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Singular Surfaces in Thermoviscoelastic Materials with Voids

S. DE CICCO and L. NAPPA*
Dipartimento di Costruzioni e Metodi Matematici in Architettura, Università di Napoli, Via Monteoliveto 3, 80134, Napoli, Italy. E-mail: nappa@unina.it

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Abstract. In the first part of the paper we derive a linear theory of thermoviscoelastic materials with voids. Then, the propagation conditions and growth equations, which govern the propagation of singular surfaces of order 1 are derived and discussed. The coupling between the discontinuities in the mechanical and thermal fields are studied.

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1. Introduction

The theory of elastic materials with voids is a recent generalization of the classical theory of elasticity. The intended applications of the theory are to geological materials and to manufactured porous materials. The nonlinear theory of elastic materials with voids has been established by Nunziato and Cowin [1]. In this theory, the bulk density is written as the product of two fields, the matrix material field and the volume fraction field. This representation introduces an additional degree of kinematic freedom. The linear theory of elastic materials with voids has been established by Cowin and Nunziato [2]. Extensions of this theory to viscoelastic materials and thermoelastic materials exist (cf. [3, 4]).

This paper is concerned with a linear theory of thermoviscoelastic materials with voids. In the first part of the paper we use the results established by Day [5] and Gurtin [6] to give the necessary and sufficient conditions that the infinitesimal entropy production be invariant under time-reversal. We present a linear theory of thermoviscoelastic materials with voids in which the heat flux is independent on the present temperature gradient, but depends upon the past history of this gradient. In the second part of the paper we study the propagation conditions and growth equations which govern the propagation of singular surfaces of order 1 in the case of linear homogeneous isotropic thermoviscoelastic materials with voids. In the

* Corresponding author.

isothermal case, the wave propagation in materials with voids has been studied in various papers (see, e.g., [7–9]).

2. Preliminaries

We consider a body which at time t^0 occupies the region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The configuration of the body at time t^0 is taken as reference configuration. The motion of the body is referred to the reference configuration and a fixed system of rectangular Cartesian axes. We use vector and Cartesian tensor notation with Latin indices having the value 1, 2, 3. Greek indices are confined to the range $(1, 2)$. Letters in boldface stand for tensors of an order $p \geqslant 1$, and if $\boldsymbol{v}$ has the order p, we write $v_{ij\ldots k}$ (p subscripts) for the components of $\boldsymbol{v}$ in the underlying rectangular Cartesian coordinate system. In all that follows, subscripts preceded by a comma denote partial differentiation with respect to corresponding material coordinate. Also, we use a superposed dot to denote differentiation with respect to t holding the material coordinates fixed. The position of a typical particle of the body at time t is $\boldsymbol{x}$. The motion of continuum is described by the mappings

$$x_i = x_i(X_j, t), \quad (X_j, t) \in B \times I, \tag{2.1}$$

where I is a time interval. The above functions are assumed to be sufficiently smooth for the ensuing analysis to be valid. The concept of a distributed body asserts that the mass density at time t has the decomposition

$$\rho = \nu\gamma, \tag{2.2}$$

where γ is the density of the matrix material and ν is the volume fraction field. The relation (2.2) also holds for the reference configuration, $\rho_0 = \nu_0\gamma_0$ where ρ_0 is the density at time t^0 and ν_0 is the volume fraction field for the reference configuration.

The law of balance of linear momentum leads to the equations

$$T_{ji,j} + \rho_0 b_i = \rho_0 \ddot{x}_i, \tag{2.3}$$

where T_{ji} is the first Piola–Kirchhoff stress tensor and $\boldsymbol{b}$ is the body force per unit mass. The local form of the law of balance of equilibrated forces can be written as

$$H_{i,i} + P + \rho_0 l = \rho_0 \kappa \ddot{\nu}, \tag{2.4}$$

where H_j is the equilibrated stress associated with surfaces in B which were originally coordinate planes perpendicular to the X_j-axes through the point $\boldsymbol{X}$, measured per unit area of these planes, P is the intrinsic equilibrated body force per unit mass, l is the extrinsic equilibrated body force per unit mass and κ is the equilibrated inertia.

The equation of moment of momentum is

$$\varepsilon_{ijk} x_{j,s} T_{sk} = 0, \tag{2.5}$$

where ε_{ijk} is the alternating symbol.

Let e denote the internal energy per unit mass, and let s denote the external heat supply per unit mass, per unit time. The balance of energy is given by

$$\rho_0 \dot{e} = T_{ki}\dot{x}_{i,k} + H_i \dot{\nu}_{,i} - P\dot{\nu} + \rho_0 s + Q_{i,i}, \tag{2.6}$$

where $\boldsymbol{Q}$ is the heat flux vector. Let η denote the entropy per unit mass, and let T (> 0) denote the absolute temperature. The entropy production rate is defined by

$$\gamma = \rho_0 \dot{\eta} - \frac{1}{T}\rho_0 s - \left(\frac{1}{T} Q_i\right)_{,i}. \tag{2.7}$$

In the remainder of this section we follow closely the definitions presented by Gurtin [6]. Let U and W be two finite-dimensional inner product spaces. We write $L(U, W)$ for the space of all linear transformations from U into W. The transpose of $\boldsymbol{A} \in L(U, W)$ is denoted by $\boldsymbol{A}^{\mathrm{T}}$. We write $\boldsymbol{A}[\boldsymbol{u}]$ for the action of $\boldsymbol{A}$ on $\boldsymbol{u} \in U$. Let $\mathbb{R}$ be the set of all reals. Let U_0 be an open subset of U. A closed path in U_0 starting from $\boldsymbol{v} \in U_0$ is a smooth function $\boldsymbol{f}\colon \mathbb{R} \to U_0$ such that $\boldsymbol{f}(t) = \boldsymbol{v}$ whenever $t < -t_0$ or $t > t_0$, for some $t_0 > 0$. We write $G(U_0, \boldsymbol{v})$ for the set of all closed paths in U_0 starting from $\boldsymbol{v} \in U_0$. The history up to time t of a closed path in U_0 is the function $\boldsymbol{f}^t\colon [0, \infty) \to U_0$ defined by $\boldsymbol{f}^t(s) = \boldsymbol{f}(t-s)$ for every $s \geqslant 0$. We write $\Phi(U_0, \boldsymbol{v})$ for the set of all histories of paths belonging to $G(U_0, \boldsymbol{v})$. The time-reversal of a closed path $\boldsymbol{f} \in G(U_0, \boldsymbol{v})$ is the function $\bar{\boldsymbol{f}}\colon \mathbb{R} \to U_0$ defined by $\bar{\boldsymbol{f}}(t) = \boldsymbol{f}(-t)$, for every $t \in \mathbb{R}$. Let $F\colon \Phi(U_0, \boldsymbol{v}) \to W$, where $\boldsymbol{v} \in U_0$ is fixed. The functional F is smooth in time if given any $\boldsymbol{f} \in G(U_0, \boldsymbol{v})$ the function $t \to F(\boldsymbol{f}^t)$ is smooth. We say that F has a derivative δF at $\boldsymbol{v}$ if given any $\boldsymbol{f} \in G(U_0, \boldsymbol{0})$ the limit

$$\delta F(\boldsymbol{f}^t) = \lim_{\alpha \to 0} \frac{1}{\alpha}[F(\boldsymbol{v} + \alpha \boldsymbol{f}^t) - F(\boldsymbol{v})],$$

exists uniformly for $t \in \mathbb{R}$ and the function $t \to \delta F(\boldsymbol{f}^t)$ is continuous. We say that F has the relaxation property at $\boldsymbol{v}$ if given any $\boldsymbol{f} \in G(U_0, \boldsymbol{v})$ we have $\lim_{t\to\infty} F(\boldsymbol{f}^t) = F(\boldsymbol{v})$. Let V be the three-dimensional vector space. We use the notations $J = L(V, V)$, $J^* = \{\boldsymbol{M} \in V \mid \boldsymbol{M} \text{ is symmetric}\}$ and $J^+ = \{\boldsymbol{M} \in J \mid \det \boldsymbol{M} > 0\}$. By a site we mean the ordered collection of functions $\sigma = \{\mathbf{F}, \nu, \mathbf{G}, T, \mathbf{g}\}$, where $\boldsymbol{F} = (F_{ij})$, $\boldsymbol{G} = (G_i)$, $\boldsymbol{g} = (g_i)$, $F_{ij} = x_{i,j}$, $G_i = \nu_{,i}$, $g_i = T_{,i}$. Clearly, $F \in J^+$, $\boldsymbol{g}, \boldsymbol{G} \in V$, $T \in \mathbb{R}^+$. We denote by T_0 the constant absolute temperature of the body in its reference state. The site $\sigma_0 = \{\mathbf{I}, \nu_0, \mathbf{0}, T_0, \mathbf{0}\}$, where $\mathbf{I} = (\delta_{ij})$, is called the equilibrium site.

We introduce the displacement vector field $\boldsymbol{u}$ defined by $u_i = x_i - X_i$. We define the temperature change ϑ by $\vartheta = T - T_0$, and the volume fraction change φ by $\varphi = \nu - \nu_0$. We say that $\boldsymbol{u}$, φ and ϑ are infinitesimal if $\boldsymbol{u} = \varepsilon \boldsymbol{u}'$, $\varphi = \varepsilon \varphi'$, $\vartheta = \varepsilon \vartheta'$, where ε is a constant small enough for squares and higher powers to be neglected, and $\boldsymbol{u}'$, φ' and ϑ' are independent of ε. By an infinitesimal site we mean the following collection $\zeta = \{\boldsymbol{L}, \varphi, \boldsymbol{G}, \vartheta, \boldsymbol{g}\}$, $\boldsymbol{L} = (u_{i,j})$, $\boldsymbol{G} = (\varphi_{,i})$, $\boldsymbol{g} = (\vartheta, i)$, where $\boldsymbol{u}$, φ and ϑ are infinitesimal.

We write Z for the set of all sites. A one-parameter family $\sigma(t) = \{\boldsymbol{F}(t), \nu(t), \boldsymbol{G}(t), T(t), \boldsymbol{g}(t)\}$ $(-\infty < t < \infty)$ of sites is called a closed process if $\sigma(\cdot)$ is a closed path in Z starting from σ_0. Let Z^* be the set of all infinitesimal sites. A one-parameter family $\zeta(t)$ $(-\infty < t < \infty)$ of infinitesimal sites is an infinitesimal closed process if $\zeta(\cdot)$ is a closed path in Z^* starting from zero.

The infinitesimal strain tensor is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.8}$$

3. Constitutive Equations

In the linearized theory of viscoelastic materials, Day [5] proved that the work done in every closed strain path starting from zero is invariant under time-reversal if and only if the stress relaxation function is symmetric. In [6], Gurtin derived an extension of Day's result within the context of thermodynamics of materials with memory. Throughout this section we use the results of Gurtin [6] to establish conditions that are both necessary and sufficient for the infinitesimal entropy production to be invariant under time-reversal.

Let S_{ij} be the second Piola–Kirchhoff stress tensor,

$$T_{ki} = S_{kj} x_{i,j}. \tag{3.1}$$

Equations (2.5) become

$$S_{ij} = S_{ji}, \tag{3.2}$$

and the balance of energy can be written in the form

$$\rho_0 \dot{e} = S_{ij} \dot{E}_{ij} + H_i \dot{\nu}_{,i} - P\dot{\nu} + \rho_0 s + Q_{i,i}, \tag{3.3}$$

where

$$E_{ij} = \frac{1}{2}(x_{m,i} x_{m,j} - \delta_{ij}). \tag{3.4}$$

With the help of (3.3), the entropy production rate becomes

$$\gamma = \rho_0 \dot{\eta} - \frac{1}{T}\rho_0 \dot{e} + \frac{1}{T}(S_{ij}\dot{E}_{ij} + H_i \dot{\nu}_{,i} - P\dot{\nu}) + \frac{1}{T^2} Q_i T_{,i}. \tag{3.5}$$

In what follows we study materials defined by constitutive relations giving the stresses $T_{ij}(t)$, $H_i(t)$, the internal energy $e(t)$, the intrinsic equilibrated body force $P(t)$, the entropy $\eta(t)$, and the heat flux $Q_i(t)$ at time t when the site-history σ^t is known,

$$\begin{aligned} T_{ij}(t) &= \widehat{T}_{ij}(\sigma^t), \qquad H_i(t) = \widehat{H}_i(\sigma^t), \qquad P(t) = \widehat{P}(\sigma^t), \\ e(t) &= \hat{e}(\sigma^t), \qquad \eta(t) = \hat{\eta}(\sigma^t), \qquad Q_i(t) = \widehat{Q}_i(\sigma^t). \end{aligned} \tag{3.6}$$

The response functionals have $\Phi(Z, \sigma_0)$ as their common domain. We assume that

(i) the response functionals are smooth in time;
(ii) $\hat{e}$, $\widehat{T}_{ij}$, $\widehat{H}_i$, $\widehat{P}$ and $\widehat{Q}_i$ have derivative $\delta\hat{e}$, $\delta\widehat{T}_{ij}$, $\delta\widehat{P}$ and $\delta\widehat{Q}_i$ at σ_0;
(iii) $\hat{e}$ and $\hat{\eta}$ have the relaxation property at σ_0;
(iv) $\widehat{\boldsymbol{T}}(\sigma_0) = \boldsymbol{0}$, $\widehat{\boldsymbol{H}}(\sigma_0) = \boldsymbol{0}$, $\widehat{P}(\sigma_0) = 0$, $\widehat{\boldsymbol{Q}}(\sigma_0) = \boldsymbol{0}$.

Let $\sigma(\cdot) = \{\boldsymbol{F}(\cdot), \nu(\cdot), \boldsymbol{G}(\cdot), T(\cdot), \boldsymbol{g}(\cdot)\}$ be a closed process. We define the entropy production $\Gamma\{\sigma(\cdot)\}$ on $\sigma(\cdot)$ by

$$\Gamma\{\sigma(\cdot)\} = \int_{-\infty}^{\infty}\left\{\rho_0\dot{\eta} - \frac{1}{T}\rho_0\dot{e} + \frac{1}{T}(S_{ij}\dot{E}_{ij} + H_i\dot{\nu}_{,i} - P\dot{\nu}) + \frac{1}{T^2}Q_iT_{,i}\right\}\mathrm{d}t. \tag{3.7}$$

Since $\sigma(\cdot)$ is a closed process, we conclude from (3.7) and (iii) that

$$\Gamma\{\sigma(\cdot)\} = \int_{-\infty}^{\infty}\frac{1}{T}\left\{S_{ij}\dot{E}_{ij} + H_i\dot{\nu}_{,i} - P\dot{\nu} - \frac{1}{T}\rho_0\dot{T}e + \frac{1}{T}Q_iT_{,i})\right\}\mathrm{d}t. \tag{3.8}$$

Let $\zeta(\cdot) = \{\boldsymbol{L}(\cdot), \varphi(\cdot), \boldsymbol{G}(\cdot), \vartheta(\cdot), \boldsymbol{g}(\cdot)\}$ be an infinitesimal closed process, and let

$$\begin{aligned} e^*(t) &= \delta\hat{e}(\zeta^t), \qquad t_{ij}(t) = \delta T_{ij}(\zeta^t), \qquad h_i(t) = \delta H_i(\zeta^t),\\ g(t) &= \delta\widehat{P}(\zeta^t), \qquad q_i(t) = \delta\widehat{Q}_i(\zeta^t). \end{aligned} \tag{3.9}$$

We define the infinitesimal entropy production $\Omega\{\zeta(\cdot)\}$ on $\zeta(\cdot)$ by (cf. [6])

$$\Omega\{\zeta(\cdot)\} = \frac{1}{T_0^2}\int_{-\infty}^{\infty}\{T_0t_{ij}\dot{e}_{ij} + T_0h_i\dot{\varphi}_{,i} - T_0g\dot{\varphi} - \rho_0e^*\dot{\vartheta} + q_i\vartheta_{,i}\}\,\mathrm{d}t. \tag{3.10}$$

We can prove that if the entropy production Γ is invariant under time-reversal, then the infinitesimal entropy production Ω is invariant under time-reversal. Following Gurtin [6] we call

$$\boldsymbol{\Lambda} = (e_{ij}, \varphi_{,i}, \varphi, \vartheta)$$

the generalized infinitesimal strain, and

$$\boldsymbol{\Sigma} = (T_0t_{ij}, T_0h_i, -T_0g, -\rho_0e^*)$$

the generalized infinitesimal stress. The quantities $\boldsymbol{\Lambda}$ and $\boldsymbol{\Sigma}$ are both elements of $W = J^* \times V \times \mathbb{R} \times \mathbb{R}$. If we introduce the notation

$$\boldsymbol{\Sigma}\cdot\dot{\boldsymbol{\Lambda}} = T_0(t_{ij}\dot{e}_{ij} + h_i\dot{\varphi}_{,i} - g\dot{\varphi}) - \rho_0e^*\dot{\vartheta},$$

then relation (3.10) can be written in the form

$$\Omega\{\zeta(\cdot)\} = \frac{1}{T_0^2}\int_{-\infty}^{\infty}(\boldsymbol{\Sigma}\cdot\dot{\boldsymbol{\Lambda}} + \boldsymbol{q}\cdot\boldsymbol{g})\,\mathrm{d}t. \tag{3.11}$$

We assume that there exist the continuous functions $\boldsymbol{M}$: $[0,\infty) \to L(W, W)$, $\boldsymbol{Y}$: $[0,\infty) \to L(V, W)$, $\boldsymbol{Z}$: $[0,\infty) \to L(W, V)$, $\boldsymbol{K}$: $[0,\infty) \to L(V, V)$, such that given any infinitesimal closed process $\zeta(\cdot) = \{\boldsymbol{L}(\cdot), \varphi(\cdot), \boldsymbol{G}(\cdot), \vartheta(\cdot), \boldsymbol{g}(\cdot)\}$ we have

$$\begin{aligned}\boldsymbol{\Sigma}(t) = \delta\widehat{\boldsymbol{\Sigma}}(\zeta^t) &= \int_{-\infty}^{t} \left\{\boldsymbol{M}(t-s)[\dot{\boldsymbol{\Lambda}}(s)] + \boldsymbol{Y}(t-s)[\boldsymbol{g}(s)]\right\} \mathrm{d}s,\\ \boldsymbol{q}(t) = \delta\widehat{\boldsymbol{Q}}(\zeta^t) &= \int_{-\infty}^{t} \left\{\boldsymbol{Z}(t-s)[\dot{\boldsymbol{\Lambda}}(s)] + \boldsymbol{K}(t-s)[\boldsymbol{g}(s)]\right\} \mathrm{d}s.\end{aligned} \tag{3.12}$$

The dependence on $\boldsymbol{L}$ only through e_{ij} is a consequence of the principle of material frame-indifference and (iv).

THEOREM 1. *A necessary and sufficient condition that the infinitesimal entropy production be invariant under time-reversal is that the following statements be true for every* $s \geqslant 0$:
(a) *the function* $\boldsymbol{M}$ *is symmetric;*
(b) *the function* $\boldsymbol{K}$ *is symmetric;*
(c) $\boldsymbol{Y}(s) = -\boldsymbol{Z}^{\mathrm{T}}(s) + $ *constant.*

The proof of the theorem is identical to that of Theorem 3 from [6].

In what follows we assume that the infinitesimal entropy production is invariant under time-reversal. Equations (3.12) can be expressed as

$$\begin{aligned}
t_{ij}(t) &= \int_{-\infty}^{t} \big\{G_{ijmn}(t-s)\dot{e}_{mn}(s) + D_{ijm}(t-s)\dot{\varphi}_{,m}(s)\\
&\quad + B_{ij}(t-s)\dot{\varphi}(s) - b_{ij}(t-s)\dot{\vartheta}(s) - L_{ijm}(t-s)\vartheta_{,m}(s)\big\}\,\mathrm{d}s,\\
h_i(t) &= \int_{-\infty}^{t} \big\{D^*_{imn}(t-s)\dot{e}_{mn}(s) + A_{ij}(t-s)\dot{\varphi}_{,j}(s)\\
&\quad + D_i(t-s)\dot{\varphi}(s) - d_i(t-s)\dot{\vartheta}(s) - N_{ij}(t-s)\vartheta_{,j}(s)\big\}\,\mathrm{d}s,\\
g(t) &= \int_{-\infty}^{t} \big\{B^*_{ij}(t-s)\dot{e}_{ij}(s) + D^*_i(t-s)\dot{\varphi}_{,i}(s) - M(t-s)\dot{\varphi}(s)\\
&\quad + m(t-s)\dot{\vartheta}(s) - M_i(t-s)\vartheta_{,j}(s)\big\}\,\mathrm{d}s,\\
\rho_0 e^*(t) &= \int_{-\infty}^{t} \big\{b^*_{ij}(t-s)\dot{e}_{ij}(s) + C^*_i(t-s)\dot{\varphi}_{,i}(s) + D(t-s)\dot{\varphi}(s)\\
&\quad + E(t-s)\dot{\vartheta}(s) + T_0 R_i(t-s)\vartheta_{,i}(s)\big\}\,\mathrm{d}s,\\
q_i(t) &= \int_{-\infty}^{t} \big\{L^*_{imn}(t-s)\dot{e}_{mn}(s) + F^*_{ij}(t-s)\dot{\varphi}_{,j}(s) + M^*_i(t-s)\dot{\varphi}(s)\\
&\quad + R^*_i(t-s)\dot{\vartheta}(s) + K_{ij}(t-s)\vartheta_{,j}(s)\big\}\,\mathrm{d}s.
\end{aligned} \tag{3.13}$$

Theorem 1 has the following immediate consequences.

COROLLARY 1. *For every $s \geqslant 0$,*

$$\begin{aligned}
&G_{ijmn}(s) = G_{mnij}(s), \qquad D^*_{imn}(s) = D^*_{mni}(s), \qquad A_{ij}(s) = A_{ji}(s),\\
&B^*_{ij}(s) = -B_{ij}(s), \qquad D^*_i(s) = -D_i(s), \qquad b^*_{ij}(s) = T_0 b_{ij}(s),\\
&C^*_i(s) = T_0 d_i(s), \qquad D(s) = T_0 m(s),\\
&L^*_{imn}(s) = T_0 L_{mni}(s) + \text{constant},\\
&F^*_{ij}(s) = T_0 N_{ji}(s) + \text{constant}, \qquad M^*_i(s) = T_0 M_i(s) + \text{constant},\\
&R^*_i(s) = T_0 R_i(s) + \text{constant}, \qquad K_{ij}(s) = K_{ji}(s).
\end{aligned} \tag{3.14}$$

The equilibrium conductivity tensor

$$\boldsymbol{K}_\infty = \int_0^\infty \boldsymbol{K}(s)\,\mathrm{d}s,$$

if it exists, is symmetric. In what follows we assume that $\boldsymbol{L}_\infty$, $\boldsymbol{N}_\infty$, $\boldsymbol{M}_\infty$, $\boldsymbol{R}_\infty$, $\boldsymbol{L}^*_\infty$, $\boldsymbol{F}^*_\infty$, $\boldsymbol{M}^*_\infty$ and $\boldsymbol{R}^*_\infty$ are equal to zero. Then, following Gurtin [6], we can prove that

$$\begin{aligned}
&L^*_{jmn}(s) = T_0 L_{mnj}(s), \qquad F^*_{ij}(s) = T_0 N_{ji}(s),\\
&M^*_i(s) = T_0 M_i(s), \qquad R^*_i(s) = T_0 R_i(s),
\end{aligned} \tag{3.15}$$

for every $s \geqslant 0$.

We assume that $\hat{\eta}$ has a derivative at σ_0. We can show that for every infinitesimal process we have

$$\begin{aligned}
\rho_0 \eta(t) &= \rho_0 \delta\hat{\eta}(\zeta^t)\\
&= \int_{-\infty}^{t} \big\{ b_{ij}(t-s)\dot{e}_{ij}(s) + d_i(t-s)\dot{\varphi}_{,i}(s)\\
&\qquad + m(t-s)\dot{\varphi}(s) + a(t-s)\dot{\vartheta}(s) + R_i(t-s)\vartheta_{,i}(s) \big\}\,\mathrm{d}s,
\end{aligned} \tag{3.16}$$

for every $t \in \mathbb{R}$, where $T_0 a(s) = E(s)$, $s \geqslant 0$.

We conclude that, in the context of the linear theory, the constitutive equations consist of (3.16) and

$$\begin{aligned}
t_{ij}(t) = \int_{-\infty}^{t} &\big\{ G_{ijmn}(t-s)\dot{e}_{mn}(s) + D_{ijm}(t-s)\dot{\varphi}_{,m}(s)\\
&+ B_{ij}(t-s)\dot{\varphi}(s) - b_{ij}(t-s)\dot{\vartheta}(s) - L_{ijm}(t-s)\vartheta_{,m}(s) \big\}\,\mathrm{d}s,
\end{aligned}$$

$$
\begin{aligned}
h_i(t) &= \int_{-\infty}^{t} \big\{ D_{mni}(t-s)\dot{e}_{mn}(s) + A_{ij}(t-s)\dot{\varphi}_{,j}(s) + D_i(t-s)\dot{\varphi}(s) \\
&\quad - d_i(t-s)\dot{\vartheta}(s) - N_{ij}(t-s)\vartheta_{,j}(s) \big\}\, \mathrm{d}s, \\
g(t) &= -\int_{-\infty}^{t} \big\{ B_{ij}(t-s)\dot{e}_{ij}(s) + D_i(t-s)\dot{\varphi}_{,i}(s) + M(t-s)\dot{\varphi}(s) \\
&\quad - m(t-s)\dot{\vartheta}(s) + M_i(t-s)\vartheta_{,j}(s) \big\}\, \mathrm{d}s, \\
\rho_0 e^*(t) &= T_0 \int_{-\infty}^{t} \big\{ b_{ij}(t-s)\dot{e}_{ij}(s) + d_i(t-s)\dot{\varphi}_{,i}(s) + m(t-s)\dot{\varphi}(s) \\
&\quad + a(t-s)\dot{\vartheta}(s) + R_i(t-s)\vartheta_{,i}(s) \big\}\, \mathrm{d}s, \\
q_i(t) &= \int_{-\infty}^{t} \big\{ T_0 L_{mni}(t-s)\dot{e}_{mn}(s) + T_0 N_{ji}(t-s)\dot{\varphi}_{,j}(s) \\
&\quad + T_0 M_i(t-s)\dot{\varphi}(s) + T_0 R_i(t-s)\dot{\vartheta}(s) + K_{ij}(t-s)\vartheta_{,j}(s) \big\}\, \mathrm{d}s.
\end{aligned}
\tag{3.17}
$$

We note that (3.2) and (3.14) imply

$$
\begin{aligned}
&G_{ijmn}(s) = G_{jimn}(s) = G_{mnij}(s), \qquad && D_{ijm}(s) = D_{jim}(s), \\
&b_{ij}(s) = b_{ji}(s), && L_{ijm}(s) = L_{jim}(s), \\
&A_{ij}(s) = A_{ji}(s), && K_{ij}(s) = K_{ji}(s),
\end{aligned}
\tag{3.18}
$$

for every $s \geqslant 0$.

In what follows we consider the initial history conditions

$$
\boldsymbol{u}(\boldsymbol{X}, t) = 0, \qquad \varphi(\boldsymbol{X}, t) = 0, \qquad \vartheta(\boldsymbol{X}, t) = 0, \quad (\boldsymbol{X}, t) \in B \times (-\infty, 0).
\tag{3.19}
$$

In the relations (3.17) the functions t_{ij}, h_i, g and e^* are independent of the history of the temperature gradient if and only if the heat flux is independent of the histories of e_{ij}, φ, $\varphi_{,j}$ and ϑ. In the case of centro-symmetric materials the constitutive moduli D_{ijr}, L_{ijm}, D_i, d_i, M_i and R_i are equal to zero. Coleman and Gurtin [10] have shown that, under the assumption of fading memory, the equilibrium heat flux vanishes when the temperature gradient vanishes. In what follows we consider constitutive equations which are consistent with this result and restrict our attention to homogeneous and isotropic materials. Thus, we have

$$
\begin{aligned}
&G_{ijrs}(\boldsymbol{X}, t) = \lambda(t)\delta_{ij}\delta_{rs} + \mu(t)(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}), \\
&B_{ij}(\boldsymbol{X}, t) = b(t)\delta_{ij}, \quad b_{ij}(\boldsymbol{X}, t) = \beta(t)\delta_{ij},
\end{aligned}
$$

$$\begin{aligned}
&D_{ijr}(\boldsymbol{X},t)=0, && L_{ijr}(\boldsymbol{X},t)=0, && D_i(\boldsymbol{X},t)=0,\\
&N_{ij}(\boldsymbol{X},t)=0, && d_i(\boldsymbol{X},t)=0, && M_i(\boldsymbol{X},t)=0,\\
&R_i(\boldsymbol{X},t)=0, && A_{ij}(\boldsymbol{X},t)=\alpha(t)\delta_{ij}, && K_{ij}(\boldsymbol{X},t)=k(t)\delta_{ij},\\
&(\boldsymbol{X},t)\in B\times(-\infty,\infty),
\end{aligned} \tag{3.20}$$

where δ_{ij} is the Kronecker delta. We assume that the relaxation functions λ, μ, β, b, α and k are of class C^2 on $(-\infty,\infty)$. With the help of (3.19) the constitutive equations become

$$\begin{aligned}
t_{ij}(\boldsymbol{X},t) &= G_{ijmn}(0)e_{mn}(\boldsymbol{X},t)+B_{ij}(0)\varphi(\boldsymbol{X},t)-b_{ij}(0)\vartheta(\boldsymbol{X},t)\\
&\quad+\int_0^t\big\{\dot{G}_{ijmn}(t-s)e_{mn}(\boldsymbol{X},s)+\dot{B}_{ij}(t-s)\varphi(\boldsymbol{X},s)\\
&\quad-\dot{b}_{ij}(t-s)\vartheta(\boldsymbol{X},s)\big\}\,\mathrm{d}s,\\
h_i(\boldsymbol{X},t) &= A_{ij}(0)\varphi_{,j}(\boldsymbol{X},t)+\int_0^t\dot{A}_{ij}(t-s)\varphi_{,j}(\boldsymbol{X},s)\,\mathrm{d}s,\\
g(\boldsymbol{X},t) &= -B_{ij}(0)e_{ij}(\boldsymbol{X},t)-M(0)\varphi(\boldsymbol{X},t)+m(0)\vartheta(\boldsymbol{X},t)\\
&\quad+\int_0^t\big\{\dot{m}(t-s)\vartheta(\boldsymbol{X},s)-\dot{B}_{ij}(t-s)e_{ij}(\boldsymbol{X},s)\\
&\quad-\dot{M}(t-s)\varphi(\boldsymbol{X},s)\big\}\,\mathrm{d}s,\\
\rho_0e^*(\boldsymbol{X},t) &= \rho_0T_0\eta(\boldsymbol{X},t)\\
&= T_0\bigg\{b_{ij}(0)e_{ij}(\boldsymbol{X},t)+m(0)\varphi(\boldsymbol{X},t)+a(0)\vartheta(\boldsymbol{X},t)\\
&\quad+\int_0^t\big[\dot{b}_{ij}(t-s)e_{ij}(\boldsymbol{X},s)+\dot{m}(t-s)\varphi(\boldsymbol{X},s)\\
&\quad+\dot{a}(t-s)\vartheta(\boldsymbol{X},s)\big]\,\mathrm{d}s\bigg\},\\
q_i(\boldsymbol{X},t) &= \int_0^t K_{ij}(t-s)\vartheta_{,j}(\boldsymbol{X},s)\,\mathrm{d}s,
\end{aligned} \tag{3.21}$$

where the constitutive moduli have the form (3.20) and we have used the notation $G_{ijrs}(0)=G_{ijrs}(\boldsymbol{X},0)$, etc.

We assume that

$$\begin{aligned}
&\lambda(0)+2\mu(0)>0, && \mu(0)>0, && \alpha(0)>0, && k(0)>0,\\
&a(0)>0, && \dot{\lambda}(0)+2\dot{\mu}(0)<0, && \dot{\mu}(0)<0, && \dot{\alpha}(0)<0.
\end{aligned} \tag{3.22}$$

The restrictions concerning λ, μ, $\dot{\lambda}$, $\dot{\mu}$, a and k have been extensively studied in the classical thermoviscoelasticity [11–14]. Let us show that the restrictions on α and

$\dot{\alpha}$ are compatible with the second law of thermodynamics. In the linear theory, the second law of thermodynamics may be written in the form

$$\int_0^t \left(\rho_0 \dot{\eta} \vartheta + t_{ij} \dot{e}_{ij} + h_i \dot{\varphi}_{,i} - g \dot{\varphi} + \frac{1}{T_0} q_i \vartheta_{,i} \right) \mathrm{d}\tau \geqslant 0, \tag{3.23}$$

for every $t \geqslant 0$. We restrict our attention only to porosity effect. In this case, the relation (3.23) reduces to the following dissipation inequality

$$U(\boldsymbol{X}, t) \geqslant 0, \quad (\boldsymbol{X}, t) \in B \times [0, \infty), \tag{3.24}$$

where the function U is defined by

$$U(\boldsymbol{X}, t) = \int_0^t [h_i(\boldsymbol{X}, \tau) \dot{\varphi}_{,i}(\boldsymbol{X}, \tau) - g(\boldsymbol{X}, \tau) \dot{\varphi}(\boldsymbol{X}, \tau)] \, \mathrm{d}\tau, \quad (\boldsymbol{X}, t) \in B \times [0, \infty). \tag{3.25}$$

Clearly, we have

$$U(t) = h_i(t)\varphi_{,i}(t) - g(t)\varphi(t) + \int_0^t [\dot{g}(\tau)\varphi(\tau) - \dot{h}_i(\tau)\varphi_{,i}(\tau)] \, \mathrm{d}\tau, \quad t \geqslant 0, \tag{3.26}$$

where, for convenience, we have suppressed the argument $\boldsymbol{X}$. In view of the constitutive equations (3.21) and the initial conditions, we find that

$$\begin{aligned} \dot{h}_i(t) &= A_{ij}(0)\dot{\varphi}_{,j}(t) + \dot{A}_{ij}(0)\varphi_{,j}(t) + \int_0^t \ddot{A}_{ij}(t-s)\varphi_{,j}(s) \, \mathrm{d}s, \\ \dot{g}(t) &= -M(0)\dot{\varphi}(t) - \dot{M}(0)\varphi(t) - \int_0^t \ddot{M}(t-s)\varphi(s) \, \mathrm{d}s. \end{aligned} \tag{3.27}$$

Thus, from (3.21), (3.27) and (3.26) we obtain

$$\begin{aligned} U(t) = {} & \frac{1}{2}[A_{ij}(0)\varphi_{,j}(t)\varphi_{,i}(t) + M(0)\varphi^2(t)] \\ & - \int_0^t [\dot{A}_{ij}(0)\varphi_{,j}(s)\varphi_{,i}(s) + \dot{M}(0)\varphi^2(s)] \, \mathrm{d}s \\ & + \varphi_{,i}(t) \int_0^t \dot{A}_{ij}(t-s)\varphi_{,j}(s) \, \mathrm{d}s + \varphi(t) \int_0^t \dot{M}(t-s)\varphi(s) \, \mathrm{d}s \\ & - \int_0^t \left\{ \varphi_{,i}(s) \int_0^s \ddot{A}_{ij}(s-\tau)\varphi_{,j}(\tau) \, \mathrm{d}\tau \right. \\ & \left. + \varphi(s) \int_0^s \ddot{M}(s-\tau)\varphi(\tau) \, \mathrm{d}\tau \right\} \mathrm{d}s, \quad t \geqslant 0. \end{aligned} \tag{3.28}$$

Now we use the following identities [15]

$$
\begin{aligned}
&2\varphi(t)\int_0^t \dot{M}(t-s)\varphi(s)\,\mathrm{d}s\\
&\quad=\int_0^t \dot{M}(t-s)\varphi^2(s)\,\mathrm{d}s\\
&\qquad-\int_0^t \dot{M}(t-s)[\varphi(t)-\varphi(s)]^2\,\mathrm{d}s+[M(t)-M(0)]\varphi^2(t),\\
&2\varphi_{,i}(t)\int_0^t \dot{A}_{ij}(t-s)\varphi_{,j}(s)\,\mathrm{d}s\\
&\quad=\int_0^t \dot{A}_{ij}(t-s)\varphi_{,i}(s)\varphi_{,j}(s)\,\mathrm{d}s\\
&\qquad-\int_0^t \dot{A}_{ij}(t-s)[\varphi_{,j}(t)-\varphi_{,j}(s)]\\
&\qquad\times[\varphi_{,i}(t)-\varphi_{,i}(s)]\,\mathrm{d}s+[A_{ij}(t)-A_{ij}(0)]\varphi_{,j}(t)\varphi_{,i}(t),\\
&2\int_0^t\!\!\int_0^s \ddot{A}_{ij}(s-\tau)\varphi_{,i}(s)\varphi_{,j}(\tau)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad=\int_0^t\!\!\int_0^t \ddot{A}_{ij}(|s-\tau|)\varphi_{,i}(s)\varphi_{,j}(\tau)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad=\int_0^t\!\!\int_0^t \ddot{A}_{ij}(|s-\tau|)\varphi_{,i}(s)\varphi_{,j}(s)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\qquad-\frac{1}{2}\int_0^t\!\!\int_0^t \ddot{A}_{ij}(|s-\tau|)[\varphi_{,i}(s)-\varphi_{,i}(\tau)][\varphi_{,j}(s)-\varphi_{,j}(\tau)]\,\mathrm{d}s\,\mathrm{d}\tau,\\
&2\int_0^t\!\!\int_0^s \ddot{M}(s-\tau)\varphi(s)\varphi(\tau)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad=\int_0^t\!\!\int_0^t \ddot{M}(|s-\tau|)\varphi(s)\varphi(\tau)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad=\int_0^t\!\!\int_0^t \ddot{M}(|s-\tau|)\varphi^2(s)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\qquad-\frac{1}{2}\int_0^t\!\!\int_0^t \ddot{M}(|s-\tau|)[\varphi(s)-\varphi(\tau)]^2\,\mathrm{d}s\,\mathrm{d}\tau,\\
&\int_0^t \ddot{M}(|s-\tau|)\,\mathrm{d}\tau=\dot{M}(s)+\dot{M}(t-s)-2\dot{M}(0).
\end{aligned}
\tag{3.29}
$$

It follows from (3.28) and (3.29) that

$$
U(t)=\frac{1}{2}[A_{ij}(t)\varphi_{,j}(t)\varphi_{,i}(t)+M(t)\varphi^2(t)]
$$

$$
\begin{aligned}
&- \frac{1}{2} \int_0^t [\dot{A}_{ij}(s)\varphi_{,j}(s)\varphi_{,i}(s) + \dot{M}(s)\varphi^2(s)]\,\mathrm{d}s \\
&- \frac{1}{2} \int_0^t \big\{\dot{A}_{ij}(t-s)[\varphi_{,i}(t) - \varphi_{,i}(s)][\varphi_{,j}(t) - \varphi_{,j}(s)] \\
&+ \dot{M}(t-s)[\varphi(t) - \varphi(s)]^2\big\}\,\mathrm{d}s \\
&+ \frac{1}{4} \int_0^t\int_0^t \big\{\ddot{A}_{ij}(|s-\tau|)[\varphi_{,i}(s) - \varphi_{,i}(\tau)][\varphi_{,j}(s) - \varphi_{,j}(\tau)] \\
&+ \ddot{M}(|s-\tau|)[\varphi(s) - \varphi(\tau)]^2\big\}\,\mathrm{d}s\,\mathrm{d}\tau, \quad t \geqslant 0.
\end{aligned} \tag{3.30}
$$

In the case of isotropic bodies the relation (3.30) reduces to

$$
\begin{aligned}
U(t) = {} & \frac{1}{2}[\alpha(t)\varphi_{,i}(t)\varphi_{,i}(t) + M(t)\varphi^2(t)] \\
&- \frac{1}{2} \int_0^t [\dot{\alpha}(s)\varphi_{,i}(s)\varphi_{,i}(s) + \dot{M}(s)\varphi^2(s)]\,\mathrm{d}s \\
&- \frac{1}{2} \int_0^t \big\{\dot{\alpha}(t-s)[\varphi_{,i}(t) - \varphi_{,i}(s)][\varphi_{,i}(t) - \varphi_{,i}(s)] \\
&+ \dot{M}(t-s)[\varphi(t) - \varphi(s)]^2\big\}\,\mathrm{d}s\,\mathrm{d}\tau \\
&+ \frac{1}{4} \int_0^t\int_0^t \big\{\ddot{\alpha}(|s-\tau|)[\varphi_{,i}(s) - \varphi_{,i}(\tau)][\varphi_{,i}(s) - \varphi_{,i}(\tau)] \\
&+ \ddot{M}(|s-\tau|)[\varphi(s) - \varphi(\tau)]^2\big\}\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned} \tag{3.31}
$$

From (3.31) we conclude that the dissipation inequality (3.24) is satisfied if $\alpha \geqslant 0$, $M \geqslant 0$, $\dot{\alpha} \leqslant 0$, $\dot{M} \leqslant 0$, $\ddot{\alpha} \geqslant 0$, $\ddot{M} \geqslant 0$. Thus, our assumptions concerning the relaxation function α are compatible with the second law of thermodynamics. For convenience, in what follows we shall denote the material coordinates by (x_1, x_2, x_3).

4. Singular Surfaces

We consider an arbitrary open region Π in the continuum, bounded by a surface $\partial\Pi$, at time t, and we suppose that ω is the corresponding region in the domain occupied by the undeformed body. We note that in the framework of the linear theory the materials with voids behaves according to the global balance law of linear momentum

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho_0 v_i \,\mathrm{d}v = \int_\omega \rho_0 b_i \,\mathrm{d}v + \int_{\partial\omega} t_{ji} n_j \,\mathrm{d}a, \tag{4.1}
$$

the law of balance of equilibrated force

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho_0 \kappa \dot{\varphi} \,\mathrm{d}v = \int_\omega (\rho_0 l + g) \,\mathrm{d}v + \int_{\partial\omega} h_i n_i \,\mathrm{d}a, \tag{4.2}
$$

and the law of balance energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega}\rho_0 e^*\,\mathrm{d}v = \int_{\omega}\rho_0 s\,\mathrm{d}v + \int_{\partial\omega} q_i n_i\,\mathrm{d}a. \tag{4.3}$$

Here $\boldsymbol{n}$ is the outward unit normal to $\partial\omega$ and $v_i = \dot{u}_i$.

Let S be a moving surface defined by the equations

$$x_i = x_i(\theta^1, \theta^2, t),$$

where θ^1, θ^2 are curvilinear coordinates on the surface. We suppose that the above functions are continuously differentiable with respect to their arguments, and that S is smooth in the sense that the matrix $(\partial x_i/\partial\theta^\alpha)$ has rank two. The metric tensor of the surface is denoted by $a_{\alpha\beta}$. In what follows we denote by n_i the unit normal to S. We note that [16]

$$\begin{aligned} &n_i n_i = 1, \qquad n_i x_{i;\alpha} = 0, \qquad x_{i;\alpha\beta} = b_{\alpha\beta} n_i,\\ &n_{i;\alpha} = -a^{\lambda\rho} b_{\rho\alpha} x_{i;\lambda}, \end{aligned} \tag{4.4}$$

where indices followed by a semicolon represent covariant partial differentiation based on the metric of S, $b_{\alpha\beta}$ is the second fundamental form of the surface and $a^{\alpha\beta}$ are the elements of the inverse of matrix $(a_{\alpha\beta})$. We have

$$a^{\alpha\beta} x_{i;\alpha} x_{j;\beta} = \delta_{ij} - n_i n_j, \qquad H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}, \tag{4.5}$$

where H is the mean curvature of the surface.

Let f be a function on $B \times (-\infty, \infty)$. We assume that f is a continuously differentiable function on each side of the moving surface S. We denote by $[f]$ the jump of the function f across S. The discontinuities in the first and second derivative of f satisfy the relations [17]:

$$\begin{aligned} [f_{,i}] &= a^{\alpha\beta} A_{;\alpha} x_{i;\beta} + B n_i, \qquad [\dot{f}] = \frac{\delta A}{\delta t} - VB,\\ [f_{,ij}] &= a^{\alpha\beta}(B_{;\alpha} + a^{\lambda\rho} b_{\alpha\lambda} A_{;\rho})(n_i x_{j;\beta} + n_j x_{i;\beta})\\ &\quad + a^{\alpha\beta} a^{\nu\rho}(A_{;\alpha\nu} - b_{\alpha\nu} B) x_{i;\beta} x_{j;\rho} + C n_i n_j,\\ [\dot{f}_{,i}] &= a^{\alpha\beta}\left(\frac{\delta A}{\delta t} - VB\right)_{;\alpha} x_{j;\beta} + \left(\frac{\delta B}{\delta t} + a^{\alpha\beta} A_{;\alpha} V_{;\beta} - CV\right) n_i,\\ [\ddot{f}] &= \frac{\delta}{\delta t}\left(\frac{\delta A}{\delta t} - VB\right) - V\left(\frac{\delta B}{\delta t} + a^{\alpha\beta} A_{;\alpha} V_{;\beta} - CV\right), \end{aligned} \tag{4.6}$$

where

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + V n_i \frac{\partial}{\partial x_i}$$

is the convected derivative for an observer moving with the surface, V is the speed of propagation of the surface, and

$$A = [f], \qquad B = [f_{,i} n_i], \qquad C = [f_{,ij} n_i n_j]. \tag{4.7}$$

In what follows we assume that the body loads b_i, l and s are continuous on $B \times (-\infty, \infty)$. Following [18], by a wave of order 1 we mean a solution (u_i, φ, θ) of equations (4.1)–(4.3), (3.21), (2.8), with the properties: (1) the functions u_i, φ and θ are continuous on $B \times (-\infty, \infty)$; (2) the first order derivative of the five-dimensional vector (u_i, φ, θ) have jump discontinuities across S, but are continuous elsewhere. We say also that S is a singular surface of order 1 and we shall refer to it as a wave surface of order 1. The balance laws (4.1)–(4.3) have the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho_0 F \,\mathrm{d}v = \int_\omega Q \,\mathrm{d}v + \int_{\partial\omega} G_i n_i \,\mathrm{d}a,$$

which, at singular surface S, is equivalent to the condition [17]

$$-V[\rho_0 F] = [G_i] n_i. \tag{4.8}$$

If we apply (4.8) to the global balance laws (4.1)–(4.3) then we obtain

$$\begin{aligned} &\rho_0 V[\dot{u}_i] + [t_{ji}] n_j = 0, \qquad \rho_0 \kappa V[\dot{\varphi}] + [h_i] n_i = 0, \\ &\rho_0 V[e^*] + [q_i] n_i = 0. \end{aligned} \tag{4.9}$$

It follows from (4.6) that

$$\begin{aligned} &[u_{i,j}] = \xi_i n_j, \qquad [\varphi_{,i}] = \zeta n_i, \qquad [\vartheta_{,i}] = \gamma n_i, \\ &[\dot{u}_i] = -V\xi_i, \qquad [\dot{\varphi}] = -V\zeta, \qquad [\dot{\vartheta}] = -V\gamma, \end{aligned} \tag{4.10}$$

where

$$\xi_i = \left[\frac{\partial u_i}{\partial n}\right], \qquad \zeta = \left[\frac{\partial \varphi}{\partial n}\right], \qquad \gamma = \left[\frac{\partial \vartheta}{\partial n}\right]. \tag{4.11}$$

We denote by n the distance measured along the normal to the wave surface. In what follows we shall use the following result established by Fisher and Gurtin [13].

LEMMA 1. *Let u and v be functions on $B \times (-\infty, \infty)$ with the following properties:* (1) *u is continuous;* (2) *v is continuous everywhere except for a possible jump discontinuity across S;* (3) *v is bounded on every compact subset of $B \times (-\infty, \infty)$. Then the function*

$$w(\boldsymbol{x}, t) = \int_0^t u(\boldsymbol{x}, t - s) v(\boldsymbol{x}, s) \,\mathrm{d}s,$$

is continuous on $B \times (-\infty, \infty)$.

In view of Lemma 1, from (3.21), (2.8) and (4.10) we obtain

$$\begin{aligned} &[t_{ij}] = G_{ijrs}(0)\xi_r n_s, \qquad [h_i] = A_{ij}(0)\zeta n_j, \\ &[g] = -B_{ij}(0)\xi_i n_j, \qquad [\rho_0 e^*] = T_0 b_{ij}(0)\xi_i n_j, \qquad [q_i] = 0. \end{aligned} \tag{4.12}$$

With the help of (4.10) and (4.12), conditions (4.9) become

$$(G_{ijrs}(0)n_s n_j - \rho_0 V^2 \delta_{ir})\xi_r = 0, \tag{4.13}$$

$$(A_{ij}(0)n_i n_j - \rho_0 \kappa V^2)\zeta = 0, \tag{4.14}$$

$$\beta(0)\xi_i n_i = 0. \tag{4.15}$$

The jumps ξ_i, ζ and γ cannot all be zero. If $\beta(0) = 0$ then equation (4.15) is identical satisfied. Equations (4.13) admit a non trivial solution for ξ_i if and only if

$$\det(G_{ijrs}(0)n_j n_s - \rho_0 V^2 \delta_{ir}) = 0.$$

Taking into account (3.20) this equation reduces to

$$(c_1^2 - V^2)(c_2^2 - V^2)^2 = 0,$$

where

$$c_1 = \left[\frac{\lambda(0) + 2\mu(0)}{\rho_0}\right]^{1/2}, \qquad c_2 = \left[\frac{\mu(0)}{\rho_0}\right]^{1/2}. \tag{4.16}$$

If $V = c_1$ the wave is longitudinal ($\xi_i = \xi n_i$). When $V = c_2$ we obtain transverse waves ($\xi_i n_i = 0$).

If $\zeta \neq 0$, the wave is a wave of compaction (or distension). The possible speed of propagation of this wave is $V = c_3$ where

$$c_3 = \left[\frac{\alpha(0)}{(\rho_0 \kappa)}\right]^{1/2}. \tag{4.17}$$

We now assume that $\beta(0) \neq 0$. Then from (4.15) we obtain $\xi_i n_i = 0$ so that in this case there are two type of singular surfaces of order 1: transverse waves and waves of compaction. We remark that the transverse mechanical waves are not coupled with compaction waves or thermal waves.

5. The Growth of Waves

The local forms of the balance laws (4.1)–(4.3) are

$$t_{ji,j} + \rho_0 b_i = \rho \ddot{u}_i, \tag{5.1}$$

$$h_{i,i} + g + \rho_0 l = \rho_0 \kappa \ddot{\varphi}, \tag{5.2}$$

$$q_{i,i} + \rho_0 s = \rho_0 \dot{e}^*, \tag{5.3}$$

respectively. Using the fact that V is constant for all waves, from (4.6) we have

$$\begin{aligned}
[u_{s,ij}] &= a^{\alpha\beta}\xi_{s;\alpha}(n_i x_{j;\beta} + n_j x_{i;\beta}) - a^{\alpha\beta}a^{\nu\rho}b_{\alpha\nu}\xi_s x_{i;\beta}x_{j;\rho} + \mu_s n_i n_j,\\
[\dot{u}_{i,j}] &= \left(-V\mu_i + \frac{\delta\xi_i}{\delta t}\right)n_j - Va^{\alpha\beta}\xi_{i;\alpha}x_{j;\beta},\\
[\ddot{u}_i] &= V^2\mu_i - 2V\frac{\delta\xi_i}{\delta t},
\end{aligned} \tag{5.4}$$

where $\mu_i = [u_{i,rs}n_r n_s]$. It follows from (5.1) that

$$[t_{ji,j}] = \rho_0[\ddot{u}_i]. \tag{5.5}$$

From (4.6) we get

$$\begin{aligned}
V[t_{ji,j}] &= V[t_{ri,j}n_j]n_r + Va^{\alpha\beta}[t_{ji}]_{;\alpha}x_{j;\beta}\\
&= -[\dot{t}_{ji}]n_j + n_j\frac{\delta}{\delta t}[t_{ji}] + Va^{\alpha\beta}[t_{ji}]_{;\alpha}x_{j;\beta}.
\end{aligned} \tag{5.6}$$

When the constitutive relations for t_{ij} is differentiated with respect to t and jumps are taken across S, we obtain

$$\begin{aligned}
[\dot{t}_{ji}] = &-VG_{ijrs}(0)\mu_r n_s - VB_{ij}(0)\zeta + V\gamma b_{ij}(0)\\
&+ G_{ijrs}(0)n_s\frac{\delta\xi_r}{\delta t} - VG_{ijrs}(0)a^{\alpha\beta}\xi_{r;\alpha}x_{s;\beta} + G^{(1)}_{ijrs}(0)\xi_r n_s.
\end{aligned} \tag{5.7}$$

Here we have used the notation $\boldsymbol{G}^{(1)} = \dot{\boldsymbol{G}}$.

With the aid of equations (5.4), (5.6) and (5.7), equation (5.5) may be written as

$$\begin{aligned}
&V\{G_{ijrs}(0)n_j n_r - \rho_0 V^2\delta_{is}\}\mu_s + VB_{ij}(0)n_j\zeta - V\gamma b_{ij}(0)n_j\\
&- G_{ijrs}(0)n_s n_j\frac{\delta\xi_r}{\delta t} + G_{ijrs}(0)n_j\frac{\delta}{\delta t}(\xi_r n_s) + 2\rho_0V^2\frac{\delta\xi_i}{\delta t}\\
&+ VG_{ijrs}(0)a^{\alpha\beta}(\xi_r n_s)_{;\alpha}x_{j;\beta} + VG_{ijrs}(0)n_j a^{\alpha\beta}\xi_{r;\alpha}x_{s;\beta}\\
&- G^{(1)}_{ijrs}(0)\xi_r n_s n_j = 0.
\end{aligned} \tag{5.8}$$

We note that

$$a^{\alpha\beta}\xi_{i;\alpha}x_{i;\beta} = a^{\alpha\beta}(x_{i;\beta}\xi_i)_{;\alpha} - 2Hn_i\xi_i, \qquad \frac{\delta n_i}{\delta t} = 0. \tag{5.9}$$

With the help of (3.20), (4.4), (4.5) and (5.9), equations (5.8) reduce to

$$\begin{aligned}
&V\{[\lambda(0) + \mu(0)]n_i n_r + \mu(0)\delta_{ir} - \rho_0V^2\delta_{ir}\}\mu_r + 2\rho_0V^2\frac{\delta\xi_i}{\delta t}\\
&+ Vb(0)n_i\zeta - V\beta(0)n_i\gamma + V[\lambda(0) + \mu(0)]\{a^{\alpha\beta}n_i(x_{r;\beta}\xi_r)_{;\alpha}\\
&+ a^{\alpha\beta}(\xi_r n_r)_{;\alpha}x_{i;\beta} - 2H\xi_r n_r n_i\} - 2HV\mu(0)\xi_i\\
&- [\lambda^{(1)}(0) + \mu^{(1)}(0)]\xi_r n_r n_i - \mu^{(1)}(0)\xi_i = 0.
\end{aligned} \tag{5.10}$$

We denote $\xi = \xi_i n_i$. If we multiply (5.10) by n_i and sum on i, we obtain

$$V\{\lambda(0) + 2\mu(0) - \rho_0 V^2\}\mu_s n_s + V\zeta b(0) + V\gamma\beta(0) + 2\rho_0 V^2 \frac{\delta\xi}{\delta t} + V[\lambda(0) + \mu(0)]a^{\alpha\beta}(x_{r;\beta}\xi_r)_{;\alpha} - 2V[\lambda(0) + 2\mu(0)]H\xi - [\lambda^{(1)}(0) + 2\mu^{(1)}(0)]\xi = 0. \quad (5.11)$$

In the case of longitudinal waves, (5.11) yields the growth equation

$$\frac{1}{c_1}\frac{\delta\xi}{\delta t} = \frac{\mathrm{d}\xi}{\mathrm{d}n} = \xi(H - J_1), \quad (5.12)$$

where we have used the notation

$$J_1 = -[\lambda^{(1)}(0) + 2\mu^{(1)}(0)](2\rho_0 c_1^3)^{-1}. \quad (5.13)$$

If, at some instant $t = t_0$, the mean and Gaussian curvatures of surfaces are H_0 and K_0, respectively, then at a subsequent time t,

$$H = \frac{H_0 - nK_0}{1 - 2nH_0 + n^2 K_0}, \quad (5.14)$$

and (5.12) may be integrated to give

$$\xi = \xi_0(1 - 2nH_0 + n^2 K_0)^{-1/2}\exp(-nJ_1), \quad (5.15)$$

where ξ_0 is the strength of the wave at time $t = t_0$.

The speed of propagation of irrotational waves is c_2 and for these waves we have $\xi_i n_i = 0$. In this case, assuming that $c_2 \neq c_3$, equation (5.11) reduces to

$$[\lambda(0) + \mu(0)][a^{\alpha\beta}(x_{r;\beta}\xi_r)_{;\alpha} + \mu_s n_s] + \beta(0)\gamma = 0.$$

Thus, from (5.10) we obtain

$$\frac{1}{c_2}\frac{\delta\xi_i}{\delta t} = \frac{\delta\xi_i}{\mathrm{d}n} = (H - J_2)\xi_i, \quad (5.16)$$

where

$$J_2 = -\mu^{(1)}(0)(2\rho_0 c_2^3)^{-1}. \quad (5.17)$$

As before, we obtain

$$\xi_i = \xi_i^0(1 - 2nH_0 + n^2 K_0)^{-1/2}\exp(-nJ_2), \quad (5.18)$$

where $\xi_i^0 = \xi_i(t_0)$.

In the case of a wave of compaction we have $V = c_3$. If we assume that $c_3 \neq c_1$ also $c_3 \neq c_2$ then from (5.10) we obtain

$$\{[\lambda(0) + \mu(0)]n_i n_s + \mu(0)\delta_{is} - \rho_0 s_3^2 \delta_{is}\}\mu_s = -b(0)n_i\zeta + \beta(0)n_i\gamma. \quad (5.19)$$

This relation implies that

$$\rho_0(c_1^2 - c_3^2)\mu_s n_s = -b(0)\zeta + \beta(0)\gamma. \tag{5.20}$$

By using (5.20) in (5.19) we obtain

$$\mu_i = -[b(0)\zeta - \beta(0)\gamma]n_i[\rho_0(c_1^2 - c_3^2)]^{-1}. \tag{5.21}$$

We see that a thermal wave or a wave of compaction induces a longitudinal mechanical acceleration discontinuity.

It follows from (5.2) that

$$[h_{i,i}] + [g] = \rho_0\kappa\left(V^2\tau - 2V\frac{\delta\zeta}{\delta t}\right), \tag{5.22}$$

where $\tau = [\varphi_{,ij}n_i n_j]$. With the help of relations

$$\begin{aligned} &V[h_{i,i}] = -[\dot{h}_i]n_i + n_i\frac{\delta}{\delta t}[h_i] + Va^{\alpha\beta}[h_j]_{;\alpha}x_{j;\beta}, \\ &[\dot{h}_i] = \alpha(0)\left\{\left(\frac{\delta\zeta}{\delta t} - V\tau\right)n_i - Va^{\alpha\beta}\zeta_{;\alpha}x_{i;\beta}\right\} + \alpha^{(1)}(0)\zeta n_i, \\ &[h_i] = \alpha(0)\zeta n_i, \qquad [g] = -b(0)\xi_i n_i, \end{aligned} \tag{5.23}$$

from (5.22) we find

$$\begin{aligned} &[\alpha(0) - \rho_0\kappa V^2]V\tau - 2HV\alpha(0)\zeta - \alpha^{(1)}(0)\zeta - Vb(0)\xi \\ &\quad + 2\rho_0\kappa V^2\frac{\delta\zeta}{\delta t} = 0. \end{aligned} \tag{5.24}$$

We assume now that $V = c_3$ with $c_3 \neq c_1$ and $c_3 \neq c_2$. Then (5.24) reduces to

$$\frac{1}{c_3}\frac{\delta\zeta}{\delta t} = (H - J_3)\zeta, \tag{5.25}$$

where

$$J_3 = -\alpha^{(1)}(0)(2\rho_0\kappa c_3^3)^{-1}. \tag{5.26}$$

It follows from (5.25) that

$$\zeta = \zeta^0(1 - 2nH_0 + n^2K_0)^{-1/2}\exp(-nJ_3), \tag{5.27}$$

where $\zeta^0 = \zeta(t_0)$. If $V = c_1 \neq c_3$ then (5.24) reduces to

$$[\alpha(0) - \rho_0\kappa c_1^2]\tau = b(0)\xi, \tag{5.28}$$

so that a longitudinal wave induces an acceleration discontinuity in the waves of compaction.

Let us consider now equation (5.3). This equation implies that

$$[q_{i,i}] = [\rho_0\dot{e}^*]. \tag{5.29}$$

In the view of relations

$$[q_i] = 0, \qquad V[q_{i,i}] = -[\dot{q}_i]n_i, \quad [\dot{q}_i] = k(0)\gamma n_i,$$
$$[\rho_0 \dot{e}^*] = T_0\left\{\beta(0)\left(n_i\frac{\delta\xi_i}{\delta t} - V\mu_i n_i - Va^{\alpha\beta}\xi_{i;\alpha}x_{i;\beta}\right) - V\zeta m(0) - V\gamma a(0) + \beta^{(1)}(0)\xi_r n_r\right\}, \tag{5.30}$$

from (5.29) we get

$$[k(0) - T_0 a(0)V^2]\gamma = T_0 V\left\{\beta(0)\left[V\mu_i n_i + Va^{\alpha\beta}\xi_{i;\alpha}x_{i;\beta} - n_i\frac{\delta\xi_i}{\delta t}\right] + V\zeta m(0) - \beta^{(1)}(0)\xi_j n_j\right\}. \tag{5.31}$$

Let us assume that $V \neq c_1$, $V \neq c_2$, $V \neq c_3$. Then from (5.11) we obtain

$$\rho_0(c_1^2 - V^2)\mu_i n_i = -\beta(0)\gamma.$$

If $\beta(0) = 0$ then we find that $\mu_i n_i = 0$ and (5.31) reduces to

$$(c_4^2 - V^2)\gamma = 0,$$

where

$$c_4 = \left[\frac{k(0)}{T_0 a(0)}\right]^{1/2}. \tag{5.32}$$

In this case we see that the possible speed of propagation of thermal waves is $V = c_4$. If $\beta(0) \neq 0$, then from (4.15) we obtain $\xi_i n_i = 0$ so that $V = c_2$ or $V = c_3$. For $V = c_2 \neq c_3$ equation (5.11) reduces to

$$\rho_0(c_1^2 - c_2^2)(a^{\alpha\beta}x_{r;\beta}\xi_{r;\alpha} + \mu_s n_s) = -\beta(0)\gamma.$$

Thus, equation (5.31) can be written in the form

$$R\gamma = 0,$$

where

$$R = a(0)\rho_0(c_1^2 - c_2^2)(c_4^2 - c_2^2) + \beta^2(0)c_2^2.$$

If $R \neq 0$, then $\gamma = 0$, so that the wave is purely mechanical.

For $V = c_3$ ($\neq c_2$) equations (5.15) and (5.31) imply that

$$\Lambda\gamma = c_3^2[\rho_0 m(0)(c_1^2 - c_3^2) - b(0)\beta(0)],$$

where

$$\Lambda = \rho_0 a(0)(c_1^2 - c_3^2)(c_4^2 - c_3^2) + c_3^2\beta^2(0).$$

Thus, in general, a compaction wave induces a thermal wave. We note that the eigenvalue problems from Section 4 govern also the propagation of thermoelastic waves studied in [3]. However, in the present paper, the amplitudes of waves contain new attenuation factors related to the relaxation functions (see (5.13), (5.15); (5.17), (5.18) and (5.26), (5.27)).

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