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NEPS OF COMPLEX UNIT GAIN GRAPHS*

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Abstract. A complex unit gain graph (or $\mathbb{T}$-gain graph) is a gain graph with gains in $\mathbb{T}$, the multiplicative group of complex units. Extending a classical construction for simple graphs due to Cvektović, suitably defined noncomplete extended p-sums (NEPS, for short) of $\mathbb{T}$-gain graphs are considered in this paper. Structural properties of NEPS like balance and some spectral properties and invariants of their adjacency and Laplacian matrices are investigated, including the energy and the possible symmetry of the adjacency spectrum. It is also shown how NEPS are useful to obtain infinitely many integral graphs from the few at hands. Moreover, it is studied how NEPS of $\mathbb{T}$-gain graphs behave with respect to the property of being nut, i.e., having 0 as simple adjacency eigenvalue and nowhere zero 0-eigenvectors. Finally, a family of new products generalizing NEPS is introduced, and their few first spectral properties explored.

Key words. Gain graph, NEPS, Products.

AMS subject classifications. 05C22, 05C25, 05C76.

1. Introduction. Let Γ be a nonempty simple graph with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and let $\vec{E}(\Gamma)$ be the set of its oriented edges. Such set contains two copies of each edge of Γ with opposite directions. We write e_{ij} for the oriented edge from v_i to v_j. Given any multiplicative group $\mathfrak{G}$, a $\mathfrak{G}$-*gain graph* is a pair $\Phi = (\Gamma, \gamma)$ consisting of an *underlying graph* Γ and a map γ from $\vec{E}(\Gamma)$ to the *gain group* $\mathfrak{G}$ such that $\gamma(e_{ij}) = \gamma(e_{ji})^{-1}$. Let $1_{\mathfrak{G}}$ denote the identity element of $\mathfrak{G}$. The gain graph Φ is said to be balanced if, for every directed cycle $\vec{C} = e_{i_1 i_2} \cdots e_{i_k i_1}$ in Γ (if any), we have $\gamma(e_{i_1 i_2})\gamma(e_{i_2 i_3}) \cdots \gamma(e_{i_k i_1}) = 1_{\mathfrak{G}}$.

In particular, a *complex unit* gain graph is a $\mathfrak{G}$-gain graph with $\mathfrak{G} = \mathbb{T}$, the multiplicative group of all complex numbers with norm 1. The theory of complex unit gain graphs incorporates those of signed graphs and mixed graphs (as defined in [17]). In fact, a signed graph (resp., mixed graph) can be seen as a particular $\mathbb{T}$-gain graph with gains in the subset $\{\pm 1\}$ (resp., $\{1, \pm i\}$) of $\mathbb{T}$. Clearly, every $\mathbb{T}_n$-gain graph, where $n \in \mathbb{N}$ and $\mathbb{T}_n$ denotes the group of nth roots of unity, can be regarded as a complex unit gain graph. Empty graphs can be thought as $\mathbb{T}$-gain graphs equipped with the empty gain function $\varnothing \to \mathbb{T}$ and are obviously balanced.

In the wake of [28], over the last decade there has been a renewed and growing interest for the Hermitian matrices associated to $\mathbb{T}$-gain graphs and their spectra (see, for instance [6, 7, 8, 9, 20, 23, 26, 31, 34, 35]).

After a section of preliminaries, we suitably define in Section 3 the *noncomplete extended p-sums* (NEPS, for short) of $\mathbb{T}$-gain graphs, originally defined by Cvetković for simple graphs [12], retrieving the *Cvetković products* of signed graphs introduced in [15] when the gains of the factors are all included in $\{\pm 1\}$. The Cartesian products, the strong product, and the direct product of $\mathbb{T}$-gain graphs all turn out to be special cases of NEPS. We prove that NEPS behave well with respect to switching equivalence and give formulæ for their adjacency eigenvalues. In Section 4, we consider the energy of an NEPS and find infinite families

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of noncospectral equienergetic $\mathbb{T}$-gain graphs. The short Section 5 is devoted to the Laplacian eigenvalues of NEPS; a formula relating the Laplacian spectrum of an NEPS and the Laplacian spectrum of its factors is given when the latter are all regular. The Cvetković products are a powerful tool to obtain infinitely many integral connected $\mathbb{T}$-gain graphs. This topic is investigated in Section 6. Afterward, we deal with complex unit gain graphs either having a symmetric spectrum or being sign-symmetric (see Section 7 for the definition). Theorem 7.4 and 7.5, two of our main results, give structural conditions characterizing those NEPS which preserve the spectral symmetry and the sign-symmetry of their factors. In Section 8, we extend to $\mathbb{T}$-gain graphs the classical notion of nut (simple) graphs: such graphs have 0 as simple eigenvalue and 0-eigenvectors without null components. In this case too, NEPS constructions show to be useful to obtain infinitely many nut $\mathbb{T}$-gain graphs from the small number one has at hand. In the final Section 9, we propose generalizations of Cvetković products, which seem promising and reasonably manageable, since the majority of the spectral results obtained for NEPS in this paper can be naturally extended to our new products.

2. Preliminaries.

2.1. Complex unit gain graphs.

Let $M_{m,n}(\mathbb{C})$ be the set of $m \times n$ complex matrices. For a matrix $A = (a_{ij}) \in M_{m,n}(\mathbb{C})$, we denote by $A^* = (a^*_{ij}) \in M_{n,m}(\mathbb{C})$ its *conjugate* (or *Hermitian*) *transpose*; i.e., $a^*_{ij} = \overline{a_{ji}}$.

The *adjacency matrix* $A(\Phi) = (a_{ij}) \in M_{n,n}(\mathbb{C})$ of a $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ is defined by

$$a_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

If v_i is adjacent to v_j, then $a_{ij} = \gamma(e_{ij}) = \gamma(e_{ji})^{-1} = \overline{\gamma(e_{ji})} = \overline{a_{ji}}$. Consequently, $A(\Phi)$ is Hermitian and its eigenvalues $\lambda_1(\Phi) \geqslant \cdots \geqslant \lambda_n(\Phi)$ are real. The *Laplacian matrix* $L(\Phi)$, defined as $D(\Gamma) - A(\Phi)$, where $D(\Gamma) = \mathrm{diag}(d(v_1), \dots, d(v_n))$ stands for the diagonal matrix of vertex degrees of Γ, is Hermitian as well, and all its eigenvalues $\lambda^L_1(\Phi) \geqslant \cdots \geqslant \lambda^L_n(\Phi)$ are nonnegative [28]. By definition, the spectrum $\mathrm{sp}(M(\Phi))$ is the multiset of eigenvalues of $M(\Phi)$, where $M \in \{A, L\}$. For brevity of notation, we shall often write $\mathrm{sp}(\Phi)$ instead of $\mathrm{sp}(A(\Phi))$ and denote by $m_\Phi(\lambda)$ the multiplicity of an eigenvalue $\lambda \in \mathrm{sp}(\Phi)$.

The *negation* of a $\mathbb{T}$-gain graph Φ is $-\Phi := (\Phi, -\gamma)$. Clearly, $A(-\Phi) = -A(\Phi)$ and $\lambda_i(-\Phi) = -\lambda_{n-i+1}(\Phi)$.

A *switching function* for a gain graph Φ is any map $\zeta : V(\Gamma) \to \mathbb{T}$. Switching a nonempty $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ means replacing γ by γ^ζ, where $\gamma^\zeta(e_{ij}) = \zeta(v_i)^{-1}\gamma(e_{ij})\zeta(v_j)$, and obtaining in this way the new $\mathbb{T}$-gain graph $\Phi^\zeta = (\Gamma, \gamma^\zeta)$. We say that $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ (and their corresponding gain functions) are *switching equivalent* if there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. By writing $\Phi_1 \sim \Phi_2$, we mean that Φ_1 and Φ_2 are switching equivalent.

To each switching function ζ we associate a diagonal matrix $D(\zeta) = \mathrm{diag}(\zeta(v_1), \dots, \zeta(v_n))$. Note that

$$M(\Phi_1^\zeta) = D(\zeta)^* M(\Phi_1) D(\zeta) \qquad \text{for } M \in \{A, L\}. \tag{2.2}$$

If Φ_2 is isomorphic (but not necessarily equal) to Φ_1^ζ for a suitable switching function ζ, then Φ_1 and Φ_2 are said to be *switching isomorphic*. If this is the case, we write $\Phi_1 \simeq \Phi_2$, and

$$M(\Phi_2) = P^{-1} M(\Phi_1^\zeta) P = (PD(\zeta))^* M(\Phi_1) PD(\zeta), \tag{2.3}$$

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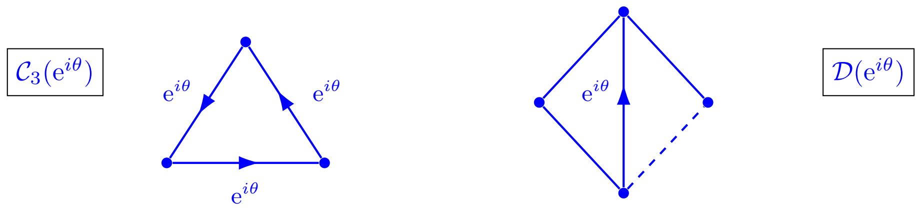

FIGURE 1. *The gain triangle* $\mathcal{C}_3(\mathrm{e}^{i\theta})$ *and the gain diamond* $\mathcal{D}(\mathrm{e}^{i\theta})$.

where P is an appropriate permutation $(0,1)$-matrix and $M \in \{A, L\}$. From (2.2) and (2.3), we easily obtain

$$\Phi_1 \sim \Phi_2 \quad \Longrightarrow \quad \Phi_1 \simeq \Phi_2 \quad \Longrightarrow \quad \mathrm{sp}(M(\Phi_1)) = \mathrm{sp}(M(\Phi_2)) \quad \text{for } M \in \{A, L\}. \tag{2.4}$$

A walk $W = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{l-1} i_l}$ is said to be *neutral*, *negative*, or *imaginary*, depending on if its *gain* $\gamma(W) := \gamma(e_{i_1 i_2})\gamma(e_{i_2 i_3}) \cdots \gamma(e_{i_{l-1} i_l})$ is 1, -1 or nonreal. We write $(\Gamma, 1)$ for the $\mathbb{T}$-gain graph with all neutral arcs.

The next proposition specializes [29, Lemma 2.2] to $\mathbb{T}$-gain graphs.

PROPOSITION 2.1. *Let* $\Phi_1 = (\Gamma, \gamma_1)$ *and* $\Phi_2 = (\Gamma, \gamma_2)$ *be* $\mathbb{T}$*-gain graphs with the same underlying graph* Γ. Φ_1 *and* Φ_2 *are switching equivalent if and only if, for every directed cycle* $\vec{C}$ *in* Γ*, we have* $\gamma_1(\vec{C}) = \gamma_2(\vec{C})$.

Neither of the two implications in (2.4) can be reversed. The counterexamples of minimal order are given in Example 2.2 and involve the gain diamonds of type $\mathcal{D}(z)$ depicted in Fig. 1, where, like in the other figures of this paper, we adopt the following drawing convention: each continuous (resp., dashed) thick undirected line represents two opposite oriented edges with gain 1 (resp., -1), whereas the arrows detect the oriented edges uv's with a nonreal gain. The value $\gamma(uv)$ is often specified near the correspondent arrow.

EXAMPLE 2.2. For each $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}$, we consider the gain diamond $\mathcal{D}(z)$ depicted on the right of Fig. 1. Fixed a suitable ordering for its vertex set, we have

$$A(\mathcal{D}(z)) = \begin{bmatrix} 0 & -1 & z & 1 \\ -1 & 0 & 1 & 0 \\ \overline{z} & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{sp}(D(z)) = \{-2, -1, 1, 2\}.$$

Hence, the graphs in the set $\{\mathcal{D}(z)) \mid z \in \mathbb{T}\}$ are all cospectral; yet, as it can easily deduced from Proposition 2.1, $\mathcal{D}(z) \simeq \mathcal{D}(z')$ only if $z' \in \{z, \overline{z}\}$ and $\mathcal{D}(z) \sim \mathcal{D}(z')$ only if $z' = z$. ■

An edge set $S \subseteq E$ is said to be *balanced* if no nonneutral directed cycles with edges in S exist. A subgraph is *balanced* if its edge set is balanced (see [1, 9, 28] for further details). It is immediately seen that the gain triangle $\mathcal{C}_3(z)$ in Fig. 1 is balanced if and only if $z^3 = 1$.

A *potential function* for γ is a function $\theta : V \to \mathbb{T}$, such that $\theta(v_i)^{-1}\theta(v_j) = \gamma(e_{ij})$ for every $e_{ij} \in \vec{E}(\Gamma)$. By Proposition 2.1, it follows that a $\mathbb{T}$-gain graph Φ is balanced if and only if all its directed cycles are neutral. This is the only laborious part along the proof of the following result.

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PROPOSITION 2.3. [28, Lemma 2.1] *Let $\Phi = (\Gamma, \gamma)$ be a $\mathbb{T}$-gain graph. The following three conditions are equivalent:*

1. *Φ is balanced.*
2. *$\Phi \sim (\Gamma, 1)$.*
3. *γ has a potential function.*

The next result, proved in [24, Theorem 4.6] when the underlying graph is connected, gives a spectral characterization of $\mathbb{T}$-gain graphs.

THEOREM 2.4. [11, Corollary 5.1] *A $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ is balanced if and only if* $\mathrm{sp}(A(\Phi)) = \mathrm{sp}(A(\Gamma))$.

The last result we recall is the computation of the adjacency spectrum of every $\mathbb{T}$-gain cycle.

THEOREM 2.5. [28, Theorem 6.1] *Let (C_n, γ) be a $\mathbb{T}$-gain cycle such that one of its directed cycles has gain $\mathrm{e}^{i\theta}$. Then,*

$$\mathrm{sp}(C_n, \gamma) = \left\{ 2\cos\left(\frac{\theta + 2\pi j}{n}\right) \,\middle|\, 0 \leqslant j \leqslant n-1 \right\}. \tag{2.5}$$

2.2. Kronecker products of matrices.

Let $A = [a_{ij}]_{k\times m}$ and $B = [b_{ij}]_{l\times n}$ be two matrices of orders $k \times m$ and $l \times n$, respectively. The Kronecker Product of A and B is by definition the $kl \times mn$ matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k1}B & a_{k2}B & \cdots & \cdots & a_{km}B \end{bmatrix}.$$

In the following proposition, we collect some results that follow more or less immediately from the definition.

PROPOSITION 2.6. [36, Theorems 4.5 and 4.6] *Let A,B,C, and D matrices of appropriate sizes. Then,*

1. *$(A+B)\otimes C = A\otimes C + B\otimes C$;*
2. *$(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$;*
3. *$A\otimes B$ is null if and only if either A or B is null.*
4. *$(A\otimes B)\otimes C = A\otimes(B\otimes C)$;*
5. *$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$;*
6. *$(A\otimes B)^{\top} = A^{\top}\otimes B^{\top}$;*
7. *$(A\otimes B)^{*} = A^{*}\otimes B^{*}$.*

Let $\mathbb{N}_{\leqslant h}$ be the subset of all positive integers from i to $h \in \mathbb{N}$. For $i \in \mathbb{N}_{\leqslant h}$, we denote by $b_{i;qr}$ the (q, r)-entry of a $k_i \times l_i$ matrix B_i. Proposition 2.6(4) implies in particular that the h-ary product $B_1 \otimes B_2 \otimes \cdots \otimes B_h$ is well defined. Its rows (resp., columns) can be indexed by the h-tuples

$$\{\mathbf{q} = (q_1, q_2, \ldots, q_h) \mid q_i \in \mathbb{N}_{\leqslant k_i}\} \qquad (\text{resp.,} \quad \{\mathbf{r} = (r_1, r_2, \ldots, r_h) \mid r_i \in \mathbb{N}_{\leqslant l_i}\})$$

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ordered lexicographically. The **qr**-entry of the h-ary product is

$$b_{\mathbf{qr}} = b_{1;q_1 r_1} b_{2;q_2 r_2} \cdots b_{h;q_h r_h}. \tag{2.6}$$

By (2.6) (or by Proposition 2.6(7)), it follows that the Kronecker product of Hermitian matrices is Hermitian. In fact,

$$b_{\mathbf{rq}} = b_{1;r_1 q_1} b_{2;r_2 q_2} \cdots b_{h;r_h q_h} = b^*_{1;q_1 r_1} b^*_{2;q_2 r_2} \cdots b^*_{h;q_h r_h} = b^*_{\mathbf{qr}}.$$

Proposition 2.7. *For $i \in \mathbb{N}_{\leqslant h}$, let P_i be an $n_i \times n_i$ permutation matrix. Then, $P := \bigotimes_{i=1}^h P_i$ is an $n \times n$ permutation matrix, where $n = \prod_{i=1}^h n_i$.*

Proof. For each $\mathbf{q} = (q_1, \ldots, q_h)$, it is well defined the h-tuple $\mathbf{r_q} = (r_{1\mathbf{q}}, \ldots, r_{h\mathbf{q}})$ such that $b_{j;q_j,r_{j\mathbf{q}}}$ is the only nonzero element on the r_jth row in P_j. From (2.6), we immediately arrive at

$$b_{\mathbf{qr}} = \begin{cases} 1 & \text{if } \mathbf{r} = \mathbf{r_q}; \\ 0 & \text{otherwise.} \end{cases}$$

The argument to show that each column of P contains just one nonzero entry, and such entry is 1, is analogous. ∎

In the following statement and throughout the paper, I_m denotes the identity matrix of order m.

Proposition 2.8. [36, Theorem 4.8] *Let B and C be square matrices of orders k and l, respectively, with eigenvalues ν_i $(1 \leqslant i \leqslant k)$ and λ_j $(1 \leqslant j \leqslant l)$. Then the kl eigenvalues of $B \otimes C$ are $\nu_i \lambda_j$, and those of $B \otimes I_l + I_k \otimes C$ are $\nu_i + \lambda_j$.*

Proposition 2.9. [13, Theorem 2.8] or [15, Lemma 2.8] *For $j \in \mathbb{N}_{\leqslant h}$ and $r \in \mathbb{N}_{\leqslant p}$, let $\lambda_{j1} \geqslant \lambda_{j2} \geqslant \cdots \geqslant \lambda_{jn_j}$ the eigenvalues of a square matrix B_j of order n_j, and let $\mathbf{q}_r = (q_{r1}, \cdots, q_{rh})$ be an h-tuple of nonnegative integers. The eigenvalues of $B := \sum_{r=1}^p B_1^{q_{r1}} \otimes \cdots \otimes B_h^{q_{rh}}$ (where $B_i^0 := I_{n_i}$) are $\lambda_{k_1,\ldots,k_h} := \sum_{r=1}^p \lambda_{1k_1}^{q_{r1}} \cdots \lambda_{hk_h}^{q_{rh}}$ for $k_j \in \mathbb{N}_{\leqslant n_j}$.*

Proof. It suffices to note that if $\{\mathbf{u}_{jk_j} \mid j \in \mathbb{N}_h\}$ is a set on nonzero vectors such that $B_j \mathbf{u}_{jk_j} = \lambda_{jk_j} \mathbf{u}_{jk_j}$, then $\mathbf{u}_{1k_1} \otimes \cdots \otimes \mathbf{u}_{hk_h}$ is nonzero and $B(\mathbf{u}_{1k_1} \otimes \cdots \otimes \mathbf{u}_{hk_h}) = \lambda_{k_1,\ldots,k_h}(\mathbf{u}_{1k_1} \otimes \cdots \otimes \mathbf{u}_{hk_h})$ by Parts (1)-(3) of Proposition 2.6. ∎

3. Cvetković products of gain graphs and their adjacency matrix.

Let $\mathfrak{B}$ be a nonempty subset of $\mathfrak{F}_h := \{0,1\}^h \setminus \{(0,...,0)\}$, the set of $\{0,1\}$-h-tuples with at least one 1 among their components. We start by recalling how Cvektović defined $\Gamma := \mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_h; \mathfrak{B})$, the noncomplete extended p-sums (or simply NEPS) of the simple graphs $\Gamma_1, \ldots, \Gamma_h$ with basis $\mathfrak{B}$ (see, for instance, [13, p. 66]): the vertex set $V(\Gamma)$ is the Cartesian product $V(\Gamma_1) \times \cdots \times V(\Gamma_h)$, and the vertices $u := (u_1, \ldots, u_h)$ and $v := (v_1, \ldots, v_h)$ are adjacent if and only if there exists a (unique) h-tuple $\mathbf{b} = (b_1, \ldots, b_h)$ in $\mathfrak{B}$ such that $u_i = v_i$ whenever $b_i = 0$, and $u_i v_i$ is an edge of Γ_i if $b_i = 1$. Note that

$$\vec{E}(\Gamma) = \bigsqcup_{\mathbf{b} \in \mathfrak{B}} \vec{E}(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_h; \{\mathbf{b}\})), \tag{3.1}$$

where the symbol $\bigsqcup$ denotes the disjoint union.

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Definition 3.1. *Let $\Phi_1 = (\Gamma_1, \gamma_1), \dots, \Phi_h = (\Gamma_h, \gamma_h)$ be h $\mathbb{T}$-gain graphs. The NEPS (or* Cvecktović product*) of $\Phi_1, \dots, \Phi_h$ with basis $\mathfrak{B}$ is the $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ defined as follows:*

- *the underlying graph Γ is $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_h; \mathfrak{B})$;*
- *for each pair of adjacent vertices $u := (u_1, \dots, u_h)$ and $v := (v_1, \dots, v_h)$ in Γ,*

$$\gamma(uv) := \prod_{j=1}^{h} \gamma_j(u_j v_j), \tag{3.2}$$

where $\gamma_j(u_j v_j)$ is understood to be 1 whenever $u_j = v_j$.

The $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ will be denoted by $\mathrm{NEPS}(\Phi_1, \dots, \Phi_h; \mathfrak{B})$. ■

The map γ really defines a $\mathbb{T}$-gain structure on $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_h; \mathfrak{B})$, in fact,

$$\gamma(vu) = \prod_{j=1}^{h} \gamma_j(v_j u_j) = \prod_{j=1}^{h} \overline{\gamma_j(u_j v_j)} = \overline{\gamma(uv)}.$$

Remark 3.2. For $h \geqslant 2$, let $p : \mathfrak{F}_h \longrightarrow \{0,1\}^{h-1}$ be the projection onto the first $h-1$ coordinates. If $b_h = 0$ for all $\mathbf{b} \in \mathfrak{B}$, then

$$\Phi := \mathrm{NEPS}(\Gamma_1, \dots, \Gamma_h; \mathfrak{B}) = \bigsqcup_{i=1}^{n_h} \mathrm{NEPS}(\Gamma_1, \dots, \Gamma_{h-1}; p(\mathfrak{B})),$$

where $n_h := |V(\Gamma_h)|$. If instead $b_h = 1$ for all $\mathbf{b} \in \mathfrak{B}$ and $n_h = 1$, then the NEPS Φ is empty. ■

In order to make each factor of Φ structurally relevant and avoid empty NEPS, we shall often assume that the following condition is fulfilled:

(3.3) for each $i \in \mathbb{N}_{\leqslant h}$, Γ_i is nonempty and there exists a $\mathbf{b}$ in $\mathfrak{B}$ whose ith component is 1.

When the set $\gamma_i(\vec{E}(\Gamma_i))$ is included in $\{\pm 1\}$ for each $i \in \mathbb{N}_{\leqslant h}$, the several Φ_i's can be regarded as signed graphs. If this is the case, $\gamma(\vec{E}(\Gamma))$ is also included in $\{\pm 1\}$, and the signature on $\mathrm{NEPS}(\Phi_1, \dots, \Phi_h; \mathfrak{B})$ (thought as a signed graph) is precisely the one proposed in [15], where NEPS of signed graphs are defined.

For $p \in \mathbb{N}_{\leqslant h}$, let $\mathfrak{B}_{h,p}$ be the subset of $\mathfrak{F}_h$ of all h-tuples containing precisely p 1's, and let $\mathbf{j}_h \in \mathfrak{F}_h$ be the all-ones h-tuple $(1, \dots, 1)$. Clearly, $\mathfrak{B}_{h,h} = \{\mathbf{j}_h\}$. Inspired by some established terminology and notation in the realm of simple graphs (see [13, Section 2.5] and [18]), we call $\mathrm{NEPS}(\Phi_1, \dots, \Phi_h; \mathfrak{B}_{h,p})$ the *(complete) p-sum* of $\Phi_1, ..., \Phi_h$, and

$$\square_{i=1}^{h} \Phi_i := \mathrm{NEPS}(\Phi_1, ..., \Phi_h; \mathfrak{B}_{h,1}), \quad \bigtimes_{i=1}^{h} \Phi_i := \mathrm{NEPS}(\Phi_1, ..., \Phi_h; \{\mathbf{j}_h\}), \quad \boxtimes_{i=1}^{h} \Phi_i := \mathrm{NEPS}(\Phi_1, ..., \Phi_h; \mathfrak{F}_h),$$

the *Cartesian* product, the *direct* or *tensor* product, and the *strong* product, respectively. Thus, the Cartesian (resp., direct) product of h $\mathbb{T}$-gain graphs is synonym of complete 1-sum (resp., h-sum).

For any simple graph Λ, it is immediate to realize that the direct products $\Phi \times (\Lambda, 1)$ and $\Phi \times (K_2, 1)$ are the $\mathbb{T}$-gain graphs considered in [30] under the names *Kronecker product of Φ and Λ* and *bipartite double of Φ*, respectively.

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For $i \in \mathbb{N}_{\leqslant h}$, let now n_i be the order of the $\mathbb{T}$-gain graph $\Phi_i = (\Gamma_i, \gamma_i)$. Fixed once for all an ordering for the sets $V(\Gamma_i)$, we denote by either u_{ij} or v_{ij} the jth vertex of Γ_i. Then, we order the vertices of $V(\Gamma) = \{(u_{1j_1}, \ldots, u_{hj_h}) \mid j_h \in \mathbb{N}_{\leqslant n_h}\}$ lexicographically. The following proposition generalizes to $\mathbb{T}$-gain graphs the correspondent results for signed graphs achieved with [15, Theorem 3.1].

Proposition 3.3. *For $i \in \mathbb{N}_{\leqslant h}$, let Φ_i be a $\mathbb{T}$-gain graph with n_i vertices. The adjacency matrix of $\Phi = \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ is given by*

$$A(\Phi) = \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} A(\Phi_1)^{b_1} \otimes \cdots \otimes A(\Phi_h)^{b_h}. \tag{3.4}$$

Moreover, if $\lambda_{i1} \geqslant \lambda_{i2} \geqslant \cdots \geqslant \lambda_{in_i}$ are the eigenvalues of $A(\Phi_i)$, then $\mathrm{sp}(\Phi) = \{\lambda_{k_1,\ldots,k_h} \mid k_i \in \mathbb{N}_{\leqslant n_i}\}$, where

$$\lambda_{k_1,\ldots,k_h} := \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} \lambda_{1k_1}^{b_1} \cdots \lambda_{hk_h}^{b_h}. \tag{3.5}$$

Proof. The argument is essentially the one used in the proof of [15, Theorem 3.1]. Let $A'(\Phi)$ temporarily denote the matrix on the right side of (3.4). As explained in Section 2.2, the rows (resp., columns) of $A'(\Phi)$ can be indexed by the h-tuples

$$\{\mathbf{q} = (q_1, q_2, \ldots, q_h) \mid q_i \in \mathbb{N}_{\leqslant n_i}\} \qquad (\text{resp.,} \quad \{\mathbf{r} = (r_1, r_2, \ldots, r_h) \mid r_i \in \mathbb{N}_{\leqslant n_i}\}). \qquad \square$$

The same is true for the rows and columns of $A(\Phi)$: the row (resp., column) indexed by $\mathbf{q}$ (resp., $\mathbf{r}$) corresponds to the vertex $u = (u_{1q_1}, u_{2q_2}, \ldots, u_{hq_h})$ (resp., $v = (v_{1r_1}, v_{2r_2}, \ldots, v_{hr_h})$). Now, it is straightforward to check that if u and v are not adjacent in Γ, then the $(\mathbf{q},\mathbf{r})$-entries of $A(\Phi)$ and $A'(\Phi)$ are both zero. If instead u and v are adjacent, by (3.1) and the definition of Γ, there exists precisely one $\mathbf{b}^{(u,v)} \in \mathfrak{B}$ such that $b_i^{(u,v)} = 1$ if and only if $q_i \neq r_i$. Recalling (2.1) and (3.2), the $(\mathbf{q},\mathbf{r})$-entry of $A(\Phi)$ is $\prod_{j=1}^h \gamma_j(u_j v_j)$, which, by (2.6), is also the $(\mathbf{q},\mathbf{r})$-entry of $A(\Phi_1)^{b_1^{(u,v)}} \otimes \cdots \otimes A(\Phi_h)^{b_h^{(u,v)}}$, the $\mathbf{b}^{(u,v)}$-summand of $A'(\Phi)$. Since the $(\mathbf{q},\mathbf{r})$-entries of the remaining summands of $A'(\Phi)$ are zero, Equality (3.4) is proved.

The second part of the statement is a direct consequence of Proposition 2.9.

In the special case of the Cartesian product, Equality (3.4) becomes

$$A\left(\square_{i=1}^h \Phi_i\right) = A(\Phi_1) \otimes I_{n_2} \otimes \cdots \otimes I_{n_h} + I_{n_1} \otimes A(\Phi_2) \otimes \cdots \otimes I_{n_h} + \cdots + I_{n_1} \otimes I_{n_2} \otimes \cdots \otimes A(\Phi_h). \tag{3.6}$$

For $p \in \mathbb{N}_{\leqslant h}$, we recall that the pth elementary symmetric polynomials in h variables $X_1, \ldots, X_h$ is

$$S_p(X_1, \ldots, X_h) := \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant h} X_{j_1} X_{j_2} \cdots X_{j_p}.$$

The following result immediately follows from Proposition 3.3.

Corollary 3.4. *The adjacency spectra of the complete p-sum, the Cartesian product, the direct product and the strong product of $\Phi_1, \ldots, \Phi_h$ read as follows:*

$$\mathrm{sp}\left(\mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B}_{h,p})\right) = \{S_p(\lambda_{1k_1}, \ldots, \lambda_{hk_h}) \mid k_i \in \mathbb{N}_{\leqslant n_i}\},$$

$$\mathrm{sp}\left(\square_{i=1}^h \Phi_i\right) = \{\lambda_{1k_1} + \cdots + \lambda_{hk_h} \mid k_i \in \mathbb{N}_{\leqslant n_i}\},$$

$$\mathrm{sp}\left(\bigtimes_{i=1}^h \Phi_i\right) = \{\lambda_{1k_1}\lambda_{2k_2} \cdots \lambda_{hk_h} \mid k_i \in \mathbb{N}_{\leqslant n_i}\}, \tag{3.7}$$

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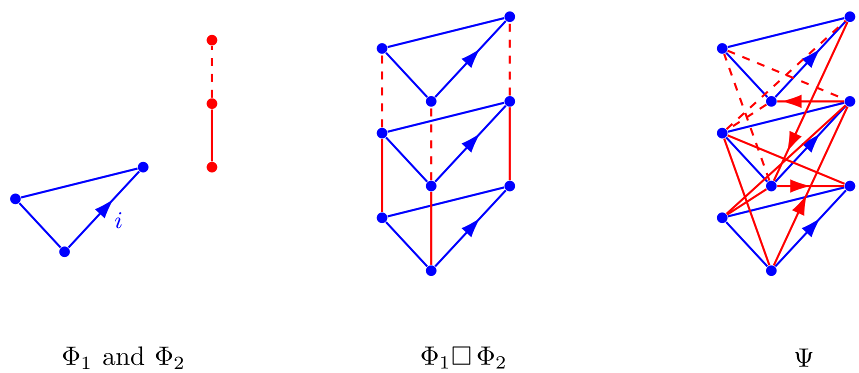

FIGURE 2. *The $\mathbb{T}$-gain graphs of Example 3.5.*

and

$$\mathrm{sp}\left(\boxtimes_{i=1}^{h} \Phi_i\right) = \left\{ \sum_{p=1}^{h} S_p(\lambda_{1k_1}, \dots, \lambda_{hk_h}) \,\middle|\, k_i \in \mathbb{N}_{\leqslant n_i} \right\},$$

where n_i is the order of Φ_i and $\lambda_{i1} \geqslant \lambda_{i2} \geqslant \cdots \geqslant \lambda_{in_i}$ are the eigenvalues of $A(\Phi_i)$.

EXAMPLE 3.5. Consider the $\mathbb{T}$-gain graphs $\Phi_1 = (C_3, \gamma_1)$ and $\Phi_2 = (P_3, \gamma_2)$ depicted on the left of Fig. 2. According to the drawing convention explained above, $\gamma_1(\vec{E}(C_3)) = \{1, \pm i\}$ and $\gamma_2(\vec{E}(P_3)) = \{\pm 1\}$ (note that, by Proposition 2.1, $\Phi_1 \sim \mathcal{C}_3(\mathrm{e}^{i\frac{\pi}{6}})$, one of gain triangles considered in Fig. 1). After fixing suitable orderings in $V(C_3)$ and $V(P_3)$, we obtain

$$A(\Phi_1) = \begin{bmatrix} 0 & i & 1 \\ -i & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad \text{and} \qquad A(\Phi_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

whose spectra are $\mathrm{sp}(\Phi_1) = \{0, \pm\sqrt{3}\}$ and $\mathrm{sp}(\Phi_2) = \{0, \pm\sqrt{2}\}$. The $\mathbb{T}$-gain graphs $\Phi_1 \square \Phi_2$ and $\Psi := \mathrm{NEPS}(\Phi_1 \times \Phi_2; \{(1,0),(1,1)\})$ are depicted in Fig. 2, where all the arrows connote arcs with gain i. By (3.4) (or directly from Definition 3.1)

$$A(\Phi_1 \square \Phi_2) = A(\Phi_1) \otimes I_3 + I_3 \otimes A(\Phi_2) \quad \text{and} \quad A(\Psi) = A(\Phi_1) \otimes I_3 + A(\Phi_1) \otimes A(\Phi_2).$$

We can use (3.5) to write down their spectra, arriving at

$$\mathrm{sp}(\Phi_1 \square \Phi_2) = \left\{0,\, \pm\sqrt{2},\, \pm\sqrt{3},\, \pm(\sqrt{2}+\sqrt{3}), \pm(\sqrt{2}-\sqrt{3})\right\}, \tag{3.8}$$

and

$$\mathrm{sp}(\Psi) = \left\{\pm\sqrt{3},\, \pm(\sqrt{3}+\sqrt{6}), \pm(\sqrt{3}-\sqrt{6}),\, 0^{(3)}\right\}, \tag{3.9}$$

where the exponent in parentheses of 0 stands for its multiplicity. ■

We now show that replacing some factors of a Cvektović product Φ with switching equivalence mates does not alter its switching equivalence class.

PROPOSITION 3.6. *For $i \in \mathbb{N}_{\leqslant h}$, let $\Phi_i = (\Gamma_i, \gamma_i)$ be a nonempty $\mathbb{T}$-gain graph, and let $\zeta_i : V(\Gamma_i) \longrightarrow \mathbb{T}$ a switching function. The $\mathbb{T}$-gain graphs $\Phi := \mathrm{NEPS}(\Phi_1, \dots, \Phi_h; \mathfrak{B})$ and $\Phi' := \mathrm{NEPS}(\Phi_1^{\zeta_1}, \dots, \Phi_k^{\zeta_h}; \mathfrak{B})$ are switching equivalent.*

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Proof. By Definition 3.1, Φ is the $\mathbb{T}$-gain graph (Γ, γ), where $\Gamma = \mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_h; \mathfrak{B})$, and γ acts as in (3.2). Similarly, Φ' is the $\mathbb{T}$-gain graph (Γ', γ'), where $\Gamma = \mathrm{NEPS}(\Gamma_1^{\zeta_1}, \ldots, \Gamma_h^{\zeta_h}; \mathfrak{B})$, and for any pair of adjacent vertices $u := (u_1, \ldots, u_h)$ and $v := (v_1, \ldots, v_h)$ in Γ,

$$\gamma'(uv) := \prod_{j=1}^{h} \gamma_j^{\zeta_j}(u_j v_j),$$

where $\gamma_j^{\zeta_j}(u_j v_j)$ has to be read as 1 if $u_j = v_j$.

We consider the following switching function for Φ:

$$\zeta : (u_1, \ldots, u_h) \in V(\Gamma) \longmapsto \zeta_1(u_1)\zeta_2(u_2) \cdots \zeta_h(u_h) \in \mathbb{T}. \tag{3.10}$$

It turns out that $\Phi' = \Phi^{\zeta}$. In fact,

$$\begin{aligned} \gamma'(uv) &= \prod_{j=1}^{h} \zeta_j^{-1}(u_j)\gamma_j(u_j v_j)\zeta_j(v_j) = \left(\prod_{j=1}^{h} \zeta_j^{-1}(u_j)\right)\left(\prod_{j=1}^{h} \gamma_j(u_j v_j)\right)\left(\prod_{j=1}^{h} \zeta_j(v_j)\right) \\ &= \zeta^{-1}(u)\gamma(uv)\zeta(v), \end{aligned}$$

as claimed. ∎

In Section 2.1, we have introduced a diagonal matrix associated to each switching function. We point out that the diagonal matrix $D(\zeta)$ associated to the map defined in (3.10) is $D(\zeta_1) \otimes \cdots \otimes D(\zeta_h)$. The following sequence of equalities can be interpreted as an alternative proof of Proposition 3.6.

$$\begin{aligned} A(\Phi') &= \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} A(\Phi_1^{\zeta_1})^{b_1} \otimes \cdots \otimes A(\Phi_h^{\zeta_h})^{b_h} && \text{(by (3.4))} \\ &= \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} D(\zeta_1)^* A(\Phi_1)^{b_1} D(\zeta_1) \otimes \cdots \otimes D(\zeta_h)^* A(\Phi_h)^{b_1} D(\zeta_h) && \text{(by (2.2))} \\ &= \left(\bigotimes_{i=1}^{h} D(\zeta_i)\right)^* \left(\sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} A(\Phi_1)^{b_1} \otimes \cdots \otimes A(\Phi_h)^{b_h}\right)\left(\bigotimes_{i=1}^{h} D(\zeta_i)\right) && \text{(by Proposition 2.6)} \\ &= D(\zeta)^* A(\Phi) D(\zeta) && \text{(again by (3.4)).} \end{aligned}$$

Proposition 3.7. *If, for every $i \in \mathbb{N}_{\leqslant h}$, the nonempty $\mathbb{T}$-gain graph $\Phi_i = (\Gamma_i, \gamma_i)$ is balanced, then $\Phi_{\mathfrak{B}} = (\Gamma_{\mathfrak{B}}, \gamma_{\mathfrak{B}}) := \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ is balanced for every nonempty $\mathfrak{B} \subseteq \mathfrak{F}_h$.*

Proof. The reader can choose his favorite argument among the following ones:

1. If $\Phi_i \sim (\Gamma_i, 1)$ for all i's, then $\Phi_{\mathfrak{B}} \sim (\Gamma_{\mathfrak{B}}, 1)$ by Proposition 3.6.
2. If Φ_i is balanced, then it has a potential $\theta_i : V(\Gamma_i) \longrightarrow \mathbb{T}$ by Proposition 2.3. The gain graph $\Phi_{\mathfrak{B}}$ is balanced again by Proposition 2.3 since it has a potential, namely

$$\theta : (u_1, \ldots, u_h) \longmapsto \prod_{i=1}^{h} \theta_i(u_i).$$

In fact, if $u = (u_1, \ldots, u_h)$ and $v = (v_1, \ldots, v_h)$ are adjacent in $\Gamma_{\mathfrak{B}}$, then

$$\theta^{-1}(u)\theta(v) = \left(\prod_{i=1}^{h} \theta_i(u_i)^{-1}\right)\left(\prod_{i=1}^{h} \theta_i(v_i)\right) = \prod_{i=1}^{h} \theta_i(u_i)^{-1}\theta_i(v_i) = \prod_{i=1}^{h} \gamma_i(u_i v_i) = \gamma(uv).$$

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3. By Proposition 3.3, $\Phi_{\mathfrak{B}}$ and $(\Gamma_{\mathfrak{B}}, 1)$ have the same eigenvalues. Hence, $\Phi_{\mathfrak{B}}$ is balanced by Theorem 2.4. ∎

An NEPS of $\mathbb{T}$-gain graphs can be balanced even though some of its factors are unbalanced. An example with the minimal number of vertices is $\mathrm{NEPS}(-(C_3, 1) \times (-(K_2, 1)); \{(1, 1)\})$ which is isomorphic to $(C_6, 1)$. More generally, as noted for the NEPS of signed graphs in [15, Section 2], if $\mathfrak{B}$ is a subset of $\mathfrak{B}_{h,p}$, then

$$\mathrm{NEPS}(-\Phi_1, \dots, -\Phi_h; \mathfrak{B}) = (-1)^p\, \mathrm{NEPS}(\Phi_1, \dots, \Phi_h; \mathfrak{B}).$$

4. Energy. The energy $\mathcal{E}(\Psi)$ of a $\mathbb{T}$-gain graph $\Psi = (\Lambda, \xi)$ is given by the formula

$$\mathcal{E}(\Psi) = \sum_{\lambda\,\in\,\mathrm{sp}(\Psi)} |\lambda|. \tag{4.1}$$

In the context of complex unit gain graphs, this numerical graph invariant has been comprehensively studied in [30], where the authors find bounds for the energy involving the spectral radius of Ψ or others parameters like the matching number, the vertex cover number, the number of odd cycles, and the largest vertex degree of Λ.

The next statement generalizes the correspondent result for signed graphs given in [15, Theorem 3.1].

PROPOSITION 4.1. *For $i \in \mathbb{N}_{\leqslant h}$, let n_i be the order of the nonempty $\mathbb{T}$-gain graph Φ_i. The energy of $\Phi := \mathrm{NEPS}(\Phi_1, \dots, \Phi_h; \mathfrak{B})$ can be computed through the formula*

$$\mathcal{E}(\Phi) = \sum_{r_1=1}^{n_1} \cdots \sum_{r_h=1}^{n_h} \left| \sum_{(b_1,\dots,b_h)\in\mathfrak{B}} \lambda_{1r_1}^{b_1} \cdots \lambda_{hr_h}^{b_h} \right|, \tag{4.2}$$

where $\lambda_{i1} \geqslant \cdots \geqslant \lambda_{in_i}$ are the eigenvalues of $A(\Phi_i)$. Moreover,

$$\frac{1}{n}\mathcal{E}(\Phi) \leqslant \sum_{\mathbf{b}\in\mathfrak{B}} \prod_{b_i=1} \frac{1}{n_i}\mathcal{E}(\Phi_i), \tag{4.3}$$

where $n := |V(\Phi)|$ and equality only holds for $\mathfrak{B} = \{\mathbf{j}_h\}$.

Proof. Equality (4.2) immediately comes from (3.5) and (4.1). In order to prove the remaining part of the statement, follow almost verbatim the final part of the proof of [15, Theorem 3.1] after replacing Σ and Σ_i with Φ and Φ_i, respectively, whenever they occur. ∎

COROLLARY 4.2. $\mathcal{E}\left(\bigtimes_{i=1}^h \Phi_i\right) = \prod_{i=1}^h \mathcal{E}(\Phi_i)$.

Proof. Use (3.7) or (4.3) with equality, since we are in the special case $\mathfrak{B} = \{\mathbf{j}_h\}$, and observe that the right side of (4.3) becomes $n^{-1}\prod_{i=1}^h \mathcal{E}(\Phi_i)$. ∎

Clearly, two switching isomorphic $\mathbb{T}$-gain graphs have the same energy. Inspired by an argument found in [33], where the energy of NEPS of simple graphs is investigated, we are about to show that each nonbipartite graph gives rise to infinitely many pairs of noncospectral equienergetic $\mathbb{T}$-graphs (signed graphs, actually) with the same underlying graph. *A fortiori*, those equienergetic pairs are not switching isomorphic.

Let Ψ be a $\mathbb{T}$-gain graph. For $0 \leqslant s \leqslant h$, we set

$$\Phi_{h,s}(\Psi) := \underbrace{(-\Psi) \times \cdots \times (-\Psi)}_{s \text{ times}} \times \underbrace{\Psi \times \cdots \times \Psi}_{h-s \text{ times}}.$$

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In the next statement, with a slight abuse of notation, Γ and $-\Gamma$ stand for $(\Gamma,1)$ and $-(\Gamma,1)$, respectively.

PROPOSITION 4.3. *The equality $\mathcal{E}(\Phi_{h,s}(\Gamma)) = \mathcal{E}(\Phi_{h,t}(\Gamma))$ holds for every pair s and t in $\{0,\dots,h\}$. Moreover, if Γ is not bipartite and the number $s-t$ is odd, then the $\mathbb{T}$-gain graphs $\Phi_{h,s}(\Gamma)$ and $\Phi_{h,t}(\Gamma)$ are not cospectral.*

Proof. A $\mathbb{T}$-gain graph Λ and its negation have the same energy since, as already noted in Section 2, they have opposite eigenvalues. By Corollary 4.2, it follows that

$$\mathcal{E}(\Phi_{h,s}(\Gamma)) = \mathcal{E}(\Phi_{h,t}(\Gamma)) = \left(\sum_{i=1}^{q}|\lambda_i|\right)^h,$$

where $\lambda_1 \geqslant \dots \geqslant \lambda_q$ are the eigenvalues of $A(\Gamma)$. In order to see that $\Phi_{h,s}(\Gamma)$ and $\Phi_{h,t}(\Gamma)$ are surely not switching isomorphic if $s-t$ is odd, it is not restrictive to assume s even and t odd. If this is the case,

$$\lambda_1(\Phi_{h,s}(\Gamma)) = \lambda_1^h > -\lambda_1^{h-1}\lambda_q = \lambda_1(\Phi_{h,t}(\Gamma)),$$

since, as a consequence of the Perron–Froebenius Theorem and [13, Theorem 3.11], we have $\lambda_1 > -\lambda_q$. Thus, $\mathrm{sp}(\Phi_{h,s}(\Gamma)) \neq \mathrm{sp}(\Phi_{h,t}(\Gamma))$. □

Proposition 4.3 shows that $\Phi_{h,s}(\Gamma)$ and $\Phi_{h,t}(\Gamma)$ with $s-t$ odd, thought as pairs of signed graphs, can be added to the list of equienergetic noncospectral signed graphs found in [10] and [27].

REMARK 4.4. The proof of Proposition 4.3 can be suitably modified to prove that each $\mathbb{T}$-gain graph Ψ with order q, provided that $\lambda_1(\Psi) \neq -\lambda_q(\Psi)$, gives rise to infinite pairs of equienergetic noncospectral graphs; namely $\Phi_{h,s}(\Psi)$ and $\Phi_{h,t}(\Psi)$ whenever the number $s-t$ is odd (see Example 4.5). It turns out that $\lambda_1(\Phi_{h,s}(\Psi)) \neq \lambda_1(\Phi_{h,t}(\Psi))$, but the values of these two numbers depend on which number between $\lambda_1(\Psi)$ and $-\lambda_q(\Psi)$ is the largest and (if $\lambda_1(\Psi) < -\lambda_q(\Psi)$) on the parity of h. ■

EXAMPLE 4.5. Let $\widetilde{\Psi} := (C_3,\widetilde{\gamma})$ be such that $\widetilde{\gamma}(\vec{C}_3) = \mathrm{e}^{\frac{\pi}{4}i}$. By (2.5), the eigenvalues of $A(\widetilde{\Psi})$ (in decreasing order) are

$$\lambda_1(\widetilde{\Psi}) = 2\cos\left(\frac{\pi}{12}\right), \quad \lambda_2(\widetilde{\Psi}) = 2\cos\left(\frac{17}{12}\pi\right) \quad \text{and} \quad \lambda_3(\widetilde{\Psi}) = -\sqrt{2}.$$

The graphs $\Phi_{2,0}(\widetilde{\Psi}) = \widetilde{\Psi}\times\widetilde{\Psi}$ and $\Phi_{2,1}(\widetilde{\Psi}) = (-\widetilde{\Psi})\times\widetilde{\Psi}$ are equienergetic, their common energy being

$$\mathcal{E}(\Phi_{2,0}(\widetilde{\Psi})) = \mathcal{E}(\Phi_{2,1}(\widetilde{\Psi})) = \left(\sqrt{2} + 2\cos\left(\frac{\pi}{12}\right) + 2\cos\left(\frac{5}{12}\pi\right)\right)^2,$$

but they are not switching isomorphic since

$$\lambda_1(\Phi_{2,0}(\widetilde{\Psi})) = 4\cos^2\left(\frac{\pi}{12}\right) > 3 > 2\sqrt{2}\cos\left(\frac{\pi}{12}\right) = \lambda_1(\Phi_{2,1}(\widetilde{\Psi})).$$

5. Laplacian eigenvalues. As in Section 2.1, we denote by $\lambda_1^L(\Psi) \geqslant \dots \geqslant \lambda_n^L(\Psi)$ the eigenvalues of the Laplacian matrix $L(\Psi) = D(\Lambda) - A(\Psi)$ of a $\mathbb{T}$-gain graph $\Psi = (\Lambda,\vartheta)$.

PROPOSITION 5.1. [15, Theorem 3.6] *For $i \in \mathbb{N}_{\leqslant h}$, let Γ_i be a simple graph of order n_i. The degree matrix of $\Gamma = \mathrm{NEPS}(\Gamma_1,\dots,\Gamma_h;\mathcal{B})$ is given by*

$$D(\Gamma) = \sum_{(b_1,\dots,b_h)\in\mathcal{B}} D(\Gamma_1)^{b_1} \otimes \cdots \otimes D(\Gamma_h)^{b_h}. \tag{5.1}$$

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For the Cartesian product, Equality (5.1) *specializes as follows:*

$$D\left(\square_{i=1}^h \Gamma_i\right) = D(\Gamma_1)\otimes I_{n_2}\otimes\cdots\otimes I_{n_h} + I_{n_1}\otimes D(\Gamma_2)\otimes\cdots\otimes I_{n_h}+\cdots+I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes D(\Gamma_h). \tag{5.2}$$

COROLLARY 5.2. *For $i\in\mathbb{N}_{\leqslant h}$, let $\lambda_{i1}^L\geqslant\cdots\geqslant\lambda_{in_i}^L$ be the Laplacian eigenvalues of a $\mathbb{T}$-gain graph Φ_i with n_i vertices. The Laplacian matrix of the Cartesian product $\Phi=\square_{i=1}^h\Phi_i$ is*

$$L(\Phi) = L(\Phi_1)\otimes I_{n_2}\otimes\cdots\otimes I_{n_h}+\cdots+I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes L(\Phi_h), \tag{5.3}$$

and

$$\mathrm{sp}(L(\Phi)) = \left\{\lambda^L_{i_1,\dots,i_h} = \lambda^L_{1i_1}+\cdots+\lambda^L_{hi_h} \mid i_j\in\mathbb{N}_{\leqslant n_j} \quad \textit{and} \quad j\in\mathbb{N}_{\leqslant h}\right\}.$$

Proof. The formula (5.3) is a straightforward consequence of Propositions 2.6, (3.6) and (5.2) (in any case, the required steps are made explicit, when the factors are signed graphs, along the proof of [15, Theorem 3.7]). The Laplacian eigenvalues are computed by taking into account Proposition 2.8. ∎

Let now $\Phi=(\Gamma,\gamma)=\mathrm{NEPS}(\Phi_1,\dots,\Phi_h;\mathfrak{B})$ for $\mathfrak{B}\subseteq\mathfrak{F}_h$. From (3.4) and (5.2), we obtain

$$L(\Phi) = D(\Gamma)-A(\Phi) = \sum_{(b_1,\dots,b_h)\in\mathcal{B}}\left(D(\Gamma_1)^{b_1}\otimes\cdots\otimes D(\Gamma_h)^{b_h} - A(\Phi_1)^{b_1}\otimes\cdots\otimes A(\Phi_h)^{b_h}\right); \tag{5.4}$$

yet, for $\mathfrak{B}\not\subseteq\mathfrak{B}_{h,1}$ there is no hope to find a general formula of type (3.5) allowing to determine $\mathrm{sp}(L(\Phi))$ from the several $\mathrm{sp}(L(\Phi_i))$'s. This fact has been known to scholars since at least the publication of [4], in which the authors gave an example of two nonisomorphic simple graphs F and H with six vertices such that $\mathrm{sp}(L(F))=\mathrm{sp}(L(H))$, and yet $\mathrm{sp}(L(F\boxtimes K_2))\neq\mathrm{sp}(L(H\boxtimes K_2))$. That is why we have to make do with the following result.

THEOREM 5.3. *For $i\in\mathbb{N}_{\leqslant h}$, let Γ_i be regular r_i graphs with n_i vertices, and let $\lambda_{i1}\geqslant\lambda_{i2}\geqslant\cdots\geqslant\lambda_{in_i}$ be the adjacency eigenvalues of the $\mathbb{T}$-gain graph $\Phi_i=(\Gamma_i,\gamma_i)$. The Laplacian eigenvalues of $\Phi=(\Gamma,\gamma)=\mathrm{NEPS}(\Phi_1,\dots,\Phi_h;\mathcal{B})$ are*

$$\lambda^L_{k_1,\dots,k_h} := \sum_{(b_1,\dots,b_h)\in\mathfrak{B}}\left(r_1^{b_1}\cdots r_h^{b_h} - \lambda_{1k_1}^{b_1}\cdots\lambda_{hk_h}^{b_h}\right) \qquad \textit{for} \quad k_j\in\mathbb{N}_{\leqslant n_j} \quad \textit{and} \quad j\in\mathbb{N}_{\leqslant h}.$$

Proof. Let $n:=\prod_{i=1}^h n_i$. By (5.4), if $\mathbf{x}_{ik_i}$ is a λ_{ik_i}-eigenvector of $A(\Phi_i)$, then $\bigotimes_{i=1}^h\mathbf{x}_{ik_i}$ is a $\lambda^L_{k_1,\dots,k_h}$-eigenvector of $L(\Phi)$, since, in our hypotheses, $D(\Gamma_1)^{b_1}\otimes\cdots\otimes D(\Gamma_h)^{b_h} = r_1^{b_1}\cdots r_h^{b_h} I_n$. ∎

6. Integral spectra. A $\mathbb{T}$-gain graph Ψ is said to be integral if such is $\mathrm{sp}(\Psi)$, i.e., if every eigenvalue of $A(\Psi)$ is an integer. Although the literature on integral simple graphs was already vast when, twenty years ago, the now renowned survey article [3] was published, a structural characterization of integral graphs still eludes us. The gain diamonds examined in Example 2.2 are all integral. Recently, some other families of integral $\mathbb{T}$-gain graphs have been detected in [2], where the authors introduce (in the restricted context of mixed graphs) the mixed asymmetric product $\Psi_1\odot\Psi_2$ of Ψ_1 and Ψ_2. From [2, Theorem 5.1], it turns out that $\Psi_1\odot\Psi_2$ is actually isomorphic to $\mathrm{NEPS}(\Psi_1,\Psi_2;\{(0,1),(1,1)\})$. As a matter of fact, NEPS operations offer a quick procedure to obtain infinitely many integral graphs from the few already at hand. In fact, (3.5) immediately yields the following result.

PROPOSITION 6.1. *For $i\in\mathbb{N}_{\leqslant h}$, let $\Phi_i=(\Gamma_i,\gamma_i)$ be a nonempty integral $\mathbb{T}$-gain graph. Then, for all nonempty $\mathfrak{B}\subseteq\mathfrak{F}_h$, the product $\mathrm{NEPS}(\Phi_1,\dots,\Phi_h;\mathfrak{B})$ is integral.*

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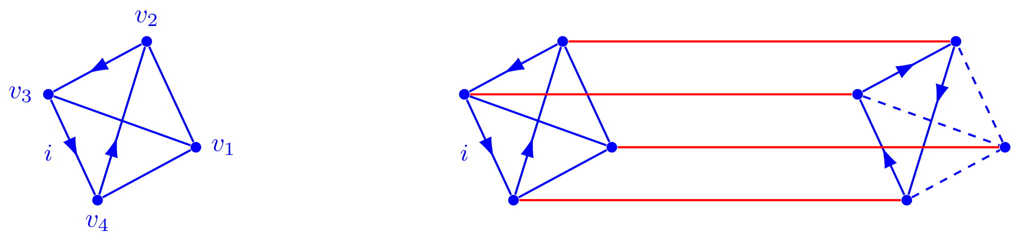

FIGURE 3. *The* $\mathbb{T}$*-gain graphs* $\widetilde{K}_4$ *and* $\mathrm{ND}(\widetilde{K}_4)$.

We now borrow some ideas from [35] in order to build infinitely many connected integral $\mathbb{T}$-gain graphs with at least one imaginary cycle. Such peculiarity ensures that their switching isomorphic class does not contain signed graphs. Following [35], the *Huang's Negative Double* $\mathrm{ND}(\Psi)$ of a $\mathbb{T}$-gain graph Ψ of order q is the $\mathbb{T}$-gain graph whose adjacency matrix is

$$\begin{bmatrix} A(\Psi) & I_q \\ I_q & -A(\Psi) \end{bmatrix}.$$

We inductively define $\mathrm{ND}^{\ell}(\Psi) := \mathrm{ND}(\mathrm{ND}^{\ell-1}(\Psi))$. Although denoted and called in another way, the sequence $\left\{\mathrm{ND}^{2\ell}(\mathcal{C}_{4,-1})\right\}_{\ell\in\mathbb{N}}$, where $\mathcal{C}_{4,-1}$ is the signed quadrangle with just one negative edge, has been recently considered in [21]. It is straightforward to check that the negative double of a connected $\mathbb{T}$-gain graph Ψ is connected; more precisely, $\mathrm{diam}(\mathrm{ND}(\Psi)) = \mathrm{diam}(\Psi) + 1$. The following lemma, already used in [35], can be proved by slightly modifying the clever implementation of the Cayley–Hamilton Theorem in the proof of [22, Lemma 2.2].

LEMMA 6.2. *Let* Ψ *be a* $\mathbb{T}$*-gain graphs with* $2q$ *vertices such that* $\mathrm{sp}(\Psi) = \left\{-\sqrt{s}^{(q)}, \sqrt{s}^{(q)}\right\}$. *Then,* $\mathrm{sp}(\mathrm{ND}(\Psi)) = \left\{-\sqrt{s+1}^{(2q)}, \sqrt{s+1}^{(2q)}\right\}$.

Fig. 3, where all the arrows denote arcs with gain i, depicts the $\mathbb{T}$-gain graphs $\widetilde{K}_4$ and $\mathrm{ND}(\widetilde{K}_4)$.

PROPOSITION 6.3. *The* $\mathbb{T}$*-gain graphs in the set* $\mathcal{ND} = \left\{\Omega_n := \mathrm{ND}^{n^2-3}(\widetilde{K}_4) \mid n \geqslant 2\right\}$ *are all connected and integral. Moreover, each* Ω_n *contains imaginary cycles, and*

$$\mathrm{sp}(\Omega_n) = \left\{-n^{(2^{n^2-2})}, n^{(2^{n^2-2})}\right\}.$$

Proof. We already noted that the operator ND preserves connectedness. Once we label the vertices of $\widetilde{K}_4$ as in Fig. 3, we see that the graph Ω_n contains copies of the (directed) cycle $\vec{C} = v_1v_2v_3$, and $\gamma(\vec{C}) = i$. The adjacency matrix of $\widetilde{K}_4$ is

$$A(\widetilde{K}_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{bmatrix},$$

whose spectrum is $\left\{-\sqrt{3}^{(2)}, \sqrt{3}^{(2)}\right\}$. The eigenvalues of Ω_n can be now computed thanks to Lemma 6.2. $\square$

COROLLARY 6.4. *For* $i \in \mathbb{N}_h$*, let* Φ_i *be a* $\mathbb{T}$*-gain graph in* $\mathcal{ND}$*. The product* $\Phi := \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ *is integral. If, additionally, the* Φ_i*'s and* $\mathfrak{B}$ *fulfill Condition* (3.3)*, then* Φ *is connected.*

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Proof. Integrality of Φ comes from (3.5) and Proposition 6.3. In order to prove the connectness of Φ, we first observe that each Ω_n is connected and nonbipartite; in fact, $\mathrm{diam}(\Omega_n) = n^2 - 2$ and its underlying graph contains 2^{n^2-3} disjoint copies of the complete graph K_4. As explained in [14, p. 33], when the factors of an NEPS are all connected with at least two vertices and for each i there exists a $\mathbf{b} \in \mathfrak{B}$ with $b_i = 1$, there should be at least one bipartite factor to possibly have a disconnected product, and this is not the case for Φ. $\square$

7. Symmetric spectra and sign-symmetry.

A gain graph Ψ has a symmetric spectrum (with respect to 0) if for each $\lambda \in \mathrm{sp}(\Psi)$, the number $-\lambda$ is also in $\mathrm{sp}(\Psi)$ and has the same multiplicity. A gain graph is said to be *sign-symmetric* if it is switching isomorphic to its negation. Since the map $\lambda \in \mathrm{sp}(\Psi) \longmapsto -\lambda \in \mathrm{sp}(-\Psi)$ is a bijection, every sign-symmetric gain graph has a symmetric spectrum. On the contrary, a symmetric spectrum does not guarantee the sign-symmetry: let us use the acronym SNS to denote those gain graphs which have a symmetric spectrum but are not sign-symmetric. In [16] it is shown that, up to isomorphism, there exists one SNS complete signed graph with 8 vertices and six SNS complete signed graphs of order 9; furthermore, it is proved that there are (noncomplete) SNS signed graphs with n vertices for all $n \geqslant 6$.

If the underlying graph of a $\mathbb{T}$-gain graph Ψ is bipartite, then Ψ is sign-symmetric (the argument given in [16, Section 2] to prove the correspondent result for signed graphs works as well in our context). The graph $\mathcal{C}_3(\mathrm{e}^{\frac{\pi}{6}i})$ (see Fig. 1) and $\mathrm{ND}(\tilde{K}_4)$ defined in Section 6 are examples of nonbipartite sign-symmetric graphs. As a matter of fact, $\mathrm{ND}(\Psi)$ is sign-symmetric for every gain graph Ψ. This is a consequence of Theorem 7.1, from which we deduce in particular that each nonbipartite gain graph of order n is an induced subgraph of several suitable connected sign-symmetric gain graphs of order $2n$.

THEOREM 7.1. *For each gain graph $\Psi = (\Lambda, \vartheta)$ with n vertices, let Ξ be a gain graph obtained from $\Psi \sqcup (-\Psi)$ by adding a positive number of edges connecting vertices of Ψ to vertices of $-\Psi$, and whose correspondent arcs have gain in $\{\pm 1\}$. Then, Ξ is sign-symmetric.*

Proof. We argue as in the proof of [16, Theorem 2.2]. With respect to a suitable ordering of $V(\Lambda \sqcup \Lambda)$, the adjacency matrix of Ξ assumes the form

$$A(\Xi) = \begin{bmatrix} A(\Phi) & C \\ C & -A(\Phi) \end{bmatrix},$$

where C is a $\{0, \pm 1\}$-matrix. The gain graphs Ξ and $-\Xi$ are switching isomorphic since their adjacency matrices are related as in (2.2). More precisely,

$$A(-\Xi) = -A(\Xi) = (PD)^* A(\Xi) PD, \qquad \text{where } P = \begin{bmatrix} O & I_n \\ I_n & O \end{bmatrix} \text{ and } D = \begin{bmatrix} -I_n & O \\ O & I_n \end{bmatrix}.$$

In other words, $-\Xi$ is switching equivalent to the gain graph obtaining from Ξ by swapping the labels of the ith vertex of Ψ and the ith vertex of $-\Psi$. $\square$

The NEPS of h gain graphs can be sign-symmetric even when not all factors are sign-symmetric; an example being $(-(C_3, 1) \times (-(K_2, 1)) = (C_6, 1)$. Furthermore, we have the following more general result.

PROPOSITION 7.2. *Let $\Psi = (\Lambda, \vartheta)$ be a gain graph. The Cartesian product $\Psi \square (-\Psi)$ is sign-symmetric.*

Proof. Let $A := A(\Psi)$, and $V(\Lambda) = \{u_1, \dots, u_n\}$. From the equalities

$$A(-(\Psi \square (-\Psi))) = -A(\Psi \square (-\Psi)) = -(A \otimes I_n + I_n \otimes (-A)) = (-A) \otimes I_n + I_n \otimes A = A((-\Psi) \square \Psi),$$

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we infer that $-(\Psi\Box(-\Psi))$ is equal to $(-\Psi)\Box\Psi$, which is isomorphic to $\Psi\Box(-\Psi)$, the suitable vertex permutation being $(u_i, u_j) \in V(\Lambda) \times V(\Lambda) \longmapsto (u_j, u_i) \in V(\Lambda) \times V(\Lambda)$. $\square$

On the contrary, it can happen that the spectrum of an NEPS is not symmetric even when all factors are sign-symmetric: the minimal example is $(K_2, 1) \boxtimes (K_2, 1) = (K_4, 1)$. An NEPS retains the property of its factors of having a symmetric spectrum or being sign-symmetric under the same assumption on $\mathfrak{B}$, as the Theorems 7.4 and 7.5 show. For their statements, we need the following definition.

Definition 7.3. [14, Definition 2.3.7] *A function in several variables is called* odd with respect to a given nonempty subset S of variables *if the function changes only in sign when all the variables in S are simultaneously changed in sign.* ■

Theorem 7.4. *For $i \in \mathbb{N}_h$, let $\Phi_i = (\Gamma_i, \gamma_i)$ be a $\mathbb{T}$-gain graph with a symmetric spectrum. If Φ_i's and $\mathfrak{B}$ fulfill Condition (3.3), then $\Phi = \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ has a symmetric spectrum if and only if there exists a subset $\{i_1, \ldots, i_p\} \subseteq \mathbb{N}_h$ with respect to which the function*

$$f_{\mathfrak{B}} : (x_1, \ldots, x_h) \in \mathbb{R}^h \longmapsto \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_h} \in \mathbb{R}, \tag{7.1}$$

is odd.

Proof. The eigenvalues of Φ are given in (3.5). By the symmetry of the several $\mathrm{sp}(\Phi_i)$'s,

$$\lambda_{1,\ldots,1} = \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} \lambda_{11}^{b_1} \cdots \lambda_{h1}^{b_h} = \max\ \mathrm{sp}(\Phi).$$

If $\mathrm{sp}(\Phi)$ is symmetric, there exists an h-tuple $(k_1, \ldots, k_h)$ such that $-\lambda_{1,\ldots,1} = \lambda_{k_1,\ldots,k_h}$. Note that none of the two inequalities in

$$-\lambda_{1,\ldots,1} = \lambda_{k_1,\ldots,k_h} = -\left|\lambda_{k_1,\ldots,k_h}\right| \geqslant -\sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} |\lambda_{1k_1}^{b_1} \cdots \lambda_{hk_h}^{b_h}| \geqslant -\sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} |\lambda_{11}^{b_1} \cdots \lambda_{h1}^{b_h}| = -\lambda_{1,\ldots,1},$$

is strict. Therefore,

$$\lambda_{1k_1}^{b_1} \cdots \lambda_{hk_h}^{b_h} = -\lambda_{11}^{b_1} \cdots \lambda_{h1}^{b_h} \qquad \text{for each } (b_1, ..., b_h) \in \mathfrak{B}. \tag{7.2}$$

Condition (3.3) ensures that the numbers $\lambda_{1k_1}, \ldots, \lambda_{hk_h}$ are all positive; moreover, $\lambda_{jk_j} \in \{\lambda_{j1}, -\lambda_{j1}\}$. We now set $F = \{j \mid \lambda_{jk_j} = -\lambda_{j1}\}$. It is clear from (7.2) that the function (7.1) is odd with respect to $\{x_j \mid j \in F\}$.

Suppose now that (7.1) is odd with respect to a certain subset $T \subseteq \{x_1, \ldots, x_h\}$. After possibly replacing Φ with a switching isomorphic $\mathbb{T}$-gain graph, we can assume $T = \{x_1, \ldots, x_p\}$ for some $p \leqslant h$. If $n_i = |V(\Gamma_i)|$, the map

$$\lambda_{i_1,\ldots,i_p,i_{p+1},\ldots,i_h} \in \mathrm{sp}(\Phi) \longmapsto \lambda_{n_1+1-i_1,\ldots,n_p+1-i_p,i_{p+1},\ldots,i_h} \in \mathrm{sp}(\Phi),$$

is a bijection and maps each eigenvalue onto its opposite. This proves the symmetry of $\mathrm{sp}(\Phi)$. $\square$

Theorem 7.5. *For $i \in \mathbb{N}_h$, let Φ_i be a sign-symmetric $\mathbb{T}$-gain graph. If Φ_i's and $\mathfrak{B}$ fulfill Condition (3.3), then $\Phi = \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ is sign-symmetric if and only if there exists a subset $\{i_1, \ldots, i_p\} \subseteq \mathbb{N}_h$ with respect to which the function (7.1) is odd.*

Proof. Let, once again, $n_i = |V(\Gamma_i)|$. The sign-symmetry of the Φ_i's is equivalent to the existence of an $n_i \times n_i$ permutation matrix P_i and a switching function $\zeta_i = V(\Gamma_i) \longrightarrow \mathbb{T}$ such that

$$S_i^* A(\Phi_i) S_i = -A(\Phi_i) \qquad \text{for} \quad S_i := P_i D(\zeta_i).$$

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If Φ is sign-symmetric, its spectrum is symmetric, and by Theorem 7.4 there exists a subset $T \subseteq \{x_1, \dots, x_h\}$ with respect to which the function (7.1) is odd. Conversely, suppose that the function (7.1) is odd with respect to a suitable $T \subseteq \{x_1, \dots, x_h\}$. As already noted in the previous proof, after possibly replacing Φ with a switching isomorphic $\mathbb{T}$-gain graph, we can assume $T = \{x_1, \dots, x_p\}$ for some $p \leqslant h$. By Proposition 2.6,

$$\hat{S}^* A(\Phi) \hat{S} = -A(\Phi) = A(-\Phi), \tag{7.3}$$

where

$$\hat{S} = \left(\bigotimes_{i=1}^{p} S_i \right) \otimes (I_{n_{p+1}} \otimes \dots \otimes I_{n_h}),$$

Proposition 2.6, together with Proposition 2.7, also shows that $\hat{S}$ is the row-by-column product between the permutation matrix $(\bigotimes_{i=1}^{p} P_i) \otimes (I_{n_{p+1}} \otimes \dots \otimes I_{n_h})$ and the diagonal matrix $(\bigotimes_{i=1}^{p} D(\zeta_i)) \otimes (I_{n_{p+1}} \otimes \dots \otimes I_{n_h})$; hence, (7.3) proves that Φ is sign-symmetric. □

Corollary 7.6. *For $i \in \mathbb{N}_h$, let Φ_i be a sign-symmetric $\mathbb{T}$-gain graph (resp., have a symmetric spectrum). Then $\square_{i=1}^{h} \Phi_i$ and $\bigtimes_{i=1}^{h} \Phi_i$ are sign-symmetric (resp., have a symmetric spectrum).*

Proof. Consider the functions

$$(x_1, \dots, x_h) \in \mathbb{R}^h \longmapsto \sum_{i=1}^{h} x_i \in \mathbb{R} \qquad \text{and} \qquad (x_1, \dots, x_h) \in \mathbb{R}^h \longmapsto \prod_{i=1}^{h} x_i \in \mathbb{R}.$$

By Theorems 7.4 and 7.5, it suffices to observe that the former is odd with respect to $\{x_1, x_2, \dots, x_h\}$, and the latter is odd with respect to $\{x_1\} \subset \{x_1, \dots, x_h\}$. □

Example 7.7. The gain graphs Φ_1 and Φ_2 in Example 3.5 are both sign-symmetric. By (3.8) and (3.9), we already know that the spectra of $\Phi_1 \square \Phi_2$ and $\Psi := \mathrm{NEPS}(\Phi_1 \times \Phi_2; \{(1,0),(1,1)\})$ are both symmetric. The sign-symmetry of the former can be deduced from Corollary 7.6. The latter is also sign-symmetric since the function

$$(x_1, x_2) \in \mathbb{R}^2 \longmapsto x_1 + x_1 x_2 \in \mathbb{R},$$

is odd with respect to $\{x_1\} \subseteq \{x_1, x_2\}$ and Theorem 7.5 holds. On the contrary, there are no subsets of $\{x_1, x_2\}$ with respect to which the function

$$f : (x_1, x_2) \in \mathbb{R}^2 \longmapsto x_1 + x_1 x_2 + x_2 \in \mathbb{R},$$

is odd. Thus, the nonsymmetry of $\mathrm{sp}\,((K_2, 1) \boxtimes (K_2, 1))$ could be predicted by Theorem 7.4. ■

Let now Ψ be a fixed SNS complete $\mathbb{T}$-gain graph. The properties of the Cartesian product allow us to determine infinitely many SNS connected $\mathbb{T}$-gain graphs containing Ψ as an induced subgraph. We recall that the clique number $\omega(\Gamma)$ of a simple graph Γ is the largest s such that K_s is a subgraph of Γ.

Theorem 7.8. *Let $\mathcal{K}_n = (K_n, \kappa)$ be a complete SNS $\mathbb{T}$-gain graph, and let $\Psi = (\Lambda, \vartheta)$ be a connected $\mathbb{T}$-gain graph such that $\mathrm{sp}(\Psi)$ is symmetric and $\omega(\Lambda) < n$. Then, $\Phi = \mathcal{K}_n \square \Psi = (K_n \times \Lambda, \varphi)$ is connected and SNS.*

Proof. Since all gain functions on K_2 give rise to switching equivalent $\mathbb{T}$-gain graphs, and all gain triangles with a symmetric spectrum are also switching isomorphic, we see that $n \geqslant 4$. Let $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_q\}$ be the vertex sets $V(K_n)$ and $V(\Lambda)$, respectively. The spectrum of Φ is symmetric by Corollary 7.6. Clearly, Φ is connected and has diameter $\mathrm{diam}(\Lambda) + 1$.

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For $i \in \mathbb{N}_q$, let H_i be the subgraph of $K_n \square \Lambda$ induced by the vertex set $V(K_n) \times \{v_i\}$. We denote by $\mathcal{H}_i$ the gain graph $(H_i, \varphi_{|_{H_i}})$. In order to show that Φ is not sign-symmetric, we first prove that $K_n \square \Lambda$ does not contain induced subgraphs isomorphic to K_n apart from $H_1, H_2, \ldots, H_q$. Assume by contradiction that H is a clique with n vertices in Φ, but $H \notin \{H_i \mid i \in \mathbb{N}_q\}$. There would exist in $E(H)$ an edge connecting two vertices of type (u_i, v_j) and (u_i, v_k). The definition of Cartesian product implies that no vertex of type (u_h, v_k) with $h \neq i$ is adjacent to (u_i, v_j). Therefore, H should be a subgraph of $\{u_i\} \times \Lambda$, but this is impossible, since $\omega(\{u_i\} \times \Lambda) = \omega(\Lambda) < n$.

So far, we have proved that each bijection $\sigma : V(K_n) \times V(\Lambda) \longrightarrow V(K_n) \times V(\Lambda)$ preserving the adjacencies of $K_n \square \Lambda$ and the gains of $\vec{E}(\Phi)$, maps $V(K_n) \times \{v_1\}$ onto $V(K_n) \times \{v_{k(\sigma)}\}$. Denoted by $\sigma(\Phi)$ the correspondent gain graph isomorphic of Φ, we have proved that if $\sigma(\Phi) \sim -\Phi$, then $\sigma(\mathcal{H}_1) \sim -\mathcal{H}_{k(\sigma)}$, implying $\mathcal{K}_n \simeq -\mathcal{K}_n$ against the hypotheses. ◻

Theorem 7.8 has its utility since, as we noted already at the beginning of this section, SNS complete gain graphs with n vertices do exist, at least for $n \in \{8, 9\}$.

In view of the next corollary, we denote by $\mathcal{C}_{k,z} = (C_k, \gamma)$ the gain cycle with k vertices $\{u_1, \ldots, u_k\}$ such that $\gamma(u_1 u_2) = z \in \mathbb{T}$, $\gamma(u_2 u_1) = \overline{z}$, and all the remaining arcs are neutral.

Corollary 7.9. *Let $\mathcal{K}_n = (K_n, \kappa)$ be a complete SNS $\mathbb{T}$-gain graph. Then, $\mathcal{K}_n \square \mathcal{C}_{k,i}$ is connected and SNS for all $k \geqslant 3$.*

Proof. The gain cycle $\mathcal{C}_{k,i}$ is switching equivalent (resp., isomorphic) to its negation if k is even (resp., odd). In any case, $\operatorname{sp}(\mathcal{C}_{k,i})$ is symmetric for all $k \geqslant 3$. Since $\omega(C_n) = 2 < n$, the gain graph $\mathcal{K}_n \square \mathcal{C}_{k,i}$ is SNS by Theorem 7.8. ◻

8. NEPS and nut $\mathbb{T}$-gain graphs.

A $\mathbb{T}$-gain graph $\Psi = (\Lambda, \vartheta)$ is said to be *singular* if such is the matrix $A(\Psi)$.

Definition 8.1. *A* nut $\mathbb{T}$-gain graph *(NTGG, for short) is a singular nonempty $\mathbb{T}$-gain graphs $\Psi = (\Lambda, \vartheta)$ such that $m_\Psi(0) = 1$ and every 0-eigenvector is full, i.e., all its components are nonzero.* ■

Nut (simple) graphs were apparently studied for the first time with this denomination in [32]. Nut signed graphs has been recently considered in [5] (where the authors settled to call them "signed nut graphs" for euphonic reasons). When $\vartheta(V(\Lambda))$ is included in $\{\pm 1\}$, a nut $\mathbb{T}$-gain graph $\Psi = (\Lambda, \vartheta)$ can be regarded as a signed nut graph as defined in [5].

For $i \in \mathbb{N}_{\leqslant h}$, let $\Phi_i = (\Gamma_i, \gamma_i)$ be a nonempty connected $\mathbb{T}$-gain graph with $n_i (\geqslant 2)$ vertices and eigenvalues $\lambda_{i1} \geqslant \cdots \geqslant \lambda_{in_i}$. For the rest of the paper, we assume that $\Phi = \operatorname{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ satisfies Condition 3.3. In this section, we shall study under which structural conditions Φ can be an NTGG. We start with a direct consequence of (3.5).

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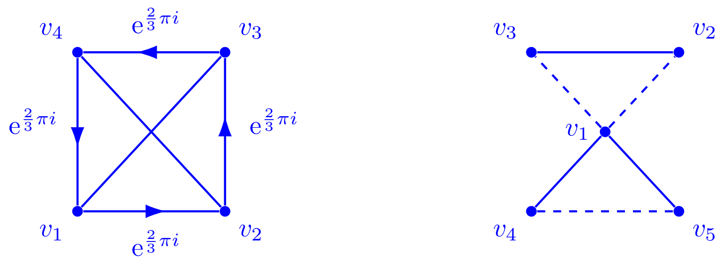

FIGURE 4. *The $\mathbb{T}$-gain graph $\widehat{\mathcal{K}}_4$ and the hourglass $\widehat{\mathcal{H}}$.*

LEMMA 8.2. *If $\Phi = \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ is an NTGG, there exists precisely one h-tuple $(k_1, \ldots, k_h)$ in $\bigtimes_{i=1}^{h} \mathbb{N}_{n_i}$ such that*

$$\lambda_{k_1,\ldots,k_h} = \sum_{(b_1,\ldots,b_h)\in\mathfrak{B}} \lambda_{1k_1}^{b_1} \cdots \lambda_{hk_h}^{b_h} = 0.$$

PROPOSITION 8.3. *If $\Phi = \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ is an NTGG, then, for each fixed $i \in \mathbb{N}_{\leqslant h}$, there exists at least one $\mathbf{b} \in \mathfrak{B}$ whose ith component is 0.*

Proof. We argue by contradiction, assuming that the first component of every $\mathbf{b} \in \mathfrak{B}$ is nonzero. Let $(k_1, \ldots, k_h)$ the h-tuple correspondent to the null eigenvalue in $\mathrm{sp}(\Phi)$. We have

$$0 = \lambda_{k_1,\ldots,k_h} = \lambda_{1k_1} \left(\sum_{(1,b_2,\ldots,b_h)\in\mathfrak{B}} \lambda_{2k_2}^{b_2} \cdots \lambda_{hk_h}^{b_h} \right).$$

Now, if $\lambda_{1k_1} = 0$ we have $\lambda_{k_1,\ell_2,\ldots,\ell_h} = 0$ for all $(\ell_2, \ldots, \ell_h) \in \bigtimes_{i=2}^{h} \mathbb{N}_{n_i}$; otherwise, $\lambda_{\ell_1,k_2,\ldots,k_h} = 0$ for all $\ell_1 \in \mathbb{N}_{n_1}$. In both cases, against the hypothesis, $m_\Psi(0)$ would be larger than 1 (recall that $n_i \geqslant 2$ from (3.3)). ◻

As a direct consequence of Proposition 8.3, there are no direct products in the class of NTGGs. It is somehow easier to find nut graphs among $\mathbb{T}$-gain graphs than among signed graphs. In fact, no signed nut graphs exist with less than 5 vertices, whereas the triangle $\mathcal{C}_3(\mathrm{e}^{i\frac{\pi}{2}}) = \mathcal{C}_3(i)$ in Fig. 1 is an NTGG (with $\mathbf{j}_3$ among its 0-eigenvectors). We also find a complete NTGG with four vertices, as the following example shows.

EXAMPLE 8.4. Let $\mathcal{C}_{2k+1}(i)$ be the gain cycle with $2k+1$ vertices obtained by assigning the gain i to all arcs running counterclockwise around it. The gain cycles $\mathcal{C}_{2k+1}(i)$ for $k \geqslant 1$ and the graph $\widehat{\mathcal{K}}_4$ depicted in Fig. 4 are all NTGG. In fact, from (2.5) we see that $m_{\mathcal{C}_{2k+1}(i)}(0) = 1$; moreover, the all-ones vector $\mathbf{j}_n$ is a 0-eigenvector since the sum of each row of $A(\mathcal{C}_{2k+1}(i))$ is $i + (-i) = 0$. In order to see that $\widehat{\mathcal{K}}_4$ is an NTGG, note that the adiacency matrix of the gain graph is

$$A(\widehat{\mathcal{K}}_4) = \begin{bmatrix} 0 & w & 1 & \overline{w} \\ \overline{w} & 0 & w & 1 \\ 1 & \overline{w} & 0 & w \\ w & 1 & \overline{w} & 0 \end{bmatrix}, \qquad \text{where} \quad w = \mathrm{e}^{\frac{2}{3}\pi i}.$$

The sum of each row is $1 + w + \overline{w} = 0$; therefore, the all-ones vector $\mathbf{j}_4$ is a 0-eigenvector, and $m_{\widehat{\mathcal{K}}_4}(0) = 1$, since $\mathrm{sp}(\widehat{\mathcal{K}}_4) = \{-1-\sqrt{3}, 0, -1+\sqrt{3}, 2\}$. ■

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An NEPS can be an NTGG even if none of the factors is an NTGG. An example of this kind is

$$\widehat{\Psi} = \mathrm{NEPS}((C_3,1),(K_2,1),(K_2,1);\widehat{\mathfrak{B}}), \qquad \text{with} \quad \widehat{\mathfrak{B}} = \{(1,1,0),(0,1,1),(0,0,1)\}\,.$$

In fact, $\mathrm{sp}(\widehat{\Psi}) = \{-3^{(2)}, -2^{(2)}, 0, 1^{(6)}, 4\}$ is computed with the aid of (3.5), and a full 0-eigenvector is given by $(1,1,1)\otimes(1,1)\otimes(1,-1)$.

In order to understand whether an NEPS of NTGG's is an NTGG, we just need to check the multiplicity of 0 in the product, since the following proposition holds.

PROPOSITION 8.5. *If, for $i \in \mathbb{N}_{\leqslant h}$, Φ_i is an NTGG, then $\Phi = \mathrm{NEPS}(\Phi_1,\ldots,\Phi_h;\mathfrak{B})$ is singular and admits a full 0-eigenvector.*

Proof. Let $(k_1,\ldots,k_h)$ be the h-tuple such that $\lambda_{ik_i} = 0$ for every $i \in \mathbb{N}_{\leqslant h}$. By (3.5), $\mathrm{sp}(\Phi)$ contains $0 = \lambda_{k_1,\ldots,k_h}$. Once you pick a (full) 0-eigenvector $\mathbf{x}_i$ of $A(\Phi_i)$ for $i \in \mathbb{N}_{\leqslant h}$, by Proposition 2.6 and (2.6) $\otimes_{i=1}^h \mathbf{x}_i$ is a full 0-eigenvector of Φ. ∎

We end this section by showing that NEPS can be useful to get infinite families of NTGG's 'built' from some known NTGG.

PROPOSITION 8.6. *Let p and q be odd coprime integers larger than 1, and let $\widehat{\mathcal{H}}$ be the hourglass in Fig. 4. The products of nut $\mathbb{T}$-gain graphs $\mathcal{N}_{pq} = \mathcal{C}_p(i) \,\Box\, \mathcal{C}_q(i)$, $\mathcal{N}_p^{\Box} = \mathcal{C}_p(i)\Box\widehat{\mathcal{H}}$, and $\mathcal{N}_p^{\boxtimes} = \mathcal{C}_p(i) \boxtimes \widehat{\mathcal{H}}$ are all nut $\mathbb{T}$-gain graphs.*

Proof. By (2.5) and a direct computation, we obtain

$$\mathrm{sp}(\mathcal{C}_p(i)) = \left\{ \mu_j := 2\cos\left(\frac{(2j+1)\pi}{2p}\right) \;\Big|\; 0 \leqslant j \leqslant p-1 \right\},$$

$$\mathrm{sp}(\mathcal{C}_q(i)) = \left\{ \mu'_k := 2\cos\left(\frac{(2k+1)\pi}{2q}\right) \;\Big|\; 0 \leqslant k \leqslant q-1 \right\},$$

and $\mathrm{sp}(\widehat{\mathcal{H}}) = \{\pm\sqrt{5}, \pm 1, 0\}$. We already observed in Example 8.4 that $\mathcal{C}_p(i)$ and $\mathcal{C}_q(i)$ are nut $\mathbb{T}$-gain graphs. The gain graph $\widehat{\mathcal{H}}$ can be regarded as a 0-net-regular signed graph, i.e., the difference between the numbers of positive and negative edges incident to a fixed vertex is always 0. This property guaranteees that the full vector $\mathbf{j}_5$ is a 0-eigenvector.

Now, $\mathrm{sp}(\mathcal{N}_{pq}) = \{\lambda_{jk} := \mu_j + \mu'_k \mid 0 \leqslant j \leqslant p-1, 0 \leqslant j \leqslant q-1\}$, and $\lambda_{jk} := \mu_j + \mu'_k = 0$ if and only if $q(2j+1) = p(2q-2k-1)$, which is equivalent to $j = (p-1)/2$ and $k = (q-1)/2$, being p and q coprime and $2j+1 \leqslant 2p-1$. Hence, the multiplicity of $0 = \lambda_{\frac{p-1}{2},\frac{q-1}{2}}$ in $\mathrm{sp}(\mathcal{N}_{pq})$ is 1, and $\mathcal{N}_{pq}$ is an NTGG by Proposition 8.5.

Turning our attention to $\mathcal{N}_p^{\Box}$, we note that for all $p \geqslant 3$ and $j \in \{0,1,\ldots,p-1\}$

$$-\sqrt{5}/2 < -1 < \cos\left(\frac{(2j+1)\pi}{2p}\right) < 1 < \sqrt{5},$$

and

$$\frac{(2j+1)\pi}{2p} \notin \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}, \tag{8.1}$$

implying that the numbers $\pm\sqrt{5}+\mu_j$ and $\pm 1+\mu_j$ are nonnull. In other words, the multiplicity of $0 \in \mathrm{sp}(\mathcal{N}_p^{\Box})$ is 1.

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Finally, we consider $\mathrm{sp}(\mathcal{N}_p^{\boxtimes}) = \{-1,\, 0,\, \nu_{1,j},\, \nu_{2,j},\, \nu_{3,j} \mid 0 \leqslant j \leqslant p-1\}$, where

$$\nu_{1,j} := 1 + 2\mu_j, \quad \nu_{2,j} := \sqrt{5} + (1+\sqrt{5})\mu_j, \quad \text{and} \quad \nu_{3,j} := -\sqrt{5} - (\sqrt{5}-1)\mu_j.$$

We need to show that the multiplicity of 0 in $\mathrm{sp}(\mathcal{N}_p^{\boxtimes})$ is 1. In view of this purpose, note that $\nu_{1,j}$ is nonzero by (8.1), whereas $\nu_{2,j}$ and $\nu_{3,j}$ are nonzero since the numbers $-\sqrt{5}/(1+\sqrt{5})$ and $-\sqrt{5}/(\sqrt{5}-1)$ cannot be eigenvalues of a $\mathbb{T}$-gain graph, their minimal polynomials

$$4x^2 + 10x + 5 \qquad \text{and} \qquad 4x^2 + 10x - 5,$$

being nonmonic. $\square$

9. Beyond Cvektović: new products of $\mathbb{T}$-gain graphs. For each $\mathbf{b} = (b_1, \ldots, b_h) \in \mathbb{R}^h$, the numbers

$$\mathrm{supp}(\mathbf{b}) := \{i \in \mathbb{N}_{\leqslant h} \mid b_i \neq 0\} \qquad \text{and} \qquad w_{\mathbf{b}}^- = \big|\{i \in \mathbb{N}_{\leqslant h} \mid b_i < 0\}\big|,$$

will be, respectively, called *support* and the *negative weight* of $\mathbf{b}$. We now explain how the set of all possible Cvektović products of a fixed h-tuple of $\mathbb{T}$-gain graphs $\Phi_1 = (\Gamma_1, \gamma_1), \ldots, \Phi_h = (\Gamma_h, \gamma_h)$ can be further enlarged by considering the nonempty subsets $\mathfrak{B}$ of the nonzero h-tuples with components in $\{0, \pm 1\}$ satisfying the following restriction:

$$\mathrm{supp}(\mathbf{b}) = \mathrm{supp}(\mathbf{b}') \Longrightarrow \mathbf{b} = \mathbf{b}' \qquad \forall\ \{\mathbf{b}, \mathbf{b}'\} \subseteq \mathfrak{B}. \tag{9.1}$$

We shall make use of the Kronecker delta symbol $\delta_{uv} = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v, \end{cases}$ with variables in suitable vertex sets.

Definition 9.1. *Let $\mathfrak{B}$ a nonempty subset of nonzero $\{0, \pm 1\}$-h-tuples satisfying (9.1). The* generalized Cvektović product *of the $\mathbb{T}$-gain graphs $\Phi_1 = (\Gamma_1, \gamma_1), \ldots, \Phi_h = (\Gamma_h, \gamma_h)$ with basis $\mathfrak{B}$ is the $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ defined as follows:*

- *the underlying graph Γ is $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_h; \mathfrak{B}^{\mathrm{abs}})$, where*

$$\mathfrak{B}^{\mathrm{abs}} := \left\{\, (|b_1|, \ldots, |b_h|) \;\middle|\; (b_1, \ldots, b_h) \in \mathfrak{B} \,\right\};$$

- *for each pair of adjacent vertices $u := (u_1, \ldots, u_h)$ and $v := (v_1, \ldots, v_h)$ in Γ,*

$$\gamma(uv) := \prod_{j=1}^{h} (-1)^{b_j} \gamma_j(u_j v_j),$$

  where $\gamma_j(u_j v_j)$ is understood to be 1 whenever $u_j = v_j$, and $(b_1, \ldots, b_h)$ is the (unique) h-tuple in $\mathfrak{B}$ such that $|b_i| = 1 - \delta_{u_i v_i}$.

The $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ will be denoted by $\mathrm{GCP}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$. ■

$\mathrm{GCP}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ is really a $\mathbb{T}$-gain graph, since $\gamma(vu) = \overline{\gamma(uv)}$ by the elementary equality $\overline{-z} = -\overline{z}$ holding for each $z \in \mathbb{C}$. Clearly, $\mathrm{GCP}(\Phi_1, \ldots, \Phi_h; \mathfrak{B}) = \mathrm{NEPS}(\Phi_1, \ldots, \Phi_h; \mathfrak{B})$ if $w^-(\mathbf{b})$ is even for all $\mathbf{b} \in \mathfrak{B}$. In particular, this happens when $\mathfrak{B}$ is a subset of $\mathfrak{F}_h$. We omit the proof of the following theorem, since it can be obtained by slightly modifying the arguments exposed in the previous sections.

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Theorem 9.2. *For $i \in \mathbb{N}_{\leqslant h}$, let $\Phi_i = (\Gamma_i, \gamma_i)$ be a $\mathbb{T}$-gain graph with n_i vertices and let $\mathfrak{B}$ a nonempty subset of nonzero $\{0, \pm 1\}$-h-tuples satisfying (9.1).*

(i) The adjacency matrix of $\Phi = \mathrm{GPC}(\Phi_1, \dots, \Phi_h; \mathfrak{B})$ is given by

$$A(\Phi) = \sum_{\mathbf{b} \in \mathfrak{B}} (-1)^{w_{\mathbf{b}}^-} A(\Phi_1)^{b_1} \otimes \cdots \otimes A(\Phi_h)^{b_h}.$$

(ii) If $\lambda_{i1} \geqslant \lambda_{i2} \geqslant \cdots \geqslant \lambda_{in_i}$ are the eigenvalues of $A(\Phi_i)$, then $\mathrm{sp}(\Phi) = \{\lambda^{\mathrm{GPC}}_{k_1,\dots,k_h} \mid k_i \in \mathbb{N}_{\leqslant n_i}\}$, where

$$\lambda^{\mathrm{GPC}}_{k_1,\dots,k_h} := \sum_{\mathbf{b} \in \mathfrak{B}} (-1)^{w_{\mathbf{b}}^-} \lambda_{1k_1}^{b_1} \cdots \lambda_{hk_h}^{b_h}. \tag{9.2}$$

(iii) For each $\zeta_i : V(\Gamma_i) \longrightarrow \mathbb{T}$, the $\mathbb{T}$-gain graph $\Phi' := \mathrm{GPC}(\Phi_1^{\zeta_1}, \dots, \Phi_k^{\zeta_h}; \mathfrak{B})$ is switching equivalent to Φ.

(iv) if the Φ_i's are all integral, then Φ is integral.

(v) if the Φ_i's all have a symmetric spectrum (resp., are sign-symmetric), then Φ has a symmetric spectrum (resp., is sign-symmetric) if and only if there exists a subset $\{i_1, \dots, i_p\} \subseteq \mathbb{N}_h$ with respect to which the function

$$f_{\mathfrak{B}} : (x_1, \dots, x_h) \in \mathbb{R}^h \longmapsto \sum_{\mathbf{b} \in \mathfrak{B}} (-1)^{w_{\mathbf{b}}^-} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_h} \in \mathbb{R},$$

is odd.

The possible presence of -1's in the defining h-tuples of $\mathfrak{B}$ does not allow to extend Proposition 3.7 to generalized Cvektović products. In fact, it is very easy to find unbalanced GPC's with balanced factors, an example being $\widehat{\Psi} := \mathrm{GPC}((C_3, 1), (K_2, 1); \{(-1, 0), (1, 0)\}) = (-(C_3, 1)) \square (K_2, 1)$. The gain graph $\widehat{\Psi}$ is a GPC which is also an NEPS with the same underlying graph, but this phenomenon does not occur in general, as the following proposition shows.

Proposition 9.3. *Let $\Phi = \mathrm{GPC}((C_3, 1), (C_3, 1); \mathfrak{B})$, with $\mathfrak{B} = \{(-1, 0), (1, 1), (1, 0)\}$. An NEPS of (at least two) $\mathbb{T}$-gain graphs cannot be switching isomorphic to Φ.*

Proof. By (9.2), we easily obtain $\mathrm{sp}(\Phi) = \{-5^{(2)}, 1^{(6)}, 4\}$. Assume by contradiction that there exists an NEPS $\Psi = (\Lambda, \vartheta)$ of at least two $\mathbb{T}$-gain graphs such that $\Psi \simeq \Phi$. The spectra of Φ and Ψ should be equal and, since the underlying graph of Φ is K_9, the factors of Ψ should necessarily be two gain triangles, say $\Psi_1 = (C_3, \gamma_1)$, and $\Psi_2 = (C_3, \gamma_2)$; moreover, $\Psi = \Psi_1 \boxtimes \Psi_2$. The graph Φ is unbalanced, but does not contain cycles with an imaginary gain; on the other hand, Ψ contains copies of Φ_1 and Φ_2 as induced subgraphs. This means that Ψ should be either switching isomorphic to $(-(C_3, 1)) \boxtimes (C_3, 1)$ or to $(-(C_3, 1)) \boxtimes (-(C_3, 1))$, but none of this two graphs have 1 in their spectrum. ∎

We end the paper by stating the following problem.

Problem. Find structural conditions on the factors and on the basis $\mathfrak{B}$ characterizing the generalized Cvektović products of $\mathbb{T}$-gain graphs which are switching isomorphic to an NEPS.

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