cs}$ as a 21_{cs} ($\overline{21}_{cs}$) for $I = 0$ and as a 15_{cs} ($\overline{15}_{cs}$) in the case of $I = 1$. States such $(qq)_{21_{cs}}Q(\overline{q}\overline{q})_{(\overline{21}_{cs})}\overline{Q}$ have I=0 and are relevant for the $\Lambda\overline{\Lambda}$ ($\Lambda_c\overline{\Lambda}_c$) channels. For shortness, we shall call them the $(21, \overline{21})$ basis. The other case, namely $(qq)_{15_{cs}}Q(\overline{q}\overline{q})_{(\overline{15}_{cs})}\overline{Q}$ is the $(15, \overline{15})$ basis and comprises hexaquarks with $I = 0, 1, 2$. This base will be used in the calculation of the $\Sigma\overline{\Sigma}$ channel.

A criterion to build the physical states, i.e. the color singlets of the six quark system, is to combine successively qq with Q (and analogously for the antiquarks) in all possible ways regarding the color group $SU(3)_c$ and then combining with those of the antiquarks. This can be easily done using the decompositions of $SU(6)_{cs} \to SU(3)_c \otimes SU(2)_s$: $21_{cs} = (\overline{3}_c, 1_s) + (6_c, 3_s)$ and $15_{cs} = (6_c, 1_s) + (\overline{3}_c, 3_s)$. Taking into account the genealogy of the states, we get for each basis, a total of 14 color singlets. They are displayed below [4]. The convention we use is the following: the composition of the baryonic (qqQ) with anti-baryonic blocks $(\overline{q}\overline{q}\overline{Q})$ is indicated by a (*), each block is enclosed by a square bracket and within each bracket we placed on the left the color-spin content of (qq) followed by that of Q (and analogously for the antiquarks).

As will be seen in the next section, we have also interest to build the basis for the system $\Lambda_c\overline{p}$.We use the ordering convention $(\overline{q}_1\overline{q}_2\overline{q}_3q_4q_5c_6)$. The $\overline{p}$, as previously, is put in a $\overline{70}_\beta$ (antisymmetric in 1,2) and the Λ_c (as the Pauli antisymmetry applies only to the pair 4 and 5) in a 70_α (symmetric with respect to 4 and 6) and a 56, which decomposes under $SU(3)_c\otimes SU(2)_s$ as: $(10, 4)+(8, 2)$. The mandatory anti-symmetrization with respect to flavor of the pair 4 and 5 implies isospin 0 for the Λ_c.

Basis $(21, \overline{21})$ for spin 1

$$\begin{aligned}
[(\overline{\mathbf{3}},\mathbf{1})(\mathbf{3},\mathbf{2})] * [(\mathbf{3},\mathbf{1})(\overline{\mathbf{3}},\mathbf{2})] \Rrightarrow\ & |1\rangle = (1,2)*(1,2) \\
& |2\rangle = (8,2)*(8,2) \\
[(\mathbf{6},\mathbf{3})(\mathbf{3},\mathbf{2})] * [(\mathbf{3},\mathbf{1})(\overline{\mathbf{3}},\mathbf{2})] \Rrightarrow\ & |3\rangle = (8_{sim},4)*(8,2) \\
& |4\rangle = (8_{sim},2)*(8,2) \\
[(\overline{\mathbf{3}},\mathbf{1})(\mathbf{3},\mathbf{2})] * [(\overline{\mathbf{6}},\mathbf{3})(\overline{\mathbf{3}},\mathbf{2})] \Rrightarrow\ & |5\rangle = (8,2)*(8_{sim},4) \\
& |6\rangle = (8,2)*(8_{sim},2) \\
[(\mathbf{6},\mathbf{3})(\mathbf{3},\mathbf{2})] * [(\overline{\mathbf{6}},\mathbf{3})(\overline{\mathbf{3}},\mathbf{2})] \Rrightarrow\ & |7\rangle = (8_{sim},4)*(8_{sim},4) \\
& |8\rangle = (8_{sim},4)*(8_{sim},2) \\
& |9\rangle = (8_{sim},2)*(8_{sim},4) \\
& |10\rangle = (8_{sim},2)*(8_{sim},2) \\
& |11\rangle = (10,4)*(\overline{10},4) \\
& |12\rangle = (10,4)*(\overline{10},2) \\
& |13\rangle = (10,2)*(\overline{10},4) \\
& |14\rangle = (10,2)*(\overline{10},2)
\end{aligned} \qquad (13)$$

[4] The representation 8_{sim} is the color octet symmetric under the exchange of the colors of the light quark pair.

Basis $(15,\overline{15})$ for spin 1

$$
\begin{array}{ll}
[(\bar{\mathbf{3}},\mathbf{3})(\mathbf{3},\mathbf{2})]*[(\mathbf{3},\mathbf{3})(\bar{\mathbf{3}},\mathbf{2})] \Rrightarrow & |1\rangle = (1,4)*(1,4) \\
& |2\rangle = (1,4)*(1,2) \\
& |3\rangle = (1,2)*(1,4) \\
& |4\rangle = (1,2)*(1,2) \\
& |5\rangle = (8,4)*(8,4) \\
& |6\rangle = (8,4)*(8,2) \\
& |7\rangle = (8,2)*(8,4) \\
& |8\rangle = (8,2)*(8,2) \\
[(\bar{\mathbf{3}},\mathbf{3})(\mathbf{3},\mathbf{2})]*[(\bar{\mathbf{6}},\mathbf{1})(\bar{\mathbf{3}},\mathbf{2})] \Rrightarrow & |9\rangle = (8,4)*(8_{sim},2) \\
& |10\rangle = (8,2)*(8_{sim},2) \\
[(\mathbf{6},\mathbf{1})(\mathbf{3},\mathbf{2})]*[(\mathbf{3},\mathbf{3})(\bar{\mathbf{3}},\mathbf{2})] \Rrightarrow & |11\rangle = (8_{sim},2)*(8,4) \\
& |12\rangle = (8_{sim},2)*(8,2) \\
[(\mathbf{6},\mathbf{1})(\mathbf{3},\mathbf{2})]*[(\bar{\mathbf{6}},\mathbf{1})(\bar{\mathbf{3}},\mathbf{2})] \Rrightarrow & |13\rangle = (8_{sim},2)*(8_{sim},2) \\
& |14\rangle = (10,2)*(\overline{10},2)
\end{array}
\tag{14}
$$

We have 5 states for spin 0 and 9 states for spin 1, they are given below:

Spin 0

$$
\begin{array}{ll}
|1\rangle = (1,2)_\beta(1,2)_\alpha & |2\rangle = (8,2)_\beta(8,2)_\alpha \\
|3\rangle = (8,4)_\beta(8,4)_\alpha & |4\rangle = (\overline{10},2)_\beta(10,2)_\alpha \\
|5\rangle = (8,2)_\beta(8,2)_{56} &
\end{array}
\tag{15}
$$

Spin 1

$$
\begin{array}{ll}
|1\rangle = (1,2)_\beta(1,2)_\alpha & |2\rangle = (8,2)_\beta(8,2)_\alpha \\
|3\rangle = (8,4)_\beta(8,4)_\alpha & |4\rangle = (\overline{10},2)_\beta(10,2)_\alpha \\
|5\rangle = (8,2)_\beta(8,4)_\alpha & |6\rangle = (8,4)_\beta(8,2)_\alpha \\
|7\rangle = (8,2)_\beta(8,2)_{56} & |8\rangle = (8,4)_\beta(8,2)_{56} \\
|9\rangle = (\overline{10},2)_\beta(10,4)_{56} &
\end{array}
\tag{16}
$$

With the introduction of appropriate color and spin projectors, it is easy to build explicitly the above basis. Symbolic expressions for the matrix elements of the chromomagnetic operator O_{CM} were obtained with the help of FORM [16]. The explicit expressions for the CM matrices for the three mentioned cases are collected in Appendix C. It was assumed for the CM matrices the same ordering as for the above states. The mass spectrum of the most interesting baryon-antibaryon states are given in appendix A.

VII. EXPERIMENTAL EVIDENCE FOR HEXAQUARKS

1) We predict a 0^-state ($qqq\overline{qqq}$), strongly coupled to the $p\bar{p}$ channel (the component along $p\bar{p}$ is 0.894), just below the threshold (1876.54 MeV), it has a mass of 1874 MeV. This is in agreement with the first observation of a narrow enhancement near $p\bar{p}$ threshold by the BES collaboration [11] in $J/\psi \to p\bar{p}\gamma$, then named $X(1859)$. Until now both the J^P assignments 0^+ or 0^- remain equally possible. It was found at a mass $m_X = 1859 \pm^{3}_{10} \pm^{5}_{25}\, MeV$ having a width smaller than 30 MeV. The state we found is slightly higher, just 7 MeV above the experimental upper limit. They estimated a branching ratio of $B(B \to \gamma X)\, B(X \to p\bar{p}) \backsimeq 7.10^{-5}$.

2) Also relevant for the light hexaquarks ($qqq\overline{qqq}$) may be a quite broad 1^- enhancement above $p\bar{p}$ threshold with mass $1935 \pm 20\, MeV$ and width $\Gamma = 215 \pm 30\, MeV$ proposed about 30 years ago [31]. We have a very good candidate for this state at a mass 1911.5 MeV with a large component (0.61) along the $p\bar{p}$ channel. However here some caution is needed, because the evidence is based on a partial wave analysis and one would have to check if the analysis is compatible with the inclusion of the additional 0^- state just mentioned above.

3) We have also a pretty good candidate for the $Y(2175)$, a 1^{--} state recently seen at the BaBar detector [32] at a mass $2170 \pm 10 \pm 15\, MeV$ (with a width $\Gamma = 58 \pm 16 \pm 20\, MeV$). We predict a singly hidden strangeness state ($qqs\overline{qqs}$) strongly coupled to the $\Lambda\bar{\Lambda}$ channel (with a component of 0.6 along this direction) with a mass 2184 MeV. Since this state is below the $\Lambda\bar{\Lambda}$ threshold (around 2231 MeV) it has to decay mostly into mesons. In fact BaBar observed this state in the decay $Y \to f_0(980)\phi$ (through $f_0 \to \pi\pi$). The $Y(2175)$ has been confirmed by the BES collaboration [33] in $J/\psi \to \eta f_0(980)\phi$ at a mass $m = 2186 \pm 10 \pm 16\, MeV$ and a width $\Gamma = 65 \pm 23\, MeV$.

4) The peak in $\Lambda_c\, \bar{p}$ seen at the mass $m = 3350^{+10}_{-20} \pm 29\, MeV$ and width $\Gamma = 70^{+40}_{-30} \pm 40\, MeV$ in $B^- \to \Lambda_c \bar{p}\pi^-$ [12] may be identified with a 0^- strange charmed hexaquark, we predict to be at 3339 MeV. There is also a 1^- at lower mass, 3274 MeV, with a component of the same order (0.35). All the states strongly coupled to $\Lambda_c\bar{p}$ are below the threshold (3225 MeV), on the other side those above the threshold, with the exception of the two above mentioned states, have negligible couplings. This implies that these two states are the only ones observable in the baryonic channel. It is useful to remark that the experiment privileges the spin 0 assignment.

5) In the singly hidden charm sector ($qqc\overline{qqc}$), the heaviest states are loosely coupled to the $\Lambda_c\bar{\Lambda}_c$, and the reasonably coupled states are just above or below the threshold (4573 MeV). We display these states and the value of the component along the baryonic channel:

$Mass(MeV)$	4533	4556	4575	4614	4642	4658	4670
comp. in $\Lambda_c\bar{\Lambda}_c$	0.41	0.21	0.52	0.42	0.48	0.16	0.24

As a matter of fact, recently, a resonance decaying into $\Lambda_c\bar{\Lambda}_c$ has been seen by the Belle detector [13, 34] at $m = 4634^{+8+5}_{-7-8}\, MeV$ and $\Gamma = 92^{+40+20}_{-24-21}\, MeV$, compatible [34, 36] with $Y(4660) \to \psi'\pi\pi$ [34, 35]. Anyway the fact that the component along the baryonic channel is not strongly dominant is welcome, since it is opportune to leave some room for the decay into $\psi'\pi\pi$. Recently it was proposed to interpret the above state as an excited L=1 tetraquark [37].

We have also calculated the spectrum of the singly hidden strangeness states ($qqs\overline{qqs}$) relevant to the $\Sigma\overline{\Sigma}$ channel, using, along the same lines, the $(15, \overline{15})$ basis. We find only two states strongly coupled to $\Sigma\overline{\Sigma}$, both are around the threshold, $2380 MeV$, one being below threshold at a mass of $2356 MeV$ the other above, at $2454 MeV$. Until now, there is no experimental evidence for these states.

VIII. CONCLUSION

The full chromo-magnetic Hamiltonian proved to be very effective in providing for an unified treatment of tetraquarks and hexaquarks. Besides reproducing the pattern of decays of currently accepted tetraquarks, it also predicts a companion for the $a_0(980)$ at a mass around $1330 MeV$, which has been confirmed by experiments, as the scalar named $a_0(Y)$ and two $cs\overline{cs}$ states, the $Y(4140)$ and the $X(4350)$. A number of candidates were compared with data for the baryon-antibaryon resonances, namely $p\overline{p}$, $\Lambda_c\overline{\Lambda}_c$, $\Lambda_c\overline{p}$ quite successfully.

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Appendix A: Spectrum of the $B\overline{B}$ states

Spectrum of some B$\overline{\text{B}}$ states

PF is the probability factor for the B$\overline{\text{B}}$ open door channel. Masses are given in *MeV*.

$\overline{BB}$ *state*	*Threshold*	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p\overline{p}$ 0^-	1876	1263	1874	2151	2407										
PF		0.15	0.80	0.001	0.05										
$p\overline{p}$ 1^-	1876	1562	1732	1911	2060	2174	2624								
PF		0.43	0.002	0.37	0	0.19	6.10^{-4}								
$\Lambda_c\overline{p}$ 0^-	3225	2653	3028	3188	3339	3595									
PF		0.15	0.28	0.42	0.13	0.02									
$\Lambda_c\overline{p}$ 1^-	3225	2740	2949	3064.	3156	3223	3274	3465	3553	3759					
PF		0.02	0.11	0.56	0.12	0.06	0.12	0.002	10^{-4}	8.10^{-5}					
$\Lambda\overline{\Lambda}$ 1^-	2231	2105	2125	2142	2184	2231	2246.8	2247.2	2274	2297	2303	2325	2343.83	2343.95	2421
PF		0.003	0	0.01	0.36	0.13	0	0.008	0.31	$3.5.10^{-5}$	0	0.17	8.10^{-4}	0	5.10^{-4}
$\Lambda_c\overline{\Lambda}_c 1^-$	4573	4468	4510	4533	4556	4575	4598	4614	4642	4654	4658	4669	4685	4689	4736
PF		0.025	0	0.17	0.044	0.27	0	0.17	0.23	0	0.026	0.06	0	0	8.10^{-5}
$\Sigma\overline{\Sigma}$ 1^-	2380	2211	2236	2270	2273	2283	2310	2334	2346	2349	2356	2415	2415.6	2434	2454
PF		2.10^{-5}	4.10^{-4}	0	0.005	4.10^{-4}	0	0.012	0.35	0	0.34	0.006	0	5.10^{-4}	0.29
$\Sigma_c\overline{\Sigma}_c$ 1^-	4910	4581	4632	4638	4646	4662	4670	4679	4702	4708	4715	4741	4742	4761	4778
PF		0.007	0.027	0	0.029	0.029	0	0.23	0.3	0	0.069	0.038	0	0.21	0.059

Appendix B: Crossing Matrices

Spin 0

$$R_{\phi\to\alpha} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B1}$$

$$R_{\phi\to\epsilon} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{\sqrt{6}} \end{pmatrix} \tag{B2}$$

Spin 1

$$R_{\psi\to\beta} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B3}$$

$$R_{\psi\to\chi} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B4}$$

Spin 2

$$R_{\xi\to\gamma} = \tfrac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix} \quad R_{\xi\to\delta} = \tfrac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & -1 \\ -1 & -\sqrt{2} \end{pmatrix} \tag{B5}$$

Appendix C: Chromo-magnetic operator for $qqq\overline{qqq}$ states

We have computed the matrices of chromo-magnetism by inserting the operator Eq. 1 between the states at Eqs. [10,11], they are given below, where $\mathbf{A_0}$ is for 0^- and $\mathbf{A_1}$ for 1^-.

$$\mathbf{A_0} = \begin{pmatrix} -2 & -\sqrt{2} & -1 & 0 \\ -\sqrt{2} & -1 & -\frac{3}{\sqrt{2}} & -\sqrt{5} \\ -1 & -\frac{3}{\sqrt{2}} & -2 & -\sqrt{\frac{5}{2}} \\ 0 & -\sqrt{5} & -\sqrt{5/2} & 0 \end{pmatrix} \tag{C1}$$

$$\mathbf{A_1} = \begin{pmatrix} 2 & -\frac{\sqrt{2}}{3} & \frac{\sqrt{5}}{3} & 0 & \frac{2\sqrt{2}}{3} & -\frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \sqrt{\frac{5}{2}} & -\frac{\sqrt{5}}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{5}}{3} & \sqrt{\frac{5}{2}} & \frac{4}{3} & \frac{5}{3\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{3} & -\frac{\sqrt{\frac{5}{2}}}{3} \\ 0 & -\frac{\sqrt{5}}{3} & \frac{5}{3\sqrt{2}} & -\frac{4}{3} & -\frac{2\sqrt{5}}{3} & \frac{2\sqrt{5}}{3} \\ \frac{2\sqrt{2}}{3} & -\frac{1}{3} & \frac{\sqrt{\frac{5}{2}}}{3} & -\frac{2\sqrt{5}}{3} & \frac{5}{6} & -\frac{1}{2} \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & -\frac{\sqrt{\frac{5}{2}}}{3} & \frac{2\sqrt{5}}{3} & -\frac{1}{2} & \frac{5}{6} \end{pmatrix} \tag{C2}$$

CM MATRICES FOR BASIS $(15,\overline{15})$

$$\begin{pmatrix} \begin{array}{|c|c|} \hline A & C \\ \hline C^T & B \\ \hline \end{array} \end{pmatrix}$$

$$A=\begin{pmatrix}
-\frac{4}{3}(r+2s) & 0 & 0 & 0 & \frac{11}{27}\sqrt{2}\,(r-2s+t) & \frac{2}{27}\sqrt{10}\,(r+s-2t) & -\frac{2}{27}\sqrt{10}\,(r+s-2t) \\
0 & -\frac{4}{3}(r-s) & 0 & 0 & \frac{2}{27}\sqrt{10}\,(r+s-2t) & \frac{5}{27}\sqrt{2}\,(2r-s-t) & -\frac{1}{27}\sqrt{2}\,(r+4(s+t)) \\
0 & 0 & -\frac{4}{3}(r-s) & 0 & -\frac{2}{27}\sqrt{10}\,(r+s-2t) & -\frac{1}{27}\sqrt{2}\,(r+4(s+t)) & \frac{5}{27}\sqrt{2}\,(2r-s-t) \\
0 & 0 & 0 & -\frac{4}{3}(r-4s) & \frac{2}{27}\sqrt{5}\,(r+4(s+t)) & -\frac{4}{27}(2r+5s+2t) & \frac{4}{27}(2r+5s+2t) \\
\frac{11}{27}\sqrt{2}\,(r-2s+t) & \frac{2}{27}\sqrt{10}\,(r+s-2t) & -\frac{2}{27}\sqrt{10}\,(r+s-2t) & \frac{2}{27}\sqrt{5}\,(r+4(s+t)) & \frac{1}{54}(5r+62s+77t) & \frac{1}{27}\sqrt{5}\,(7r-2(s+7t)) & \frac{1}{27}\sqrt{5}\,(-7r+2(s+7t)) \\
\frac{2}{27}\sqrt{10}\,(r+s-2t) & \frac{5}{27}\sqrt{2}\,(2r-s-t) & -\frac{1}{27}\sqrt{2}\,(r+4(s+t)) & -\frac{4}{27}(2r+5s+2t) & \frac{1}{27}\sqrt{5}\,(7r-2(s+7t)) & \frac{1}{54}(-2r+s-35t) & \frac{1}{54}(-7r+8s-28t) \\
-\frac{2}{27}\sqrt{10}\,(r+s-2t) & -\frac{1}{27}\sqrt{2}\,(r+4(s+t)) & \frac{5}{27}\sqrt{2}\,(2r-s-t) & \frac{4}{27}(2r+5s+2t) & \frac{1}{27}\sqrt{5}\,(-7r+2(s+7t)) & \frac{1}{54}(-7r+8s-28t) & \frac{1}{54}(-2r+s-35t)
\end{pmatrix}$$

$$B=\begin{pmatrix}
\frac{1}{54}(-100r-7(4s+t)) & \frac{1}{9}\sqrt{2}\,(r+8s) & \frac{1}{18}(2r+31s) & -\frac{1}{9}\sqrt{2}\,(r+8s) & \frac{1}{18}(2r+31s) & -\frac{r}{2} & -\frac{\sqrt{5}\,r}{3} \\
\frac{1}{9}\sqrt{2}\,(r+8s) & \frac{1}{18}(-30r+23s-5t) & -\frac{2}{9}\sqrt{2}\,(2s+t) & -\frac{r}{2} & \sqrt{2}\,r & -\frac{2\sqrt{2}\,s}{3} & \frac{2\sqrt{10}\,s}{3} \\
\frac{1}{18}(2r+31s) & -\frac{2}{9}\sqrt{2}\,(2s+t) & \frac{1}{18}(-30r-14s-t) & -\sqrt{2}\,r & -\frac{r}{2} & \frac{11s}{6} & -\frac{\sqrt{5}\,s}{3} \\
-\frac{1}{9}\sqrt{2}\,(r+8s) & -\frac{r}{2} & -\sqrt{2}\,r & \frac{1}{18}(-30r+23s-5t) & \frac{2}{9}\sqrt{2}\,(2s+t) & \frac{2\sqrt{2}\,s}{3} & -\frac{2\sqrt{10}\,s}{3} \\
\frac{1}{18}(2r+31s) & \sqrt{2}\,r & -\frac{r}{2} & \frac{2}{9}\sqrt{2}\,(2s+t) & \frac{1}{18}(-30r-14s-t) & \frac{11s}{6} & -\frac{\sqrt{5}\,s}{3} \\
-\frac{r}{2} & -\frac{2\sqrt{2}\,s}{3} & \frac{11s}{6} & \frac{2\sqrt{2}\,s}{3} & \frac{11s}{6} & \frac{1}{2}(-4r-t) & -\frac{\sqrt{5}\,t}{3} \\
-\frac{\sqrt{5}\,r}{3} & \frac{2\sqrt{10}\,s}{3} & -\frac{\sqrt{5}\,s}{3} & -\frac{2\sqrt{10}\,s}{3} & -\frac{\sqrt{5}\,s}{3} & -\frac{\sqrt{5}\,t}{3} & -\frac{2}{3}(3r+t)
\end{pmatrix}$$

$$C = \begin{pmatrix} \frac{2}{27}\sqrt{5}\,(r+4(s+t)) & \frac{2}{9}\sqrt{10}\,(-r+s) & -\frac{2}{9}\sqrt{5}\,(r+2s) & \frac{2}{9}\sqrt{10}\,(r-s) & -\frac{2}{9}\sqrt{5}\,(r+2s) & \frac{2\sqrt{5}\,r}{3} & 0 \\ -\frac{4}{27}(2r+5s+2t) & \frac{5}{9}\sqrt{2}\,(r-s) & -\frac{4}{9}(r+2s) & \frac{1}{9}\sqrt{2}\,(r+2s) & \frac{4}{9}(2r+s) & \frac{4r}{3} & 0 \\ \frac{4}{27}(2r+5s+2t) & \frac{1}{9}\sqrt{2}\,(r+2s) & -\frac{4}{9}(2r+s) & \frac{5}{9}\sqrt{2}\,(r-s) & \frac{4}{9}(r+2s) & -\frac{4r}{3} & 0 \\ -\frac{1}{27}\sqrt{2}\,(4r+4s+t) & -\frac{4}{9}(r+2s) & -\frac{1}{9}\sqrt{2}\,(2r+s) & \frac{4}{9}(r+2s) & -\frac{1}{9}\sqrt{2}\,(2r+s) & -\frac{\sqrt{2}\,r}{3} & 0 \\ \frac{1}{27}\sqrt{\frac{5}{2}}\,(7r-8s+28t) & \frac{1}{9}\sqrt{5}\,(r-4s) & \frac{1}{9}\sqrt{\frac{5}{2}}\,(r+8s) & -\frac{1}{9}\sqrt{5}\,(r-4s) & \frac{1}{9}\sqrt{\frac{5}{2}}\,(r+8s) & \sqrt{\frac{5}{2}}\,r & \frac{5\sqrt{2}\,r}{3} \\ -\frac{2}{27}\sqrt{2}\,(7r-5s+7t) & \frac{1}{18}(-5r+47s) & \frac{1}{9}\sqrt{2}\,(r+8s) & \frac{1}{18}(-r-8s) & -\frac{2}{9}\sqrt{2}\,(r+2s) & \sqrt{2}\,r & \frac{2\sqrt{10}\,r}{3} \\ \frac{2}{27}\sqrt{2}\,(7r-5s+7t) & \frac{1}{18}(-r-8s) & \frac{2}{9}\sqrt{2}\,(r+2s) & \frac{1}{18}(-5r+47s) & -\frac{1}{9}\sqrt{2}\,(r+8s) & -\sqrt{2}\,r & -\frac{2\sqrt{10}\,r}{3} \end{pmatrix}$$

CM MATRICES FOR BASIS $(15,\overline{15})$

$$\left(\begin{array}{|c|c|} \hline A & C \\ \hline C^T & B \\ \hline \end{array}\right)$$

Parameters are

$C_{qq} = r,\ C_{qc} = s,\ C_{c\bar{c}} = t$

CM MATRICES FOR BASIS $(21, \overline{21})$

$$\left(\begin{array}{c|c} A & C \\ \hline C^T & B \end{array}\right)$$

$$A = \begin{pmatrix}
4r & -\frac{\sqrt{2}\,t}{3} & -\frac{4s}{3} & \frac{\sqrt{2}\,s}{3} & \frac{4s}{3} & \frac{\sqrt{2}\,s}{3} & \frac{2\sqrt{5}\,r}{3} \\
-\frac{\sqrt{2}\,t}{3} & 4r-\frac{7t}{6} & \frac{4\sqrt{2}\,s}{3} & \frac{5s}{6} & -\frac{4\sqrt{2}\,s}{3} & \frac{5s}{6} & \sqrt{\frac{5}{2}}\,r \\
-\frac{4s}{3} & \frac{4\sqrt{2}\,s}{3} & \frac{7r}{3}+\frac{35s}{18}-\frac{5t}{18} & -\frac{10\sqrt{2}\,s}{9}-\frac{2\sqrt{2}\,t}{9} & -\frac{r}{2} & \sqrt{2}\,r & -\frac{5\sqrt{5}\,r}{9}+\frac{2\sqrt{5}\,s}{9} \\
\frac{\sqrt{2}\,s}{3} & \frac{5s}{6} & -\frac{10\sqrt{2}\,s}{9}-\frac{2\sqrt{2}\,t}{9} & \frac{7r}{3}+\frac{5s}{9}-\frac{t}{18} & -\sqrt{2}\,r & -\frac{r}{2} & -\frac{5}{9}\sqrt{\frac{5}{2}}\,r-\frac{2\sqrt{10}\,s}{9} \\
\frac{4s}{3} & -\frac{4\sqrt{2}\,s}{3} & -\frac{r}{2} & -\sqrt{2}\,r & \frac{7r}{3}+\frac{35s}{18}-\frac{5t}{18} & \frac{10\sqrt{2}\,s}{9}+\frac{2\sqrt{2}\,t}{9} & \frac{5\sqrt{5}\,r}{9}-\frac{2\sqrt{5}\,s}{9} \\
\frac{\sqrt{2}\,s}{3} & \frac{5s}{6} & \sqrt{2}\,r & -\frac{r}{2} & \frac{10\sqrt{2}\,s}{9}+\frac{2\sqrt{2}\,t}{9} & \frac{7r}{3}+\frac{5s}{9}-\frac{t}{18} & -\frac{5}{9}\sqrt{\frac{5}{2}}\,r-\frac{2\sqrt{10}\,s}{9} \\
\frac{2\sqrt{5}\,r}{3} & \sqrt{\frac{5}{2}}\,r & -\frac{5\sqrt{5}\,r}{9}+\frac{2\sqrt{5}\,s}{9} & -\frac{5}{9}\sqrt{\frac{5}{2}}\,r-\frac{2\sqrt{10}\,s}{9} & \frac{5\sqrt{5}\,r}{9}-\frac{2\sqrt{5}\,s}{9} & -\frac{5}{9}\sqrt{\frac{5}{2}}\,r-\frac{2\sqrt{10}\,s}{9} & \frac{67r}{18}-\frac{5s}{3}+\frac{11t}{18}
\end{pmatrix}$$

$$B = \begin{pmatrix}
\frac{31r}{9}+\frac{5s}{6}-\frac{5t}{18} & -\frac{5r}{18}-\frac{2t}{9} & -\frac{10\sqrt{2}\,r}{9}-\frac{2\sqrt{2}\,t}{9} & \frac{10r}{27}+\frac{10s}{27}-\frac{20t}{27} & \frac{10\sqrt{5}\,r}{27}-\frac{5\sqrt{5}\,s}{27}-\frac{5\sqrt{5}\,t}{27} & -\frac{\sqrt{5}\,r}{27}-\frac{4\sqrt{5}\,s}{27}-\frac{4\sqrt{5}\,t}{27} & -\frac{4\sqrt{10}\,r}{27}-\frac{10\sqrt{10}\,s}{27}-\frac{4\sqrt{10}\,t}{27} \\
-\frac{5r}{18}-\frac{2t}{9} & \frac{31r}{9}+\frac{5s}{6}-\frac{5t}{18} & \frac{10\sqrt{2}\,r}{9}+\frac{2\sqrt{2}\,t}{9} & -\frac{10r}{27}-\frac{10s}{27}+\frac{20t}{27} & -\frac{\sqrt{5}\,r}{27}-\frac{4\sqrt{5}\,s}{27}-\frac{4\sqrt{5}\,t}{27} & \frac{10\sqrt{5}\,r}{27}-\frac{5\sqrt{5}\,s}{27}-\frac{5\sqrt{5}\,t}{27} & \frac{4\sqrt{10}\,r}{27}+\frac{10\sqrt{10}\,s}{27}+\frac{4\sqrt{10}\,t}{27} \\
-\frac{10\sqrt{2}\,r}{9}-\frac{2\sqrt{2}\,t}{9} & \frac{10\sqrt{2}\,r}{9}+\frac{2\sqrt{2}\,t}{9} & -\frac{4r}{9}+\frac{10s}{3}-\frac{t}{18} & \frac{5\sqrt{2}\,r}{27}+\frac{20\sqrt{2}\,s}{27}+\frac{20\sqrt{2}\,t}{27} & -\frac{4\sqrt{10}\,r}{27}-\frac{10\sqrt{10}\,s}{27}-\frac{4\sqrt{10}\,t}{27} & \frac{4\sqrt{10}\,r}{27}+\frac{10\sqrt{10}\,s}{27}+\frac{4\sqrt{10}\,t}{27} & -\frac{4\sqrt{5}\,r}{27}-\frac{4\sqrt{5}\,s}{27}-\frac{\sqrt{5}\,t}{27} \\
\frac{10r}{27}+\frac{10s}{27}-\frac{20t}{27} & -\frac{10r}{27}-\frac{10s}{27}+\frac{20t}{27} & \frac{5\sqrt{2}\,r}{27}+\frac{20\sqrt{2}\,s}{27}+\frac{20\sqrt{2}\,t}{27} & \frac{106r}{27}+\frac{124s}{27}+\frac{22t}{27} & \frac{16\sqrt{5}\,r}{27}-\frac{8\sqrt{5}\,s}{27}-\frac{8\sqrt{5}\,t}{27} & -\frac{16\sqrt{5}\,r}{27}+\frac{8\sqrt{5}\,s}{27}+\frac{8\sqrt{5}\,t}{27} & \frac{8\sqrt{10}\,r}{27}-\frac{16\sqrt{10}\,s}{27}+\frac{8\sqrt{10}\,t}{27} \\
\frac{10\sqrt{5}\,r}{27}-\frac{5\sqrt{5}\,s}{27}-\frac{5\sqrt{5}\,t}{27} & -\frac{\sqrt{5}\,r}{27}-\frac{4\sqrt{5}\,s}{27}-\frac{4\sqrt{5}\,t}{27} & -\frac{4\sqrt{10}\,r}{27}-\frac{10\sqrt{10}\,s}{27}-\frac{4\sqrt{10}\,t}{27} & \frac{16\sqrt{5}\,r}{27}-\frac{8\sqrt{5}\,s}{27}-\frac{8\sqrt{5}\,t}{27} & \frac{98r}{27}+\frac{2s}{27}-\frac{10t}{27} & -\frac{8r}{27}+\frac{16s}{27}-\frac{8t}{27} & -\frac{32\sqrt{2}\,r}{27}+\frac{40\sqrt{2}\,s}{27}-\frac{8\sqrt{2}\,t}{27} \\
-\frac{\sqrt{5}\,r}{27}-\frac{4\sqrt{5}\,s}{27}-\frac{4\sqrt{5}\,t}{27} & \frac{10\sqrt{5}\,r}{27}-\frac{5\sqrt{5}\,s}{27}-\frac{5\sqrt{5}\,t}{27} & \frac{4\sqrt{10}\,r}{27}+\frac{10\sqrt{10}\,s}{27}+\frac{4\sqrt{10}\,t}{27} & -\frac{16\sqrt{5}\,r}{27}+\frac{8\sqrt{5}\,s}{27}+\frac{8\sqrt{5}\,t}{27} & -\frac{8r}{27}+\frac{16s}{27}-\frac{8t}{27} & \frac{98r}{27}+\frac{2s}{27}-\frac{10t}{27} & \frac{32\sqrt{2}\,r}{27}-\frac{40\sqrt{2}\,s}{27}+\frac{8\sqrt{2}\,t}{27} \\
-\frac{4\sqrt{10}\,r}{27}-\frac{10\sqrt{10}\,s}{27}-\frac{4\sqrt{10}\,t}{27} & \frac{4\sqrt{10}\,r}{27}+\frac{10\sqrt{10}\,s}{27}+\frac{4\sqrt{10}\,t}{27} & -\frac{4\sqrt{5}\,r}{27}-\frac{4\sqrt{5}\,s}{27}-\frac{\sqrt{5}\,t}{27} & \frac{8\sqrt{10}\,r}{27}-\frac{16\sqrt{10}\,s}{27}+\frac{8\sqrt{10}\,t}{27} & -\frac{32\sqrt{2}\,r}{27}+\frac{40\sqrt{2}\,s}{27}-\frac{8\sqrt{2}\,t}{27} & \frac{32\sqrt{2}\,r}{27}-\frac{40\sqrt{2}\,s}{27}+\frac{8\sqrt{2}\,t}{27} & -\frac{14r}{27}-\frac{56s}{27}-\frac{2t}{27}
\end{pmatrix}$$

$$
C = \begin{pmatrix}
\frac{4r}{3} & -\frac{4r}{3} & -\frac{\sqrt{2}\,r}{3} & 0 & 0 & 0 & 0 \\
\sqrt{2}\,r & -\sqrt{2}\,r & -\frac{r}{2} & \frac{5\sqrt{2}\,r}{3} & \frac{2\sqrt{10}\,r}{3} & -\frac{2\sqrt{10}\,r}{3} & -\frac{\sqrt{5}\,r}{3} \\
\frac{25r}{18}+\frac{17s}{18} & \frac{5r}{18}+\frac{2s}{9} & -\frac{5\sqrt{2}\,r}{9}-\frac{4\sqrt{2}\,s}{9} & \frac{10r}{9}-\frac{10s}{9} & -\frac{5\sqrt{5}\,r}{9}+\frac{5\sqrt{5}\,s}{9} & -\frac{\sqrt{5}\,r}{9}-\frac{2\sqrt{5}\,s}{9} & \frac{2\sqrt{10}\,r}{9}+\frac{4\sqrt{10}\,s}{9} \\
-\frac{5\sqrt{2}\,r}{9}-\frac{4\sqrt{2}\,s}{9} & -\frac{10\sqrt{2}\,r}{9}-\frac{2\sqrt{2}\,s}{9} & -\frac{5r}{9}+\frac{25s}{18} & \frac{5\sqrt{2}\,r}{9}+\frac{10\sqrt{2}\,s}{9} & \frac{2\sqrt{10}\,r}{9}+\frac{4\sqrt{10}\,s}{9} & \frac{4\sqrt{10}\,r}{9}+\frac{2\sqrt{10}\,s}{9} & \frac{2\sqrt{5}\,r}{9}+\frac{\sqrt{5}\,s}{9} \\
\frac{5r}{18}+\frac{2s}{9} & \frac{25r}{18}+\frac{17s}{18} & \frac{5\sqrt{2}\,r}{9}+\frac{4\sqrt{2}\,s}{9} & -\frac{10r}{9}+\frac{10s}{9} & -\frac{\sqrt{5}\,r}{9}-\frac{2\sqrt{5}\,s}{9} & -\frac{5\sqrt{5}\,r}{9}+\frac{5\sqrt{5}\,s}{9} & -\frac{2\sqrt{10}\,r}{9}-\frac{4\sqrt{10}\,s}{9} \\
\frac{10\sqrt{2}\,r}{9}+\frac{2\sqrt{2}\,s}{9} & \frac{5\sqrt{2}\,r}{9}+\frac{4\sqrt{2}\,s}{9} & -\frac{5r}{9}+\frac{25s}{18} & \frac{5\sqrt{2}\,r}{9}+\frac{10\sqrt{2}\,s}{9} & -\frac{4\sqrt{10}\,r}{9}-\frac{2\sqrt{10}\,s}{9} & -\frac{2\sqrt{10}\,r}{9}-\frac{4\sqrt{10}\,s}{9} & \frac{2\sqrt{5}\,r}{9}+\frac{\sqrt{5}\,s}{9} \\
\frac{5\sqrt{5}\,r}{9}-\frac{2\sqrt{5}\,t}{9} & -\frac{5\sqrt{5}\,r}{9}+\frac{2\sqrt{5}\,t}{9} & \frac{5}{9}\sqrt{\frac{5}{2}}\,r+\frac{2\sqrt{10}\,t}{9} & \frac{11\sqrt{5}\,r}{27}-\frac{22\sqrt{5}\,s}{27}+\frac{11\sqrt{5}\,t}{27} & \frac{10r}{27}+\frac{10s}{27}-\frac{20t}{27} & -\frac{10r}{27}-\frac{10s}{27}+\frac{20t}{27} & \frac{5\sqrt{2}\,r}{27}+\frac{20\sqrt{2}\,s}{27}+\frac{20\sqrt{2}\,t}{27}
\end{pmatrix}
$$

CM MATRICES FOR BASIS $(21,\overline{21})$

$$
\left(\begin{array}{|c|c|} \hline A & C \\ \hline C^T & B \\ \hline \end{array}\right)
$$

Parameters are

$C_{qq} = r,\ C_{qc} = s,\ C_{c\bar{c}} = t$

CM MATRICES FOR BASIS $\Lambda_c \bar{p}$

SPIN 0

$$
\begin{pmatrix}
16\,r & 4\sqrt{2}\,(-s+r) & -8\,r & 0 & 4\sqrt{2}\,(s+r) \\
4\sqrt{2}\,(-s+r) & 12\,(s+r) & \frac{8}{3}\sqrt{2}\,(-s+r) & \frac{4}{3}\sqrt{5}\,(-s+r) & 0 \\
-8\,r & \frac{8}{3}\sqrt{2}\,(-s+r) & 16\,r & \frac{4}{3}\sqrt{10}\,(2\,s+r) & -\frac{4}{3}\sqrt{2}\,(2\,s+7\,r) \\
0 & \frac{4}{3}\sqrt{5}\,(-s+r) & \frac{4}{3}\sqrt{10}\,(2\,s+r) & -8\,s+8\,r & -\frac{4}{3}\sqrt{5}\,(s+5\,r) \\
4\sqrt{2}\,(s+r) & 0 & -\frac{4}{3}\sqrt{2}\,(2\,s+7\,r) & -\frac{4}{3}\sqrt{5}\,(s+5\,r) & 8\,r
\end{pmatrix}
$$

SPIN 1

$$
\begin{pmatrix}
16\,r & \frac{2\sqrt{2}\,s}{3}-\frac{2\sqrt{2}\,r}{3} & -\frac{4\sqrt{5}\,r}{3} & 0 & -\frac{8\sqrt{2}\,r}{3} & \frac{4\sqrt{2}\,s}{3}-\frac{4\sqrt{2}\,r}{3} & -\frac{2\sqrt{2}\,s}{3}-\frac{2\sqrt{2}\,r}{3} & -\frac{4\sqrt{2}\,s}{3}-\frac{4\sqrt{2}\,r}{3} & 0 \\
\frac{2\sqrt{2}\,s}{3}-\frac{2\sqrt{2}\,r}{3} & \frac{76s}{9}+\frac{44r}{9} & -\frac{8\sqrt{10}\,s}{9}+\frac{8\sqrt{10}\,r}{9} & \frac{4\sqrt{5}\,s}{9}-\frac{4\sqrt{5}\,r}{9} & -\frac{8s}{9}+\frac{8r}{9} & \frac{56s}{9}+\frac{112r}{9} & \frac{40s}{9}-\frac{40r}{9} & -\frac{40s}{9}+\frac{40r}{9} & -\frac{16\sqrt{10}\,s}{9}-\frac{32\sqrt{10}\,r}{9} \\
-\frac{4\sqrt{5}\,r}{3} & -\frac{8\sqrt{10}\,s}{9}+\frac{8\sqrt{10}\,r}{9} & -\frac{8s}{9}+\frac{104r}{9} & \frac{40\sqrt{2}\,s}{9}+\frac{20\sqrt{2}\,r}{9} & -\frac{8\sqrt{10}\,s}{9}+\frac{20\sqrt{10}\,r}{9} & -\frac{4\sqrt{10}\,s}{9}+\frac{4\sqrt{10}\,r}{9} & -\frac{8\sqrt{10}\,s}{9}-\frac{28\sqrt{10}\,r}{9} & -\frac{4\sqrt{10}\,s}{9}+\frac{16\sqrt{10}\,r}{9} & \frac{40s}{9}-\frac{40r}{9} \\
0 & \frac{4\sqrt{5}\,s}{9}-\frac{4\sqrt{5}\,r}{9} & \frac{40\sqrt{2}\,s}{9}+\frac{20\sqrt{2}\,r}{9} & -\frac{40s}{9}-\frac{56r}{9} & -\frac{32\sqrt{5}\,s}{9}-\frac{16\sqrt{5}\,r}{9} & -\frac{8\sqrt{5}\,s}{9}+\frac{8\sqrt{5}\,r}{9} & \frac{4\sqrt{5}\,s}{9}+\frac{20\sqrt{5}\,r}{9} & -\frac{8\sqrt{5}\,s}{9}-\frac{40\sqrt{5}\,r}{9} & -\frac{32\sqrt{2}\,s}{9}+\frac{32\sqrt{2}\,r}{9} \\
-\frac{8\sqrt{2}\,r}{3} & -\frac{8s}{9}+\frac{8r}{9} & -\frac{8\sqrt{10}\,s}{9}+\frac{20\sqrt{10}\,r}{9} & -\frac{32\sqrt{5}\,s}{9}-\frac{16\sqrt{5}\,r}{9} & -\frac{20s}{9}+\frac{80r}{9} & \frac{8s}{9}-\frac{8r}{9} & -\frac{8s}{9}+\frac{32r}{9} & \frac{8s}{9}+\frac{28r}{9} & \frac{20\sqrt{10}\,s}{9}-\frac{20\sqrt{10}\,r}{9} \\
\frac{4\sqrt{2}\,s}{3}-\frac{4\sqrt{2}\,r}{3} & \frac{56s}{9}+\frac{112r}{9} & -\frac{4\sqrt{10}\,s}{9}+\frac{4\sqrt{10}\,r}{9} & -\frac{8\sqrt{5}\,s}{9}+\frac{8\sqrt{5}\,r}{9} & \frac{8s}{9}-\frac{8r}{9} & \frac{124s}{9}+\frac{104r}{9} & -\frac{40s}{9}+\frac{40r}{9} & -\frac{20s}{9}+\frac{20r}{9} & -\frac{4\sqrt{10}\,s}{9}-\frac{8\sqrt{10}\,r}{9} \\
-\frac{2\sqrt{2}\,s}{3}-\frac{2\sqrt{2}\,r}{3} & \frac{40s}{9}-\frac{40r}{9} & -\frac{8\sqrt{10}\,s}{9}-\frac{28\sqrt{10}\,r}{9} & \frac{4\sqrt{5}\,s}{9}+\frac{20\sqrt{5}\,r}{9} & -\frac{8s}{9}+\frac{32r}{9} & -\frac{40s}{9}+\frac{40r}{9} & -\frac{32s}{9}+\frac{56r}{9} & \frac{56s}{9}-\frac{32r}{9} & -\frac{16\sqrt{10}\,s}{9}+\frac{16\sqrt{10}\,r}{9} \\
-\frac{4\sqrt{2}\,s}{3}-\frac{4\sqrt{2}\,r}{3} & -\frac{40s}{9}+\frac{40r}{9} & -\frac{4\sqrt{10}\,s}{9}+\frac{16\sqrt{10}\,r}{9} & -\frac{8\sqrt{5}\,s}{9}-\frac{40\sqrt{5}\,r}{9} & \frac{8s}{9}+\frac{28r}{9} & -\frac{20s}{9}+\frac{20r}{9} & \frac{56s}{9}-\frac{32r}{9} & \frac{16s}{9}+\frac{44r}{9} & -\frac{4\sqrt{10}\,s}{9}+\frac{4\sqrt{10}\,r}{9} \\
0 & -\frac{16\sqrt{10}\,s}{9}-\frac{32\sqrt{10}\,r}{9} & \frac{40s}{9}-\frac{40r}{9} & -\frac{32\sqrt{2}\,s}{9}+\frac{32\sqrt{2}\,r}{9} & \frac{20\sqrt{10}\,s}{9}-\frac{20\sqrt{10}\,r}{9} & -\frac{4\sqrt{10}\,s}{9}-\frac{8\sqrt{10}\,r}{9} & -\frac{16\sqrt{10}\,s}{9}+\frac{16\sqrt{10}\,r}{9} & -\frac{4\sqrt{10}\,s}{9}+\frac{4\sqrt{10}\,r}{9} & \frac{64s}{9}+\frac{56r}{9}
\end{pmatrix}
$$

Parameters are

$C_{qq} = r$, $C_{qc} = s$