On the ring of inertial endomorphisms of an abelian p-group *

Ulderico Dardano[†] - Silvana Rinauro[‡]

Abstract. An endomorphisms φ of a group G is said inertial if $\forall H \leq G$ $|\varphi(H) : (H \cap \varphi(H))| < \infty$. Here we study the ring of inertial endomorphisms of an abelian torsion group and the group of its units. Also the case of vector spaces is considered.
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1 Introduction and statement of main results

Recently there has been interest for inert subgroups of groups (see [9], [4], [5], for example). A subgroup is said inert if it is commensurable to each conjugate of its. Here we consider *inertial endomorphims*, that is endomorphims mapping setwise subgroups to commensurable ones.

More precisely, if φ is an endomorphism of an abelian group A (from now on always in additive notation) we say:
(RIN) φ is right-inertial iff $\forall H \leq A \;\; |\varphi(H) : (H \cap \varphi(H))| < \infty$,
(LIN) φ is left-inertial iff $\forall H \leq A \;\; |H : (H \cap \varphi(H))| < \infty$.

In [2] we considered automorphisms of abelian group A and showed that in this case (RIN) and (LIN) are equivalent, when A is periodic. This generalized previous results from [1] and [6]. On the other hand, in [5] authors consider (RIN) only, which seems to be more adequate for non-invertible

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†dardano@unina.it
‡silvana.rinauro@unibas.it

maps. Moreover, if A is periodic (LIN) implies (RIN), see Theorem 1. Let us call RIN-endomorphims *inertial*.

Fact *Inertial endomorphisms of any abelian group A fill a subring $\mathcal{I}End(A)$ of the full ring $End(A)$ of endomorphisms of A.*

Clearly $\mathcal{I}End(A)$ contains the ideal $FEnd(A)$ of endomorphisms with finite image and the subring $PEnd(A)$ of power endomorphisms (say *multiplications*) of A.

Here we have a characterization of inertial endomorphisms of torsion abelian groups.

Theorem 1 *Let A be an abelian periodic group and $\varphi \in \mathrm{End}(A)$. Then φ is inertial iff there is a finite index subgroup $B = D \oplus E \oplus L$ of A such that:*
i) $D \oplus E$ and L are coprime,
ii) D is divisible with finite total rank and E has finite exponent,
iii) φ is power on D, E and L .

Thus φ is inertial iff :

$$(FS) \qquad \exists n \ \forall H \leq A \ |H^{(\varphi)}/H_{(\varphi)}| \leq n.$$

Moreover, φ is LIN iff it is inertial and there are subgroups B, D, E, L as above such that φ is non-zero on D and invertible on E and L

Thus the picture of $\mathcal{I}End(A)$, when A is periodic, can be described in Corollary 1. Note that *an endomorphism of an abelian torsion group is inertial iff it is such on all primary components and multiplication on all but finitely many of them.* Notice also that from Theorem 1 it follows that inertial endomorphisms of an abelian p-group are *elementary*, in the sense they act as a multiplication on a finite index subgroup (they are close to be multiplications, see later), unless *the maximum divisible subgroup $D := div(A)$ of A is non-trivial and has finite rank while A/D has infinite rank and finite exponent.* For short, say that such an A is *critical.* To describe the ring $\mathcal{I}End(A)$ we also need consideration of the, say, *essential exponent* $eexp(A)$ of an infinite p-group A (with finite exponent) , that is the smallest power p^e such that $p^e A$ is finite or, equivalently, the maximum p^e such that $A[p^e]/A[p^{e-1}]$ is infinite. Clearly, the above e is the least finite Ulm-Kaplansky invariant of A. Denote by $\mathcal{J}_p$ the ring of p-adics. For terminology and elementary facts see [7] and [8].

Corollary 1 *Let A be an abelian p-group and $D := div(A)$. Then*
1) If A is non-critical:

$$\mathcal{I}End(A) = PEnd(A) + FEnd(A)$$

and, according to $exp(A) = \infty$ or $exp(A) = p^m$ and $p^e = eexp(A)$, we have

$$PEnd(A) \cap FEnd(A) = 0 \text{ or } = p^e PEnd(A) \simeq p^e\mathbb{Z}/p^m\mathbb{Z}$$

$$\frac{\mathcal{I}End(A)}{FEnd(A)} \simeq \quad \mathcal{J}_p \text{ or } \mathbb{Z}/p^e\mathbb{Z}.$$

2) If A is critical, $p^m := exp(A/D)$ and $p^e := eexp(A/D)$:

$$\mathcal{I}End(A) = PEnd(A) \oplus (FEnd(A) + R)$$

where $\quad R \simeq PEnd(A/D) \simeq \mathbb{Z}(p^m)$ *and* $\quad FEnd(A) \cap R = p^e R.$ *Moreover*

$$\frac{\mathcal{I}End(A)}{FEnd(A)} \simeq \mathbb{Z}(p^e) \oplus \mathcal{J}_p.$$

Concerning invertible inertial endomorphisms of a periodic abelian group, note that these fill a group $\mathcal{I}Aut(A)$. Theorem 1 lead us to the consideration of the normal subgroup filled by the so called *finitary* automorphisms, that is $FAut(A) := \{\gamma \in \mathrm{Aut}(A) \mid [A, \gamma] \textit{ is finite}\}$, and the group $PAut(A)$ of invertible multiplications.

Corollary 2 *Let A be an abelian p-group and $D := div(A)$. Then*
1) If A is non-critical,

$$\mathcal{I}Aut(A) = PAut(A) \cdot FAut(A)$$

where $PAut(A) \cap FAut(A) = 1$ if $exp(A) = \infty$.
Otherwise, if $p^m := exp(A)$ and $p^e := eexp(A)$, we have

$$PAut(A) \cap FAut(A) = \{x \mapsto rx \mid r \equiv 1 \, mod \, p^e\} \simeq \{\bar{r} \in \mathbb{Z}(p^m) \mid r \equiv 1 \, mod \, p^e\}.$$

2) If A is critical, $p^m := exp(A/D)$ and $p^e := eexp(A/D)$,

$$\mathcal{I}Aut(A) = PAut(A) \times (FAutA \cdot \Gamma)$$

with $FAutA \cdot \Gamma = \{\varphi \in \mathcal{I}Aut(A) \mid \varphi_{|D} = 1\}$, $\Gamma \simeq \mathcal{U}(\mathbb{Z}(p^m))$ and

$$FAut(A) \cap \Gamma \simeq \{\bar{r} \in \mathbb{Z}(p^m) \mid r \equiv 1 \, mod \, p^e\}.$$

One may ask a similar question about vector spaces and get a similar picture, without critical case. Let V be a K-vector space and denote by $FEnd(V)$ the ring of K-linear maps which are finitary, that is have image with finite dim. Note that these are precisely the linear maps acting as the zero-map on a finite codimension subspace.

Theorem 2 *Let φ be an endomorphism of an infinite dimension K-vector space A. Then $dim(\varphi(H)/(\varphi(H) \cap H)) < \infty$ for each K-subspace H of V iff φ acts as a scalar multiplication on a finite codimension subspace.*

Therefore the above endomorphisms fill the following subring of $End(V)$:

$$\bar{K} \oplus FEnd(V)$$

where $\bar{K}$ is the field of scalar multiplication and $FEnd(V)$ is the ideal of endomorphisms whose image has finite dimension.

On the other hand, $H \cap \varphi(H)$ has finite codimension in H for each K-subspace H of V it iff φ acts as a scalar non-zero multiplication on a finite codimension subspace. Thus such a φ has the above property as well.

2 Proofs

We first prove the above stated Fact. The corresponding statement for vector spaces has a similar proof and we omit it.

Proposition 1 *1) If φ and ψ are LIN-endomorphism (risp. RIN) of any group G. Then $\varphi\psi$ is LIN (risp. RIN).*
2) RIN-endomorphism of an abelian group A fill a subring $\mathcal{I}End(A)$ of $End(A)$, containing the ideal $FEnd(A)$ of endomorphism with finite image.

Proof. 1) If H is any subgroup of G then from $|H/(H \cap \varphi(H))| < \infty$ it follows $|(H \cap \psi(H))/(H \cap \psi(H) \cap \varphi\psi(H))| \leq |\psi(H)/(\psi(H) \cap \varphi\psi(H))| < \infty$.

2) If φ and ψ both have RIN, then $|(H + \varphi(H))/H| < \infty$ and $|(H + \varphi(H) + \psi(H) + \varphi\psi(H))/(H + \varphi(H)| < \infty$. □

Now we prove Theorem 2, which will serve also for the proof of Theorem 1 in the case A of prime exponent. For a subset X of V and $\varphi \in End(V)$, we denote respectively by $\langle X \rangle = KX$ and $X^{(\varphi)} = K[\varphi]X$ the K-subspace and the $K[\varphi]$-submodule of V spanned by X.

Proof. By contradiction, assume φ is multiplication on no quotient space. We claim that: for all finite dimension subspaces $X \leq A$ such that $X \cap \varphi(X) = 0$ there exists a subspace $X' > X$ with finite dim such that

$$X' \cap \varphi(X') = 0 \;\text{ and }\; \varphi(X') > \varphi(X).$$

Therefore, starting at $X_0 = 0$, by transfinite recursion we define $X_{i+1} := X_i'$ and $X_\omega := \cup_i X_i$. We get that both X_ω and $\varphi(X_\omega)$ have infinite dimension and $X_\omega \cap \varphi(X_\omega) = 0$, a contradiction.

To prove the claim, we first prove that if $a \in V$, then

$$dim(Ka^{(\varphi)}) < \infty.$$

This is true as we can consider the natural epimorphism

$$F : K[x] \mapsto Ka^{(\varphi)}$$

mapping 1 to a and x to $\varphi(a)$. If F is injective, we can replace V by $K[x]$ and φ by multiplication by x. If $H := K[x^2]$, then both H and $\varphi(H) = xH$ are infinite dim, while $H \cap xH = 0$, a contradiction. Therefore $(Ka)^{(\varphi)} = im(F)$ has finite dim and the same holds for $Z = X^{(\varphi)}$.

Since φ does not act as a scalar multiplication on A/Z, we can choose $a \in V$ such that $\varphi(a) \notin \langle a, Z\rangle$ and define $X' := \langle a\rangle + X$. If now $y \in X' \cap \varphi(X')$, then $\exists n, s \in K$, $\exists x, x_0 \in X$ such that $y = na + x = s\varphi(a) + \varphi(x_0)$. Thus $s\varphi(a) \in \langle a\rangle + Z$ while $\varphi(a) \notin \langle a\rangle + Z$. Therefore $s = 0$ and $na = 0$ as well. It follows $y = x = \varphi(x_0) \in X \cap \varphi(X) = 0$, as claimed.

Finally, we have seen that if for each K-subspace H, $H \cap \varphi(H)$ has finite codimension in H then φ is multiplication on a finite codim subspace B of A. In particular $\varphi(B) \neq 0$. $\square$

Recall that $\varphi \in \mathrm{End}(A)$ is said to be power or multiplication iff
(PW) $\forall H \leq A \;\; \varphi(H) \subseteq H$,
and that PW endomorphisms of an abelian p-group are *locally universal* that is have form $x \mapsto \alpha_p x$ for a p-adic α_p, when A is a p-group. Also, if $C \leq B \leq A$, we say that φ is PW on B/C iff $C \leq H \leq B$ implies $H^\varphi \subseteq H$.

We say that endomorphisms φ_1 and $\varphi_2 \in \mathrm{End}(A)$ are **close** iff the image of $\varphi_1 - \varphi_2$ is finite, that is they act the same way on a finite index subgroup or -equivalently- modulo a finite order subgroup. This is the congruence in $End(A)$ whose kernel is the ideal $FEnd(A)$ of endomorphisms with finite image. An endomorphism which is close to a (RIN) (resp. LIN) one remains such, clearly. We *say that an endomorphisms is PF iff it is close to a multiplication.* Let us sum up basic facts.

Proposition 2 *PF-endomorphisms of an abelian group A fill a subring of $End(A)$,*

$$PEnd(A) + FEnd(A)$$

where the sum is direct, provided $exp(A) = \infty$. Otherwise, if A is a p-group with $p^m = exp(A) < \infty$ and $p^e = eexp(A)$, there is a natural ring isomorphisms

$$PEnd(A) \cap FEnd(A) \simeq p^e\mathbb{Z}/p^n\mathbb{Z}.$$

Moreover, if φ is PF, then

$$(FS) \qquad \exists n \; \forall X \leq A \;\; |X^{(\varphi)}/X_{(\varphi)}| \leq n.$$

□

Here by $X^{(\varphi)}$ (resp. $X_{(\varphi)}$) we mean the smallest (resp. largest) φ-(invariant) subgroup of A containing X (resp. contained in X).

Proof. This is quite elementary. If φ acts as $\alpha \in PEnd(A)$ on $B \leq A$ with $|A : B| < \infty$, then $\varphi - \alpha \in FEnd(A)$. Moreover if $C := ker\ (\varphi - \alpha)$, we have that for each $X \leq A$ it holds $(X \cap B) \leq X_{(\varphi)}$ and $X^{(\varphi)} \leq (X + C)$. Thus $|X^{(\varphi)}/X_{(\varphi)}| \leq |A/B| \cdot |C| \leq |A/B|^2$.

If $0 \neq \alpha \in PEnd(A) \cap FEnd(A)$ we have that exists i such that $ker\ \alpha = A[p^i]$ (clearly p^i is the maximal power of p dividing α). If $A[p^i]$ has finite index in A, then $exp(A) < \infty$ and $e \geq i$. Conversely, if p^e divides α it is plain that $\alpha \in FEnd(A)$. □

Let us now have a look at PW-endomorphisms which are LIN too. Recall that an abelian group A with the *minimal condition* (Min) is just a group with shape $A = F \oplus D$, where F is finite and D is divisible with finite total rank.

Proposition 3 *Let φ be a PW endomorphism of an abelian periodic group. Then φ is LIN iff $A = A_\pi \oplus A_{\pi'}$ coprime summands where A_π has Min and $\varphi_{|A'_\pi}$ is invertible.*

Proof. Assume φ is PW and LIN and let π be the set of primes p such that φ is not invertible on A_p. Then π is finite. Now p divides φ_p for any $p \in \pi$, and hence $\varphi(A[p]) = 0$. It follows that A_p has Min and so A_π has Min as well. Conversely, if $A = A_\pi \oplus A_{\pi'}$ coprime summands where A_π has Min and $\varphi_{|A'_\pi}$ is invertible, then for any $H \leq A$ the quotient $H/\varphi(H)$ is finite, as it has finite rank and exponent, and φ is LIN. □

We prove now a Lemma which extends a result due to D.Robinson [9].

Lemma 1 *Let $a \in A$ be an abelian p-group and $\varphi \in \mathrm{End}(A)$.*
(1) If φ either LIN or RIN, then the torsion subgroup of the φ-submodule $\langle a\rangle^{(\varphi)}$ of A generated by a is finite.
(2) If $|X/X_{(\varphi)}| < \infty$ for all $X \leq A$, then $|X^{(\varphi)}/X| < \infty$ for all $X \leq A$.
(3) If $|X/X_{(\varphi)}| \leq p^m$ for all $X \leq A$, then $|X^{(\varphi)}/X| \leq p^{m^2}$ for all $X \leq A$.

Proof. (1) We may assume $A = \langle a\rangle^{(\varphi)}$. Suppose first a has order prime p and regard A as $\mathbb{Z}_p[x]$-module (where x acts as φ) and consider the natural epimorphism mapping 1 to a and x to $\varphi(a)$:

$$F : \mathbb{Z}_p[x] \mapsto A.$$

If F is injective, we can replace A by $\mathbb{Z}_p[x]$ and φ by multiplication by x. If $H := \mathbb{Z}_p[x^2]$, then $\varphi(H) = xH$ is infinite, while $H \cap xH = 0$, a contradiction. If now a has order p^ϵ, then A/pA is finite, by the above. Moreover, pA is finite by induction on ϵ.

(2) This can be proved in a similar way as case (3)

(3) We claim that *if $a \in A$ has order p^ϵ, then $|\langle a\rangle^{(\varphi)}| \leq p^{(m+1)\epsilon}$.*
Assume first $\epsilon = 1$, that is a has order p and $A_0 := \langle a\rangle^{(\varphi)}$ is elementary abelian. Suppose, by contradiction, the above F is injective. As above, let $H := \mathbb{Z}_p[x^2]$. Then $H_{(\varphi)} = (g(x^2))$ for some polynomial g. Since $|H/H_{(\varphi)}| = p^m < \infty$, we have $g \neq 0$. Then $(g(x^2)) \not\subseteq H$, a contradiction. Therefore, for some $f \in \mathbb{Z}_p[x]$ with degree say n, we have

$$\frac{\mathbb{Z}_p[x]}{(f)} \simeq_\gamma \langle a\rangle^{(\varphi)} = A_0$$

Thus the minimal φ-invariant subgroups of A_0 correspond $1-1$ to the irreducible monic factors of f, which are at most n. Consider a $\mathbb{Z}_p$-basis X of A containing an element in each subgroup of them. The the hyperplane H of equation $x_1 + x_2 + \cdots + x_n = 0$ has index p in $\langle a\rangle^{(\varphi)}$ and $H_{(\varphi)} = 0$ as $H \cap X = \emptyset$. Therefore $|\langle a\rangle^{(\varphi)}| \leq p^{m+1}$.

If $\epsilon > 1$, by induction $B := \langle p^{\epsilon-1}a\rangle^{(\varphi)}$ has order at most $p^{(m+1)(\epsilon-1)}$ and $\langle a\rangle^{(\gamma)}/B$ has order at most p^{m+1} by case $\epsilon = 1$. Therefore $|a^{(\varphi)}| \leq p^{(m+1)\epsilon}$, as claimed.

In the general case let X be any subgroup of A and $X_{(\varphi)} = 0$. Thus $|X| =: p^{\epsilon} \leq p^m$. Write $X = \langle a_1\rangle \oplus \cdots \oplus \langle a_r\rangle$ with a_i of order p^{ϵ_i} and $\epsilon_1 + \cdots + \epsilon_r = \epsilon$. Since $|\langle a_i\rangle^{(\varphi)}| \leq p^{(m+1)\epsilon_i}$ by the above, we have $|X^{(\varphi)}| \leq p^{(m+1)\epsilon}$. So that $|X^{(\varphi)}/X| \leq p^{(m+1)\epsilon-\epsilon} \leq p^{m^2}$. □

Lemma 2 *Let $\varphi \in \mathrm{End}(A)$ and $D \leq A$ divisible and primary. If φ is either LIN or RIN, then φ is PW on D, that is there is a p-adic α that is $\varphi(a) = \alpha a \ \ \forall a \in D$.*

Proof. Without loss of generality, let D have rank 1. If φ is LIN, then $D \leq \varphi(D)$ and thus $D = \varphi(D)$. Therefore in both cases LIN or RIN, we have $\varphi(D) \leq D$. Thus φ is PW on D. □

Proof of Theorem 1 We may assume A is a p-group with $D := div(A)$ and note that *if A is an elementary abelian*, the statement follows from Theorem 2.

We claim that for any RIN or LIN-endomorphism φ of any p-group A:
(fs) $\quad \forall H \leq A \ \ |H^{(\varphi)}/H_{(\varphi)}| < \infty$. Therefore (LIN) $\Rightarrow$ (RIN).
To this aim we may suppose $H_{(\varphi)} = 0$. Thus, since φ is PW on the divisible radical D of A, (see Lemma 2), we have $D \cap H = 0$ and H is reduced. Moreover, by the elementary abelian case, φ is PW on a subgroup of finite index of $A[p]$, we get that $H[p]$ is finite. It follows that H is finite. Then (fs) holds by Lemma 1.

Let now A be any residually finite abelian p-group and assume, by contradiction, that φ acts as a multiplication on no quotient with finite kernel. As in the proof argument of Theorem 2, we note that if φ is LIN (resp. RIN), then *there is no sequence of subgroups X_i with the property* that if we denote $Y_i := X_i \cap \varphi(X_i)$ then we have:
(1) $Y_{i+1} \cap X_i = Y_i$
(2) the sequence $|X_i/Y_i|$ (resp. $|\varphi(X_i)/Y_i|$) is *strictly increasing.*
This is true since otherwise there would exists a subgroup $X_\omega := \cup_i X_i$ with the properties that $|X_\omega/X_\omega \cap \varphi(X_\omega)| \geq |X_i/Y_i| \geq i$ (resp. $|\varphi(X_\omega)/X_\omega \cap \varphi(X_\omega)| \geq |\varphi(X_i)/Y_i| \geq i$) for each i. On the other hand, we will construct now a prohibited sequence X_i, a contradiction. Let X be any finite subgroup

of A. As above the subgroup $K := X^{(\varphi)}$ is finite by Lemma 1. By (fs) there is a φ-subgroup B with finite index such that $B \cap K = 0$. Now, as φ is not PW on $(B+K)/K$, there is $a \in B$ such that $\varphi(a) \notin \langle a, K\rangle$. Let $X' := \langle a\rangle + X$ and $Y' := X' \cap \varphi(X')$. Let us check that
(1) $X \cap Y' = Y$;
(2′) $X' > X + Y'$ and $\varphi(X') > \varphi(X) + Y'$.
In fact, on one hand we have $X \cap Y' = Y$, as if $x \in X \cap Y'$ then $x = s\varphi(a) + \varphi(x_0)$ with $s \in \mathbb{Z}$, $x_0 \in X$ and $s\varphi(a) = x - \varphi(x_0) \in B \cap K = 0$, hence $x = \varphi(x_0) \in Y$ and (1) holds. On the other hand $Y' \leq \langle pa\rangle + Y \not\ni a$ and $Y' \leq \langle p\varphi(a)\rangle + Y \not\ni \varphi(a)$. Indeed if $y' \in Y' = X' \cap \varphi(X')$, then $\exists n, s \in \mathbb{Z}$, $\exists x, x_0 \in X$ such that $y' = na + x = s\varphi(a) + \varphi(x_0)$ where $na = s\varphi(a)$ and $x = \varphi(x_0) \in Y := X \cap \varphi(X)$. It follows that p divides s, hence p divides n as well. Then (2′) holds.

Thus we can define by induction a prohibited sequence as above, since from (1) and (2′) it follows $|X'/Y'| > |X/Y|$ and $|\varphi(X')/Y'| > |\varphi(X)/Y|$.

Let now A be any reduced p-group and let R be a basic subgroup. By (fs), $R^{(\varphi)}/R$ is finite and so $H := R^{(\varphi)}$ is residually finite as well. Also, A/H is divisible. By the above there are a p-adic $\alpha \in \mathcal{J}_p$ and a finite φ-invariant subgroup C of H such that $\varphi = \alpha$ on H/C. As the kernel K/C of $(\varphi - \alpha)_{|A/C}$ contains H/C and its image is reduced, while A/H is divisible, it is clear that $K = A$ and φ is close to α, as wished.

Finally, *assume A is any p-group* and φ is not close to any multiplication. As (fs) holds, at the expense of substituting A with a finite index φ-subgroup we have $A = D \oplus E$ with D divisible and E reduced and both φ-invariant. Now φ is multiplication on both D (see Lemma 2) and a finite index subgroup of E (see above). So we may also assume φ is power on E. Say $\varphi_{|D} = \alpha_1$ and $\varphi_{|E} = \alpha_2$, where $\alpha_1 \neq \alpha_2$ are p-adics.

If E has finite exponent, we may substitute it by $E[p^e]$ where $p^e = eexp(E)$. By the reduced case, φ is power on a subgroup A' of finite index of $A[p^e]$. Then if D has infinite rank, $\alpha_1 \equiv \alpha_2 \ mod\ p^e$ and φ it is multiplication on $D \oplus (E \cap A')$ which has finite index in A, a contradiction. Thus *D has finite rank.*

If by contradiction E has infinite exponent (and by our assumption $\alpha_1 \neq \alpha_2$), then there is a quotient E/S of its which is a Prüfer group (and infinite). By (fs) we can assume S to be φ-invariant and consider $\bar{A} := A/S$. This a divisible group on which φ acts as a (universal) multiplication by Lemma 2, contradicting $\alpha_1 \neq \alpha_2$.

Finally let us show that (FS) holds. If φ is a multiplication on some $B \leq A$ take $n = |A/B|^2$. In the other (critical) case, observe that $H_0 := (D \cap H) + (E \cap H)$ is φ-invariant and the group $(H \cap B)/H_0$ has exponent $\leq exp(E) =: p^m$ and finite rank $r <$ rank of D. Thus $|H/H_{(\varphi)}| \leq np^{mr}$. Then apply Lemma 1. Conversely, it is plain that (FS) implies that φ is inertial. $\square$

Proof of Corollary 1. Let $\varphi \in \mathcal{I}End(A)$. If A is non-critical, apply Theorem 1 and Proposition 2.

If $A = D \oplus E$ is critical, there is a φ-invariant finite index subgroup $E_1 \leq E$. Let $E_2 := E_1[p^e]$. By the above φ acts as multiplication by r on a finite index subgroup of E_2. For each r we may consider $\bar{r} \in PEnd(A)$ acting as the zero map on D and multiplication by r on E. Let $\alpha \in \mathcal{J}_p$ represent the action of φ in D (which is power by Lemma 2). Thus $\varphi - \alpha - \overline{r - \alpha} \in FEnd(A)$ and so $\mathcal{I}End(A) = PEnd(A) + (FEnd(A) + R)$, where $R = \{\bar{r} \mid r \in \mathbb{Z}\} \simeq \mathbb{Z}(p^m)$. Further, if $\alpha = \varphi_0 + \bar{r} \in PEnd(A) \cap (FEnd(A) + R)$, then α act as $\bar{r}$ on a subgroup with finite index and therefore on D. Thus $\alpha = 0$. To prove $FEnd(A) \cap R = p^e R$ apply Proposition 2 to E. $\square$

Proof of Corollary 2. Let $\gamma \in \mathcal{I}Aut(A)$. Suppose A non-critical. Then, according to Theorem 1, there exists $\alpha \in PEnd(A)$ and a finite index subgroup $B \leq A$ such that $\gamma_B = \alpha$ and $\gamma^{-1}\alpha$ acts on B as the identity map. This guarantees that p does not divide α, which is therefore invertible. Further if $\alpha \in PAut(A) \cap FAut(A)$ then $\alpha_B = 1$ on a finite index subgroup $B \leq A$. Then $\alpha = 1$, provided $exp(A) = \infty$. Otherwise, if α acts as the identity map on a finite index B subgroup of A, then $B \geq A[p^e]$ and $\alpha \equiv 1 \ mod \ p^e$ (see Proposition 2).

Suppose now A is critical. Fix E with finite exponent such that $A = D \oplus E$. Consider $\Gamma := \{\zeta \in \mathcal{I}Aut(A) \mid \zeta_D = 1 \ and \ \zeta = \zeta_r \ is \ power \ r \ on \ E\ \}$. By Theorem 1, there exists an invertible p-adic α such that $\gamma_D = \alpha$ and $r \in \mathbb{Z}$ such that $\gamma\alpha^{-1}$ acts by means of power r on a finite index subgroup of E. Also $\bar{r} \in \mathcal{U}(\mathbb{Z}(p^m))$, as p does not divide r. Thus $\gamma\alpha^{-1}\zeta_r^{-1} \in FAut(A)$, as wished. The final part of the statement follows from the above argument. $\square$

References

[1] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, *Rend. Sem. Mat. Univ. Padova* **81** (1981), 107-149.

[2] U. Dardano, S. Rinauro, Inertial automorphisms of an abelian group, *Rend. Sem. Mat. Univ. Padova* **127** (2012) 213-233.

[3] S. Franciosi, F. de Giovanni and M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Alg.* **23(14)** (1995), 5483-5497.

[4] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, Intrinsic algebraic entropy, preprint.

[5] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, Fully inert subgroups of Abelian groups, preprint.

[6] S. Franciosi, F. de Giovanni and M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Alg.* **23(14)** (1995), 5483-5497.

[7] L. Fuchs "Infinite Abelian Groups", Academic Press, New York - London, 1970 - 1973.

[8] D.J.S. Robinson, "A Course in the Theory of Groups", Springer V., Berlin, 1982.

[9] D.J.S. Robinson, On inert subgroups of a group, *Rend. Sem. Mat. Univ. Padova* **115** (2006), 137-159.

Ulderico Dardano, Dipartimento di Matematica e Applicazioni "R.Caccioppoli", Università di Napoli "Federico II", Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy. `dardano@unina.it`

Silvana Rinauro, Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata, Via dell'Ateneo Lucano 10 - Contrada Macchia Romana, I-85100 Potenza, Italy. `silvana.rinauro@unibas.it`