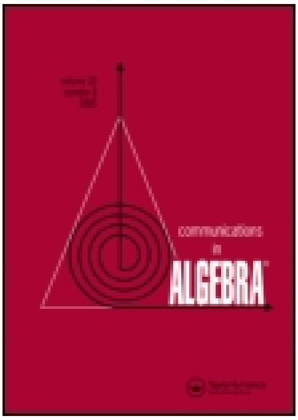

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A Family of Algebraically Closed Fields Containing Polynomials in Several Variables

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A FAMILY OF ALGEBRAICALLY CLOSED FIELDS CONTAINING POLYNOMIALS IN SEVERAL VARIABLES

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We introduce a family of fields of series with support in strongly convex rational cones. All these fields contain polynomials in several variables. We prove that they are algebraically closed with a construction that is analogous to the Newton polygon for algebraic curves. As a corollary, we show the existence of fractional power solutions with support in cones for systems of equations.

Key Words: Newton polygon; Parametrization; Puiseux series.

2000 Mathematics Subject Classification: Primary 32S05, 32B10; Secondary 14M25.

1. INTRODUCTION

To extend Puiseux's theorem from one to several variables, the usual approach is to consider the field of multi-Laurent series with support in well-ordered sets for the lexicographic order. This field is used in Sathaye (1983) to extend the Abhyankar–Moh Semigroup theorem. The same construction can be done with any compatible total order in $\mathbb{Q}^N$ (see Rayner, 1974 or Ribenboim, 1992). These fields are called *fields of generalized power series*.

McDonald (1995) shows that a polynomial in y with coefficients polynomials in N variables has a root in the ring of Puiseux series with support in some strongly convex polyhedral cone. Then González Pérez (2000) notes that McDonald's proof may be extended to consider coefficients in the ring of Puiseux series with support in a strongly convex polyhedral cone.

Series with support in cones have many interesting properties: They can be transformed in series with support in an orthant by a chain of monomial transformations (Aranda, 2002; Soto and Vicente, 2006), an extension of Abel's theorem holds (Aroca, 2004), they appear in the solutions of holonomic systems (Saito et al., 2000).

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In this note, we introduce a family of fields (called *ω-positive Puiseux series*) that arise as an infinite union of rings of series with support in strongly convex polyhedral cones. Our fields are strictly smaller than the ones described in the first paragraph.

We prove the following theorem.

Theorem 1. *The field of ω-positive Puiseux series is algebraically closed.*

McDonald (2002) extends his explorations in McDonald (1995) to systems of equations. He gives a construction that works for "general" systems but he cannot characterize the systems for which solutions do exist. Once Theorem 1 is established, the existence of solutions is just a consequence of Hilbert's Nullstellensatz. We see this in the last section of this note.

In an article in preparation with López de Medrano, we will give a method to find these solutions using the tropicalization of the ideal generated by the system.

Theorem 1 follows from a general theorem proved in Rayner (1974) using valuation theory. It can also be seen as a consequence of a construction for partial differential equations given in Aroca et al. (2003).

Here we give a simple constructive proof using two geometrical objects: the ω-Newton polygon and the ω-barrier wedge.

2. THE FIELDS

We will work on an algebraically closed field $\mathbb{K}$ of characteristic zero. For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Q}^N$ set $x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}$. With this notation a fractional power series φ in N variables is expressed as

$$\varphi = \sum_{\alpha \in \mathbb{Q}^N} c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{K}.$$

The *set of exponents* of φ is the set

$$\mathscr{E}(\varphi) := \{\alpha \in \mathbb{Q}^N \mid c_\alpha \neq 0\}.$$

A fractional power series φ is a *Puiseux series* when its set of exponents is contained in a lattice. That is, there exists $K \in \mathbb{N}$ such that $\mathscr{E}(\varphi) \subset \frac{1}{K}\mathbb{Z}^N$. The set $\mathscr{P}$ of Puiseux series is a group with addition but multiplication is not always defined.

A *convex rational polyhedral cone* is a subset of $\mathbb{R}^N$ of the form

$$\sigma = \{\lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0\}, \tag{1}$$

where $v_1, \ldots, v_r \in \mathbb{Q}^N$ are vectors. A cone is said to be *strongly convex* if it contains no nontrivial linear subspaces.

Let $\sigma \subset \mathbb{R}^N$ be a strongly convex cone. Then $\mathbb{Q}^N \cap \sigma$ is a semigroup, and the set of Puiseux series with support in σ

$$\mathbb{K}[\sigma] = \{\varphi \in \mathscr{P} \mid \mathscr{E}(\varphi) \subset \sigma\}$$

is a ring.

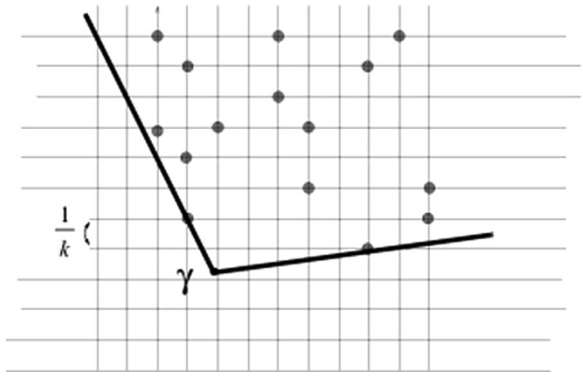

Figure 1 The exponents of a Puiseux series with support in a translate of σ.

The set of Puiseux series with support in translates of σ (see Fig. 1)

$$\mathbb{K}[\sigma]^T = \{\varphi \in \mathcal{P} \mid \mathcal{E}(x^{-\gamma}\varphi) \subset \sigma \text{ for some } \gamma \in \mathbb{Q}^N\}$$

is also a ring. When $N = 1$, $\mathbb{K}[\sigma]^T$ is a field but for $N \geq 2$ it is not.

Given $\omega \in \mathbb{R}^N$, we say that a cone σ is ω-positive when it is contained in the half space $\mathcal{H}_\omega = \{v \in \mathbb{R}^N \mid \omega \cdot v \geq 0\}$.

Remark 1. The union of ω-positive rational cones is always contained in a ω-positive rational cone. When ω is of rationally independent coordinates a ω-positive rational cone is strongly convex.

We say that a Puiseux series φ is *ω-positive* when there exists $\gamma \in \mathbb{Q}^N$ and a ω-positive rational cone σ such that $\mathcal{E}(x^{-\gamma}\varphi) \subset \sigma$.

The set of ω-positive Puiseux series will be denoted by $\mathcal{S}_\omega$. When ω is of rationally independent coordinates, by Remark 1, $\mathcal{S}_\omega$ is a ring. We will show that $\mathcal{S}_\omega$ is an algebraically closed field.

We end the section with a remark that we will need later on.

Remark 2. Let φ be a ω-positive Puiseux series, and let $\gamma \in \mathbb{Q}^N$ be such that $\omega \cdot \gamma \leq \omega \cdot \alpha$ for all $\alpha \in \mathcal{E}(\varphi)$. Then there exists a ω-positive rational cone σ such that $\mathcal{E}(x^{-\gamma}\varphi) \subset \sigma$.

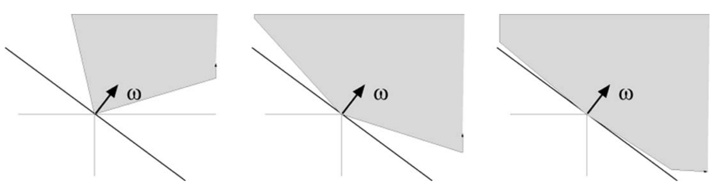

Figure 2 ω-Positive rational cones.

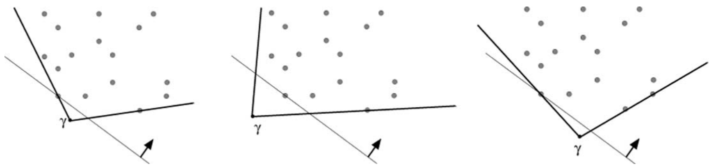

Figure 3 Different cones for the same ω-positive Puiseux series.

3. NEWTON POLYGON

From now on, ω will stand for a vector of rationally independent coordinates, and by cone we will mean strongly convex rational polyhedral cone.

We will be dealing with polynomials

$$f = \sum_{i=0}^{d} \varphi_i(x) y^i, \tag{2}$$

where φ_i are ω-positive Puiseux series. Since d is finite f may be written as

$$f = \sum_{\alpha \in (\gamma+\sigma)\cap \frac{1}{K}\mathbb{Z}^N, i=0,\ldots,d} a_{\alpha,i} x^\alpha y^i, \tag{3}$$

where σ is a w-positive rational cone, $\gamma \in \mathbb{Z}^N$, $K \in \mathbb{N}$, and $a_{\alpha,i} \in \mathbb{K}$.

The set of exponents of f is the subset of $\mathbb{R}^{N+1}$

$$\mathscr{E}(f) = \{(\alpha, i) \in \mathbb{Q}^N \times \mathbb{N} \mid a_{\alpha,i} \neq 0\}.$$

We will say that the last coordinate of a point in $\mathbb{R}^{N+1}$ is its *height*. We say that a set is *horizontal* when it has constant height. Given two points P and P' in $\mathbb{R}^{N+1}$, if the height of P' is smaller than that of P, we say that P is *higher* than P' and that P' is *lower* than P.

Let π_ω be the projection from $\mathbb{R}^{N+1}$ to $\mathbb{R}^2$ given by

$$\begin{aligned} \pi_\omega : \mathbb{R}^N \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ (\alpha, h) &\mapsto (\omega \cdot \alpha, h). \end{aligned}$$

The preimage of a point $P \in \mathbb{R}^2$ is an horizontal linear space of codimension 2 orthogonal to $(\omega, 0)$.

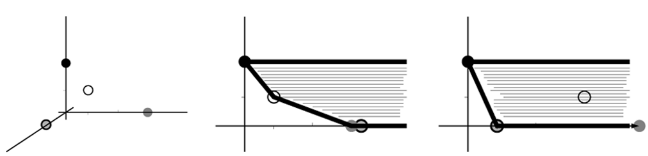

Figure 4 The sets $\mathscr{E}(f)$, $\mathrm{NP}_{(1,\pi)}f$ and $\mathrm{NP}_{(\pi,1)}f$ for $f = y^2 + x_1 y - x_1^3 - x_2$.

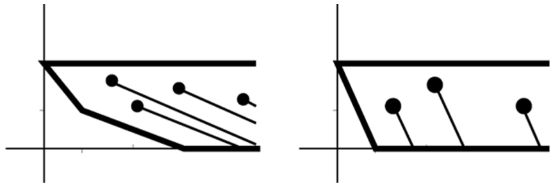

Figure 5 Remark 3.

Proposition 1. *Given $P = (q, h) \in \mathbb{R}^2$ the linear space $\pi_\omega^{-1}(P)$ has at most one point of rational coordinates.*

Proof. Suppose that (α, h) and (α', h) are in $\pi_\omega^{-1}(P)$, then

$$\omega \cdot \alpha = q = \omega \cdot \alpha' \Rightarrow \omega \cdot (\alpha - \alpha') = 0$$

so, if $\alpha \neq \alpha'$ are rational $(\alpha - \alpha')$ is a rational combination of the coordinates of ω. □

As a consequence of Proposition 1 we have that the correspondence

$$\pi_w : \mathscr{E}(f) \longrightarrow \pi_w(\mathscr{E}(f))$$

is one to one.

If f is as in (3) then the set $\pi_w(\mathscr{E}(f))$ is contained in the half band

$$\{(q, h) \in \mathbb{R}^2 \mid q \geq \omega \cdot \gamma, 0 \leq h \leq d\}.$$

Definition 1. Given $\omega \in \mathbb{R}^N$ of rationally independent coordinates and f, a polynomial with coefficients in $\mathscr{S}_\omega$. The *ω-Newton polygon* of f is the convex hull of the set $\pi_\omega(\mathscr{E}(f)) + (\mathbb{R}_{\geq 0} \times \{0\})$ that is

$$\mathrm{NP}_\omega f = \mathrm{conv}\left(\bigcup_{P \in \mathscr{E}(f)} \pi_\omega(P) + (\mathbb{R}_{\geq 0} \times \{0\}) \right).$$

The ω-Newton polygon has a finite number of edges. Two of the edges are horizontal and unbounded and the rest are bounded. $\mathrm{NP}_\omega f$ touches the axis of abscissas if and only if $y = 0$ is not a root of f.

We state here a remark that we will use later on:

Remark 3. Let $\eta \in \mathbb{R}$ be such that the vector $(\eta, -1)$ is parallel to the lowest finite edge of $\mathrm{NP}_\omega f$ and let $P = (a, h)$ be a point in $\mathrm{NP}_\omega f$. The segment $\{P + \lambda(\eta, -1) \mid 0 \leq \lambda \leq h\}$ is contained in $\mathrm{NP}_\omega f$ (see Fig. 5).

4. BARRIER WEDGE

Let σ be an N-dimensional cone, and let ℓ be a nonhorizontal line in $\mathbb{R}^{N+1}$. The *wedge of spine* ℓ *and openness* σ is the set

$$W = \bigcup_{x\in\ell}\{x + (\beta, 0) \mid \beta \in \sigma\}.$$

A wedge is just a translate of a (not strongly) convex cone.

Remark 4. Let P be a point in the spine of a wedge W and let $v \in \mathbb{R}^{N+1}$ be a vector different than zero. If the point $P + v$ is contained in W, then the half line $\{Q + \lambda v \mid \lambda \in \mathbb{R}_{\geq}0\}$ is in W for all $Q \in W$.

Remark 5. Let P be a point in the spine of a wedge W, then

$$\gamma + \sigma = \{\mu \in \mathbb{R}^N \mid P + (\mu, -1) \in W\},$$

where $(\gamma, -1)$ is parallel to the spine of W and σ is its openness.

A wedge of ω-positive rational openness will be called a ω-*barrier wedge* of f if its spine contains a point of the set of exponents of f, and all the set is contained in W.

Proposition 2. *Let Q be a point in $\mathscr{E}(f)$. If $\pi_\omega(Q)$ is a vertex of* $\mathrm{NP}_\omega f$, *then there exists a ω-barrier wedge of f containing Q in its spine.*

Proof. Let l be a supporting line of $\mathrm{NP}_\omega f$ with $\pi_\omega(Q) \in l$, and let ℓ be a non horizontal line contained in $\pi_\omega^{-1}(l)$ passing through Q.

Since $\mathrm{NP}_\omega f \subset \pi_\omega(\ell) + (\mathbb{R}_{\geq 0} \times \{0\})$, then $\mathscr{E}(f) \subset \ell + (\mathscr{H}_w \times \{0\})$. In particular, if f is as in (2) and, for $i = 0, \ldots, d$, (γ_i, i) is the point of ℓ of height i, then $\omega \cdot \gamma_i \leq \omega \cdot \alpha$ for all $\alpha \in \mathscr{E}(\varphi_i)$. So, by Remark 2 and the fact that d is finite, there exists a ω-positive rational cone σ such that $\mathscr{E}(f) \subset \ell + (\sigma \times \{0\})$. $\square$

5. THE CONSTRUCTION

Let $\omega \in \mathbb{R}^N$ be of rationally independent coordinates, and let f be a polynomial with coefficients in $\mathscr{S}_\omega$ such that $y = 0$ is not a root of f.

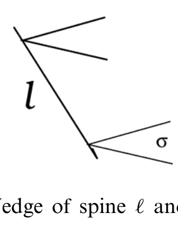

Figure 6 Wedge of spine ℓ and openness σ.

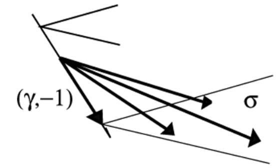

Figure 7 Remark 5.

Let $\underline{V}$ and $\overline{V}$ be the vertex of height zero and the vertex of smallest positive height of $\mathrm{NP}_\omega f$, respectively. Set (Proposition 1)

$$\underline{Q} = (\underline{\beta}, 0) = \pi_\omega^{-1}(\underline{V}) \cap \mathscr{E}(f) \qquad \text{and} \qquad \overline{Q} = (\overline{\beta}, h) = \pi_\omega^{-1}(\overline{V}) \cap \mathscr{E}(f). \tag{4}$$

The segment joining $\underline{Q}$ and $\overline{Q}$ will be called the ω-*segment* of f, and the N-tuple

$$\mu = \frac{\underline{\beta} - \overline{\beta}}{h}, \tag{5}$$

will be called the ω-*slope* of f.

Remark 6. The slope of the line containing $\pi_\omega(\overline{Q})$ and $\pi_\omega(\underline{Q})$ is $\frac{-1}{\omega \cdot \mu}$.

Given a subset $\mathscr{C} \subset \mathbb{R}^{N+1}$ and f as in (3), we denote

$$f|_{\mathscr{C}} := \sum_{\substack{\alpha \in \gamma + \sigma, i = 0, \dots, d \\ (\alpha, i) \in \mathscr{C}}} a_{\alpha,i} x^\alpha y^i.$$

Let L be the ω-segment of f, let $\overline{Q}$ be its upper vertex, and let μ be the ω-slope of f. We have

$$L \cap (\mathbb{Q}^N \times \mathbb{N}) = \{\overline{Q} + i(\mu, -1) \mid i = 0, \dots, h\}$$

and then

$$f|_L = \sum_{i=0}^{h} a_{\overline{Q} + i(\mu, -1)} x^{\bar{\beta} + i\mu} y^{h-i}. \tag{6}$$

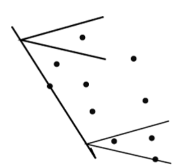

Figure 8 Barrier wedge.

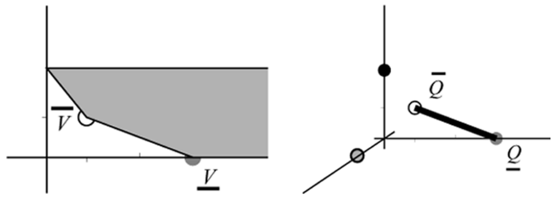

Figure 9 The $(1, \pi)$-segment of $f = y^2 + x_1 y - x_1^3 - x_2$.

The ω-*characteristic polynomial* of f is the polynomial defined by

$$\Psi(c) = f|_L(\underline{1}, c),$$

that is,

$$\Psi(c) = \sum_{i=0}^{h} a_{\overline{Q}+i(\mu,-1)} c^{h-i}. \tag{7}$$

The ω-characteristic polynomial is a polynomial of degree h with coefficients in $\mathbb{K}$ so it has a root.

Construction. Set $f^{(0)} = f$ and define inductively $f^{(i+1)} = f^{(i)}(x, y + c_i x^{\mu^{(i)}})$. Where

(i) When $y = 0$ is not a root of $f^{(i)}$, $\mu^{(i)}$ is the ω-slope of $f^{(i)}$, c_i a root of the ω-characteristic polynomial of $f^{(i)}$;
(ii) When $y = 0$ is a root of $f^{(i)}$, $\mu^{(i)} = \underline{0}$ and $c_i = 0$.

Set

$$\varphi(x) = \sum_{i=0}^{\infty} c_i x^{\mu^{(i)}}.$$

Example. Set $f = y^2 + x_1 y - x_1^3 - x_2$. For $\omega = (1, \pi)$ the construction gives

$$\varphi = x_1^2 + x_1^{-1} x_2 - x_1^3 - 2x_2 - x_1^{-3} x_2^2 + 6x_1 x_2 + 6x_1^{-2} x_2^2 + \cdots$$

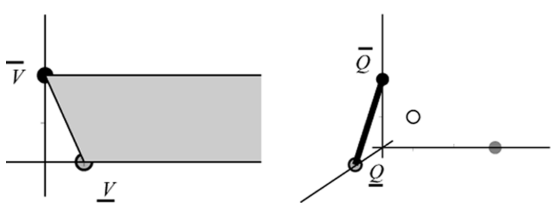

Figure 10 The $(\pi, 1)$-segment of $f = y^2 + x_1 y - x_1^3 - x_2$.

and for $\omega = (\pi, 1)$, we get

$$\varphi = x_2^{\frac{1}{2}} - \frac{1}{2}x_1 + \frac{1}{8}x_1{}^2x_2{}^{-\frac{1}{2}} + \frac{1}{2}x_1^3x_2{}^{-\frac{1}{2}} - \frac{1}{128}x_1^4x_2{}^{-\frac{3}{2}} - \frac{1}{16}x_1^5x_2{}^{-\frac{3}{2}} + \cdots.$$

When the construction gives a finite sum of monomials $y = 0$ is a root of $f^{(i)}(y) = 0$ for some i and then φ is a root of f.

We claim that even when the process is infinite $\varphi(x)$ is a ω-positive Puiseux series and that it is a root of f.

6. FROM $\mathscr{E}(f^{(i)})$ TO $\mathscr{E}(f^{(i+1)})$

Let f be a polynomial in y with coefficients in $\mathscr{S}_w$ such that y does not divide f. Construct

$$\tilde{f}(x, y) = f(x, y + cx^{\mu}),$$

where c is a root of the ω-characteristic equation of f and μ is its ω-slope. The result is a new polynomial in y with coefficients in $\mathscr{S}_w$. In this section, we study the relation between the set of exponents of f, and the set of exponents of $\tilde{f}$.

Let's start with a monomial

$$\widetilde{x^{\alpha}y^{i}} = x^{\alpha}(y + cx^{\mu})^{i} = x^{\alpha}\sum_{j=0}^{i}\binom{i}{j}c^{j}x^{j\mu}y^{i-j},$$

since $\mathbb{K}$ is of characteristic zero,

$$\mathscr{E}(\widetilde{x^{\alpha}y^{i}}) = \{(\alpha, i) + j(\mu, -1) \mid j = 0, \ldots, i\}. \tag{8}$$

So, for any polynomial f the set of exponents of $\tilde{f}$ is contained in segments parallel to the ω-segment whose upper vertex is an exponent of f. That is

$$\mathscr{E}(\tilde{f}) \subset \bigcup_{(\alpha,h)\in\mathscr{E}(f)} \{(\alpha, h) + \lambda(\mu, -1) \mid 0 \leq \lambda \leq h\}. \tag{9}$$

Remark 7. Given a point $P \in \mathbb{Q}^N \times \mathbb{N}$, let l_P^+ be the half-line over P parallel to the ω-segment. That is, $l_P^+ = \{P + \lambda(\mu, -1) \mid \lambda \in \mathbb{R}_{\leq 0}\}$. From (8) it follows:

(i) If $l_P^+ \cap \mathscr{E}(f) = \emptyset$, then $P \notin \mathscr{E}(\tilde{f})$;
(ii) If $l_P^+ \cap \mathscr{E}(f)$ is only one point, then $P \in \mathscr{E}(\tilde{f})$.

Proposition 3. *The lower vertex of the ω-segment of f is never an exponent of $\tilde{f}$.*

Proof. Let L be the ω-segment of f. Note that $\widetilde{f|_L} = \tilde{f}|_L$, so $\underline{Q} \notin \mathscr{E}(\tilde{f})$ if and only if $y = 0$ is a root of $\widetilde{f|_L}$. Now

$$\widetilde{f|_L}(x, 0) = f|_L(x, cx^{\mu}) \overset{\text{by (6) and (7)}}{=} x^{\underline{\beta}+h\mu}f|_L(\underline{1}, c)$$

is zero when c is a root of the characteristic polynomial. $\square$

From Remark 4 and Eq. (9) the following remark follows.

Remark 8. Let W be a ω-barrier wedge of f containing $\overline{Q}$ in its spine. The set of exponents of $\tilde{f}$ is contained in W.

7. THE PIVOT

The projection π_ω induces an injective map between lines parallel to the ω-segment containing a point in $\mathbb{Q}^{N+1}$ and lines in $\mathbb{R}^2$ parallel to the lower finite edge of $\mathrm{NP}_\omega f$ (i.e., of slope $\frac{-1}{\omega\cdot\mu}$).

Equation (9) gives

$$\mathrm{NP}_\omega f^{(i+1)} \subset \bigcup_{(a,h)\in \mathrm{NP}_\omega f^{(i)}} \{(a, h) + \lambda(\omega\cdot\mu^{(i)}, -1) \mid 0 \le \lambda \le h\}.$$

By Remarks 3 and 6, this is equivalent to

$$\mathrm{NP}_\omega f^{(i+1)} \subset \mathrm{NP}_\omega f^{(i)}. \tag{10}$$

Proposition 3 translates into the following remark.

Remark 9. The lower vertex of $\mathrm{NP}_\omega f^{(i)}$ is never in $\mathrm{NP}_\omega f^{(i+1)}$.

Then Remark 7 together with (10) implies the following remark.

Remark 10. All vertexes of positive height of $\mathrm{NP}_\omega f^{(i)}$ are again vertexes of $\mathrm{NP}_\omega f^{(i+1)}$.

Suppose that $f^{(i)}(x, 0) \ne 0$ and $f^{(i+1)}(x, 0) \ne 0$. Let $\overline{Q}^{(i)}$ and $\overline{Q}^{(i+1)}$ be the upper vertex of the ω-segment of $f^{(i)}$ and $f^{(i+1)}$, respectively.

By Remark 10, either $\overline{Q}^{(i)} = \overline{Q}^{(i+1)}$ or $\overline{Q}^{(i)}$ is higher than $\overline{Q}^{(i+1)}$. Since the height is a natural number there exists $k \in \mathbb{N}$ such that the upper vertex of the w-segment of $f^{(i)}$ is the same for all $i \ge k$. This point is called the *pivot* of the construction.

Proposition 4. *Let $k \in \mathbb{N}$ be such that the upper vertex of the ω-segment of $f^{(k)}$ is the pivot. There exists a wedge W that is a ω-barrier wedge of $f^{(i)}$ for all $i \ge k$.*

Proof. Let W be a ω-barrier wedge of $f^{(k)}$ containing the pivot in its spine (Proposition 2).

From Remark 4 and Eq. (9) it follows that $\mathscr{E}(f^{(k+1)}) \subset W$. The upper vertex of the ω-segment of $f^{(k+1)}$ is again the pivot, and the result follows by induction. □

8. THE THEOREM

Remarks 6 and 9 imply $\frac{-1}{\omega\cdot\mu^{(i)}} < \frac{-1}{\omega\cdot\mu^{(i+1)}}$ or equivalently

$$\omega\cdot\mu^{(i)} < \omega\cdot\mu^{(i+1)}.$$

In particular, $\mu^{(i)} \neq \mu^{(j)}$ for $i \neq j$, and the construction of Section 5 is a well defined fractional power series $\varphi = \sum_{i=0}^{\infty} c_i x^{\mu^{(i)}}$ with

$$\mathscr{E}(\varphi) = \{\mu^{(i)} \mid i \in \mathbb{N}\}.$$

To see that $\mathscr{S}_\omega$ is algebraically closed, we need to see that φ is an element of $\mathscr{S}_\omega$ and that it is actually a root of f.

Proposition 5. *Let φ be a result of the construction of Section* 5. *There exists γ and a ω-positive rational cone such that $\mathscr{E}(x^{-\gamma}\varphi) \subset \sigma$.*

Proof. Let $k \in \mathbb{N}$ be such that the upper vertex of the ω-segment of $f^{(k)}$ is the pivot. Let W be a ω-barrier wedge of $f^{(i)}$ for all $i \geq k$ (Proposition 4). By definition, the vector $(\mu^{(i)}, -1)$ is parallel to the ω-segment of $f^{(i)}$. For $i \geq k$, the ω-segment of $f^{(i)}$ is contained in W, and its upper vertex is contained in its spine (it is the pivot). Then, by Remark 5

$$\mu^{(i)} \in \gamma + \sigma \qquad \forall i \geq k,$$

where σ is the openness of W, and $(\gamma, -1)$ is parallel to the spine of W.

Since $\sum_{i=0}^{k} c_i x^{\mu^{(i)}}$ is finite, we have the result. □

Proposition 6. *Let φ be a result of the construction of Section* 5. *There exists $K \in \mathbb{N}$ such that $\mathscr{E}(\varphi) \subset \frac{1}{K}\mathbb{Z}^N$.*

Proof. Let $k \in \mathbb{N}$ be such that the upper vertex of the ω-segment of $f^{(k)}$ is the pivot. Let $K \in \mathbb{N}$ be such that $\{\mu_i \mid i = 0, \ldots, k-1\} \subset \frac{1}{K}\mathbb{Z}^N$. Then $\mathscr{E}(f^{(k)}) \subset \frac{1}{K}\mathbb{Z}^N \times \mathbb{N}$.

Since $\overline{Q}^{(k)} = \overline{Q}^{(k+1)} = \text{Pivot}$, then $\overline{Q}^{(k)} + (\mu^{(k)}, -1) \notin \mathscr{E}(f^{(k+1)})$ and then, by Remark 7, $\overline{Q}^{(k)} + (\mu^{(k)}, -1) \in \mathscr{E}(f^{(k)})$. Therefore, $\mu^{(k)} \in \frac{1}{K}\mathbb{Z}^N$ and, by induction, $\mu^{(i)} \in \frac{1}{K}\mathbb{Z}^N$ for all $i \in \mathbb{N}$. □

Proof of Theorem 1. By Propositions 5 and 6, the result of the construction is in $\mathscr{S}_\omega$. It rests to see that it is actually a solution.

For each step (i) in the construction, define $\underline{\beta}^{(i)}$ as in (4). We have that $\pi_\omega((\underline{\beta}^{(i)}, 0))$ is the lowest vertex of $\mathrm{NP}_\omega f^{(i)}$. Then, Remark 9 together with Eq. (10) implies

$$\omega \cdot \underline{\beta}^{(i)} < \omega \cdot \underline{\beta}^{(i+1)}.$$

By Proposition 6, there exists a rational lattice that contains $\mathscr{E}(f^{(i)}(x, y))$ for all $i \in \mathbb{N}$. And, by Proposition 4, there exists a wedge W that is a ω-barrier wedge of $f^{(i)}$ for any i sufficiently large. In particular, there exists $K \in \mathbb{Z}^N$, $\gamma \in \mathbb{Q}^N$, and a ω-positive cone, σ, such that $\underline{\beta}^{(i)} \in \frac{1}{K}\mathbb{Z}^N \cap (\gamma + \sigma)$. Therefore,

$$\lim_{i\to\infty} \omega \cdot \underline{\beta}^{(i)} = \infty.$$

By construction,

$$f\left(x, \sum_{j=0}^{i-1} c_j x^{\mu^{(j)}}\right) = f^{(i)}(x, 0)$$

and, since $(\omega \cdot \underline{\beta}^{(i)}, 0)$ is the lowest vertex of $\mathrm{NP}_\omega f^{(i)}$,

$$\mathscr{E}(f^{(i)}(x, 0)) \subset \{\alpha \in \mathbb{R}^N \mid \omega \cdot \alpha \geq \omega \cdot \underline{\beta}^{(i)}\} \times \{0\}.$$

And then $\mathscr{E}\big(f\big(x, \sum_{j=0}^{\infty} c_j x^{\mu^{(j)}}\big)\big)$ is the empty set. □

9. SYSTEMS OF EQUATIONS

Given r algebraic equations in $N + M$ variables

$$f_i(x_1, \ldots, x_N, y_1, \ldots, y_M) = 0 \qquad \text{for } i = 1, \ldots, r. \tag{11}$$

We want to find a strongly convex rational cone $\sigma \subset \mathbb{R}^N$ and an M-tuple of fractional power series with exponents in a translate of σ

$$\Phi = (\varphi_1(x_1, \ldots, x_N), \ldots, \varphi_M(x_1, \ldots, x_N))$$

such that

$$f_i(x, \Phi) = 0, \qquad i = 1, \ldots, r. \tag{12}$$

Let $\mathscr{I}$ be the ideal generated by the f_i's. The equalities (12) hold if and only if

$$f(x, \Phi) = 0, \qquad \forall f \in \mathscr{I}.$$

This explains the following remark.

Remark 11. If $\mathscr{I} \cap \mathbb{K}[x] \neq \{0\}$ then the system does not have a solution.

Let $\mathbb{K}(x)$ denote the field of fractions of $\mathbb{K}[x]$, and let $\mathscr{I}^e$ be the extension of $\mathscr{I}$ to $\mathbb{K}(x)[y]$ via the natural inclusion

$$\mathbb{K}[x, y] = \mathbb{K}[x][y] \subset \mathbb{K}(x)[y].$$

Suppose that there exists $f \in \mathscr{I}^e \cap (\mathbb{K}(x)\backslash\{0\})$; then

$$f = \frac{g}{h} = \frac{g_1}{h_1} f_1 + \cdots + \frac{g_r}{h_r} f_r \Rightarrow f \prod_{i=1}^{r} h_i = g_1 f_1 \prod_{i=2}^{r} h_i + \cdots + g_r f_r \prod_{i=1}^{r-1} h_i \in \mathbb{K}[x]\backslash\{0\}.$$

Remark 12. If $\mathcal{I} \cap \mathbb{K}[x] = \{0\}$ then $\mathcal{I}^e \cap \mathbb{K}(x) = \{0\}$.

By Remark 12, if $\mathcal{I} \cap \mathbb{K}[x] = \{0\}$, then $\mathcal{I}^e \cap \mathbb{K}(x)[y]$ is a proper ideal of $\mathbb{K}(x)[y]$. Given $\omega \in \mathbb{R}^N$ of rationally independent coordinates $\mathcal{S}_\omega$ is an algebraically closed extension of $\mathbb{K}(x)$. Then, by the Nullstellensatz, the zero locus of $\mathcal{I}^e$ in $\mathcal{S}_\omega^M$ is not empty.

In particular, there exists $\Phi \in \mathcal{S}_\omega^M$ such that the equalities (12) hold.

We get the following corollary.

Corollary 1. *Given r polynomials in $N + M$ variables $f_1, \ldots, f_r \in \mathbb{K}[x, y]$ and $\omega \in \mathbb{R}^N$ of rationally independent coordinates. Let $\mathcal{I}$ be the ideal of $\mathbb{K}[x, y]$ generated by the f_is. There exists a ω-positive cone σ and an M-tuple Φ of Puiseux power series with support in a translate of σ such that $y = \Phi$ solves the system $\{f_i = 0\}$ if and only if $\mathcal{I} \cap \mathbb{K}[x] = 0$.*

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