Noncommutative gauge theory and symmetry breaking in matrix models

Harald Grosse,[1,*] Fedele Lizzi,[2,3,†] and Harold Steinacker[1,‡]

[1]*Department of Physics, University of Vienna, Boltzmanngasse5, A-1090 Vienna, Austria*
[2]*Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN, Sezione di Napoli, Via Cintia, 80126 Napoli, Italy*
[3]*High Energy Physics Group, Departament d'Estructura i Constituents de la Matèria and Institut de Ciències del Cosmos Universitat de Barcelona Barcelona, Catalonia, Spain*

(Received 25 January 2010; published 23 April 2010)

We show how the fields and particles of the standard model can be naturally realized in noncommutative gauge theory. Starting with a Yang-Mills matrix model in more than four dimensions, an $SU(n)$ gauge theory on a Moyal-Weyl space arises with all matter and fields in the adjoint of the gauge group. We show how this gauge symmetry can be broken spontaneously down to $SU(3)_c \times SU(2)_L \times U(1)_Q$ [resp. $SU(3)_c \times U(1)_Q$], which couples appropriately to all fields in the standard model. An additional $U(1)_B$ gauge group arises which is anomalous at low energies, while the trace-$U(1)$ sector is understood in terms of emergent gravity. A number of additional fields arise, which we assume to be massive, in a pattern that is reminiscent of supersymmetry. The symmetry breaking might arise via spontaneously generated fuzzy spheres, in which case the mechanism is similar to brane constructions in string theory.

DOI: 10.1103/PhysRevD.81.085034 PACS numbers: 11.10.Nx, 02.40.Gh, 11.15.Ex

I. INTRODUCTION

While no one knows how to describe physics at the Planck scale, there are suggestions that it may be described by some generalization of ordinary spaces which goes under the generic name of *noncommutative geometry* [1–4]. Regardless of the details of such a construction, the noncommutative generalization of the coordinate functions is given by matrices x^μ which satisfy commutation relations of the type

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{1.1}$$

where $\theta^{\mu\nu}$ is a quantity of the order of the square of the Planck length. An action is then naturally defined as some kind of matrix model in terms of these noncommutative coordinates, such as the models introduced in [5–7]. These matrix models are known to describe noncommutative gauge theory [8,9], and contain gravity as an emergent phenomenon [10] *à la* Sakharov [11,12]. Thus they are promising candidates for a quantum theory of fundamental interactions. However, the noncommutative gauge theories obtained in this manner are quite restrictive [13]: only $U(n)$ gauge groups (or possibly products thereof) are consistent, fermions can be introduced only in the adjoint or possibly (anti)fundamental representation, and the trace-$U(1)$ sector is afflicted with the notorious UV/IR mixing [14–17]. Hence these models are often thought to be incompatible with particle physics. There are proposals for how to circumvent this restriction using additional structures such as open Wilson lines or "Higgsac" fields [18]. This, however, leads to other problems, e.g. with unitarity [19], and a hypercharge sector suffering from UV/IR mixing. The latter issue has been addressed in [16], in which a model is constructed where the problematic trace-$U(1)$ sectors of $U(4) \times U(3) \times U(2)$ are separated from the hypercharge, but nevertheless they lead to unwanted low-energy fields which are not understood. A different proposal [20] is based on a Seiberg-Witten expansion in θ, which leads to models which can be viewed as the low-energy effective action of some underlying noncommutative theory. Here the restriction for the gauge group does not apply; however, such θ-expanded models are typically not renormalizable [21], and a different approach is needed for a fundamental theory.

The main point of this paper is to demonstrate that the simple matrix models for noncommutative gauge theory may nevertheless lead to low-energy gauge theories which are extensions of the standard model. In particular, we show how all fermions in the standard model with their appropriate charges can be accommodated. The principal idea is to consider a matrix model which describes not only the usual Moyal plane $\mathbb{R}^4_\theta$, but also extra dimensions encoded by additional matrices. These matrices corresponding to extra dimensions can be equivalently interpreted as scalar fields on $\mathbb{R}^4_\theta$, and can acquire nontrivial vacuum expectation values leading, via the usual Higgs effect, to spontaneous symmetry breaking. The extra-dimensional matrices are assumed to have a finite spectrum and no massless modes, similar in a sense to Connes' approach to the standard model in noncommutative geometry [1,22,23]. The mechanism is essentially the same as the generation of fuzzy extra dimensions in ordinary gauge theory [24]; cf. [25]. The trace-$U(1)$ components and its

*harald.grosse@univie.ac.at
†fedele.lizzi@na.infn.it
‡harold.steinacker@univie.ac.at

UV/IR mixing were understood in [10] to be part of the gravity sector and are not part of the low-energy gauge theory. This allows us to resolve the problems with the $U(1)$ sector found in previous formulations of the standard model on the Moyal-Weyl plane [16,17] based on (products of) $U(N)$ gauge groups.

The models we will describe below have some key features of the standard model, mainly regarding symmetry breaking, but are not yet phenomenologically viable, in the sense that there are still several features which are unrealistic. However, the basic mechanism based on spontaneous symmetry breaking of the underlying noncommutative $SU(N)$ [resp. $U(N)$] gauge theory is rather general, and it is quite conceivable that more sophisticated versions might be realistic. In particular, we will see that a promising line of development is to consider the internal space as fuzzy spheres, similar to [24]. Then the pattern which emerges is quite similar to string-theoretical constructions of (extensions of the) standard model [26–28], based on strings stretching between branes. These modes are recovered here as bi-modules of $SU(n_i)$ subgroups of the spontaneously broken $SU(N)$ gauge group. One of the main open problems is the origin of chirality, and we only discuss some possible avenues here. This problem is similar to the commutative case [29], and can probably be solved by invoking more sophisticated geometrical structures such as orbifolds [30].

This paper is structured as follows. After recalling the basic constructions of matrix models and noncommutative gauge theory, we discuss in Sec. III the symmetry breaking of $SU(n)$ to products of $SU(n_i)$ via extra dimensions. We consider both a simplified effective treatment involving only the low-energy degrees of freedom and a more sophisticated realization in terms of fuzzy spheres in extra dimensions. Section IV contains the main results of the paper, namely, the embedding of the standard model particles and fields in the basic matrices which are in the adjoint of $SU(N)$, focusing on $N = 7$. Electroweak symmetry breaking is discussed in Sec. V, as well as the structure of the Yukawa couplings. Here we only exhibit some qualitative aspects and discuss possible avenues for further studies. Parts of the present paper have been presented in the proceedings [31].

II. THE MATRIX MODEL

We start with a Yang-Mills matrix model which involves[1] $D = 4 + n$ matrices X^a and a set of fermions:

$$S_{\rm YM} = -(2\pi)^2 \frac{\Lambda_{\rm NC}^4}{g^2} {\rm Tr}([X^a, X^b][X^{a'}, X^{b'}]\eta_{aa'}\eta_{bb'} + \bar{\Psi}\Gamma_a[X^a, \Psi]), \tag{2.1}$$

where X^a are infinite-dimensional Hermitian matrices ${\rm Mat}(\infty, \mathbb{C})$, or operators in a Hilbert space $\mathcal{H}$. Γ_a generates the $SO(1, D-1)$ Clifford algebra, the metric $\eta_{aa'}$ is the flat Minkowski (or Euclidean), and Ψ is a corresponding (Grassmann-valued) spinor taking values in ${\rm Mat}(\infty, \mathbb{C})$. We introduced a scale parameter $\Lambda_{\rm NC}$ which will be identified with the scale of noncommutativity below, and g will be identified as a gauge coupling constant. This model is invariant under the symmetry

$$X^a \to UX^aU^{-1}, \qquad U \in U(\mathcal{H}). \tag{2.2}$$

The equations of motion of the bosonic part of the model are

$$[X^a, [X^b, X^{a'}]]\eta_{aa'} = 0. \tag{2.3}$$

We will discuss the fermions later. There are several solutions for this classical equation of motion, which we will often call *vacua* in the following. Apart from the trivial one ($X^a = 0$) and the case in which all of the X's commute, a relevant vacuum for our model is the "scalar Moyal-Weyl" vacuum:

$$[X_0^a, X_0^b] = i\theta^{ab} \tag{2.4}$$

with θ^{ab} constant. The functions of the X_0's in this case generate an algebra isomorphic (under appropriate regularity conditions) to the algebra of functions on a D-dimensional space multiplied with the Grönewold-Moyal product. That is, given two functions f and g, consider $f(x)$, $g(x)$ as ordinary functions on the plane, and $f(X)$, $g(X)$ operators; then

$$f(X)g(X) = (f \star g)(X) \tag{2.5}$$

with

$$(f \star g)(x) = {\rm e}^{-({\rm i}/2)\theta^{ab}\partial_{x^a}\partial_{y^b}} f(x)g(y)|_{x=y}. \tag{2.6}$$

We interpret this as the fact that the vacuum (2.4) describes a noncommutative space where the coordinates have a nontrivial constant commutator, the noncommutative space $\mathbb{R}^D_\theta$. The bosonic action has a gauge invariance defined by conjugation with unitary elements of the algebra; since we are considering functions of X, we consider these unitary elements as unitary matrix functions $U(X)$ to which corresponds a function of x which is, as usual, a phase. We call this association of functions of the matrices with functions on an ordinary space the *Moyal-Weyl limit*.

Another vacuum of interest is

$$\bar{X}^a = X_0^a \otimes \mathbb{1}_n. \tag{2.7}$$

In this case the Moyal-Weyl limit is given by matrix-valued functions on $\mathbb{R}^D_\theta$ and the gauge symmetry is given by

[1]The case $D = 10$ is of particular interest. In this case it is possible to impose a Majorana-Weyl condition on Ψ, and the model admits an extended supersymmetry [6]. On a four-dimensional Moyal-Weyl background as discussed below, the model then reduces to the $N = 4$ super-Yang-Mills on $\mathbb{R}^4_\theta$, which is expected to be well behaved upon quantization.

unitary elements of the algebra of $n \times n$ matrices of functions of the X_0. We say that this theory has a noncommutative $U(n)$ gauge symmetry because in the semiclassical limit it corresponds to a non-Abelian gauge theory.

A. Moyal-Weyl, gauge theory, and extra dimensions

Let us now consider the case in which not all dimensions have the same significance. Split the D matrices as

$$X^a = (X^\mu, \mathfrak{X}^i), \qquad \mu = 0, \ldots, 3, \qquad i = 1, \ldots, n \tag{2.8}$$

into four "spacetime" generators X^μ which will be interpreted as (quantized) coordinate functions, and n generators $\mathfrak{X}^i$ which are interpreted as extra dimensions. More specifically, we consider a background (i.e. a solution) of the matrix model where four spacetime generators X^μ generate the Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$,

$$\bar{X}^\mu = X_0^\mu \otimes \mathbb{1}_N, \qquad \bar{\mathfrak{X}}^i = 0 \tag{2.9}$$

which satisfies

$$[\bar{X}^\mu, \bar{X}^\nu] = \mathrm{i}\theta^{\mu\nu} \otimes \mathbb{1}_N, \qquad [\bar{X}^\mu, \bar{\mathfrak{X}}^i] = 0. \tag{2.10}$$

Here we assume $\theta^{\mu\nu} = \text{const}$ for simplicity. This background preserves the symmetry $SU(N)$ which commutes with X^μ.

Now consider small fluctuations around this solution,

$$X^\mu = \bar{X}^\mu + \mathcal{A}^\mu, \qquad \Phi^i = \Lambda^2_{\mathrm{NC}} \mathfrak{X}^i, \tag{2.11}$$

so that X^μ has dimension *length* and Φ^i has dimension length^{-1} and can be considered (also dimensionally) a field from the four-dimensional point of view. As shown in [10] and recalled in Sec. II B, the trace-$U(1)$ fluctuations give rise to the gravitational degrees of freedom which lead to an effective ("emergent") metric and gravity. For the sake of the present paper we will ignore these $U(1)$ degrees of freedom and concentrate on traceless fluctuations, assuming a flat Moyal-Weyl background with Minkowski signature. The remaining $SU(N)$-valued fluctuations

$$\mathcal{A}^\mu = -\theta^{\mu\nu} A^\alpha_\mu(x) \otimes \lambda_\alpha \tag{2.12}$$

then correspond to $SU(N)$-valued gauge fields, while the fluctuations in the internal degrees of freedom,

$$\Phi^i = \Phi^{i,\alpha}(x) \otimes \lambda_\alpha, \tag{2.13}$$

correspond to scalar fields in the adjoint. The matrix model action (2.1) therefore describes $SU(N)$ gauge theory on $\mathbb{R}^4_\theta$ coupled to n scalar fields. From now on we will drop the $\otimes$ sign whenever there is no risk of confusion.

Noncommutative gauge theory is obtained from the matrix model using the following basic observation:

$$[\bar{X}^\mu + \mathcal{A}^\mu, f] = i\theta^{\mu\nu}\left(\frac{\partial}{\partial \bar{x}^\nu} + i[A_\nu, .]\right) f \equiv i\theta^{\mu\nu} D_\nu f. \tag{2.14}$$

The matrix model action (2.1) can then be written as

$$\begin{aligned} S_{\mathrm{YM}} = \frac{1}{g^2} \int d^4\bar{x}\, \mathrm{tr}(& G^{\mu\mu'} G^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} \\ &+ 2G^{\mu\nu} D_\mu \Phi^i D_\nu \Phi^i \delta_{ij} - [\Phi^i, \Phi^j][\Phi^{i'}, \Phi^{j'}]\delta_{ij}\delta_{i'j'} \\ &+ \bar{\Psi} \not{D} \Psi + \bar{\Psi}\Gamma_i[\Phi^i, \Psi]). \end{aligned} \tag{2.15}$$

This is the action of an $SU(N)$ gauge theory[2] on $\mathbb{R}^4_\theta$, with the effective metric given by

$$G^{\mu\nu} = \rho\theta^{\mu\mu'}\theta^{\nu\nu'}\eta_{\mu'\nu'}, \qquad \rho = (\det\theta^{\mu\nu})^{-1/2} =: \Lambda^4_{\mathrm{NC}}, \tag{2.16}$$

which satisfies $\sqrt{|G|} = 1$. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength on $\mathbb{R}^4_\theta$,

$$D_\mu \equiv \partial_\mu + i[A_\mu, .] \tag{2.17}$$

is the covariant derivative for fields in the adjoint, and tr () denotes the trace over the $SU(N)$ components. The effective Dirac operator is given by

$$\not{D}\Psi = \Gamma_\mu[X^\mu, \Psi] \sim i\gamma^\mu D_\mu \Psi \tag{2.18}$$

where [32]

$$\gamma^\mu = \sqrt{\rho}\Gamma_\nu\theta^{\nu\mu}, \qquad \{\gamma^\mu, \gamma^\nu\} = 2G^{\mu\nu}. \tag{2.19}$$

The fermions have been rescaled appropriately, and a constant shift as well as total derivatives in the action are dropped. Note that g is now identified as the coupling constant for the non-Abelian gauge fields on $\mathbb{R}^4_\theta$.

B. Fluctuations of the vacuum, emergent gravity, and gauge theory

As explained above, fluctuations of X^a can be parametrized in terms of gauge fields A_μ and scalar fields ϕ^i on $\mathbb{R}^4_\theta$. At first sight, this might suggest that the action (2.15) describes $U(n)$ gauge theory on $\mathbb{R}^4_\theta$. However, this interpretation is not quite correct: it turns out [10] that the trace-$U(1)$ fluctuations of both $\mathcal{A}^\mu$ and Φ^i describe gravitational degrees of freedom which modify the geometry of $\mathbb{R}^4_\theta$, defining an effective (emergent) geometry given by

$$\begin{aligned} G^{\mu\nu}(x) &= e^{-\sigma}\theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x) g_{\mu'\nu'}(x), \\ i\theta^{\mu\nu}(x) &= i\{x^\mu, x^\nu\} \sim [X^\mu, X^\nu], \\ g_{\mu\nu} &= \eta_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi_i, \\ e^\sigma &= \sqrt{\det\theta^{\mu\nu}(x)\det g_{\mu\nu}(x)}. \end{aligned} \tag{2.20}$$

The $SU(n)$-valued components of $\mathcal{A}^\mu$ and Φ^i describe non-Abelian gauge fields (resp. scalar fields) which in the semiclassical limit (denoted by $\sim$) couple to the effective metric $G^{\mu\nu}(x)$. Note that the "would-be $U(1)$ gauge fields"

[2]The $U(1)$ components are gravitational degrees of freedom and will be ignored here.

$A_\mu \mathbb{1}$ are absorbed in the Poisson structure $\theta^{\mu\nu}(x) = \bar{\theta}^{\mu\nu}(\bar{x}) - \bar{\theta}^{\mu\mu'}\bar{\theta}^{\nu\nu'}F_{\mu'\nu'}$, and similarly the $U(1)$ degrees of freedom of ϕ^i are absorbed in $g_{\mu\nu}(x)$. In the following we will ignore the gravitational degrees of freedom and only keep the $SU(n)$-valued components of $\mathcal{A}^\mu$ and ϕ^i, focusing on the flat Moyal-Weyl space $\mathbb{R}^4_\theta$.

III. SPONTANEOUS SYMMETRY BREAKING IN EXTRA DIMENSIONS

In this section we will present two mechanisms to break the gauge symmetry from $SU(n)$ down to a smaller group—the first with one constant extra dimension and the second with inner fuzzy spheres. The two mechanisms are not really different because the former can be seen as an effective model of the latter. Only the bosonic part will be discussed in this section as well. Both models are somewhat analogous to a grand unified theory (GUT)-like model, where the breaking is realized through a Higgs in the adjoint.

A. Single-coordinate effective breaking

The mechanism for how to obtain nontrivial low-energy gauge groups and particle spectra can be understood in a simple way as follows. Consider a model with $\bar{X}^\mu$ as in (2.10), but this time with a single extra dimension which we call $\mathfrak{X}^\Phi$, which in a suitable vacuum takes the form

$$\langle \mathfrak{X}^\Phi \rangle = \begin{pmatrix} \alpha_1 \mathbb{1}_2 & & \\ & \alpha_2 \mathbb{1}_2 & \\ & & \alpha_3 \mathbb{1}_3 \end{pmatrix}. \tag{3.1}$$

Here α's are constant[3] quantities with the dimensions of length, which are assumed to be distinct. These new coordinates are still solutions of the equations of motion because $[X^\mu, \langle \mathfrak{X}^\Phi \rangle] = 0$ i.e. $\theta^{\mu\Phi} = 0$, which in turn implies that $G^{\mu\Phi} = 0$ regardless of the value of $\eta^{\Phi\Phi}$. Therefore the extra coordinate is not geometric and does not correspond to propagating degrees of freedom from the four-dimensional point of view. The new coordinate is not invariant for the transformation

$$\langle \mathfrak{X}^\Phi \rangle \to U \langle \mathfrak{X}^\Phi \rangle U^\dagger \neq \langle \mathfrak{X}^\Phi \rangle \tag{3.2}$$

for a generic $U \in SU(7)$. The traceless generators commuting with $\langle \mathfrak{X}^\Phi \rangle$ generate the surviving gauge group $SU(2) \times SU(2) \times SU(3) \times U(1) \times U(1)$.

In the bosonic action as in Sec. II A the spacetime $(\mu\nu)$ part of the action remains unchanged, while for the $\mu\phi$ components we obtain, in the Moyal-Weyl background,

$$\begin{aligned} &[\bar{X}^\mu + \mathcal{A}^\mu, \mathfrak{X}^\Phi] = i\theta^{\mu\nu} D_\nu \mathfrak{X}^\phi = i\theta^{\mu\nu}(\partial_\nu + iA_\nu)\mathfrak{X}^\phi, \\ &-(2\pi)^2 \operatorname{Tr}[X^\mu, \mathfrak{X}^\phi][X^\nu, \mathfrak{X}^\phi]\eta_{\mu\nu} = \int d^4x G^{\mu\nu}(\partial_\mu \mathfrak{X}^\Phi \partial_\nu \mathfrak{X}^\Phi - [A_\mu, \mathfrak{X}^\Phi][A_\nu, \mathfrak{X}^\Phi]). \end{aligned} \tag{3.3}$$

Note that the mixed terms $\int \partial^\mu \mathfrak{X}^\Phi [A_\mu, \mathfrak{X}^\Phi] = -\frac{1}{2}\int \mathfrak{X}^\phi[\partial^\mu A_\mu, \mathfrak{X}^\Phi] = 0$ vanish, assuming the Lorentz gauge $\partial^\mu A_\mu = 0$.

Now consider the vacuum (3.1). Since X^μ and $\langle \mathfrak{X}^\Phi \rangle$ commute, this means $\langle \mathfrak{X}^\Phi \rangle = \text{const}$ and the first term in the integral above vanishes. We can therefore separate the fluctuations of this extra dimension which are a field, the (high-energy) Higgs field. In the action the first term is nothing but the derivative of it. The second term instead is

$$[A^\mu, \langle \mathfrak{X}^\Phi \rangle] = \begin{pmatrix} 0 & (\alpha_2 - \alpha_1)A^\mu_{12} & (\alpha_3 - \alpha_1)A^\mu_{13} \\ (\alpha_1 - \alpha_2)A^\mu_{21} & 0 & (\alpha_3 - \alpha_2)A^\mu_{23} \\ (\alpha_1 - \alpha_3)A^\mu_{31} & (\alpha_2 - \alpha_3)A^\mu_{32} & 0 \end{pmatrix}, \tag{3.4}$$

where we consider the block form of A^μ,

$$A^\mu = \begin{pmatrix} A^\mu_{11} & A^\mu_{12} & A^\mu_{13} \\ A^\mu_{21} & A^\mu_{22} & A^\mu_{23} \\ A^\mu_{31} & A^\mu_{32} & A^\mu_{33} \end{pmatrix}. \tag{3.5}$$

Therefore (3.3) leads to the mass terms for the off-diagonal gauge fields,

$$\begin{aligned} &-(2\pi)^2 \operatorname{Tr}[X^\mu, \langle \mathfrak{X}^\Phi \rangle][X^\nu, \langle \mathfrak{X}^\Phi \rangle]\eta_{\mu\nu} \\ &= \int d^4x G^{\mu\nu}\Big(\sum (\alpha_i - \alpha_j)^2 A_{\mu,ij} A_{\nu,ji}\Big), \end{aligned} \tag{3.6}$$

which is nothing but the usual Higgs effect. If we now assume that the differences $\alpha_i - \alpha_j$ are large, say of the grand-unification scale, it is easy to see that all nondiagonal blocks of A^μ acquire large masses $m^2_{ij} \sim (\alpha_i - \alpha_j)^2$, thus effectively decoupling.

In order to approach the standard model, we will assume the following version of the above mechanism:

$$\langle \mathfrak{X}^\Phi \rangle = \begin{pmatrix} \alpha_1 \mathbb{1}_2 & & \\ & \alpha_2 \sigma_3 & \\ & & \alpha_3 \mathbb{1}_3 \end{pmatrix} \tag{3.7}$$

with $\alpha_1 \neq \alpha_2 \neq \alpha_3$. Then the surviving traceless gauge group is given by $SU(3) \times SU(2) \times U(1) \times U(1) \times U(1)$, and the off-diagonal gauge fields $A_{\mu,ij}$ for i, j labeling the

[3]Actually it is sufficient that they commute with the $\bar{X}$'s.

four blocks acquire a mass as in (3.6). Again, this may simply be a crude picture of some more sophisticated mechanism involving fuzzy spheres (branes) in extra dimensions, as discussed below. This then comes very close to some of the proposals for how to recover the standard model using branes and strings stretching between them; cf. [26,27]. Thus in a sense we show how such a mechanism can be realized in the matrix model framework.

B. Fuzzy sphere breaking

According to the splitting of the matrices into a noncommutative spacetime $\mathbb{R}^4_\theta$ and "extra" generators $\mathfrak{X}^i$, it is quite natural to add extra terms to the potential and add a potential term involving quadratic and cubic terms in the fluctuations Φ^i defined in (2.11):

$$V_{\mathrm{soft}}(\Phi^i) = 2\,\mathrm{Tr}(c_2\Phi^i\Phi^j\delta_{ij} + ic_3\varepsilon_{ijk}\Phi^i\Phi^j\Phi^k) = 2\Lambda^4_{\mathrm{NC}}\int d^4x\,\mathrm{tr}(c_2\Phi^i\Phi^j\delta_{ij} + ic_3\varepsilon_{ijk}\Phi^i\Phi^j\Phi^k). \tag{3.8}$$

From the point of view of field theory on $\mathbb{R}^4_\theta$, these amount to soft (resp. relevant) terms which may (partially) break the global $SO(n)$ symmetry, as well as supersymmetry if applicable. In particular, they may be generated upon quantization. The full one-loop effective potential can have more complicated effective potentials, but for the present work we will limit our considerations to these terms.

The bosonic part of the action (2.1) now becomes the following gauge theory action on $\mathbb{R}^4_\theta$,

$$\begin{aligned} S_{\mathrm{YM}} = &\int d^4x \frac{1}{g^2}\,\mathrm{tr}F_{\mu\nu}F_{\mu'\nu'}G^{\mu\mu'}G^{\nu\nu'} \\ &+ 2\Lambda^4_{\mathrm{NC}}\int d^4x\,\mathrm{tr}\Big(\frac{1}{g^2}G^{\mu\nu}D_\mu\Phi^i D_\nu\Phi^i \\ &- \frac{\Lambda^4_{\mathrm{NC}}}{2g^2}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}]\delta_{ii'}\delta_{jj'} + c_2\Phi^i\Phi^j\delta_{ij} \\ &+ ic_3\varepsilon_{ijk}\Phi^i\Phi^j\Phi^k\Big) \end{aligned} \tag{3.9}$$

with $G^{\nu\nu'}$ as in (2.16). We omit here surface terms (such as $\int d^4xF_{\mu\nu}\theta_{\mu\nu}$), as well as all trace-$U(1)$ degrees of freedom which are part of the gravitational sector.

Scalar fields and spontaneous symmetry breaking.—Assuming that the extra coordinates commute with the spacetime ones, $[\mathfrak{X}^i, X^\mu] = [\Phi^i, X^\mu] = 0$, or that, in other words, in the Moyal-Weyl vacuum they commute with the x's, the above action leads to the following equation of motion for the scalar fields:

$$\frac{2\Lambda^4_{\mathrm{NC}}}{g^2}[\Phi^j,[\Phi^i,\Phi^{j'}]]\delta_{jj'} + 2c_2\Phi^i + \frac{3}{2}\mathrm{i}c_3\varepsilon_{ijk}[\Phi^j,\Phi^k] = 0. \tag{3.10}$$

This equation has the solution

$$\langle\Phi^i\rangle = aJ^i_N, \tag{3.11}$$

where J^i_N are generators of the $N\times N$ representation of $SU(2)$,

$$[J^i_N, J^j_N] = \mathrm{i}\varepsilon_{ijk}J^k_N, \qquad J^i_N J^i_N = \frac{N^2-1}{4}. \tag{3.12}$$

This solution is interpreted as a fuzzy sphere [33] with radius

$$\alpha = \frac{a}{2}\sqrt{N_i^2-1} \tag{3.13}$$

as in [24]. The equations of motion then reduce to

$$\frac{2\Lambda^4_{\mathrm{NC}}}{g^2}a^2 + 2c_2 - 3c_3a = 0, \tag{3.14}$$

which generically has two solutions. It is important to note that one of them really is a global minimum of the potential for Φ^i (3.9):

$$\begin{aligned} V[\Phi^i] &= \mathrm{tr}\Big(-\frac{\Lambda^4_{\mathrm{NC}}}{2g^2}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}]\delta_{ii'}\delta_{jj'} + c_2\Phi^i\Phi^j\delta_{ij} \\ &\quad + ic_3\varepsilon_{ijk}\Phi^i\Phi^j\Phi^k\Big) \\ &= -N\Big(\frac{\alpha^4\Lambda^4_{\mathrm{NC}}}{2g^2} + c_2\alpha^2 - c_3\alpha^3\Big). \end{aligned} \tag{3.15}$$

This potential has either one or two degenerate minima as a function of α, and (3.11) is the physical vacuum of the model. The $SU(N)$ gauge symmetry is then broken completely by the presence of a fuzzy sphere in the internal space.

A physically more relevant vacuum could be the one corresponding to a "stack" of fuzzy spheres as proposed in [24], in particular,[4]

$$\langle\Phi_i\rangle = \begin{pmatrix} a_1J^i_{N_1}\otimes\mathbb{1}_2 & 0 & 0 \\ 0 & a_2J^i_{N_2}\otimes\sigma_3 & 0 \\ 0 & 0 & a_3J^i_{N_3}\otimes\mathbb{1}_3 \end{pmatrix}, \tag{3.16}$$

which gives an explicit realization of (3.7). Each of the blocks corresponds to a fuzzy sphere $S^2_{N_i}$ with radius α_i. More precisely, the last block has a threefold multiplicity, which can be interpreted as a stack of three coinciding fuzzy spheres with radius α_3. If these fuzzy spheres are large, then the fluctuations around this vacuum effectively "see" only the radius of the fuzzy spheres. Thus at low energies we are very close to the case of the previous subsection. The symmetry in the vacuum (3.16) is broken down to $SU(3)\times SU(2)\times U(1)\times U(1)\times U(1)$, which is very close to what we want. Note that by setting e.g.

[4]More sophisticated versions are of course conceivable [34].

$\alpha_1 = \alpha_2$ and $N_1 = N_2$ the symmetry is enhanced, and more sophisticated symmetry breaking processes with several steps (resp. scales) are conceivable.

An important bonus compared with standard Higgs scenarios is that the above Higgs fields have a natural geometrical interpretation in terms of compact fuzzy spaces. The double commutator has an interpretation in terms of a higher-dimensional curvature, and the additional potential (3.8) is cubic. This should lead to milder renormalization properties and less fine-tuning compared with the standard ϕ^4 case, as observed in [24].

Massive vector bosons and Higgs effect.—The masses of the four-dimensional non-Abelian gauge bosons A_μ in the presence of such a fuzzy sphere vacuum were studied in [24]. The result is essentially the same as in Sec. III A; i.e. the off-diagonal components $A_{\mu,ij}$ for $i \neq j$ acquire a mass due to the term $\mathrm{tr} G^{\mu\nu} D_\mu \Phi^i D_\nu \Phi^i$, provided $(\alpha_i, N_i) \neq (\alpha_j, N_j)$. For the diagonal blocks, only the $l = 0$ mode of the decomposition of $\mathrm{Mat}(N_i) = \oplus_{l=0}^{2N_i-1} Y_{l,m}$ into fuzzy spherical harmonics remains massless, while all higher Kaluza-Klein modes acquire a mass $m^2 \sim \alpha^2 \Lambda_{\rm NC}^4 l(l+1)$. This is nothing but a geometrical version of the usual Higgs effect. Therefore the low-energy sector of such a fuzzy sphere vacuum is essentially captured by the effective single-variable description in Sec. III A. However, the fuzzy sphere scenario provides a natural origin of a Higgs potential with a nontrivial minimum, which is not seen in the single-variable description.

IV. PARTICLE ASSIGNMENTS, CHARGES, AND SYMMETRIES

In this section we show how the fermions in the standard model can be naturally accommodated in the framework of matrix models. This is nontrivial because the fermions in the matrix model are necessarily in the adjoint of some basic $SU(N)$ gauge group. In a later section we will also show how electroweak symmetry can be broken through a somewhat modified Higgs sector, and the Yukawa couplings are obtained.

We start with the matrix model of Sec. III in $D = 4 + n$ dimensions. Thus the fermions are realized as D-dimensional spinors Ψ in the adjoint of $SU(N)$, and there are n scalar fields Φ^i in the adjoint of $SU(N)$ as well as the four-dimensional gauge fields[5] A_μ.

The basic idea is to assume that the fundamental $SU(N)$ gauge group is spontaneously broken in several steps down to the low-energy gauge group $SU(3) \times U(1)$ of the standard model. It is natural to assume that the various (intermediate and low-energy) gauge groups are realized as block-diagonal subgroups of $SU(N)$. We will show in a later section how this can be realized by spontaneous compactification on fuzzy internal spaces. In this section, we simply assume that in a first step (at very high energy) the block-matrix decomposition in Sec. III has occurred and that, therefore, the symmetry is broken to $SU(3) \times SU(2) \times U(1) \times U(1) \times U(1)$ as in (3.7). We assign the fermions accordingly by the matrix

$$\Psi = \begin{pmatrix} \mathcal{L}_{4\times 4} & \mathcal{Q} \\ \mathcal{Q}' & 0_{3\times 3} \end{pmatrix}. \tag{4.1}$$

Here the 4×4 block $\mathcal{L}$ will contain the leptons which are color-blind, and Q (resp. Q') will contain the quarks [which we assume to be in $(\bar{3})$ here for convenience]. We drop all fermions in the adjoint of an unbroken gauge group; i.e. we assume that they are very massive. This is plausible if this block arises from Kaluza-Klein modes on some fuzzy sphere as discussed above; in principle, such fermions would correspond to gauginos. We denote the off-diagonal blocks according to (3.7) as

$$\mathcal{L} = \begin{pmatrix} 0_{2\times 2} & & L_L \\ L'_L & 0 & e_R \\ & e'_R & 0 \end{pmatrix}, \qquad L_L = (\tilde{l}_L \quad l_L), \tag{4.2}$$
$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad \tilde{l}_L = \begin{pmatrix} \tilde{e}_L \\ \tilde{\nu}_L \end{pmatrix}.$$

Here l_L will be the standard (left-handed) leptons, and e_R the right-handed electron. Fields with a prime may either be related to the unprimed ones through some conjugation, or be independent new fields, or they may vanish for some reason; this will be discussed below. In particular, $\tilde{l}$ will correspond to additional leptons, which may be present at some energy, or which may be null; at present, the model allows them and we will keep the term. The quarks split accordingly as

$$\mathcal{Q} = \begin{pmatrix} Q_L \\ Q_R \end{pmatrix}, \qquad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad Q_R = \begin{pmatrix} d_R \\ u_R \end{pmatrix}, \tag{4.3}$$

which will again correspond to the standard quarks. This gives the following general fermionic matrix,

$$\Psi = \begin{pmatrix} 0_{2\times 2} & & L_L & Q_L \\ L'_L & 0 & e_R & Q_R \\ & e'_R & 0 & \\ Q'_L & & Q'_R & 0_{3\times 3} \end{pmatrix}. \tag{4.4}$$

The correct hypercharge, electric charge, and baryon number are then reproduced by the following traceless generators:

$$Y = \begin{pmatrix} 0_{2\times 2} & & \\ & -\sigma_3 & \\ & & -\frac{1}{3}\mathbb{1}_{3\times 3} \end{pmatrix} + \frac{1}{7}\mathbb{1}, \tag{4.5}$$

[5] Which in turn are obtained as fluctuations of the covariant coordinates in the matrix model.

$$Q = T_3 + \frac{Y}{2} = \frac{1}{2}\begin{pmatrix} \sigma_3 & & \\ & -\sigma_3 & \\ & & -\frac{1}{3}\mathbb{1}_{3\times 3} \end{pmatrix} + \frac{1}{14}\mathbb{1}, \tag{4.6}$$

$$B = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -\frac{1}{3}\mathbb{1}_{3\times 3} \end{pmatrix} + \frac{1}{7}\mathbb{1}, \tag{4.7}$$

which act in the adjoint. For easy reference we display the charges (Q, Y, B) of these block matrices for the above generators:

$$(Q, Y, B)\Psi = \begin{pmatrix} 0_{2\times 2} & \begin{pmatrix} (1,1,0) \\ (0,1,0) \end{pmatrix} & \begin{pmatrix} (0,-1,0) \\ (-1,-1,0) \end{pmatrix} & \begin{pmatrix} (\frac{2}{3},\frac{1}{3},\frac{1}{3}) \\ (-\frac{1}{3},\frac{1}{3},\frac{1}{3}) \end{pmatrix} \\ * & 0 & (-1,-2,0) & \begin{pmatrix} (-\frac{1}{3},-\frac{2}{3},\frac{1}{3}) \\ (\frac{2}{3},\frac{4}{3},\frac{1}{3}) \end{pmatrix} \\ & * & 0 & \\ * & & * & 0_{3\times 3} \end{pmatrix}, \tag{4.8}$$

as it should be, omitting the obvious lower-diagonal entries. In particular, the charges of the exotic leptons $\tilde{l}$ are those of Higgsinos.

The main result is that all particles of the standard model with the correct quantum numbers fit naturally into this framework, based on matrices in the adjoint of a fundamental $SU(N)$ gauge symmetry. This is very important because the only representations which can be realized in noncommutative gauge theory[6] are fundamental, antifundamental, and adjoint representations of $U(N)$ gauge groups. Matrix models provide a natural framework to study the quantization of NC gauge theory in a nonperturbative way. In order to be free of UV/IR mixing, it appears that only the IKKT matrix model, or close relatives, is consistent at the quantum level, which contains only matrices in the adjoint. This is a strong and predictive constraint, which restricts the freedom in model building, with the added bonus of an intrinsic gravity sector [10]. Perhaps the main result of this paper is that realistic models for particle physics appear feasible within this framework.

There is some freedom in relation to the primed fermions appearing in the lower block of the matrix (4.1), and we need to understand the relation among the upper- and lower-diagonal blocks. It is likely that a supersymmetric version of this model can be built, and in that framework some of the zeros of the matrix can be filled by supersymmetric partners, and the additional leptons $\tilde{l}$ could be identified with Higgsinos.

Let us discuss some possibilities for how fermions may arise in the off-diagonal blocks. We first need to understand the relation between the upper-diagonal and the lower-diagonal components. We note that the upper and lower triangles in the matrix are exchanged under Hermitian conjugation, which is part of charge conjugation. Therefore, as we will see in detail in the following, the role of primed and unprimed elements is exchanged in $\bar{\Psi}$. There are three obvious choices for the primed fermions.

(1) If $\Psi = \Psi^C$ is a Majorana-Weyl fermion in the fundamental matrix model, then the lower-diagonal components are related with the upper-diagonal ones directly by charge conjugation. This choice may be natural in the presence of ten dimensions, in which case it is possible to have fermions which are both Majorana and Weyl.

(2) One can set the primed fermions equal to zero, so that Ψ is an upper-triangular matrix. The lower part of the matrix will appear in $\bar{\psi}$, which will be lower triangular. This choice has several advantages, as we will see in the following, but it seems "*ad hoc*," without an explanation at the present. This may be related to the presence of a magnetic flux [29].

(3) If both upper- and lower-diagonal components are nonvanishing and not related via conjugation, the model is nonchiral, corresponding to a mirror model. Then one has to explain why each single sector has an independent cancellation of anomalies, which would be canceled by the mirrors anyway; we refrain from speculating on a possible relevance to dark matter, given the present state of the art.

In the latter two cases the lower-triangular one can be seen as an instance of the presence of fermion doubling, which is a known phenomenon in noncommutative geometry [23,35]. At this stage it is really a matter of taste if one prefers to eliminate the lower triangle of the matrix, setting it to zero, or to keep it as a mirror world. With the former choice one is setting to zero a sector which is, in principle, present, but which can give unwanted couplings.

The correct chirality assignment is put in by hand here. There is also a slot which could naturally accommodate additional leptons $\tilde{l}_L$. Notice also that the scheme is natu-

[6]At a fundamental level i.e. without resorting to an effective Seiberg-Witten expansion.

rally suited for supersymmetry, since all particles and fields arise from matrices in the adjoint of $SU(N)$.

The full $U(N)$ model is certainly free of anomalies. After symmetry breaking, the $U(1)_B$ gauge symmetry may turn out to be anomalous, as it often does in string theory [28], and we assume that the corresponding gauge boson becomes massive through some version of the Green-Schwarz mechanism. Furthermore, the additional leptons $\tilde{l}$ lead to an anomaly unless their lower-diagonal partners $\tilde{l}$ are also present; this strongly suggests that $\tilde{l}$ should be set to zero.

These ambiguities indicate that an additional mechanism is required to single out the correct physical result. In particular, it is very interesting that the above scheme is very similar to constructions in string theory [26–28], where the standard model is realized in terms of four stacks of branes with exactly the above gauge groups, and particles realized as strings stretched between these branes. The latter correspond precisely to the off-diagonal blocks, and there seems to be a correspondence between the possibilities indicated above and the different versions of this construction in string theory. This suggests that additional structures such as intersecting branes should be considered in the matrix model framework. This is probably possible, and e.g. branes with fluxes were recently realized in [29]. In particular, fuzzy orbifolds [30] appear to realize the above structures in a chiral model. We will not investigate these in the present paper.

Here we do not claim to have the final answer; rather, we want to point out possible directions which should be pursued elsewhere. We take this similarity with string theory as additional encouragement. However, we want to stress that our approach offers advantages over string theory, simply because the matrix model is a very specific and predictive framework. For example, the branes realized as fuzzy spheres are naturally obtained as stable minima of the potential (3.15). Furthermore, this result shows clearly that there is no obstacle to describe (an extension of) the standard model within the framework of noncommutative gauge theory. The mechanism is applicable to models which are expected to be well defined at the quantum level, in particular, the IKKT model [6].

V. ELECTROWEAK BREAKING

Now we show how electroweak symmetry breaking might be realized in this framework. To explain the idea we will first present a simplified version where the Higgs is realized in terms of a single extra coordinate (resp. scalar) field. In Sec. V B we then discuss a more elaborate version involving extra coordinates (resp. Higgs fields), which form an "electroweak" fuzzy sphere. This is again not intended as a realistic model, but it shows that a suitable Higgs potential can naturally arise within the present framework.

A. Electroweak Higgs and Yukawa coupling

The Higgs field connects the left and right sectors of leptons, and is otherwise color-blind; it is therefore natural to consider, along the lines of Sec. III A, another extra coordinate which will necessarily have to be off diagonal. The following matrix has the correct characteristics:

$$\mathcal{X}^{\phi} = \Lambda_{\mathrm{NC}}^{-2}\begin{pmatrix} 0_{2\times 2} & \phi & 0_{2\times 3} \\ \phi^{\dagger} & 0_{2\times 2} & 0_{2\times 2} \\ 0_{3\times 2} & 0_{3\times 2} & 0_{3\times 3} \end{pmatrix}, \tag{5.9}$$

where we use again the notations of Sec. III A and consider the extra variable $\mathcal{X}$, its vacuum expectation value, and the fluctuations which are a physical field. The Higgs ϕ is a 2×2 matrix which is actually composed of two doublets (which form the Higgs content of the minimal supersymmetric standard model), i.e. two scalar doublets with opposite hypercharges:

$$\phi = \begin{pmatrix} \tilde{\varphi} & \varphi \end{pmatrix}. \tag{5.10}$$

The vacuum expectation value of ϕ is an off-diagonal matrix:

$$\langle \phi \rangle = \begin{pmatrix} 0 & v \\ \tilde{v} & 0 \end{pmatrix}. \tag{5.11}$$

All other components (possibly even some of the components of ϕ) are assumed to be very massive, e.g. due to the commutator with the high-energy breaking discussed before.

Now consider the fermionic part of the action (2.1), which can be written on $\mathbb{R}^4_\theta$ in the form (2.15). The part involving X^μ gives the usual Dirac action as in (2.15), and the part involving $\mathcal{X}^\phi$ yields the Yukawa couplings

$$S_Y = \mathrm{Tr}\bar{\Psi}\gamma_\phi[\mathcal{X}^\phi, \Psi] \tag{5.12}$$

giving mass to the fermions. Here γ_ϕ is an extra-dimensional gamma matrix corresponding to $\mathcal{X}^\phi$. We write

$$\bar{\Psi} = \Psi^\dagger\gamma_0 = \begin{pmatrix} 0_{2\times 2} & \bar{L}'_L & \bar{Q}'_L \\ \bar{L}_L & \begin{matrix} 0 & \bar{e}'_R \\ \bar{e}_R & 0 \end{matrix} & \bar{Q}'_R \\ \bar{Q}_L & \bar{Q}_R & 0_{3\times 3} \end{pmatrix}. \tag{5.13}$$

Then the full Yukawa term without any omissions or further assumptions is

$$\begin{aligned} S_Y = \mathrm{Tr}\Big(&-\bar{L}'\gamma_\phi\begin{pmatrix} 0 & e_R \\ e'_R & 0 \end{pmatrix}\phi^\dagger - \bar{Q}'_L\gamma_\phi Q'_R\phi^\dagger \\ &+ \bar{L}_L\gamma_\phi\phi\begin{pmatrix} 0 & e_R \\ e'_R & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{e}'_R \\ \bar{e}_R & 0 \end{pmatrix}\gamma_\phi(\phi^\dagger L - L'\phi) \\ &- \bar{Q}'_R\gamma_\phi Q'_L\phi + \bar{Q}_L\gamma_\phi\phi Q_R + \bar{Q}_R\gamma_\phi\phi^\dagger Q_L\Big). \end{aligned} \tag{5.14}$$

We now impose $\gamma_\phi = \gamma_5$, which is natural since in this way the five-dimensional Clifford algebra is closed, and

$$\gamma_5 L_L = +L_L, \qquad \gamma_5 Q_L = +Q_L,$$
$$\gamma_5 Q_R = -Q_R, \qquad \gamma_5 e_R = -e_R, \tag{5.15}$$

with this assumption, and we see that the couplings turn out to be the correct ones. Only opposite chiralities couple in such a Yukawa term.

The construction is quite solid and works in all three cases for the primed and unprimed fermions. In the case for which the primed fermions vanish, only a few terms will survive. In the case of Majorana fermions the couplings are the ones needed to give Dirac masses to Majorana fermions. In the case of mirror fermions there is no coupling among mirror and ordinary fermions, so that the mirror world effectively decouples.

The extra fermion doublet $\tilde{l}$ does not couple with the remaining leptons with the option of setting the primed fermions to zero. If the primed sector is the conjugate sector of Majorana fermions, there is a coupling of e_R and $\tilde{e}_L$ which may cause problems. Note that in this model the masses and the Yukawa couplings of leptons and quarks turn out to be the same and there is no way to differentiate them. A breaking with fuzzy spheres discussed below gives more structure to the extra dimensions and may create differences.

Assuming

$$\langle\phi\rangle = \begin{pmatrix} 0 & v \\ \tilde{v} & 0 \end{pmatrix} \tag{5.16}$$

with v and $\tilde{v}$ real, we get that the quark contribution to the action is

$$\begin{aligned}\bar{\mathcal{Q}}\gamma_5\begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix}\mathcal{Q} - \bar{\mathcal{Q}}'\gamma_5\mathcal{Q}'\begin{pmatrix} 0 & \phi^\dagger \\ \phi & 0 \end{pmatrix} \\ = -v(\bar{d}_R d_L - \bar{d}_L d_R - \bar{d}'_R d'_L + \bar{d}'_L d'_R) \\ - \tilde{v}(\bar{u}_R u_L - \bar{u}_L u_R - \bar{u}'_R u'_L + \bar{u}'_L u'_R).\end{aligned} \tag{5.17}$$

The lepton part of the action is instead

$$\begin{aligned}\bar{\mathcal{L}}\gamma_\phi[\mathcal{L},\phi] = -\tilde{v}(-\bar{e}_L e_R + \bar{e}_R e_L + \bar{e}'_L e_R - \bar{e}_R e'_L) \\ - v(-\bar{e}_R\tilde{e}'_L + \bar{e}'_R\tilde{e}_L - \bar{\tilde{e}}_L e'_R + \bar{\tilde{e}}'_L e_R).\end{aligned} \tag{5.18}$$

Note that with the choice (5.16) the leptons of the $\tilde{l}$ doublet do not have mass terms, but have spurious coupling to the ordinary leptons. It is possible to set them to zero in this scheme, but then we get that the mass of the electron is the same as the one of the up quark. Note also that if we relax the reality requirement on the v's, then the coefficients of $\bar{e}_L e_R$ and $\bar{e}_R e_L$ are complex conjugates of each other, and the same will hold for quarks. There is no problem in setting the primed fermions to zero; the Hermitian conjugates appear naturally because of $\bar{\Psi}$, and if we set $\tilde{l} = 0$ then there also is no problem for Majorana spinors. Mirror fermions again have no problems, except that it is still not clear how to give them large mass. It is still too early for a complete analysis of the various choices for the couplings since we are not yet at the stage to be building a completely realistic model. For example, there are no generations, nor different couplings, for the different gauge groups, and this points to the necessity of the refinement of the model.

B. Electroweak breaking by fuzzy spheres

Consider fuzzy sphere breaking at high energies (at the GUT scale, say) described in Sec. III B and, in particular, the stack of fuzzy sphere breaking described in (3.16). The residual gauge symmetry in this case is $SU(3) \times SU(2) \times U(1)_Q \times U(1)_Y \times U(1)_B$. We will later discuss the splitting into $SU(4) \times SU(3) \times U(1)$ corresponding to the 4×4 lepton block plus the 3×3 color block.

As in Sec. III B, we assume that there are additional quadratic and cubic terms as in (3.8) in the effective potential

$$V_H(\mathcal{X}_i^\phi) = \mathrm{Tr}\Big(-\frac{1}{g^2}[\mathcal{X}_i^\phi, \mathcal{X}_j^\phi]^2 + c_2\mathcal{X}_i^\phi\mathcal{X}_j^\phi\delta^{ij} + ic_3\varepsilon^{ijk}\mathcal{X}_i^\phi\mathcal{X}_j^\phi\mathcal{X}_k^\phi\Big) \tag{5.19}$$

at the electroweak scale. A possible minimum of this potential is given by the following fuzzy sphere,[7] S^2_{EW-I}:

$$\begin{aligned}\langle\mathcal{X}_1^\phi\rangle &= \alpha_H\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} = \mathbb{1}\otimes\sigma_3, \\ \langle\mathcal{X}_2^\phi\rangle &= \alpha_H\begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = \sigma_1\otimes\sigma_2, \\ \langle\mathcal{X}_3^\phi\rangle &= \alpha_H\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} = \sigma_1\otimes\sigma_1.\end{aligned} \tag{5.20}$$

This breaks the symmetry $SU(3) \times SU(2) \times U(1)_Q \times U(1)_Y \times U(1)_B$ of (3.16) down to $SU(3) \times U(1)_Q \times U(1)_B$, and we have indeed achieved the desired electroweak symmetry breaking. Observe that $\langle\mathcal{X}_2^\phi\rangle$ and $\langle\mathcal{X}_3^\phi\rangle$ are very similar to the two Higgs doublets H, $\tilde{H}$ in the minimal supersymmetric standard model, with an additional third Higgs $\langle\mathcal{X}_1^\phi\rangle$ in the diagonal blocks. Since the off-diagonal blocks are assumed to have definite chirality as in the standard model, this diagonal Higgs does not contribute to the Yukawa couplings. However, it does contribute to the mass of the $W^\pm$ and Z bosons. This will be discussed below.

Alternatively, if we start from a vacuum

$$\langle\Phi_i\rangle = \begin{pmatrix} a_1 J^i_{N_1}\otimes\mathbb{1}_4 & 0 \\ 0 & a_3 J^i_{N_3}\otimes\mathbb{1}_3 \end{pmatrix}, \tag{5.21}$$

with $SU(4) \times SU(3) \times U(1)$ symmetry, then the single

[7] Here we indicate only the relevant 4×4 block in square brackets and drop the color blocks.

fuzzy sphere (5.20) is not sufficient, since it commutes with the generators $\mathcal{X}'^{\phi}{}_i = Q\mathcal{X}^{\phi}_i$,

$$\langle \mathcal{X}'^{\phi}{}_1\rangle = \alpha'_H\begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{bmatrix} = \sigma_3 \otimes \mathbb{1},$$
$$\langle \mathcal{X}'^{\phi}{}_2\rangle = \alpha'_H\begin{bmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0\end{bmatrix} = \sigma_2 \otimes \sigma_1, \tag{5.22}$$
$$\langle \mathcal{X}'^{\phi}{}_3\rangle = \alpha'_H\begin{bmatrix} 0 & -i\sigma_1 \\ -i\sigma_2 & 0\end{bmatrix} = -\sigma_2 \otimes \sigma_2.$$

The above six matrices generate an $SO(4)$ Lie algebra

$$\begin{aligned} [\langle \mathcal{X}^{\phi}_i\rangle, \langle \mathcal{X}^{\phi}_j\rangle] &= -2i\varepsilon_{ijk}\alpha_H\langle \mathcal{X}^{\phi}_k\rangle, \\ [\langle \mathcal{X}^{\phi}_i\rangle, \langle \mathcal{X}'^{\phi}{}_j\rangle] &= 2i\varepsilon_{ijk}\alpha'_H\langle \mathcal{X}'^{\phi}_k\rangle, \\ [\langle \mathcal{X}'^{\phi}{}_i\rangle, \langle \mathcal{X}'^{\phi}{}_j\rangle] &= -2i\varepsilon_{ijk}\frac{\alpha'^2_H}{\alpha_H}\langle \mathcal{X}^{\phi}_k\rangle, \\ \frac{1}{\alpha_H^2}\langle \mathcal{X}^{\phi}_i\rangle\langle \mathcal{X}^{\phi}_i\rangle &= \frac{1}{\alpha'^2_H}\langle \mathcal{X}'^{\phi}{}_i\rangle\langle \mathcal{X}'^{\phi}{}_i\rangle = \mathbb{1}_{4\times 4}. \end{aligned} \tag{5.23}$$

The two commuting $SO(3)$ algebras are then

$$\langle \mathcal{X}^{\pm\phi}{}_i\rangle = \frac{1}{2\alpha_H}\langle \mathcal{X}^{\phi}_i\rangle \pm \frac{1}{2\alpha'_H}\langle \mathcal{X}'^{\phi}{}_i\rangle \tag{5.24}$$

and they represent two fuzzy spheres which commute with Q (and of course the identity).

Thus in that case we can achieve the desired symmetry breaking down to $SU(3) \times U(1)_Q \times U(1)_B$ using these two fuzzy spheres. There may be important differences between these scenarios depending on the energy scales of these spheres, and more detailed work is required before claiming any direct phenomenological relevance. In any case, our main point is that it seems feasible to obtain a (near-)realistic extension of the standard model by these or similar mechanisms. The essential ingredients, notably the stacks of various fuzzy spheres, are similar to string-theoretical constructions of (extensions of the) standard model using branes in extra dimensions. Similar ideas are also used in [24,30].

The coordinates of the fuzzy spheres couple with the fermions in the action (2.15) via the term $\bar{\Psi}\Gamma_i[\Phi^i, \Psi]$, with the Φ's proportional to the $\mathcal{X}$'s as in (2.11) and the γ's of the internal dimensions represented as diagonal matrices in the 7×7 gauge matrix space. The Yukawa couplings for the fuzzy sphere (5.20) are then (omitting the proportional factor α_H)

$$\begin{aligned} \mathrm{tr}\bar{\Psi}[\mathcal{X}_1{}^{\phi}, \Psi] &= -\bar{d}_L d_L + \bar{d}_R d_R + 2\bar{e}_R e_R + \bar{u}_L u_L - \bar{u}_R u_R \\ &\quad + 2\bar{\nu}\nu - 2\bar{\tilde{\nu}}\tilde{\nu} + \bar{d}'_L d'_L - \bar{d}'_R d'_R - 2\bar{e}'_R e'_R \\ &\quad - \bar{u}'_L u'_L + \bar{u}'_R u'_R - 2\bar{\nu}'\nu' + 2\bar{\tilde{\nu}}'\tilde{\nu}', \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}_2{}^{\phi}, \Psi] &= -i\bar{d}_R d_L + i\bar{d}_L d_R + i\bar{e}_L e_R + i\bar{u}_R u_L \\ &\quad - i\bar{u}_L u_R - i\bar{\tilde{e}}'_L e_R - i\bar{d}'_R d'_L + i\bar{d}'_L d'_R \\ &\quad + i\bar{e}'_R \tilde{e}_L - i\bar{e}'_R e_L - i\bar{e}_R e_L + i\bar{e}_R \tilde{e}'_L \\ &\quad - i\bar{\tilde{e}}_L e'_R + i\bar{e}'_L e'_R + i\bar{u}'_R e'_L - i\bar{u}'_L u'_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{\phi}_3, \Psi] &= \bar{d}_R d_L + \bar{d}_L d_R + \bar{e}_L e_R + \bar{u}_R u_L + \bar{u}_L u_R \\ &\quad - \bar{\tilde{e}}_L{}' e_R - \bar{d}'_R d'_L - \bar{d}'_L d'_R + \bar{e}'_R \tilde{e}_L - \bar{e}'_R e_L \\ &\quad + \bar{e}_R e_L - \bar{e}_R \tilde{e}'_L - \bar{\tilde{e}}_L e'_R - \bar{e}'_L e'_R - \bar{u}'_R e'_L \\ &\quad - i\bar{u}'_L u'_R, \end{aligned} \tag{5.25}$$

while if one considers the pair of spheres (5.23) (setting $\alpha_H = \alpha'_H = 1$ for simplicity) one obtains

$$\begin{aligned} \mathrm{tr}\bar{\Psi}[\mathcal{X}^{+\phi}_1, \Psi] &= \bar{e}_L e_L + \bar{\tilde{e}}_L \tilde{e}_L + \bar{e}_R e_R + \bar{u}_L u_L - \bar{u}_R u_R \\ &\quad + 2\bar{\nu}\nu - \bar{e}'_L e'_L - \bar{\tilde{e}}'_L \tilde{e}'_L - \bar{e}'_R e'_R - \bar{u}'_L u'_L \\ &\quad + \bar{u}'_R u'_R - 2\bar{\nu}'\nu', \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{+\phi}_2, \Psi] &= i\bar{u}_R u_L - i\bar{u}_L u_R - i\bar{\tilde{e}}'_L e_R + i\bar{e}_R \tilde{e}_L + i\bar{e}_R \tilde{e}'_L \\ &\quad - i\bar{\tilde{e}}_L e'_R + i\bar{u}'_R u'_L - i\bar{u}'_L u'_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{+\phi}_3, \Psi] &= \bar{u}_R \bar{u}_L + \bar{u}_L \bar{u}_R - \bar{\tilde{e}}'_R e_R + \bar{e}'_R \tilde{e}_L - \bar{e}_R \tilde{e}'_L \\ &\quad + \bar{\tilde{e}}_L e'_R - \bar{u}'_R u'_L - \bar{u}'_L u'_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{-\phi}_1, \Psi] &= -\bar{d}_L d_L + \bar{d}_R d_R - \bar{e}_L e_L - \bar{\tilde{e}}_L \tilde{e}_L + \bar{e}_R e_R \\ &\quad - 2\bar{\tilde{\nu}}\tilde{\nu} + \bar{d}'_L d'_L - \bar{d}'_R d'_R + \bar{e}'_L e'_L + \bar{\tilde{e}}'_L \tilde{e}'_R \\ &\quad - \bar{e}'_R e'_R + 2\bar{\tilde{\nu}}'\tilde{\nu}', \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{-\phi}_2, \Psi] &= -i\bar{d}_R d_L + i\bar{d}_L d_R - i\bar{e}_R e_L + i\bar{e}_L e_R \\ &\quad - i\bar{d}'_R d'_L + i\bar{d}'_L d'_R - i\bar{e}'_R e'_L + i\bar{e}'_L e'_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{-\phi}_3, \Psi] &= \bar{d}_R d_L + \bar{d}_L d_R + \bar{e}_R e_L + \bar{e}_L e_R - \bar{d}'_R d'_L \\ &\quad - \bar{d}'_L d'_R - \bar{e}'_R e'_L - \bar{e}'_L e'_R. \end{aligned} \tag{5.26}$$

The couplings which appear are all "reasonable," meaning that they are either Majorana or Dirac masses, or coupling among the primed particles or the spurious leptons. Setting all of these to zero we obtain

$$\begin{aligned} \mathrm{tr}\bar{\Psi}[\mathcal{X}^{+\phi}_1, \Psi] &= \bar{e}_L e_L + \bar{e}_R e_R + \bar{u}_L u_L - \bar{u}_R u_R + 2\bar{\nu}\nu, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{+\phi}_2, \Psi] &= i\bar{u}_R u_L - i\bar{u}_L u_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{+\phi}_3, \Psi] &= \bar{u}_R \bar{u}_L + \bar{u}_L \bar{u}_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{-\phi}_1, \Psi] &= -\bar{d}_L d_L + \bar{d}_R d_R - \bar{e}_L e_L + \bar{e}_R e_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{-\phi}_2, \Psi] &= -i\bar{d}_R d_L + i\bar{d}_L d_R - i\bar{e}_R e_L + i\bar{e}_L e_R, \\ \mathrm{tr}\bar{\Psi}[\mathcal{X}^{-\phi}_3, \Psi] &= \bar{d}_R d_L + \bar{d}_L d_R + \bar{e}_R e_L + \bar{e}_L e_R. \end{aligned} \tag{5.27}$$

These are the couplings of the standard model in the absence of right-handed neutrinos. Some of these terms may vanish depending on the specific chirality assignment, as discussed before. As it is, the model does not allow for different masses, apart from some freedom afforded by the tuning of α_H and α'_H.

Note that the remaining eight generators of Mat$(4, \mathbb{C})$ do not commute with these two fuzzy spheres; thus the gauge symmetry is indeed broken to Q and the generators of color and baryon number. According to what we explained in the previous sections this implies that they are massive.

It is worthwhile to elaborate in some detail the explicit form of the low-energy electroweak Higgs. Consider first the vacuum without fluctuations. Using

$$\langle \mathcal{X}_i^\phi \rangle \langle \mathcal{X}_j^\phi \rangle \delta^{ij} = \alpha_H^2, \qquad \varepsilon^{ijk} \langle \mathcal{X}_i^\phi \rangle \langle \mathcal{X}_j^\phi \rangle \langle \mathcal{X}_k^\phi \rangle = -2\mathrm{i}\alpha_H^3, \tag{5.28}$$

the effective potential for α_H becomes

$$V_H(\langle \mathcal{X}_i^\phi \rangle) = \mathrm{Tr}\bigg(\frac{4}{g^2}\alpha_H^4 + c_2\alpha_H^2 + 2c_3\alpha_H^3\bigg). \tag{5.29}$$

This has a nontrivial minimum in $\alpha_H \neq 0$ provided $c_3 \neq 0$ or $c_2 < 0$, and the sign of α_H depends on the sign of c_3. Note that these terms have a geometrical interpretation in terms of field strength on S_N^2, leading to some protection from quantum corrections [24].

The vacuum expectation values (VEV's) of $\mathcal{X}_2^\phi$ and $\mathcal{X}_3^\phi$ contain the expected degrees of freedom of the two Higgs doublets ϕ as in (5.10), parametrizing one complex scalar. This is as in the minimal supersymmetric standard model; however, the two doublets are related to each other. They become independent in the presence of the second fuzzy sphere (5.22). The VEV of $\mathcal{X}_1^\phi$ contains scalar degrees of freedom which are in the adjoint of the electroweak $SU(2)$, with the same VEV. This is different from the standard model and should have observable signatures. We therefore obtain an interesting geometrical interpretation of these scalar Higgs fields.

Now consider fluctuations around this vacuum. Again, these include fluctuations of the two Higgs doublets ϕ as in (5.10), and also fluctuations of $\mathcal{X}_1^\phi$ which is in the adjoint of the electroweak $SU(2)$. More generally, fluctuations on the fuzzy sphere can be interpreted as scalar (resp. gauge fields) on S_N^2.

Vector boson masses.—As explained in Sec. III B, the vector bosons corresponding to the 4×4 leptonic block will acquire particular mass terms in the presence of these electroweak fuzzy spheres. For example, $\mathcal{X}_1^\phi$ will contribute a mass proportional to α_H^2 to the W bosons.

Furthermore, note that $\mathcal{X}_1'^\phi$ breaks $SU(4)$ into $SU(2) \times SU(2)$, which seems quite appealing; this suggests that α'_H should have a higher scale than α_H; on the other hand, $\mathcal{X}_{2,3}'^\phi$ lead to electroweak symmetry breaking, which is strange. This suggests some interplay between the two spheres.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have shown how the matrix model which gives rise to noncommutative spaces and emergent gravity can also accommodate a gauge theory with the features of the standard model. We have seen that a simple solution with extra (nonpropagating) dimensions contains all the necessary fundamental fermionic degrees of freedom, with a few extra particles which can be set to zero without prejudice to the model. Also, the basic gauge symmetries can be accommodated and, with the use of extra dimensions, the pattern of symmetry breakings can be substantially reproduced. The breaking happens in two stages, first some sort of grand-unification breaking and then the electroweak breaking. Both stages can be accomplished either with the presence of a simple (effective) extra matrix dimension or with the use of fuzzy spheres. The former mechanism can be considered an effective version of the latter.

There are several gaps in the constructions, and several lines of development which hopefully can fill the gaps. The list of shortcomings includes the fact that there are some extra $U(1)$ symmetries, the lack of generations, and the fact that couplings do not differentiate between fermions and bosons. Clearly the solutions discussed here are not phenomenologically viable, but we find it rather inspiring that we have a semirealistic matrix model from which gravity and gauge theories of a realistic kind emerge naturally. Among the lines of development there is the possibility to have a supersymmetric version of the model. This type of matrix model is in fact well suited for supersymmetric extensions, the most notable example being the IKKT model [6]. Moreover, the introduction of additional geometrical structures such as intersecting branes and orbifolds [30] is likely to resolve at least some of these problems, in particular, the issue of chirality. Another line of development is a better understanding of the connections among the extra dimensions in the guise of fuzzy spheres and the results obtained in string and brane theories. This supports the hope that the framework of matrix models might be suitable to approach the goal of a consistent quantum theory of fundamental interactions including gravity.

ACKNOWLEDGMENTS

H. S. would like to thank Athanasios Chatzistavrakidis and George Zoupanos for many discussions and related collaboration. F. L. would like to thank the Department of Estructura i Constituents de la materia, and the Institut de Ciències del Cosmos, Universitat de Barcelona for hospitality. His work has been supported in part by CUR Generalitat de Catalunya under Project No. 2009SGR502. The work of H. S. was supported in part by FWF Project No. P18657 and No. P21610.

[1] A. Connes, *Noncommutative Geometry* (Academic Press, New York, 1994).
[2] G. Landi, *An Introduction to Noncommutative Spaces and Their Geometry* (Springer, New York, 1997).
[3] J. M. Gracia-Bondia, J. C. Varilly, and H. Figueroa, *Elements of Noncommutative Geometry* (Birkhaeuser, Boston, 2000).
[4] J. Madore, Lond. Math. Soc. Lect. Note Ser. **257**, 1 (2000).
[5] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982).
[6] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. **B498**, 467 (1997).
[7] A. Y. Alekseev, A. Recknagel, and V. Schomerus, J. High Energy Phys. 05 (2000) 010.
[8] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, Nucl. Phys. **B565**, 176 (2000).
[9] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); R. J. Szabo, Phys. Rep. **378**, 207 (2003).
[10] H. Steinacker, J. High Energy Phys. 12 (2007) 049.
[11] A. Sakharov, Dokl. Akad. Nauk SSSR **177**, 70 (1967) [Sov. Phys. Dokl. **12**, 1040 (1968)]; Sov. Phys. Usp. **34**, 394 (1991). Reprinted in Usp. Fiz. Nauk **161** 64 (1991) [Sov. Phys. Usp. **34**, 394 (1991)].
[12] M. Visser, Mod. Phys. Lett. A **17**, 977 (2002).
[13] S. Terashima, Phys. Lett. B **482**, 276 (2000); K. Matsubara, Phys. Lett. B **482**, 417 (2000); M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **526**, 132 (2002).
[14] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020.
[15] A. Matusis, L. Susskind, and N. Toumbas, J. High Energy Phys. 12 (2000) 002.
[16] V. V. Khoze and J. Levell, J. High Energy Phys. 09 (2004) 019.
[17] J. Jaeckel, V. V. Khoze, and A. Ringwald, J. High Energy Phys. 02 (2006) 028.
[18] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Eur. Phys. J. C **29**, 413 (2003); M. Chaichian, A. Kobakhidze, and A. Tureanu, Eur. Phys. J. C **47**, 241 (2006).
[19] J. L. Hewett, F. J. Petriello, and T. G. Rizzo, Phys. Rev. D **66**, 036001 (2002).
[20] X. Calmet, B. Jurco, P. Schupp, J. Wess, and M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002).
[21] R. Wulkenhaar, J. High Energy Phys. 03 (2002) 024.
[22] A. Connes and J. Lott, Nucl. Phys. B, Proc. Suppl. **18**, 29 (1991).
[23] A. H. Chamseddine, A. Connes, and M. Marcolli, Adv. Theor. Math. Phys. **11**, 991 (2007).
[24] P. Aschieri, T. Grammatikopoulos, H. Steinacker, and G. Zoupanos, J. High Energy Phys. 09 (2006) 026.
[25] P. Aschieri, J. Madore, P. Manousselis, and G. Zoupanos, J. High Energy Phys. 04 (2004) 034.
[26] I. Antoniadis, E. Kiritsis, J. Rizos, and T. N. Tomaras, Nucl. Phys. **B660**, 81 (2003).
[27] R. Blumenhagen, B. Kors, D. Lust, and S. Stieberger, Phys. Rep. **445**, 1 (2007).
[28] E. Kiritsis, Fortschr. Phys. **52**, 200 (2004); Phys. Rep. 105 **421** (2005); **429**, 121(E) (2006).
[29] H. Steinacker and G. Zoupanos, J. High Energy Phys. 09 (2007) 017.
[30] A. Chatzistavrakidis, H. Steinacker, and G. Zoupanos, arXiv:1002.2606.
[31] H. Grosse, F. Lizzi, and H. Steinacker, arXiv:1001.2706.
[32] D. Klammer and H. Steinacker, J. High Energy Phys. 08 (2008) 074.
[33] J. Madore, Classical Quantum Gravity **9**, 69 (1992).
[34] A. Chatzistavrakidis, H. Steinacker, and G. Zoupanos, arXiv:0909.5559.
[35] F. Lizzi, G. Mangano, G. Miele, and G. Sparano, Phys. Rev. D **55**, 6357 (1997).