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The complete Steenrod algebra at odd primes

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Abstract We study the complete Steenrod algebra $\hat{\mathscr{A}}$ for an odd prime p and its relations with the generalized Dickson algebra on infinitely many generators, as a $\mathbb{Z}[\frac{1}{p}]$-graded algebra.

Keywords Steenrod algebra · Invariants algebras

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1 Introduction

Let p be an odd prime number, and let $\mathscr{A}$ be the algebra of reduced power operations, which is the quotient of the mod p Steenrod algebra with respect to the ideal generated by β, the Bockstein coboundary operator associated with the exact coefficient sequence

$$0 \to \mathbb{F}_p \to \mathbb{F}_{p^2} \to \mathbb{F}_p \to 0.$$

It is the graded associative algebra generated by the elements P^i of degree $2i(p-1)$, subject to the Adem relations

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$$P^aP^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j$$

if $a < pb$, and $P^0 = 1$. A monomial in $\mathscr{A}$ can be written in the form $P^{i_1}P^{i_2}\cdots P^{i_k}$, where $i_1, \ldots, i_k \in \mathbb{N}$. We denote this monomial by P^I, where I is the multiindex $(i_1, \ldots, i_{1k})$. If $i_s \geq pi_{s+1}$ for any $s = 1, \ldots, k-1$, the monomial P^I will be called *admissible*. The admissible monomials form a basis for A as an $\mathbb{F}_p$-vector space [9].

The Steenrod algebra is an important tool in algebraic topology. Its action on polynomials commutes with the action of the general linear group and this property has revealed important in invariant theory [7]. Because of its connection with the stable homotopy groups of spheres, it would be nice to have a complete knowledge of the cohomology of $\mathscr{A}$. In [8] the author asks some questions about the interaction between the map $\widetilde{P^0}$ (the map induced in cohomology by the Frobenius on $\mathscr{A}^*$) and the multiplicative structure of $\mathrm{Ext}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p)$ and he points out that these questions and conjectures can be reformulated in terms of the (dual) complete Steenrod algebra $\hat{\mathscr{A}}_*$. This point of view rises the natural interest on the cohomology of $\hat{\mathscr{A}}$, which seems easier to understand than $\mathscr{A}$.

In this paper we study the complete Steenrod algebra and show that its dual is isomorphic to the p-root closure of the Dickson algebra on infinitely many generators. The case $p = 2$ is treated by Llerena and Hu'ng [5]. In order to make this paper self-contained, we recall all definitions we need, even if they are quite natural generalizations of those given for the mod 2 case.

2 The p-complete Steenrod algebra

We consider objects graded over $\mathbb{Z}[\frac{1}{p}]$.

Definition 1 A *large* $\mathbb{Z}[\frac{1}{p}]$*-graded algebra* A is an algebra $A \subset \Pi_{n\in\mathbb{Z}[\frac{1}{p}]}A_n$, whose multiplication is defined by maps $A_m \otimes A_n \to A_{m+n}$, for $m, n \in \mathbb{Z}[\frac{1}{p}]$. If $A = \oplus_{n\in\mathbb{Z}[\frac{1}{p}]}A_n$, then A is called a $\mathbb{Z}[\frac{1}{p}]$*-graded algebra* A.

A (*large*) $\mathbb{Z}[\frac{1}{p}]$*-graded Hopf algebra* A is a Hopf algebra which is a (large) $\mathbb{Z}[\frac{1}{p}]$-graded algebra with respect to its multiplication; the comultiplication is defined by maps $A_m \to A_n \otimes A_{m-n}$ for $m, n \in \mathbb{Z}[\frac{1}{p}]$.

A $\mathbb{Z}[\frac{1}{p}]$*-graded module* M over a (large) $\mathbb{Z}[\frac{1}{p}]$-graded algebra A is an A-module $M = \oplus_{n\in\mathbb{Z}[\frac{1}{p}]}M_n$ where the action of A on M is given by maps $A_m \otimes M_n \to M_{m+n}$, for $m, n \in \mathbb{Z}[\frac{1}{p}]$.

If M is a (large) $\mathbb{Z}[\frac{1}{p}]$-graded object, we define p^tM as the (large) $\mathbb{Z}[\frac{1}{p}]$-graded object which is isomorphic to M as ungraded object and has $(p^tM)_n = M_{\frac{n}{p^t}}$ for any $t \in \mathbb{Z}$. For example, we can think of the Steenrod algebra $\mathscr{A}$ as a $\mathbb{Z}[\frac{1}{p}]$-graded Hopf algebra which is zero in negative and fractional degrees. The homomorphism

$$d : \frac{1}{p}\mathscr{A} \to \mathscr{A}$$

defined by $d(P^a) = P^{a/p}$ if $a \equiv 0$ modulo p, and $d(P^a) = 0$ if $a \not\equiv 0$ modulo p, is a homomorphism of Hopf algebras which preserves the degree, while $d' : \mathscr{A} \to \mathscr{A}$, $P^a \mapsto P^{a/p}$ is a map of Hopf algebras which divides degrees by p.

Definition 2 The complete Steenrod algebra is the inverse limit

$$\hat{\mathscr{A}} = \lim_{\longleftarrow} \left\{ \cdots \xrightarrow{d} \frac{1}{p^{t+1}}\mathscr{A} \xrightarrow{d} \frac{1}{p^t}\mathscr{A} \xrightarrow{d} \cdots \right\},$$

where $d : \frac{1}{p^{t+1}}\mathscr{A} \to \frac{1}{p^t}\mathscr{A}$ is the map of Hopf algebras which divides degrees by p. The tensor product $\hat{\mathscr{A}} \otimes \hat{\mathscr{A}}$ is defined as

$$\hat{\mathscr{A}} \otimes \hat{\mathscr{A}} = \lim_{\longleftarrow} \left(\frac{1}{p}\mathscr{A} \otimes \frac{1}{p}\mathscr{A} \right).$$

In a way similar to the case $p = 2$, $\hat{\mathscr{A}}$ can be endowed with a structure of $\mathbb{Z}[\frac{1}{p}]$-graded Hopf algebra: the product and coproduct in $\mathscr{A}$ induce operations

$$\phi_t : \frac{1}{p^t}\mathscr{A} \otimes \frac{1}{p^t}\mathscr{A} \to \frac{1}{p^t}\mathscr{A}, \quad \psi_t : \frac{1}{p^t}\mathscr{A} \to \frac{1}{p^t}\mathscr{A} \otimes \frac{1}{p^t}\mathscr{A}$$

compatible with the homomorphism d, so it is possible to define a product and a coproduct in $\hat{\mathscr{A}}$ as

$$\phi = \lim_{\longleftarrow} \phi_t : \hat{\mathscr{A}} \otimes \hat{\mathscr{A}} \to \hat{\mathscr{A}}, \quad \psi = \lim_{\longleftarrow} \psi_t : \hat{\mathscr{A}} \to \hat{\mathscr{A}} \otimes \hat{\mathscr{A}}.$$

Let us denote by P^{r/p^t} the element of $\hat{\mathscr{A}}$ represented by the sequence $\{P^{p^n r} \in \frac{1}{p^{t+n}}\mathscr{A} | n \in \mathbb{N}\}$, that is, P^{r/p^t} is the sequence having P^r in $\frac{1}{p^t}\mathscr{A}$, hence P^{rp} in $\frac{1}{p^{t+1}}\mathscr{A}$, $\ldots$, P^{rp^n} in $\frac{1}{p^{t+n}}\mathscr{A}$ and so on. Since $d(P^{rp}) = P^r$, we have $P^{r/p^t} = P^{rp/p^{t+1}}$. The operation P^{r/p^t} has degree r/p^t. Given a finite sequence $I = (i_1, \ldots, i_k)$ of elements in $\mathbb{N}[\frac{1}{p}]$, let us denote by P^I the operation $P^{i_1} P^{i_2} \cdots P^{i_k}$.

The complete Steenrod algebra $\hat{\mathscr{A}}$ is not of finite type: for every $a = 1, \ldots, p-1$ the operations $P^{\frac{p^n - a}{p^n}} P^{\frac{a}{p^n}}$, $n > 1$, are of degree 1 and are linearly independent: $P^{\frac{p^n - a}{p^n}} P^{\frac{a}{p^n}}$ represents the sequence having $P^{p^n - a} P^a$ in $\frac{1}{p^n}\mathscr{A}$.

The complete Steenrod algebra $\hat{\mathscr{A}}$ is a large $\mathbb{Z}[\frac{1}{p}]$-graded Hopf algebra, but not a $\mathbb{Z}[\frac{1}{p}]$-graded Hopf algebra: given $a = 1, \ldots, p-1$ the element $\theta_a = \sum_{n \in \mathbb{N}} P^{a/p^n}$ belongs to $\hat{\mathscr{A}}$, but it does not belong to $\oplus_{n \in \mathbb{Z}[\frac{1}{p}]} \hat{\mathscr{A}}$ (think of P^{a/p^n} as of the sequence with P^a in $\frac{1}{p^n}\mathscr{A}$; then θ_a is the sequence having $\sum_{k=0}^{n} P^{p^k a}$ in $\frac{1}{p^{n+1}}\mathscr{A}$). Furthermore, $\hat{\mathscr{A}} \neq \Pi_{n\in\mathbb{Z}[\frac{1}{p}]}\hat{\mathscr{A}_n}$. In fact, $\alpha = \sum_{n\in\mathbb{N}} P^n \notin \hat{\mathscr{A}}$, but $\alpha \in \Pi_{n\in\mathbb{Z}[\frac{1}{p}]}\hat{\mathscr{A}_n}$. A sum $\sum_{I\in\mathscr{I}} P^I$ belongs to $\hat{\mathscr{A}}$ if and only if, for each $t \in \mathbb{Z}$, the number of sequences $I = (i_1, \ldots, i_m) \in \mathscr{I}$ such that $p^t i_q \in \mathbb{N}$ for all i_q, $q = 1, \ldots, m$ is finite.

We define the binomial coefficient $\binom{a}{b}$ for $a, b \in \mathbb{N}[\frac{1}{p}]$ as the residue class modulo p of $\binom{p^N a}{p^N b}$, where N is an integer such that $p^N a, p^N b \in \mathbb{Z}$. It is well defined since it does not depend on N. The following result gives the Adem relations and the coproduct formula in $\hat{\mathscr{A}}$.

Proposition 1 *The Adem relations in $\hat{\mathscr{A}}$ are*

$$P^a P^b = \sum_{i\in\mathbb{N}[\frac{1}{p}]} \binom{(p-1)(b-i) - p^{-N(a,b,i)}}{a-pi} P^{a+b-i} P^i \tag{1}$$

for $0 < a < pb$. Here $0 \leq i \leq a/p$ and $N = N(a, b, i)$ is a big enough integer such that $p^N a, p^N b, p^N i \in \mathbb{Z}$.

The coproduct formula is

$$\psi(P^a) = \sum_{i\in\mathbb{N}[\frac{1}{p}]} P^i \otimes P^{a-i}$$

for any $a \in \mathbb{N}[\frac{1}{p}]$.

Proof Let us recall that if $A = a_0 + a_1 p + \cdots + a_n p^n$ and $B = b_0 + b_1 p + \cdots + b_n p^n$ are the p-adic expansions of A and B, respectively, then $\binom{A}{B} \equiv \prod_i \binom{a_i}{b_i} \bmod p$ [9]. This proves that the binomial coefficient does not depend on N. Since $pA - 1 = p(A-1) + (p-1)$, we get

$$\binom{pA-1}{pB} = \binom{p(A-1)}{pB} \cdot \binom{p-1}{0} = \binom{A-1}{B}.$$

Hence,

$$\begin{aligned}\binom{(p-1)(b-i) - p^{-(N+1)}}{a-pi} &= \binom{p(p-1)(p^N b - p^N i) - 1}{p(p^N a - p^N pi)} \\ &= \binom{(p-1)(p^N b - p^N i) - 1}{p^N a - p^N pi} = \binom{(p-1)(b-i) - p^{-N}}{a-pi}.\end{aligned}$$

Now let $\pi_t : \hat{\mathscr{A}} \to \frac{1}{p^t}\mathscr{A}$ denote the projection. We prove that the images under π_t of both sides are equal for any t big enough, and this suffices to say that (1) holds. Take t such that $p^t a, p^t b \in \mathbb{Z}$. We have $\pi(P^i) = P^{p^t i}$ if $p^t i \in \mathbb{N}$ and it is 0 otherwise.

Applying π_t to both sides of the formula in (1), we have

$$P^{p^t a} P^{p^t b} = \sum_{0 \le i \le \frac{a}{p}} \binom{(p-1)(b-i) - p^{-t}}{a - pi} P^{p^t a + p^t b - p^t i} P^{p^t i},$$

where the sum is taken over those indices i such that $p^t i \in \mathbb{N}$. The previous sum is finite. Taking $a' = p^t a$, $b' = p^t b$, $i' = p^t i$, we get the Adem relations in $\mathscr{A}$:

$$P^{a'} P^{b'} = \sum_{i'=0}^{[\frac{a'}{p}]} \binom{(p-1)(b'-i') - 1}{a' - pi'} P^{a'+b'-i'} P^{i'},$$

for any $a', b', i' \in \mathbb{N}$ and $0 < a' < pb'$. A similar argument can be applied to prove the coproduct formula. □

Given a sequence $I = (i_1, \ldots, i_m)$ with elements in $\mathbb{N}[\frac{1}{p}]$, we say that the operation P^I is admissible if $i_k \ge p i_{k+1}$ for any $k = 1, \ldots, m-1$: for t big enough, its image $P^{p^t i_1} P^{p^t i_2} \ldots P^{p^t i_m}$ by π_t is an admissible operation in $\frac{1}{p^t}\mathscr{A}$. Then the admissibles in $\hat{\mathscr{A}}$ are linearly independent.

3 Root algebras over $\hat{\mathscr{A}}$

For every $\mathscr{A}$-algebra M, it is possible to define the root closure of M so that it becomes an $\hat{\mathscr{A}}$-algebra.

Definition 3 A root algebra B is a $\mathbb{Z}[\frac{1}{p}]$-graded commutative $\mathbb{F}_p$-algebra such that the degree preserving homomorphism $\hat{\delta} : pB \to B$ defined by $\hat{\delta}(x) = x^p$ is an isomorphism.

Think of a root algebra B as an algebra where any element $x \in B$ has a unique p-th root $y = \sqrt[p]{x}$, i.e. the element y with $\hat{\delta}(y) = x$. In general, set $\sqrt[p^t]{x} = \sqrt[p]{\sqrt[p^{t-1}]{x}}$.

Example 1 Let $\mathbb{F}_p[x_1, \ldots, x_n]$ be the polynomial algebra over $\mathbb{F}_p$ in n indeterminates $x_1, \ldots, x_n$ of fixed degree m and let

$$\delta : \frac{1}{p^{t-1}}\mathbb{F}_p[x_1, \ldots, x_n] \to \frac{1}{p^t}\mathbb{F}_p[x_1, \ldots, x_n]$$

be defined by $\delta(x_i) = x_i^p$. Then

$$R[x_1, \ldots, x_n] = \lim_{\overrightarrow{\delta}} \frac{1}{p^t}\mathbb{F}_p[x_1, \ldots, x_n]$$

is the *free root algebra* generated by $x_1, \ldots, x_n$. This construction can be extended to polynomial algebras with an infinite number of generators.

Definition 4 (1) Given a $\mathbb{Z}$-graded algebra M, the *p-root closure* of M is the $\mathbb{Z}[\frac{1}{p}]$-graded algebra

$$M^{\sqrt[p]{\ }} = \varinjlim_{\delta} \left(\frac{1}{p^t} M\right),$$

with respect to the homomorphism $\delta : \frac{1}{p^t} M \to \frac{1}{p^{t+1}} M$ given by $\delta(x) = x^p$. It is a root algebra.

(2) Let $f : M \to N$ be a homomorphism of $\mathbb{Z}$-graded algebras. Since f commutes with the homomorphism δ, it induces a homomorphism of algebras

$$f^{\sqrt[p]{\ }} : M^{\sqrt[p]{\ }} \to N^{\sqrt[p]{\ }}.$$

In particular, we have $f^{\sqrt[p]{\ }}(\sqrt[p]{u}) = \sqrt[p]{f^{\sqrt[p]{\ }}(u)}$, $u \in M^{\sqrt[p]{\ }}$.

We start from a $\mathbb{Z}$-graded $\mathscr{A}$-algebra M and define an $\hat{\mathscr{A}}$-module structure on the root closure $M^{\sqrt[p]{\ }}$.

Proposition 2 *If θ is an operation in $\frac{1}{p^{t+1}}\mathscr{A}$ and x is an element in $\frac{1}{p^t}M$, then $\theta\delta x = \delta(d\theta x)$.*

Proof It is enough to check the relation for $\theta = P^i$, i a positive integer. Let x be any element in M. Recall that M is graded commutative. Then, by an iterated use of the Cartan formula for $\mathscr{A}$-modules, we obtain

$$P^i(x^p) = \sum \binom{p}{m_1 \cdots m_k} (P^{i_1}(x))^{m_1} \cdots (P^{i_k}(x))^{m_k}, \tag{*}$$

where the sum is taken over all p-partitions of i, i.e. $i = \sum_{j=1}^k m_j i_j$, $1 \le k \le p$, $i_1, \ldots, i_k$ are pairwise distinct and $\sum_{j=1}^k m_j = p$. It means that m_j is the multiplicity of i_j as a summund of i (that is m_j says how many times i_j appears in the sum $i = \sum_{j=1}^k i_j$). Here $\binom{p}{m_1 \cdots m_k}$ is the multinomial coefficient $\frac{p!}{m_1! \cdots m_k!}$. It is always divisible by p but the case $m_{j_0} = p$ for some j_0 (hence $m_j = 0$ for any $j \neq j_0$). The last case occurs iff $i \equiv 0 \bmod p$. Thus, $P^i x^p = 0$ if $i \not\equiv 0 \bmod p$ (every multinomial coefficient in $(*)$ is 0 mod p). When $i = pi'$, the sum $(*)$ has just one term with non-zero coefficient, namely $(P^{i'}(x))^p$. So, $P^{pi'}(x^p) = (P^{i'}(x))^p$. If $i \not\equiv 0 \bmod p$, then $d(P^i) = 0$ and $P^i \delta x = P^i x^p = 0$. If $i = pi'$, $P^i(\delta x) = P^{pi'}(x^p) = (P^{i'}(x))^p = \delta(d(P^{pi'})x)$. □

This shows that the action of $\hat{\mathscr{A}}$ on $M^{\sqrt[p]{\ }}$, extending that of $\mathscr{A}$ on M, which we are going to define, does not depend on the representative.

Definition 5 Let M be an $\mathscr{A}$-algebra, $\xi \in \hat{\mathscr{A}}$ and $u \in M^{\sqrt[p]{\ }}$. Then $\xi \cdot u$ is the element in $M^{\sqrt[p]{\ }}$ represented by $(\pi_t \xi) \cdot x$, where $x \in \frac{1}{p^t} M$ is a representative of u.

In particular, for $\xi \in \hat{\mathscr{A}}$, $u \in M^{\sqrt[p]{}}$, we have $\sqrt[p]{\xi u} = \hat{d}(\xi)\sqrt[p]{u}$, where $\hat{d} : \frac{1}{p}\hat{\mathscr{A}} \to \hat{\mathscr{A}}$ is the homomorphism which divides the degree by p.

We say that an $\mathscr{A}$-module M verifies the *condition of finiteness* (f.c. for short) if $P^i(x) = 0$ for all but a finite number of integers i. A similar definition holds for $\hat{\mathscr{A}}$-modules $\mathscr{M}$, where $i \in \mathbb{N}[\frac{1}{p}]$. If the f.c. holds for M, then so it does for $M^{\sqrt[p]{}}$.

If an $\hat{\mathscr{A}}$-module $\mathscr{M}$ is an algebra, satisfies the f.c. and

$$P^l(xy) = \sum_{h+k=l} P^h(x)P^k(y)$$

for $l, h, k \in \mathbb{N}[\frac{1}{p}]$ and for every $x, y \in \mathscr{M}$, then we sat that $\mathscr{M}$ is an $\hat{\mathscr{A}}$-algebra. It is easy to prove that if M is an $\mathscr{A}$- algebra satisfying the f.c., then $M^{\sqrt[p]{}}$ is an $\hat{\mathscr{A}}$-algebra and $P(\sqrt[p^t]{x}) = \sqrt[p^t]{P(u)}$ for any $x \in M^{\sqrt[p]{}}$ and $t \in \mathbb{Z}$, having set $P = \sum_{i\in\mathbb{N}[\frac{1}{p}]} P^i$ to mean the *total power* in $\hat{\mathscr{A}}$ (i.e. the total power in every $\frac{1}{p^t}\mathscr{A}$). We observe that the action of P on a module $\mathscr{M}$ makes sense if the f.c. holds for $\mathscr{M}$. The root closure of $\mathscr{A}$-algebras defines an exact functor which can be applied in particular to the cohomology of topological spaces, since their unstability as $\mathscr{A}$-algebras ensure that the f.c. holds.

Remark 1 Suppose in the mod p Steenrod algebra we do not disregard the Bockstein β. We could think of $\mathscr{A}$ as a bigraded algebra by assigning $\beta^\varepsilon P^i$ bidegree $(2i(p-1), \varepsilon)$. Then $(\frac{1}{p^t}\mathscr{A})_{(l,e)} = \mathscr{A}_{(l/p,e)}$. We could define $d : \frac{1}{p^t}\mathscr{A} \to \mathscr{A}$ by letting $d(\beta^\varepsilon P^i) = \beta^\varepsilon P^{i/p}$ if $i \equiv 0 \bmod p$ and zero otherwise, and take $\hat{\mathscr{A}}$ as the inverse limit with respect to d.

Now, let M be the cohomology ring with coefficients in $\mathbb{F}_p$ of $L_p = S^\infty/(\mathbb{Z}/p\mathbb{Z})$. Then $M = E[x] \otimes \mathbb{F}_p[y]$, where x and y are cohomolgy classes of degree 1 and 2, respectively, and $\beta(x) = y$. If we want to define the root closure of M, we have to declare what $\delta(x)$ is. Since $x^2 = 0$, the isomorphism δ should act as the identity on x. This means that the root closure of M could be thought of as the tensor product of $E[x]$ with the root closure of the polynomial part $\mathbb{F}_p[y]$:

$$\delta(x) = x, \quad \delta(y) = y^p.$$

According to the previous proposition, $\theta\delta z = \delta(d\theta z)$, where $\theta \in \frac{1}{p^{t+1}}\mathscr{A}$ and $z \in \frac{1}{p^t}M$. For $\theta = P^i$, we have $P^i\delta(xy^k) = P^i(xy^{kp}) = xP^i(y^{kp})$ and $\delta dP^i(xy^k) = \delta P^{i/p}(xy^k) = x((P^{i/p}(y^k))^p$: they are equal in any case. When $\theta = \beta$, then $\beta\delta(x) = \beta(x) = y$, while $\delta d\beta(x) = \delta\beta(x) = \delta(y) = y^p$.

This shows that the Bockstein β in the mod p Steenrod algebra is an obstruction to the extension of the $\mathscr{A}$-action on M to an $\hat{\mathscr{A}}$-action on $M^{\sqrt[p]{}}$. It depends on the following fact: the polynomial part of $H^*(L_p^n)$ is closed under the action of those operations in $\mathscr{A}_p$ involving the powers P^i only; further the action of the P^i's on the external part of $H^*(L_p^n)$ is zero. The Bockstein β acts as zero on the polynomial part of $H^*(L_p^n)$, but the external part is not closed under its action: $\beta(x_i) = y_i, i = 1, \ldots, n$.

For these reasons we confine the attention to $\mathscr{A}$-algebras M which are trivial in odd degrees (for example, the cohomology rings of CW-complexes with no cells in odd dimensions).

Remark 2 Let us consider the $\mathbb{Z}$-graded algebra $M = \mathbb{F}_2[x]/(x^2)$. Its root closure is simply $\mathbb{F}_2$ for the universal property of direct limits. Hence, in general, a $\mathbb{Z}$-graded algebra M does not canonically embed into its root closure (a mistake in part (2) of Remark 2.4., p. 276, [Llerena-Hung]). M embeds into its root closure if it does not contain nilpotent elements (if there was a nilpotent element x of height k, then x^{2^i} would annihilate for every i such that $2^i > k$; by the property of direct limits, there could not be roots for x). We can say that the quotient of any $\mathbb{Z}$-graded algebra M by its p-torsion part can be embedded into its root closure.

4 The Root closure of the Dickson algebra

Let $GL_n = GL(n, \mathbb{F}_p)$ be the general linear group acting on the polynomial algebra $P_n = \mathbb{F}_p[y_1, \ldots, y_n]$ on generators $y_1, \ldots, y_n$ of degree 2. The Dickson algebra D_n is the algebra of invariants $P_n^{GL_n}$. It is a polynomial algebra on n generators

$$D_n = \mathbb{F}_p[Q_{n,0}, Q_{n,1}, \ldots, Q_{n,n-1}],$$

where $Q_{n,s}$ is the Dickson invariant of degree $2(p^n - p^s)$. They are inductively defined by the following formula

$$Q_{n,s} = Q_{n-1,s} V_n^{p-1} + Q_{n-1,s-1}^p,$$

where $Q_{n,n} = 1$, $Q_{n,s} = 0$ if $s < 0$ or $s > n$, $V_n = \prod_{\lambda_i \in \mathbb{F}_p} (\lambda_1 y_1 + \cdots + \lambda_{n-1} y_{n-1} + y_n)$.

We call $D_n^{\sqrt[p]{}}$ the *generalized Dickson algebra* over $\mathbb{F}_p$, according to the name given by Arnon to the root closure of D_n for $p = 2$. Observe that $\delta : \frac{1}{p^t} P_n \to \frac{1}{p^{t+1}} P_n$ is a GL_n-homomorphism:

$$\begin{aligned} \delta(g \cdot p(y_1, \ldots, y_n)) &= (g \cdot p(y_1, \ldots, y_n))^p \\ &= g \cdot (p(y_1, \ldots, y_n))^p \\ &= g \cdot \delta(p(y_1, \ldots, y_n)), \end{aligned}$$

since $(a+b)^p = a^p + b^p$ in an $\mathbb{F}_p$-algebra. This means that δ brings invariant elements to invariant elements, hence

$$D_n^{\sqrt[p]{}} := \left(P_n^{GL_n}\right)^{\sqrt[p]{}} = \left(P_n^{\sqrt[p]{}}\right)^{GL_n}.$$

In [1] Arnon observed that the root closure of the subalgebra $D_n \subset P_n$ of invariants is the same as the invariants of the root closure of P_n, for the $p = 2$ case, using particular elements ${}_k\omega_n$ defined by Peterson. Dealing with odd primes we can define the corresponding elements to the ${}_k\omega_n$'s only for $k = 1$ and $k = 2$, namely

$$ {}_1\omega_{n(p-1)} = \left(y_1^{p-1}\right)^n = z_1^n \in D_1^{\sqrt[p]{\ }} $$

for $n \in \mathbb{N}[\frac{1}{p}]$;

$$ {}_2\omega_{n(p-1)} = \sum_{s_1+s_2=n} z_1^{s_1} z_2^{s_2} \in D_2^{\sqrt[p]{\ }}, $$

where $z_i = y_i^{p-1}$, $i = 1, 2$, $n \in \mathbb{N}[\frac{1}{p}]$, $s_i = p^m k$ and $k = 0, 1, \ldots, p-1$. For example,

$$ {}_2\omega_{p^2-1} = \sum_{j=0}^{p-1} z_1^{j+1} z_2^{p-j} = Q_{2,0}, $$

$$ {}_2\omega_{p^2-p} = \sum_{j=0}^{p} z_1^{j} z_2^{p-j} = Q_{2,1}. $$

We just recall that

$$ Q_{2,0} = \begin{vmatrix} y_1 & y_2 \\ y_1^p & y_2^p \end{vmatrix}^{p-1} = y_1^{p-1} y_2^{p-1} \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} y_1^{j(p-1)} y_2^{(p-1)(p-1-j)}. $$

Now, $\binom{p-1}{k}_p = 1$ if k is even and it is equal to $-1 = p-1$ if k is odd (it follows from the relation $0 = \binom{p}{k}_p = \binom{p-1}{k}_p + \binom{p-1}{k-1}_p$). Then

$$ Q_{2,0} = z_1 z_2 \sum_{j=0}^{p-1} z_1^j z_2^{p-1-j}. $$

The other Dickson invariant is

$$ Q_{2,1} = \frac{\begin{vmatrix} y_1 & y_2 \\ y_1^{p^2} & y_2^{p^2} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{p} & y_2^{p} \end{vmatrix}} = \frac{z_2^{p+1} - z_1^{p+1}}{z_2 - z_1} = \sum_{j=0}^{p} z_1^j z_2^{p-j}. $$

The most natural way to generalise what has already been done for $n = 2$ would be defining elements ${}_k\omega_n$, $n \equiv 0$ modulo $p-1$, in such a way that $Q_{n,i} = {}_n\omega_{p^n-p^i}$. Looking at the recursive formula defining the Dickson invariants, we deduce that they are polynomials in $z_1, z_2, \dots, z_i, \dots$, where $z_i = y_i^{p-1}$. The degree of $Q_{n,s}$ with respect to the z_i's is $\frac{p^n - p^i}{p-1} = p^i + p^{i+1} + \cdots + p^{n-1}$.

Let us recall that

$$L_n = [0, 1, \dots, n-1] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^p & y_2^p & \cdots & y_n^p \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{p^{n-1}} & y_2^{p^{n-1}} & \cdots & y_n^{p^{n-1}} \end{vmatrix}$$

$$\begin{aligned} L_{n,s} &= [0, 1, \dots, \hat{s}, \dots, n], \quad 0 \le s \le n, \\ Q_{n,s} &= L_{n,s}/L_n, \quad V_n = L_n/L_{n-1}, \end{aligned}$$

in particular

$$Q_{n,0} = L_{n,0}/L_n = \frac{[1, 2, \dots, n]}{[0, 1, \dots, n-1]} = \frac{[0, 1, \dots, n-1]^p}{[0, 1, \dots, n-1]} = L_n^{p-1}.$$

The explicit expression for $Q_{n,0}$ is

$$Q_{n,0} = z_1 z_2 \cdots z_n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ z_1^{1+p} & z_2^{1+p} & \cdots & z_n^{1+p} \\ \cdots & & & \cdots \\ z_1^{1+p+\cdots+p^{n-2}} & z_2^{1+p+\cdots p^{n-2}} & \cdots & z_n^{1+p+\cdots p^{n-2}} \end{vmatrix}^{p-1};$$

it is the $(p-1)$th power of a sum of $n!$ terms. When we explicitly compute this power, multinomial coefficients $\binom{p-1}{\alpha_1, \cdots, \alpha_{n!}}$, $\alpha_1 + \cdots + \alpha_{n!} = p-1, 0 \le \alpha_i \le p-1$ appear and they are not always equal to 1 as it occurs in the $p = 2$ case. Thus the generalization of the ${}_k\omega_n$'s to the odd primes case is not clear. What we can do is defining elements ${}_k\omega_n$ such that ${}_k\omega_n^p = {}_k\omega_{pn}$, only for particular integers n, namely

$${}_k\omega_{p^i-1} := \sqrt[p^{k-i}]{Q_{k,k-i}}.$$

Its degree is $\deg({}_k\omega_{p^i-1}) = 2(p^i - 1)$, which does not depend on k. We get $Q_{k,s} = {}_k\omega_{p^k - p^s}$.

Proposition 3 *$D_n^{\sqrt[p]{}}$ is the free root algebra generated by $\{{}_n\omega_{p^i-1}\}_{i=1}^n$.*

Proof For every $h \in \mathbb{Z}$, $Q_{n,s}^{p^h} = ({}_n\omega_{p^n - p^s})^{p^h} = ({}_n\omega_{p^{n-s}-1})^{p^{s+h}}$. Conversely, for any $k \in \mathbb{Z}$, $({}_n\omega_{p^i-1})^{p^k} = ({}_n\omega_{p^n - p^{n-i}})^{p^{k-n+i}} = Q_{n,n-i}^{p^{k-n+i}} = (\sqrt[p^{n-i}]{Q_{n,n-i}})^{p^k}$. □

Definition 6 Let $i_n : D_n^{\sqrt[p]{}} \to D_{n+1}^{\sqrt[p]{}}$ be the degree preserving monomorphism of algebras given by $i_n({}_n\omega_{p^i-1}) = {}_{n+1}\omega_{p^i-1}$, for $1 \leq i \leq n$. Then we define

$$D_\infty^{\sqrt[p]{}} = \varinjlim_n D_n^{\sqrt[p]{}} \quad \text{and} \quad \omega_{p^i-1} = \varinjlim_n {}_n\omega_{p^i-1}.$$

$D_\infty^{\sqrt[p]{}}$ is the free root algebra generated by $\{\omega_{p^i-1}\}_{i=1}^\infty$ with $\deg(\omega_{p^i-1}) = 2(p^i - 1)$.

Remark 3 i_n is not a homomorphism of $\hat{\mathscr{A}}$-algebras. Indeed, $P^1 \cdot Q_{n,0}^p = 0$ by the Cartan formula and the relations $P^1 \cdot Q_{n,0} = 0$, $P^1 \cdot Q_{n+1,1} = Q_{n+1,0}$, $i_n(Q_{n,0}^p) = i_n({}_n\omega_{p^n-1}^p) = {}_{n+1}\omega_{p^n-1}^p = {}_{n+1}\omega_{p^{n+1}-p} = Q_{n+1,1}$. Hence

$$P^1 \cdot i_n(Q_{n,0}^p) = Q_{n+1,0} \neq 0 = i_n(P^1 \cdot Q_{n,0}^p).$$

There is a relation between the Dickson invariants and the *dual* of the complete Steenrod algebra. $\hat{\mathscr{A}}$ is not of finite type, so we first need to make precise what we mean by its dual $\hat{\mathscr{A}}_*$.

Definition 7 Let $d^* \colon \frac{1}{p^t}\mathscr{A}_* \longrightarrow \frac{1}{p^{t+1}}\mathscr{A}_*$ be the dual of the homomorphism $d : \frac{1}{p^{t+1}}\mathscr{A} \to \frac{1}{p^t}\mathscr{A}$ sending P^{pa} to P^a. The *graded dual* of $\hat{\mathscr{A}}$ is the direct limit

$$\hat{\mathscr{A}}_* = \varinjlim_{d^*} \left(\frac{1}{p^t}\mathscr{A}_*\right),$$

where $\mathscr{A}_*$ is the graded dual of the finite type algebra $\mathscr{A}$.

Lemma 1 *If $\xi \in \mathscr{A}_*$, then*

$$\langle \xi^p, P^I \rangle = \begin{cases} \langle \xi, P^J \rangle & \text{if } I = pJ \\ 0 & \text{otherwise} \end{cases}$$

Proof We have $\langle \xi^p, P^I \rangle = \langle \psi^*(\xi \otimes \cdots \otimes \xi), P^I \rangle = \langle (\xi \otimes \cdots \otimes \xi), \psi^{p-1}(P^I) \rangle = \langle \xi \otimes \cdots \otimes \xi, \sum_{I_1 + \cdots I_p = I} P^{I_1} \otimes \cdots \otimes P^{I_p} \rangle = \sum_{I_1 + \cdots I_p = I} \langle \xi, P^{I_1} \rangle \cdots \langle \xi, P^{I_p} \rangle$. The terms in this summation cancel mod p, unless $I_1 = \ldots = I_p = J$ when $I = pJ$. Now $\langle \xi, P^J \rangle \in \mathbb{F}_p$, therefore $\langle \xi, P^J \rangle^p = \langle \xi, P^J \rangle$. The lemma follows. □

Theorem 1 *$\hat{\mathscr{A}}_*$ and $D_\infty^{\sqrt[p]{}}$ are isomorphic as $\mathbb{Z}[\frac{1}{p}]$-graded algebras.*

Proof The dual $\mathscr{A}_*$ of the Steenrod algebra is isomorphic to $\mathbb{F}_p[\xi_1, \xi_2, \ldots]$, where ξ_i is dual to $P^{p^{i-1}} \cdots P^p P^1$ with respect to the basis of admissible monomials in $\mathscr{A}$. Its degree is $2(p^i - 1)$. Now we prove that $d^*(\xi_i) = \xi_i^p$ for any $i \in \mathbb{N}$. According to the definition of d,

$$d(P^I) = \begin{cases} P^J & \text{if } I = pJ \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$d\left(P^{p^i}\cdots P^{p^2}P^p\right) = P^{p^{i-1}}\cdots P^pP^1.$$

Passing to the dual,

$$d^*(\xi_i) = d^*\left(P^{p^{i-1}}\cdots P^pP^1\right)^* = \left(P^{p^i}\cdots P^{p^2}P^p\right)^*.$$

By the previous Lemma,

$$\xi_i^p = \left(P^{p^i}\cdots P^{p^2}P^p\right)^* = d^*(\xi_i).$$

Now, by the Milnor isomorphism,

$$\hat{\mathscr{A}}_* = \varinjlim_{d^*}\left(\frac{1}{p^t}\mathscr{A}_*\right) \cong \varinjlim_{d^*}\left(\frac{1}{p^t}\mathbb{F}_p[\xi_1,\xi_2,\ldots]\right) = R[\xi_1,\xi_2,\ldots].$$

On the other hand, $D_\infty^{\sqrt[p^\infty]{}}$ is also a free root algebra generated by the elements ω_{p^i-1} of degree $2(p^i-1)$. Therefore the correspondence $\xi_i \mapsto \omega_{p^i-1}$ establishes the following isomorphism of $\mathbb{Z}[\frac{1}{p}]$-graded algebras

$$R[\xi_1,\xi_2,\ldots] = \mathbb{F}_p[\xi_1,\xi_2,\ldots]^{\sqrt[p^\infty]{}} \cong D_\infty^{\sqrt[p^\infty]{}}.$$

□

Let us consider the algebra $\Gamma_n = \mathbb{F}_p[Q_{n,0}^{\pm 1}, Q_{n,1},\ldots,Q_{n,n-1}]$, the Dickson algebra with the Euler class inverted. The action of $\mathscr{A}$ on D_n can be extended in a natural way to Γ_n through the Cartan formula applied to $1 = Q_{n,0}^{-1}Q_{n,0}$. For example, when $p=2$, we have in Γ_2 that $Sq^i(Q_{2,0}) = Q_{2,0}Q_{2,t}$ for $i = 4-2^t$, $0\le t\le 2$, and vanishes otherwise (see [3] and the extension [4] to the case of odd primes). Then we get the following recursive formula for $Q_{2,0}^{-1}$:

$$Sq^0\left(Q_{2,0}^{-1}\right) = Q_{2,0}^{-1},\quad Sq^i\left(Q_{2,0}^{-1}\right) = Q_{2,1}Sq^{i-2}\left(Q_{2,0}^{-1}\right) + Q_{2,0}Sq^{i-3}\left(Q_{2,0}^{-1}\right),\quad i>0.$$

Let Γ_n^- be the subalgebra of Γ_n generated by the elements $Q_{n,0}^{-1}Q_{n,i}$, $1\le i\le n$. Again, by the Cartan formula and the action of $\mathscr{A}$ on D_n [3], we get, for example in $\Gamma_2 = \mathbb{F}_2[Q_{2,0}^{\pm 1}, Q_{2,1}]$,

$$Sq^i\left(Q_{2,0}^{-1}Q_{2,1}\right) = Q_{2,1}^2Sq^{i-2}\left(Q_{2,0}^{-1}\right) + Q_{2,0}Sq^{i-1}\left(Q_{2,0}^{-1}\right) + Q_{2,1}Sq^i\left(Q_{2,0}^{-1}\right).$$

It turns out that the action of Sq^2, Sq^3 and Sq^5 on $Q_{2,0}^{-1}Q_{2,1}$ is zero and

$$Sq^1\left(Q_{2,0}^{-1}Q_{2,1}\right) = 1,\quad Sq^4\left(Q_{2,0}^{-1}Q_{2,1}\right) = Q_{2,0},$$

so that Γ_n^- is not an $\mathscr{A}$-submodule of Γ_n.

Let i_n be the homomorphism of Definition 4.2. We have:

$$i_n(Q_{n,i}) = i_n\left({}_n\omega^{p^i}_{p^{n-i}-1}\right) = {}_{n+1}\omega^{p^i}_{p^{n-i}-1} = \sqrt[p]{{}_{n+1}\omega^{p^{i+1}}_{p^{n+1-(i+1)}-1}} = \sqrt[p]{Q_{n+1,i+1}},$$

then

$$i_n(Q^{-1}_{n,0}Q_{n,i}) = \sqrt[p]{Q_{n+1,i+1}}/\sqrt[p]{Q_{n+1,1}},$$

and it does not make sense in any object we are dealing with (D_n, Γ_n, Γ^-_n and their root closure); in particular $i_n(\Gamma^-_n) \not\subset \Gamma^-_{n+1}$.

Let j_n be the following morphism:

$$j_n({}_n\omega_{p^i-1}) = {}_{n+1}\omega_{p^{i+1}-1}.$$

It does not preserve the degree and $j_n(Q_{n,i}) = Q_{n+1,i}$; indeed,

$$j_n(Q_{n,i}) = j_n\left({}_n\omega^{p^i}_{p^{n-i})-1}\right) = {}_{n+1}\omega^{p^i}_{p^{n+1-i)}-1} = Q_{n+1,i}.$$

Hence,

$$j_n\left(Q^{-1}_{n,0}Q_{n,i}\right) = Q^{-1}_{n+1,0}Q_{n+1,i},$$

so the restriction $\overline{j}_n$ of j_n to Γ^-_n is well defined and preserves the degree. Since Γ^-_n is not an $\mathscr{A}$-module, there is no point in checking if $\overline{j}_n$ is an $\mathscr{A}$-module homomorphism.

Let us denote by Γ^- the direct limit

$$\Gamma^- = \lim_{\overrightarrow{\overline{j}_n}} \Gamma^-_n$$

with respect to $\overline{j}_n$, and by η_i the element

$$\eta_i = \lim_{\overrightarrow{\overline{j}_n}} Q^{-1}_{n,0}Q_{n,i}$$

of degree $2(1-p^i)$. Then

$$\mathscr{A}_* \cong \mathbb{F}_p[\eta_1, \eta_2, \ldots],$$

i.e. $\mathbb{F}_p[\eta_1, \eta_2, \ldots]$ is another invariant theoretic description of $\mathscr{A}_*$, and its root closure $R[\eta_1, \eta_2, \ldots]$ is another invariant theoretic description of $\hat{\mathscr{A}}_*$.

5 The complete iterated total power operation

Let X be a CW-complex with no cells in odd dimensions and let $H^*(X)$ be the cohomology ring with coefficients in $\mathbb{F}_p$. For every integer $m \geq 1$ we have the iterated total power operation

$$T_m : H^*(X) \to H^*((\mathbb{Z}/p\mathbb{Z})^m)^{\widetilde{SL_m}} \otimes H^*(X),$$

where $\widetilde{SL_m}$ is the subgroup of all non-singular matrices M such that $(det M)^{(p-1)/2} = 1$.

Mui gave an expression of T_m in terms of the Milnor bases, showing how the elements of this basis can be derived from the Dickson invariants. Because of our assumption on X, Mui's expression reduces to the following:

$$T_m(z) = \mu(q)^m \widetilde{L_m}^q \sum_R (-1)^{r(R)} Q_{m,0}^{r_0} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \otimes St^R(z),$$

where $z \in H^q(X)$, $R = (r_1, \dots, r_m)$, $r_0 = -r_1 - \cdots - r_m$, $St^R = (\xi_1^{r_1} \cdots \xi_m^{r_m})^*$, $r(R) = r_1 + 2r_2 + \cdots mr_m$.

In [2] the normalized version S_m of T_m for odd primes is studied (see [6] for $p = 2$):

$$S_m : H^*(X) \to \Phi_m^{GL_m} \otimes H^*(X),$$

where Φ_m is the localization of the polynomial part of $H^*(B(\mathbb{Z}/p)^m)$ out of its Euler class.

It has been proved that

$$S_m(z) = \sum_R (-1)^{r(R)} Q_{m,0}^{r_0} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \otimes St^R(z),$$

or

$$S_m(z) = \sum_R (-1)^{r(R)} \left(\frac{Q_{m,1}}{Q_{m,0}}\right)^{r_1} \cdots \left(\frac{Q_{m,m}}{Q_{m,0}}\right)^{r_m} \otimes St^R(z).$$

Further,

$$\frac{Q_{m,k}}{Q_{m,0}} = \sum_J w^{-J} = (-1)^k \delta_m(\xi_k),$$

where J is a multiindex of the form $(0, \dots, 0, p^{k-1}, \dots, p, 0, \dots, 1, 0, \dots)$ with $m - k$ zeros inserted. See [2], Definition 5, for the map $\delta_m : \mathscr{A}^* \to \Delta_m = \Phi_m^{B_m}$. Here $\Phi_m^{B_m}$ is the subalgebra of invariants under the action of the Borel subgroup B_n of $GL(n, \mathbb{F}_p)$.

Definition 8 Let $\mathscr{M}$ be an $\hat{\mathscr{A}}$-module which the f.c. holds for. Let us define

$$\mathscr{S}_1 : \mathscr{M} \to \Phi_1^{\sqrt[p]{}} \otimes \mathscr{M}$$

as $\mathscr{S}_1(u) = \sum_{i \in \mathbb{N}[\frac{1}{p}]} y_1^{-i} \otimes P^i(u)$, $u \in \mathscr{M}$. The we define inductively

$$\mathscr{S}_m(u) = \mathscr{S}_1(\mathscr{S}_{m-1}(u)).$$

By the Cartan formula, $\mathscr{S}_m$ is an $\hat{\mathscr{A}}$-module homomorphism. By the definition of the action of $\hat{\mathscr{A}}$ on $H^{\sqrt[p]{}}(X)$, it is not difficult to prove the following result:

Proposition 4 *Given a topological space X,*

$$\mathscr{S}_m : H^{\sqrt[p]{}}(X) \to (\Phi_m^{GL_m})^{\sqrt[p]{}} \otimes H^{\sqrt[p]{}}(X)$$

is an algebra homomorphism which coincides with $S_m^{\sqrt[p]{}}$ *and, for every* $u \in: H^{\sqrt[p]{}}(X)$*,*

$$\mathscr{S}_m(u) = \sum_{R=(r_1,\ldots,r_m)} {}_m\omega_{p^m-1}^{-(r_1+\cdots+r_m)} \; {}_m\omega_{p^{m-1}-1}^{pr_1} \cdots \; {}_m\omega_{p-1}^{p^{m-1}r_{m-1}} \otimes St^R(u),$$

where $i \in \mathbb{N}[\frac{1}{p}]$.

This shows how the operations in the complete Steenrod algebra $\hat{\mathscr{A}}$ can be derived from the root closure of $\Gamma_n = \mathbb{F}_p[Q_{n,0}^{\pm 1}, Q_{n,1}, \ldots, Q_{n,n-1}]$.

References

1. Arnon, D.: Generalized Dickson invariants. Isr. J. Math. **118**, 183–205 (2000)
2. Ciampella, A.: Modular invariant theory and the iterated total power operation. Boll. Un. Mat. Ital. **3-B**(8), 325–335 (2000)
3. Hu'ng, N.H.V.: The action of the Steenrod squares on the modular invariants of linear groups. Proc. AMS **113**, 1097–1104 (1991)
4. Hu'ng, N.H.V., Minh, P.A.: The action of the mod p Steenrod operations on the modular invariants of linear groups. Vietnam J. Math. **23**, 39–56 (1995)
5. Llerena I., Hu'ng N.H.V. (1996) The complete Steenrod algebra and the generalized Dickson algebra. Algebraic topology: new trends in localization and periodicity (Sant Feliu de Giúxols, 1994). In: Progr. Math., vol. 136, pp. 271–284. Birkhäuser, Basel (1996)
6. Lomonaco, L.A.: The iterated total squaring operation. Proc. AMS **115**, 1149–1155 (1992)
7. Meyer, D.M., Smith, L.: Poincaré duality algebras, Macaulay's dual systems, and Steenrod operations. In: Cambridge Tracts in Mathematics, vol. 167. Cambridge University Press, London (2005)
8. Palmieri, J.H.: Some quotient Hopf algebras of the dual Steenrod algebra. Trans. Am. Math. Soc. **358**, 671–685 (2006)
9. Steenrod, N.E., Epstein, D.B.A.: Cohomology operations. In: Annals of Mathematical Studies N., vol. 50. Princeton University Press, Princeton (1962)