UNIQUENESS THEOREM FOR MIXTURES WITH MEMORY

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Abstract

This work concerns the theory of viscoelastic composites which are modeled as mixtures of two interpenetrating solid continua. First it is given a characterization of the boundary-initial value problem in which the initial conditions are incorporated into the field equations. Then a uniqueness theorem for the dynamic theory is established avoiding the use of Laplace transform.

Key words and phrases: Binary mixtures, viscoelastic composites, materials with memory, uniqueness theorem.

AMS subject classification: 74D05, 74H20, 74E30, 74F20, 74M25

1. Introduction

The theory of mixtures has undergone significant developments in recent decades. The theoretical progress in the field is discussed in detail in the review articles by Bowen [1], Atkin and Craine [2], Bedford and Drumheller [3] and in the books of Samohyl [4] and Rajagopal and Tao [5]. The significance of the theory has been demonstrated amply for his applications to a variety of different fields of physics and engineering. It provide a mathematical model to study material aggregates composed of various ingredients such as slurries and suspensions, porous rocks and soil infused with water or oil, biological tissues and muscles, plasmas and gaseous mixtures. Moreover, by appealing to this theory we can overcome the inadequacy of Darcy's law in providing some important information on diffusion processes.

The idea of employing interpenetrating continua as a model of composite materials has been introduced by Bedford and Stern [6]. Their approach is based on the Lagrangian description of motion. The theory developed in [6] has been extended by Pop and Bowen [7], who established a thermodynamic theory of mixtures with long-range spatial interactions. The model of interpenetrating solid continua was applied by Tiersten and Jahanmir [8] to derive a theory of composites where the relative displacement of the

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individual constituents is infinitesimal. The influence of viscous dissipation is included in the general treatment. The theory of viscoelastic composites modelled as interpenetrating solid continua with memory has been investigated by several authors (see e.g. [9]-[12]). In [11], McCarthy has studied the propagation of shock and higher order waves in a binary viscoelastic mixture.

In this paper we consider the linear theory of viscoelastic composites modelled as interpenetrating solid continua with memory presented in [10, 12]. First, we give a characterization of the boundary-initial value problem in which the initial conditions are incorporated into the field equations. Then we use the results established by Gurtin, McCamy and Murphy [13] to derive a uniqueness theorem for the mixed problem. This result is obtained avoiding the use of Laplace transform.

A minimum principle in the theory of viscoelastic mixtures has been presented in [14].

2. Basic equations

We consider a body which is made up of two solid interpenetrating continua with memory. We assume that the body occupies at time t_0 the properly regular region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The motion of the body is referred to a fixed system of rectangular Cartesian axes $Ox_i (i = 1, 2, 3)$. We designate by $\mathbf{n}$ the outward unit normal of ∂B. Letters in boldface stand for tensor of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots s}$(p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate system. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

We consider the basic equations for the mechanical behavior of a binary mixture in the framework of the linearized theory. We assume that the constituents s_1 and s_2 are each viscoelastic bodies. We denote by $\mathbf{u}$ and $\mathbf{w}$ the displacement vector fields associated with the constituents s_1 and s_2, respectively. Let $\mathbf{t}$ and $\mathbf{s}$ be the partial stress tensors associated with the constituents s_1 and s_2, respectively. The equations of motion can be expressed as

$$t_{ji,j} - p_i + F_i = \rho_1^0 \ddot{u}_i, \quad s_{ji,j} + p_i + G_i = \rho_2^0 \ddot{w}_i, \tag{2.1}$$

where $\mathbf{p}$ is the internal body force, $\mathbf{F}$ is the body force per unit volume acting on the constituent $s_1, \mathbf{G}$ is the body force per unit volume acting on s_2, and ρ_α^0 is the mass density of the constituent s_α at time t_0. We introduce the measures of deformation e_{ij}, g_{ij} and d_i (see [2, 15])

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad g_{ij} = u_{j,i} + w_{i,j}, \quad d_i = u_i - w_i. \tag{2.2}$$

The constitutive equations are

$$\begin{aligned} t_{ji}(\mathbf{x},t) &= \int_{-\infty}^{t} \{[A_{jimn}(\mathbf{x},t-s) + B_{mnji}(\mathbf{x},t-s)]\dot{e}_{mn}(\mathbf{x},s) + \\ &\quad +[B_{jimn}(\mathbf{x},t-s) + C_{jimn}(\mathbf{x},t-s)]\dot{g}_{mn}(\mathbf{x},s) + \\ &\quad +[D_{jim}(\mathbf{x},t-s) + E_{jim}(\mathbf{x},t-s)]\dot{d}_m(\mathbf{x},s)\}ds, \\ s_{ij}(\mathbf{x},t) &= \int_{-\infty}^{t} \{B_{mnji}(\mathbf{x},t-s)\dot{e}_{mn}(\mathbf{x},s) + C_{jimn}(\mathbf{x},t-s)\dot{g}_{mn}(\mathbf{x},s) + \\ &\quad +E_{jim}(\mathbf{x},t-s)\dot{d}_m(\mathbf{x},s)\}ds, \\ p_i(\mathbf{x},t) &= \int_{-\infty}^{t} \{D_{mni}(\mathbf{x},t-s)\dot{e}_{mn}(\mathbf{x},s) + E_{mni}(\mathbf{x},t-s)\dot{g}_{mn}(\mathbf{x},s) + \\ &\quad +a_{ij}(\mathbf{x},t-s)\dot{d}_j(\mathbf{x},s)\}ds, \end{aligned} \tag{2.3}$$

where the relaxation functions are twice continuously differentiable on $\overline{B} \times [0,\infty)$ and possess the following symmetry properties [16, 10, 12]

$$\begin{aligned} A_{ijmn} = A_{jimn} = A_{mnij}, \quad B_{ijmn} = B_{jimn}, \quad C_{ijmn} = C_{mnij}, \\ D_{ijm} = D_{jim}, \quad a_{ij} = a_{ji} \text{ on } \overline{B} \times (0,\infty). \end{aligned} \tag{2.4}$$

Let $\mathbf{T}$ and $\mathbf{S}$ be the partial tractions associated with the constituents s_1 and s_2, respectively, acting at a point $\mathbf{x}$ on the surface Σ. Then

$$T_i = t_{ji} n_j, \quad S_i = s_{ji} n_j, \tag{2.5}$$

where $\mathbf{n}$ is the outward unit normal to Σ at $\mathbf{x}$.

Let f be a function on $\overline{B} \times (-\infty,\infty)$. We say that $f \in C^{M,N}$ if

$$\frac{\partial^m}{\partial x_i \partial x_j \dots \partial x_p} \left(\frac{\partial^n f}{\partial t^n} \right),$$

exists and is continuous on $\overline{B} \times (-\infty,\infty)$ for $m = 0,1,\dots,M, n = 0,1,\dots,N$, and $m + n \le \max(M,N)$. We write C^N for $C^{N,N}$.

We introduce the notion of an admissible process

$$\pi = \{u_i, w_i, e_{ij}, g_{ij}, d_i, t_{ij}, s_{ij}, p_i\}$$

by which we mean an ordered array of functions $u_i, w_i, e_{ij}, g_{ij}, t_{ij}, s_{ij}$ and p_i defined on $\overline{B} \times (-\infty,\infty)$ with the following properties:

(i) $u_i, w_i, \dot{u}_i, \dot{w}_i, \ddot{u}_i, \ddot{w}_i, e_{ij}, g_{ij}, d_i, \dot{e}_{ij}, \dot{g}_{ij}, \dot{d}_i$, and p_i are continuous on $\overline{B} \times (-\infty, \infty)$; (ii) $e_{ij} = e_{ji}, t_{ji} - s_{ij} = t_{ij} - s_{ji}$; (iii) t_{ij} and s_{ij} are of class $C^{1,0}$ on $B \times (-\infty, \infty)$; (iv) $t_{ij}, t_{ji,j}, s_{ij}$ and $s_{ji,j}$ are continuous on $\overline{B} \times (-\infty, \infty)$. If we define addition and multiplication of an admissible process by a scalar through

$$\pi + \pi' = \{u_i + u_i', w_i + w_i', \dots, p_i + p_i'\}, \quad \lambda\pi = \{\lambda u_i, \lambda w_i, \dots, \lambda p_i\},$$

then the set of all admissible processes is a linear vector space.

A viscoelastic material remembers its past history so that we must prescribe the functions $u_i, w_i, e_{ij}, g_{ij}, d_i, t_{ij}, s_{ij}$ and p_i up to some instant t_1. The initial data consists of the functions $\{u_i^*, w_i^*, e_{ij}^*, g_{ij}^*, d_i^*, t_{ij}^*, s_{ij}^*, p_i^*\} = \pi^*$, defined on $\overline{B} \times (-\infty, t_1)$ which satisfy the field equations. The initial history condition is

$$\pi^{(i)} = \pi^*, \tag{2.6}$$

where $\pi^{(i)}$ is the restriction of the admissible process π to $\overline{B} \times (-\infty, t_1)$. Without loss of generality, we take $t_1 = 0$. If π is an admissible process that satisfies the initial history condition (2.6), then $u_i, w_i, \dot{u}_i$ and $\dot{w}_i$ automatically satisfy the initial conditions

$$\begin{aligned} &u_i(\mathbf{x}, 0) = \lim_{t\to 0} u_i^*(\mathbf{x}, t) \equiv \alpha_i^0, \quad \dot{u}_i(\mathbf{x}, 0) \lim_{t\to 0} \dot{u}_i^*(\mathbf{x}, t) = \beta_i^0, \\ &w_i(\mathbf{x}, t) = \lim_{t\to 0} w_i^*(\mathbf{x}, t) \equiv \gamma_i^0, \quad \dot{w}_i(\mathbf{x}, 0) = \lim_{t\to 0} \dot{w}_i^*(\mathbf{x}, t) = \delta_i^0, \ \ \mathbf{x} \in \overline{B}. \end{aligned} \tag{2.7}$$

The boundary conditions in the theory of mixtures have been discussed in [1]-[5]. Let S_1 and S_2 be subsets of ∂B so that $\overline{S} \cup S_2 = \partial B$ and $S_1 \cap S_2 = \emptyset$. We consider the following boundary conditions

$$\begin{aligned} &u_i = \widetilde{u}_i, \quad w_i = \widetilde{w}_i \text{ on } \overline{S}_1 \times I, \\ &(t_{ji} + s_{ji})n_j = \widetilde{\sigma}_i, \ \ d_i = \widetilde{d}_i \text{ on } S_2 \times I, \end{aligned} \tag{2.8}$$

where $\widetilde{u}_i, \widetilde{w}_i, \widetilde{\sigma}_i$ and $\widetilde{d}_i$ are prescribed functions, and $I = (0, \infty)$. We assume that: (α) $\mathbf{F}$ and $\mathbf{G}$ are continuous on $B \times I$; (β) ρ_1^0 and ρ_2^0 are of class C^1 and strictly positive on $\overline{B}$; (γ) $\widetilde{u}_i$ and $\widetilde{w}_i$ are continuous on $S_1 \times I$; (δ) $\widetilde{\sigma}_i$ are piecewise regular on $S_2 \times I$, and $\widetilde{d}_i$ are continuous on $S_2 \times I$.

By a viscoelastic process corresponding to the body loads $\{\mathbf{F}, \mathbf{G}\}$ we mean an admissible process that satisfies the equations (2.1)-(2.3). By a solution of the mixed problem we mean a viscoelastic process corresponding to the body loads $\{\mathbf{F}, \mathbf{G}\}$ that satisfies the initial history condition (2.6) and the boundary conditions (2.8).

Let u and v be scalar fields on $B \times I$ that are continuous in time. We denote by $u * v$ the convolution

$$(u * v)(\mathbf{x}, t) = \int_0^t u(\mathbf{x}, t - \tau) v(\mathbf{x}, \tau) d\tau, \ \ \mathbf{x} \in B, t \in I.$$

The constitutive equations (2.3) can be expressed in the form

$$\begin{aligned} t_{ji} &= T_{ji} + \frac{d}{dt}[(A_{jimn} + B_{mnji}) * e_{mn} + (B_{jimn} + C_{jimn}) * g_{mn} + \\ &\quad +(D_{jim} + E_{jim}) * d_m], \\ s_{ij} &= S_{ij} + \frac{d}{dt}(B_{mnji} * e_{mn} + C_{jimn} * g_{mn} + E_{jim} * d_m), \\ p_i &= P_i + \frac{d}{dt}(D_{mni} * e_{mn} + E_{mni} * g_{mn} + a_{ij} * d_j), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} T_{ji} &= \int_0^\infty \{[\dot{A}_{jimn}(t+s) + \dot{B}_{mnji}(t+s)]e_{mn}(-s) + \\ &\quad +[\dot{B}_{jimn}(t+s) + \dot{C}_{jimn}(t+s)]g_{mn}(-s) + \\ &\quad +[\dot{D}_{jim}(t+s) + \dot{E}_{jim}(t+s)]d_m(-s)\}ds, \\ S_{ij} &= \int_0^\infty [\dot{B}_{mnji}(t+s)e_{mn}(-s) + \dot{C}_{jimn}(t+s)g_{mn}(-s) + \\ &\quad +\dot{E}_{jim}(t+s)d_m(-s)]ds, \\ P_i &= \int_0^\infty [\dot{D}_{mni}(t+s)e_{mn}(-s) + \dot{E}_{mni}(t+s)g_{mn}(-s) + \\ &\quad +\dot{a}_{ij}(t+s)d_j(-s)]ds, \end{aligned} \tag{2.10}$$

and, for convenience, we have suppressed the argument $\mathbf{x}$. Thus, $\pi = \{u_i, w_i, e_{ij}, g_{ij}, t_{ij}, s_{ij}, p_i\}$ is a viscoelastic process corresponding to the body loads $\{\mathbf{F}, \mathbf{G}\}$, with the initial history π^*, if and only if π is admissible and satisfies the equations (2.1), (2.2), (2.9) and the condition (2.6).

3. A uniqueness result

The uniqueness question in the dynamic linear theory of viscoelasticity has been considered in various works (see, e.g., [17]-[20]). Uniqueness results in the theory of mixture of elastic solids have been presented in [1, 21, 22]. In this section we use the results established by Gurtin, McCamy and Murphy [13] to derive a uniqueness theorem for the mixed problem presented in Section 2.

We denote

$$\begin{aligned} &e^*_{ij}(\mathbf{x}, \alpha, \beta) = e_{ij}(\mathbf{x}, \alpha) - e_{ij}(\mathbf{x}, \beta), \quad g^*_{ij}(\mathbf{x}, \alpha, \beta) = g_{ij}(\mathbf{x}, \alpha) - g_{ij}(\mathbf{x}, \beta), \\ &d^*_i(\mathbf{x}, \alpha, \beta) = d_i(\mathbf{x}, \alpha) - d_i(\mathbf{x}, \beta), \quad \mathbf{x} \in \overline{B}, \quad \alpha, \beta \in I. \end{aligned} \tag{3.1}$$

Let us introduce the notations

$$
\begin{aligned}
\Phi(\alpha,\beta;\rho) = & \frac{1}{2}A_{ijmn}(\rho)e^*_{ij}(\alpha,\beta)e^*_{mn}(\alpha,\beta)+ \\
& + B_{mnji}(\rho)e^*_{mn}(\alpha,\beta)g^*_{ji}(\alpha,\beta) + \frac{1}{2}C_{ijmn}(\rho)g^*_{ij}(\alpha,\beta)g^*_{mn}(\alpha,\beta)+ \\
& + D_{mni}(\rho)e^*_{mn}(\alpha,\beta)d^*_i(\alpha,\beta) + E_{mni}(\rho)g^*_{mn}(\alpha,\beta)d^*_i(\alpha,\beta)+ \\
& + \frac{1}{2}a_{ij}(\rho)d^*_i(\alpha,\beta)d^*_j(\alpha,\beta), \\
\Psi(\alpha,\beta;\rho) = & \frac{1}{2}\dot{A}_{ijmn}(\rho)e^*_{ij}(\alpha,\beta)e^*_{mn}(\alpha,\beta)+ \\
& + \dot{B}_{mnji}(\rho)e^*_{mn}(\alpha,\beta)g^*_{ji}(\alpha,\beta) + \frac{1}{2}\dot{C}_{ijmn}(\rho)g^*_{ij}(\alpha,\beta)g^*_{mn}(\alpha,\beta)+ \\
& + \dot{D}_{mni}(\rho)e^*_{mn}(\alpha,\beta)d^*_i(\alpha,\beta) + \dot{E}_{mni}(\rho)g^*_{mn}(\alpha,\beta)d^*_i(\alpha,\beta)+ \\
& + \frac{1}{2}\dot{a}_{ij}(\rho)d^*_i(\alpha,\beta)d^*_j(\alpha,\beta), \\
\Gamma(\alpha,\beta;\rho) = & \frac{1}{2}\ddot{A}_{ijmn}(\rho)e^*_{ij}(\alpha,\beta)e^*_{mn}(\alpha,\beta)+ \\
& + \ddot{B}_{mnji}(\rho)e^*_{mn}(\alpha,\beta)g^*_{ji}(\alpha,\beta) + \frac{1}{2}\ddot{C}_{ijmn}(\rho)g^*_{ij}(\alpha,\beta)g^*_{mn}(\alpha,\beta)+ \\
& + \ddot{D}_{mni}(\rho)e^*_{mn}(\alpha,\beta)d^*_i(\alpha,\beta) + \ddot{E}_{mni}(\rho)g^*_{mn}(\alpha,\beta)d^*_i(\alpha,\beta)+ \\
& + \frac{1}{2}\ddot{a}_{ij}(\rho)d^*_i(\alpha,\beta)d^*_j(\alpha,\beta), \quad \alpha,\beta,\rho \in I,
\end{aligned}
\tag{3.2}
$$

where, for convenience, we have suppressed the argument $\mathbf{x}$.

Theorem 3.1. *If $\pi = \{u_i, w_i, e_{ij}, g_{ij}, t_{ij}, s_{ij}, p_i\}$ is an admissible process that corresponds to null initial history and satisfies the constitutive equations, then*

$$
\begin{aligned}
\int_0^t (t_{ji}\dot{u}_{i,j} + s_{ji}\dot{w}_{i,j} + p_i\dot{d}_i)ds = & \Phi(t,0;t) - \int_0^t \Psi(\tau,0;\tau)d\tau - \\
& - \int_0^t \Psi(t,\tau;t-\tau)d\tau + \frac{1}{2}\int_0^t\int_0^t \Gamma(r,s;|r-s|)drds.
\end{aligned}
\tag{3.3}
$$

Proof. In the case of the null initial history we have

$$
e^*_{ij}(t,0) = e_{ij}(t), \quad g^*_{ij}(t,0) = g_{ij}(t), \quad d^*_i(t,0) = d_i(t), \quad t \in I.
$$

Since π corresponds to null initial history, the constitutive equations (2.3) reduce to

$$\begin{aligned} t_{ji}(t) &= [A_{jimn}(0) + B_{mnji}(0)]e_{mn}(t) + [B_{jimn}(0) + C_{jimn}(0)]g_{mn}(t)+ \\ &+ [D_{jim}(0) + E_{jim}(0)]d_m(t) + \int_0^t \{[\dot{A}_{jimn}(t-s)+ \\ &+ \dot{B}_{mnji}(t-s)]e_{mn}(s) + [\dot{B}_{jimn}(t-s) + \dot{C}_{jimn}(t-s)]g_{mn}(s)+ \\ &+ [\dot{D}_{jim}(t-s) + \dot{E}_{jim}(t-s)]d_m(s)\}ds, \\ s_{ij}(t) &= B_{mnji}(0)e_{mn}(t) + C_{jimn}(0)g_{mn}(t) + E_{jim}(0)d_m(t)+ \\ &+ \int_0^t \{\dot{B}_{mnji}(t-s)e_{mn}(s) + \dot{C}_{jimn}(t-s)g_{mn}(s)+ \\ &+ \dot{E}_{jim}(t-s)d_m(s)\}ds, \\ p_i(t) &= D_{mni}(0)e_{mn}(t) + E_{mni}(0)g_{mn}(t) + a_{ij}(0)d_j(t)+ \\ &+ \int_0^t \{\dot{D}_{mni}(t-s)e_{mn}(s) + \dot{E}_{mni}(t-s)g_{mn}(s) + \dot{a}_{ij}(t-s)d_j(s)\}ds. \end{aligned} \tag{3.4}$$

In view of the initial conditions we get

$$\int_0^t (t_{ji}\dot{u}_{i,j} + s_{ji}\dot{w}_{i,j} + p_i\dot{d}_i)ds = t_{ji}u_{i,j} + s_{ji}w_{i,j}+ \\ +p_i d_i - \int_0^t (\dot{t}_{ji}u_{i,j} + \dot{s}_{ji}w_{i,j} + \dot{p}_i d_i)ds.$$

By (2.2), (2.4) and (3.4),

$$\begin{aligned} t_{ji}u_{i,j} &+ s_{ji}w_{i,j} + p_i d_i = 2\Phi(t,0;0) + \int_0^t \{\dot{A}_{ijmn}(t-s)e_{ij}(t)e_{mn}(s)+ \\ &+ \dot{B}_{mnji}(t-s)[e_{mn}(s)g_{ji}(t) + e_{mn}(t)g_{ji}(s)]+ \\ &+ \dot{C}_{jimn}(t-s)g_{ji}(t)g_{mn}(s) + \dot{D}_{jim}(t-s)[e_{ji}(t)d_m(s) + e_{ji}(s)d_m(t)]+ \\ &+ \dot{E}_{jim}(t-s)[g_{ji}(t)d_m(s) + g_{ji}(s)d_m(t)] + \dot{a}_{ij}(t-s)d_i(t)d_j(s)\}ds. \end{aligned} \tag{3.5}$$

It follows from (3.4) that

$$\begin{aligned} \dot{t}_{ji}u_{i,j} &+ \dot{s}_{ji}w_{i,j} + \dot{p}_i d_i = \frac{d}{dt}\Phi(t,0;0) + 2\Psi(t,0;0)+ \\ &+ \int_0^t \{\ddot{A}_{ijmn}(t-s)e_{ij}(t)e_{mn}(s)+ \\ &+ \ddot{B}_{mnji}(t-s)[e_{mn}(s)g_{ji}(t) + e_{mn}(t)g_{ji}(s)]+ \\ &+ \ddot{C}_{jimn}(t-s)g_{ji}(t)g_{mn}(s)+ \\ &+ \ddot{D}_{jim}(t-s)[e_{ji}(t)d_m(s) + e_{ji}(s)d_m(t)]+ \\ &+ \ddot{E}_{jim}(t-s)[g_{ji}(t)d_m(s) + g_{ji}(s)d_m(t)] + \ddot{a}_{ij}(t-s)d_i(t)d_j(s)\}ds. \end{aligned} \tag{3.6}$$

In view of (3.5) and (3.6) we obtain

$$\int_0^t (t_{ji}\dot{u}_{i,j} + s_{ji}\dot{w}_{i,j} + p_i\dot{d}_i)ds = \Phi(t,0;0) - 2\int_0^t \Psi(s,0;0)ds + \Omega(t), \quad (3.7)$$

where

$$\begin{aligned}\Omega(t) \;=\; & \int_0^t \{\dot{A}_{ijmn}(t-s)e_{ij}(t)e_{mn}(s)+ \\ & + \dot{B}_{mnji}(t-s)[e_{mn}(s)g_{ji}(t) + e_{mn}(t)g_{ji}(s)]+ \\ & + \dot{C}_{jimn}(t-s)g_{ji}(t)g_{mn}(s) + \dot{D}_{jim}(t-s)[e_{ji}(t)d_m(s) + e_{ji}(s)d_m(t)]+ \\ & + \dot{E}_{jim}(t-s)[g_{ji}(t)d_m(s) + g_{ji}(s)d_m(t)] + \dot{a}_{ij}(t-s)d_i(t)d_j(s)\}- \\ & - \int_0^t\int_0^s \{\ddot{A}_{ijmn}(s-\tau)e_{ij}(s)e_{mn}(\tau) + \ddot{B}_{mnji}(s-\tau)[e_{mn}(\tau)g_{ji}(s)+ \\ & + e_{mn}(s)g_{ji}(\tau)] + \ddot{C}_{jimn}(s-\tau)g_{ji}(s)g_{mn}(\tau)+ \\ & + \ddot{D}_{jim}(s-\tau)[e_{ji}(s)d_m(\tau)+ \\ & + e_{ji}(\tau)d_m(s)] + \ddot{E}_{jim}(s-\tau)[g_{ji}(s)d_m(\tau) + g_{ji}(\tau)d_m(s)]+ \\ & + \ddot{a}_{ij}(s-\tau)d_i(s)d_j(\tau)\}dsd\tau, \quad t \in I.\end{aligned} \quad (3.8)$$

Following [13] we find that

$$\begin{aligned}&\int_0^t \dot{B}_{mnji}(t-s)[e_{mn}(s)g_{ji}(t) + e_{mn}(t)g_{ji}(s)]ds = \\ &\quad = [B_{mnji}(t) - B_{mnji}(0)]e_{mn}(t)g_{ji}(t) + \int_0^t \dot{B}_{mnji}(t-s)e_{mn}(s)g_{ji}(s)ds- \\ &\quad - \int_0^t \dot{B}_{mnji}(t-s)[e_{mn}(t) - e_{mn}(s)][g_{ji}(t) - g_{ji}(s)]ds, \\ &2\int_0^t\int_0^s \ddot{a}_{ij}(s-\tau)d_i(s)d_j(\tau)d\tau ds = \int_0^t\int_0^t \ddot{a}_{ij}(|s-\tau|)d_i(\tau)d_j(\tau)d\tau ds- \\ &\quad -\frac{1}{2}\int_0^t\int_0^t \ddot{a}_{ij}(|s-\tau|)[d_i(s) - d_i(\tau)][d_j(s) - d(\tau)]d\tau ds, \\ &\int_0^t\int_0^s \ddot{B}_{mnji}(s-\tau)[e_{mn}(\tau)g_{ji}(s) + e_{mn}(s)g_{ji}(\tau)]d\tau ds = \\ &\quad = \int_0^t\int_0^t \ddot{B}_{mnji}(|s-\tau|)e_{mn}(s)g_{ji}(\tau)d\tau ds = \\ &\quad = \int_0^t\int_0^\infty \ddot{B}_{mnji}(|s-\tau|)e_{mn}(\tau)g_{ji}(\tau)d\tau ds- \\ &\quad -\frac{1}{2}\int_0^t\int_0^t \ddot{B}_{mnji}(|s-\tau|)[e_{mn}(s) - e_{mn}(\tau)][g_{ji}(s) - g_{ji}(\tau)]d\tau ds,\end{aligned}$$

$$\int_0^t \ddot{a}_{ij}(|s-\tau|)ds = \dot{a}_{ij}(\tau) + \dot{a}_{ij}(t-\tau) - 2\dot{a}_{ij}(0). \quad (3.9)$$

If we use (3.9) we get

$$\begin{aligned}\Omega(t) &= -\Phi(t,0;0)+\Phi(t,0;t)-\int_0^t \Psi(t,\tau;t-\tau)d\tau-\int_0^t \Psi(\tau,0;\tau)d\tau+\\ &+2\int_0^t \Psi(\tau,0;0)d\tau+\frac{1}{2}\int_0^t\int_0^t \Gamma(s,\tau;|s-\tau|)dsd\tau, \quad t\in I.\end{aligned} \tag{3.10}$$

From (3.7) and (3.10) we obtain the desired result.□

Theorem 3.2. *Assume that*
(i) ρ_1^0 *and* ρ_2^0 *are strictly positive;*
(ii) $\Phi \geq 0, \Psi \leq 0, \Gamma \geq 0,$ *on* $B\times I$, *for any* $e^*_{ij}, g^*_{ij}, d^*_i$ *with* $e^*_{ij}=e^*_{ji}$.
Then the mixed problem has at most one solution.

Proof. Let $\pi=\{u_i,w_i,e_{ij},g_{ij},t_{ij},s_{ij},p_i\}$ be the difference of two solutions of the mixed problem. Then π corresponds to null data. In view of the equations of motion, we have

$$t_{ji}\dot{u}_{i,j}+s_{ji}\dot{w}_{i,j}+p_i\dot{d}_i=(t_{ji}\dot{u}_i+s_{ji}\dot{w}_i)_{,j}-\frac{1}{2}\frac{\partial}{\partial t}(\rho_1^0\dot{\mathbf{u}}^2+\rho_2^0\dot{\mathbf{w}}^2).$$

By using the divergence theorem, we find that

$$\int_B (t_{ji}\dot{u}_{i,j}+s_{ji}\dot{w}_{i,j}+p_i\dot{d}_i)dv$$

$$=\int_{\partial B}(t_{ji}\dot{u}_i+s_{ji}\dot{w}_i)n_j da-\tfrac{1}{2}\tfrac{d}{dt}\int_B(\rho_1^0\dot{\mathbf{u}}^2+\rho_2^0\dot{\mathbf{w}}^2)dv. \tag{3.11}$$

If we take into account the boundary conditions we can write

$$(t_{ji}\dot{u}_i+s_{ji}\dot{w}_i)n_j=\frac{1}{2}[(t_{ji}+s_{ji})n_j(\dot{u}_i+\dot{w}_i)+(t_{ji}-s_{ji})n_j\dot{d}_i]=0 \text{ on } \partial B\times I. \tag{3.12}$$

From (3.3), (3.11), (3.12) and the initial data we obtain

$$\begin{aligned}\int_B\{&\frac{1}{2}(\rho_1^0\dot{\mathbf{u}}^2+\rho_2^0\dot{\mathbf{w}}^2)+\Phi(t,0;t)-\int_0^t\Psi(\tau,0;\tau)d\tau-\\ &-\int_0^t \Psi(t,\tau;t-\tau)d\tau+\frac{1}{2}\int_0^t\int_0^t\Gamma(r,s;|r-s|)drds\}dv=0.\end{aligned} \tag{3.13}$$

It follows from (3.13) and the hypotheses of the theorem that $\dot{\mathbf{u}}=\mathbf{0}, \dot{\mathbf{w}}=\mathbf{0}$ on $B\times I$. In view of the initial data we conclude that $\mathbf{u}$ and $\mathbf{w}$ vanish on $B\times I$. □

The existence of a generalized solution can be studied by using the method given by Dafermos in [23].

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