RESONANCE: A KEY WORD IN MATHEMATICS TEACHING-RESEARCH

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ABSTRACT
In this paper a model of cognitive dynamics is proposed, for interpreting classroom as well as teacher training processes, nowadays supported also by some recent neuroscience results. Core relevance in the model is assigned to a basic resonance dynamics, assumed to work at the root of all the modulations, from perception to abstract thinking, and interferences characterizing knowledge construction. On the basis of this theoretical framework, teachers are seen as "resonance mediators," i. e. experts who favor the resonance process in their students; correspondingly, any teacher training process has to accomplish this result. Finally, we will briefly examine a school episode, showing how a teacher, playing this role, acts effectively both on students' understanding and on their motivation.

1. INTRODUCTION

Many cognitive theories have been developed in recent years. For a general frame, concerning in particular mathematics education, we refer to Tall (2004). Among other things, Tall notices an emerging strand of researches into brain activity, where a lot of experimental data have been collected from those concerning innate numerical competences to those based on Brain Imaging Techniques, applied to subjects engaged in elementary arithmetic tasks (Dehaene, 1997; Butterworth, 1999).

In this paper a particular model[1] of cognitive dynamics is employed, in order to draw some consequences for teachers' role and for their training strategies. The model is presented in depth in Iannece and Tortora (2007a, 2007b), Mellone (2007), and is outlined in Guidoni et al. (2005a), so we will present here only some of its essential features. This model has been developed mainly on the basis of our classroom experience but nowadays it is supported also by the above quoted neurophenomenological studies.

The importance of focusing the attention on a cognitive model arises when we recognize how teachers' beliefs about the essence of mathematics, the ways of knowledge transmission and the ways people learn and understand are relevant in teachers' behaviors. Very often these beliefs are just embodied in daily practice and partially unconscious but notwithstanding they determine automatic small and large scale decisions in the conduct of class activities. For instance, teachers can act as knowledge transmitters just because they are merged in the Piagetian milieu, where three basic assumptions operate: scientific concepts – inserted as they are in a structured

[1] We use the word "model" in a weaker sense than the one used in hard sciences, but more suitable for a quite unformalized domain like math education.

cultural system – are the only ones that allow valid knowledge; they are obtained by substitution of the originary everyday concepts; the reasoning of any educated mind follows the formal rules of logic. Therefore, a first task in any teacher training is to let these beliefs emerge, in order to be compared and discussed (Mason, 2002). Then, usually only as a consequence of a shared agreement, a necessity can arise to modify these beliefs, and to assume new views that turn to be more effective in designing, managing and assessing learning environments and didactic actions.

This paper is arranged as follows: in the second section we present our model of the understanding process together with some related results from neuroscience; in the third section we derive our views about the role of teachers and some strategic lines for teacher training; the fourth and last one is devoted to an illustration by means of an example of the potential impact and the possible outcomes of our assumptions.

A distinctive feature of our cognitive modeling is the key word *resonance*, used as a powerful metaphor for learning processes. We borrow it from physics,[2] where it denotes the increase of the amplitude of enforced vibrations, taking place whenever the frequency of the source is approaching one of the proper frequencies of the receiving system. We can see resonance phenomena in a lot of fields of common experience, sometimes with catastrophic consequences, like when a bridge can be destroyed by virtue of a prolonged sequence of quite weak earthquake waves. Or everyone who has been on a swing knows very well how movements of the legs can expand or extinguish the swing amplitude, depending on their accordance with the swinging frequency.

Much in the same way, we say that a learning process, which is always the result of a complex interaction between different variables, is driven by a resonance dynamics when different cognitive items (an idea, a mental construct, an image, an action, etc.) are simultaneously activated whenever one is evoked, producing by reciprocal interference a mutual reinforcement effect. But, as we will see in the sequel, what is even more important for educational goals in mathematics is to put in resonance: (1) the actual potentialities of individual cognitive structures; (2) the patterns supplied by codified cultures; and (3) the constraints of the real world. Finally, the circumstance that in evolutionary theories of the brain, the term “resonance” appears also as a key word in modeling human cognitive behavior, which, as we will show in the next section, does not seem to us a simple coincidence.

To activate resonance, or, even better, to guide children to consciously look for resonance, is one of the necessary conditions that allow them to be in the mood to succeed in understanding. This is a hard task for any teacher, while the task of research is to investigate how this can be accomplished at best. The importance of the pivotal role of teacher-researchers – whatever this means – is quite evident.

2. OUTLINE OF A COGNITIVE MODEL

In every didactic action we recognize very schematically at least four basic “model-ingredients:” (i) a realistic, even rough, model of “natural” cognitive dynamics; (ii) a global, epistemologically founded view of mathematics as an internally structured scientific discipline; (iii) a modulated view of the variety of interferences of mathematical thinking with other cultural fields (mainly scientific and technological ones), and with everyday culture(s); and (iv) a model of cultural transmission in

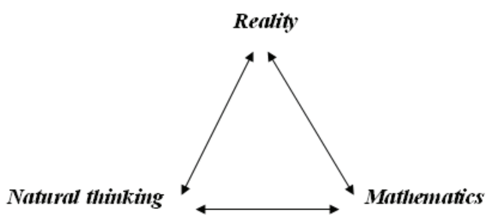
[2] The word resonance appears not only in physics but in many other contexts with different meanings. In an educational context, for example, it is employed by Comiti et al. (1996), as a measure of the responsitivity of teachers to students’ interventions.

knowledge areas, in particular, scientific ones. Such ingredients, obviously crucial to teaching profession, are clearly correlated to each other: in particular, the basic framing role assumed by i) with respect to other aspects is clear.

In order to characterize our cognitive modeling, we feel it necessary to root it within the complex landscape of the cognitive theories available nowadays. In particular, we draw attention to the fact that many critical aspects of cognition have been variously regarded in the course of time within different (often reciprocally contrasting) cognitive theories and/or epistemological positions. Actually, our basic research finding is that most of such aspects appear relevant in interpreting experimental teaching/learning evidences: and this directly implies their reciprocal complementarity. For example:

It is now quite common to refer to Vygotsky's views about the crucial role assumed by a natural language in the development of knowledge, since the earliest age (Vygotsky, 1934). However, such a role is only in a minor part an automatic, passive one: careful observations of cognitive transactions show that an early, active adult mediation plays a key role in fostering resonances and preventing dissonances ("misconceptions" appear as the result of missing/wrong/misleading mediations between developing cognition and culture): Apart from the "stages" machinery, some insights by Piaget appear to be crucial to outline features of cognitive dynamics. Assimilation, accommodation, and (temporary) equilibration lively define the main modes of any resonance process.

The point is to correlate such views within a comprehensive dynamical model: and the resonance dynamics frame, as described in Guidoni et al. (2005b), actually lends itself to account for many crucial correlation aspects as we want to show. So, we assume that any true learning in scientific/mathematical field is a result of the process of resonance between individual cognition, social culture and reality structures along cognitive paths efficiently addressed and controlled in their meaning-driven dynamics. It requires, at any level, also resonance between various "dimensions" of natural thinking (Guidoni, 1985): perception, language, action, representation, planning, interpretation, looking for sense, etc. We can schematize our view by means of a triangular schema, resembling the famous Chevallard triangle but with different variables on its vertices.



We want to stress: (a) the complexity of every "vertex," in particular of the "natural thinking" because of its multidimentional and time-dependent character; (b) the two-way direction of the arrows that marks the impossibility to uniquely determine the thinking process. Looking for resonance is a very useful and flexible tool for teachers in driving a didactic action and in reflecting on it, and for children in understanding how they understand. Let us go into some clarifying details.

In a Vygotskian perspective, we know that the roots of a large number of culturally sophisticated concepts, like those of mathematics, can be found in the ways children spontaneously face the complexity of the surrounding world, far before they are involved in school contexts. For example, very early on it is necessary to correlate things and properties changing in time: a typical cognitive strategy is to give causal explications to these phenomena, trying to interpret, often forcefully, a variation of a property as a cause or an effect of the variation of another. Here, we clearly recognize the "natural" cognitive root for the dependance of a variable from another that is the mathematical concept of function. Of course, things are all but easy: a natural thought strategy cannot go very far in managing mathematical complex situations, where abstract structures come into play typically neglecting semantical counterparts. In this case, a search for resonance guides teachers in addressing the development of natural strategies toward a goal of reification and nominalization; just like in the sense of Vigotsky.

Again, let us suppose that children work with numbers, their operations and properties. For natural numbers (natural not by chance!) it is easy to see a correspondence between the things around us, our perceptions and actions, and finally the mathematical structures. So, a teacher can utilize the correspondence between reality and our natural ways of thinking to build suitable mathematical structures, again according to a Vygotskian evolutionary perspective.

But the problems arise when new kinds of numbers come into play. Trying to root again multiplication by zero (see Section 4, below) or between negative numbers in the reality can be the first spontaneous attempt both for teachers and for learners, but the circumstance that the correspondence with reality, already observed for natural numbers, does not work anymore could provoke irreparable consequences in children's minds, like severe separation between intuition and mathematical knowledge, if not the first refusal of being involved in active learning. All this can be avoided if teachers wisely guide their students in the complex play of resonance. This time the need for a rich and sound mathematical structure must be invoked, pointing for instance to natural strategies of generalization or to a natural ability to structure a new game. Reality comes only afterwards, perhaps when we are able to recognize that the multiplicative structure of integers offers a powerful model for a more complex physics phenomenology.

A more detailed example will be discussed in Section 4 below. For other examples, see Iannece and Tortora (2007b) and Mellone (2007), and also some of the papers contained in this book and in the PDTR *PISA Handbook*, e.g. De Blasio et al. (2008) and Pezzia et al. (2008): In all these examples, referring to various school grades, several positive outcomes of our resonance model can be observed, among others: (a) the ability to autonomously utilize one's informal knowledge to support the construction of formal knowledge, and, in the opposite direction, the ability to give a sense to formal knowledge interpreting it within informal contexts; (b) the ability to select linguistic tools according to specific objectives; and (c) a marked growth of self-esteem.

All these examples and, more generally, the goal of promoting resonance between the mathematical constructs and natural cognitive structures suggest deepening of knowledge of the latter. We will try to do this in the next section using some recent neuroscience research results.

The development of human brain, prerepresentations, mirror neurons

Leron (2004) says that "Human nature [can be seen] as a collection of universal, reliably-developing cognitive and behavioral abilities – such as walking on two feet, face recognition or the use of language – that are spontaneously acquired and effortlessly used by all people under normal development. Common sense is a cognitive part of human nature, the collection of abilities people are spontaneously and naturally good-at."

Many research streams deepen the Leron's notion of human nature, showing the existence of other universal behavioral abilities, like the ability to formulate hypotheses (Changeux), or to control the coherence of argumentation (Houdé), or to use metaphors (Lakoff & Núñes). And there are still a lot of things to discover, as it is easy to recognize just observing, as we do everyday, the learning process of students of every age. Using, as we do, an extensive meaning for the word "natural," our construct of resonance appears well expained by Vygotsky's dychotomies between natural/scientific concepts and natural/higher psychic functions (Vygotsky, 1934).

Some models of the working and cognitive behavior of the brain, recently devised on the basis of experimental neurophysiology data, highlight how our synaptic structures develop according to a continuous learning process. We mainly refer to Changeux (2000) for these results. Today, they offer new experimental confirmations to Vygotsky's hypotheses about the social nature of learning and the evolutionary character of concepts and of the psychic functions mediated by culture: these ideas are part of our own modeling.

According to Changeux, our brain is characterized by a marked "structural plasticity", due to a continuous interaction with the external environment and to a likely continuous internal reworking (e.g. dreams, thoughts, and imagination). It develops according to two distinct but related processes: on a biological scale, it is the outcome of a Darwinian selection of the most advantageous representations of the external world (for this idea, see also Edelman, 1987), on an individual scale, it changes according to a never-ending learning process. Spontaneously, the neurons generate impulses and transitory synaptic connections are activated, giving rise to "prerepresentations" of the external world. In a sense, as foreseen by many authors, like e.g. Neisser (1981), our brain does not simply receive information from outside but throws its own interpretative schemata in. In this way, knowledge origins as a result of selection and stabilization of prerepresentations, guided by a "cognitive relevance" principle, similar to that studied by Sperber and Wilson (1993) in communication theory. The relevance is marked by a correspondence with reality: "The answer coming from outside is decisive. It constitutes a test of how the prerepresentation fits in the environment."[3]

Therefore, the spontaneous activity of our neurons can be seen as a natural aptitude to explore and modelize the physical world. Recalling also Galilei's words: "*Ma io stimerei prima la natura aver fatto le cose a modo suo e poi i pensieri degli uomini atti a capirla*,"[4] a possible answer can be found to the ancient question about the more or less innate nature of mathematical concepts, in the sense that the innate numerical abilities (Dehaene, 2000; Devlin, 2002) can also be seen as the result of an epigenetic selection of neural networks, stabilized as the most effective to mankind survivance.

A noteworthy amount of recent neurophenomenological results concern the primacy of the perceptual-motory brain system also in the processing of higher functions, with a central role assigned to the so-called mirror neurons (see, e. g. Kohler et al., 2002; Gallese & Metzinger, 2003; Gallese & Lakoff, 2005): For these results we mainly refer to Rizzolatti and Sinigaglia (2006) where many of them are collected and supported by experimental data. The basic hypothesis is that the roots of any cognitive process stand in the motory cortex that is to say in our *actions*. "Processes that are

[3] Changeux, 2000, 65; our translation from the Italian edition. A lot of data supporting these assertions can be found there.
[4] "I would suppose that nature [comes] before made things and then men's thoughts capable to understand them," our translation from Galilei (1964).

usually considered of a higher order, like perception or recognition of the acts of other people, imitation and also gestural and verbal forms of communication, can be referred to the motory system, where they find their neural background."[5]

The physiological bases of this integration are the so-called "canonical neurons" and "mirror neurons." The former codify motory acts, i.e. not just movements in themselves but movements having a specific goal (see Iannece & Tortora, 2007a, for more details). In other words, there is an area (F5) in our brain where a kind of *vocabulary of actions* is stored. Some of these canonical neurons are excited when we act for a particular goal, as well as when we just observe an object on which an action can be done (e.g., seeing a cup on a table stimulates the potential – complex – act of picking it): So, what happens is that the vision of an object generates in our brain a prerepresentation: "A brain that acts is, first of all, a brain that understands," in the words of Rizzolatti and Sinigaglia (2006):[6]

The mirror neurons are so called because they reproduce actions of people in the brain of other people: e. g., they are excited when we move our hand to pick something as much as when we see someone moving their hand for the same goal. Due to them, our vocabulary of actions can be viewed as our cognitive budget to understand and interpret actions of other people. Now, this budget is strongly modified by a learning process, therefore, the things that we understand "without effort" vary according to how our own experience and knowledge develop. The notion of elementary (effortless) tasks is crucial. They are accomplished by means of innate brain circuits (typically those deputed to treat sensory-motor information), automatically activated when the exigence of guaranteeing survivance occurs: to drink when thirsty, to escape in front of a danger, to recognize a face, etc. On the other hand, an effort is necessary when other areas and functions (e.g. memory and imagination) of the brain are involved. It is more or less the same Vygotskian distinction between elementary and higher psychic functions. Of course, most of the tasks involved in the learning of mathematics are of the second type.

In an interesting experiment Houdé et al. (2000; see also Iannece et al., 2006, for details) show that activating emotions can be crucial even for complex reasoning,[7] since effortless (perceptual) strategies are inhibited in favor of rational ones, more suitable for the specific goal (of a logical type, in the case studied): This confirms that two different kinds of "reasonings" are both natural for us, although the former is more "spontaneous" (in the sense of Vygotsky), while the latter requires an effort. We claim in Guidoni et al. (2005a) that in mathematics education, like in any complex learning environment, the key is not that of inhibiting a kind of reasoning strategy in favor of another, but to put them in resonance, i. e. to consciously pass from a cognitive dimension to another in a continuous reinforcing game. Houdé himself says: "*Le cerveau de l'homme est una sorte de jungle où le competence du bébé, de l'enfant e de l'adulte, sont à tout moment susceptibles de se télescoper, d'entrer en competition, en même temps qu'elles se construisent*" (2000).

In eveyday life, our way of understanding is a process of continuously projecting outside prerepresentations looking for a feedback from the external world. In our opinion, the same thing happens within an abstract cultural context: attempts and errors, conjectures validated or refused are the ingredients of the dynamics by which our

[5] Rizzolatti & Sinigaglia, 2006, 122; our translation.
[6] *ibidem*, 3, our translation.
[7] According to Damasio (1994) there is a very strict relation between rationality and emotions, at the level of brain circuits.

brain works in order to understand. Therefore, in any education context, it is necessary to favor this process: our resonance model and our didactic choices are made following these assumptions.

Coherently, we look at the formal structures of mathematics as at one of the two principal cultural tools by which the phenomena of the external world are interpreted and described (another one being physics): Today, the prevailing view of mathematics, within most curricula, emphasizes its *a priori separateness* from other scientific areas, thus conflicting with natural cognitive processes, and causing many students' difficulties. On the contrary, if mathematics is perceived as an *a posteriori abstraction* coming through a modelization process, its "cognitive" resonance stimulates students' motivated interest toward its structural development and allows them to reach quite high levels of formalization.

We are not saying that mathematics should be reduced to a mere language for physics nor physics to a simple field of examples of mathematical structures. We are saying that it is necessary to recognize that: a) the same cognitive process underlies both disciplines; b) both can be viewed as *discourses* (in the sense of Sfard, 2000), characterized by different rules but complementary in their role of cognitive reconstructions of the reality; c) this complementarity is a precious resource from an educational point of view. In this sense, the cultural constructs of physics are a bridge between the perception and the more abstract notions of mathematics: for example, the physical concept of motion fills the gap between the variety of experienced movements and the symbolic mathematical treatment. We take it a step further: introducing abstract structures as linguistic tools to describe and reason about things is the strongest way to motivate even their autonomous disciplinary development.

But what does it mean to put individual cognition and reality structures in resonance? We suggest that attention should always be paid to all those models and strategies that have been developed by mankind as a whole, and are developed by each individual in order to interpret and manage often unconsciously the daily experience. We are thinking of Rizzolatti and Sinigaglia's "vocabulary of actions" or the "schemata" described in Lakoff e Núñez (2005): When observing everyday human actions and reasonings, it is easy to recognize how such models and strategies are generally employed, and also how complex and sophisticated they can be.[8] But what is essential for educational purposes, is to observe them at work in students' behaviors.

In the first section we have already proposed the example of the relationships between two changing variables. But also the order relation, the direct proportionality (Guidoni et al., 2003) are all models that are employed very early, independently from one's linguistic tools, and the same happens for thinking strategies like the dychotomy concrete/abstract (in children's words: "by truth or by fiction"): see Iannece and Tortora (2007b) for more examples.

3. TEACHERS' ROLE

In Section 1, we have recalled how, according to Piaget, the scientific concepts are constructed. This view assigns to teachers the role of knowledge transmitters. On the contrary, in our model of cognitive dynamics, where the focus is on the integration

[8] For example, when crossing the street with heavy car traffic, a human being (but also, say, a cat) must uncounsciously and rapidly activate sophisticated controls of distance and speed, and of their variations, in order to avoid danger.

(resonance) among natural thought, real world and culture, teachers play a completely different role which we call *resonance mediation* (Guidoni et al., 2005b):

"Pick them up where they are then find a path which guides them to the place you want them to reach." According to this famous Wittgenstein's mot, teachers must recognize in every class context the "space of cognitive configurations" (let us use again a metaphor from physics), in order to design possible learning trajectory paths, drawn on all the available resources. In general, they will adopt teaching strategies that are not imposing but supportive of potentialities. Teachers should create, on a local level, the many possible links between individual cognition, social culture and reality structure through the use of dynamics of abstraction and de-abstraction (modeling and de-modeling) with graduality, coherence, flexibility and competence.

In other words, what teachers have to do is first of all to make explicit and to foster all natural models and strategies of their students. This means, for instance, that, when recognizing proportional thinking as a spontaneous strategy for interpreting real phenomena, the right thing to do is not to reinvent *ex novo* the corresponding mathematical notion, but to favor the development of the language which appears appropriate to express it, exploiting suitable learning environments (Iannece & Tortora, 2004):

According to the above-mentioned Houdé's words, many ages always coexist in our minds. Therefore, teachers could and should exploit the synergy of all those cognitive dimensions, rather than being cast down for the so frequent *cognitive regressions* of their students.

A critical awareness and a responsible assumption of such a role surely make teachers reflexive and also in some cases turn them into true researchers. In working with in-service teachers, where teachers and students are simultaneously involved, this awareness and this assumption of role are supported by the immediate and long term interaction with the cognitive processes of learners. Several years of research on our part in the formation of school teachers, based on didactic strategies gradually validated, have convinced us that the guided collective participation in modelization or problem solving processes makes up a privileged entrance into the world of the combined acquisition of knowledge and professionalism. It is important to highlight that the cognitive dynamics put into play by in-service teachers (and by pre-service teachers in training as well) in substance correspond to what takes place in class; likewise the crucial role played by a meta-cognitive attitude is analogous both on an individual and group scale. We have also noticed that it is important in all situations to alternate auto-directed work of manipulation and interpretation either individually or in small groups (including substantial homework) with collective guided work of comparison and analysis of partial results, yet leaving to the individual the final systemizing of results and interpretation of the processes being adopted.

As said above, two kinds of activities appear as critical keys for both teachers' formation and students' learning: modelization processes from every day experience contexts and word problems. For word problems, we refer to Guidoni et al. (2003) and Tortora, (2001), where some of our views are reported; a similar analysis can be found in Mason (2001a). Here, we want to say a few more words about the modelization process.

In mathematics education literature, not to mention other fields, the meanings assigned to the word "modelization" considerably vary (see, e.g. Mason, 2001b and Verschaffel, 2002). Therefore, it is necessary to begin with an explanation of what "modelization" means for us (contrasting it for example with Verschaffel's definition). The data on the functioning of the brain show that our way of interaction with the external world lays on powerful, often unconscious, neural mechanisms for interpreting it: so, in a very general sense, a “model” is nothing more than a linkage between the things that happen and the brain that tries to understand them. According to this, we interpret modelization as a very complex process, neither deterministic nor one-way, where the formal structures are seen as one of the different correlated ways into which the cognitive reconstruction of external world structures takes form. This process cannot be reduced to guiding students toward abstraction, through a standard hierarchy of multi-representations (actions, words, graphs, and so on) whose top is identifiable by the algebraic formulation of a physical law. Due to the subtended cognitive dynamics, what is most effective is a continuous shifting from one cognitive dimension to another in a mutual progressive enhancement. And the process itself requires very lenghty didactic paths, even extended over the whole school curriculum.

A systematic resort to modelization processes in mathematics education, because of their resonance with natural learning strategies, enhances also students’ motivation allowing them to actively participate in the construction of culturally validated theories of course within the cognitive and linguistic bounds of every age. At the same time both teachers and students can distinguish between a substantial continuity of the natural and scientific ways of organizing knowledge, and an essential discontinuity in terms of systematicity and inner coherence (a distinction which recalls again Vygotsky’s dychotomy between a natural and scientific thought):

Therefore, it is important to make the first moves in the abstraction process starting from perceptual-motory experiences, which allow the involved notions to develop better and to be transferred to other situations. The choice of the contexts to be explored is always addressed by some conditions. Some contexts are surely privileged, like, e. g., *motion* (Balzano, 2007), *springiness* (Guidoni et al., 2003; 2005b), *shadows* (Boero et al., 1995), since they contribute to approaches characterized, since the beginning of the cognitive path, by direct manipulations guided by reflection on what is being observed. A good context should be at the same time simple enough to allow for an exploration not too rigidly guided, and complex enough to demand a careful, previous individuation of interacting systems and of pertinent variables, and to allow formulating non trivial conjectures.

We conclude emphasizing once again that collective and guided modelization activities bear an intrinsic value, independently from the mathematical content that they allow to build: a cognitive, a metacognitive and also an emotional value, inasmuch they are resonant with the natural way of functioning of our minds, and provided they are accompanied by the awareness of the development of our thought processes.

4. SOME EXCERPTS OF A CASE STUDY

Here we present a brief account of a class episode, in order to illustrate how the construct of resonance can help teachers in designing and managing class activity, as well as the interaction teacher-tutor in the training process. We refer to situations faced by some teachers of our team that are presented in De Blasio et al., (2008), where more details of the activity can be found.

Teacher (Nicoletta) with a mainly pedagogical background wanted to explain to a third-grade class, the role of zero in the multiplication of natural numbers. She tried to put reality and disciplinary structures in resonance, using the “linguistic” mediation of an action procedure. This worked very well until the zero was not involved:

As usual for me, I looked for "real" stories to support children in their construction of meaning for the operation. So, I proposed a movement activity where children were required to go back and forth on the number line then I said: "Go forward 4 steps 0 times"

Children's reactions were very interesting:

Anna: I am not able to do this, I can't move... What is the sense of the words "4 steps zero times"? I would never ask anyone to make 4 steps zero times.

Giovanni: 4×0 is not really a multiplication! A multiplication needs repetition of an action; it needs the 'times.'

Alessia: The word 'multiplication' is obtained by putting together two words: 'multi,' that means a lot and 'action.' Then multiplication means to carry on a lot of actions. But when there is zero there is no action, so we have to choose a new name... Perhaps we can still decide to call it multiplication but with a different meaning.

Of course, the children encountered serious difficulties in extending the familiar model of multiplication of natural numbers to this new "strange" situation. Till now, the cognitive strategies based on actions, strongly embodied according to Rizzolatti and Sinigaglia (2006), supported the conceptualization of the multiplicative structure, and, coherently, Anna and Giovanni tried to refer to those strategies, in order to understand. On the contrary, Alessia felt a dissonance between teacher's request and the "action" meaning of multiplication and, rather than forcing the motor-perceptual metaphor, displayed a different cognitive behavior which turned out to be resonant with a typical mathematical process: to enlarge the meaning of an operation. To solve the problem, Alessia resorted to a direct resonance between the mathematical structures and her own cognitive resources – the aptitude to change the rules in a game, in order to respect coherence constraints, – while neglecting reality, the third pole of our triangle. This kind of reasoning, of course more refined than the previous one, was still natural (Iannece et al., 2006), in fact it was spontaneously activated.

Alessia's intervention opened a way toward a higher level of shared understanding. Nicoletta, though at first bewildered, was able to recognize this opportunity by virtue of her participation in the project and of her exposition to teachers' formation activities, as described in her diary:

In a sense, my students were trying to convince me about the uselessness of my searching for a concrete situation that constituted a metaphor for this operation. To my great surprise, the students revealed a natural aptitude to change their point of view, jumping from reasoning supported by observation to logic argumentation. Alessia even analyzed the structure of the word "multiplication," looking for the sense of the operation.

Analyzing with mentors my students' reasoning, I became aware that not all mathematics can be discovered starting from observation of the reality. In fact, there are some rules that can be justified only by the necessity of an internal coherence of mathematics as a discipline.

But a still greater surprise was that my difficulties in leaving concrete motivations for mathematics rules were not shared by my students: for children the acceptance of a sort of game rules led to a generalization of already established meanings.

The sequel of the activity designed and guided in collaboration with my colleagues in the project and with my mentors successfully brought me and my students into encounter with a problem that I had never solved before: why is it impossible to divide by zero?

A new story begins…

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