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Fidelity analysis of topological quantum phase transitions

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We apply the fidelity metric approach to analyze two recently introduced models that exhibit a quantum phase transition to a topologically ordered phase. These quantum models have a known connection to classical statistical mechanical models; we exploit this mapping to obtain the scaling of the fidelity metric tensor near criticality. The topological phase transitions manifest themselves in divergences of the fidelity metric across the phase boundaries. These results provide evidence that the fidelity approach is a valuable tool to investigate novel phases lacking a clear characterization in terms of local order parameters.

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INTRODUCTION

This is an exciting period for condensed-matter physics, when novel phases of matter that defy traditional understanding are being observed and predicted. Examples include topological phases [1], which cannot be described by the Landau-Ginzburg-Wilson paradigm [2]. Absence of local order parameters and symmetry-breaking mechanisms are among the most remarkable features of these systems. These novel phases arise, for example, in collective phenomena exhibited in strongly correlated systems of two-dimensional electrons at very low temperature, as in the fractional quantum Hall effect [3,4]. In such systems, the motion of electrons is highly constrained, and the fluctuations are entirely quantum in nature. In this situation, Landau's theory, which is essentially a theory of classical order, can fail.

It is compelling to find new ways to analyze such phases. Using tools from quantum information, it has been possible to characterize topological order using the concept of topological entropy [5–7]. Here, we call for a new information-theoretic tool for studying quantum phase transitions (QPTs) [8] to topological phases. The new notion is the fidelity of ground states, whose role in the study of QPTs has been developed in [9–33]. The basic idea is that near a quantum critical point, there is a drastic enhancement in the degree of distinguishability between two ground states, corresponding to slightly different values of the parameters that define the Hamiltonian. This distinguishability can be quantified by the fidelity, which for pure states reduces to the amplitude of the inner product or overlap. This approach is suitable for detecting QPTs and analyzing topological phases, since the method does not rely on constructing an order parameter, nor does it rely on the symmetries of the system. The overlap of two nearby ground states is a global quantity of the system that does not depend on local features such as the existence of a local order parameter. Therefore, it should contain all the information that describes topological order. The capability of fidelity to spot a topological QPT has been shown in [22] by numerical analysis. Since topological order is a property of the ground-state wave function alone, knowledge of the ground state of the system is sufficient in order to carry out this analysis.

In this work, we analyze two models that exhibit topological order: an extension of the toric code [34] and the quantum eight-vertex model [35]. Both present a transition from a nontopologically ordered phase to a topological phase. Moreover, they exhibit a close connection to classical statistical models. Indeed, the fidelity and its second derivative, the so-called fidelity metric, are related to the partition function [36] and correlation functions of the corresponding classical model, respectively.

A TOPOLOGICAL QPT IN AN EXTENSION OF THE TORIC CODE MODEL

In this section, we apply the fidelity approach to analyze the quantum phase transition to a topologically ordered phase for an extension of the toric code, expressed in stochastic matrix form decomposition [34]. In [37], the authors showed their model had a transition from a magnetically ordered state to a topological ordered phase, with the topological entropy having a jump to a nonzero value at the transition. We now apply the fidelity approach to this topological QPT and find the scaling of the fidelity metric near criticality. Let us start by briefly reviewing the model. Given a square lattice with periodic boundary conditions and spins $\frac{1}{2}$ on the bonds, consider the following Hamiltonian:

$$H = -\lambda_0 \sum_p B_p - \lambda_1 \sum_s A_s + \lambda_1 \sum_s e^{-\beta \sum_{i \in s} \hat{\sigma}_i^z} = H_{\text{Kitaev}} + \lambda_1 \sum_s e^{-\beta \sum_{i \in s} \hat{\sigma}_i^z}, \quad (1)$$

where $A_s = \prod_{i \in s} \hat{\sigma}_i^x$ and $B_p = \prod_{i \in p} \hat{\sigma}_i^z$ are the star and plaquette operators of the Kitaev model [38].

The ground state of this Hamiltonian can be computed exactly. In particular, the ground state corresponding to the topological sector containing the fully magnetized state ($|0\rangle$) is given by [37]

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$$|gs\rangle = \sum_{g\in G} \frac{e^{-\beta\Sigma_i \sigma_i^z(g)/2}}{\sqrt{Z(\beta)}} g|0\rangle, \tag{2}$$

$$Z(\beta) = \sum_{g\in G} e^{-\beta\Sigma_i \sigma_i^z(g)}. \tag{3}$$

Here, G is the Abelian group generated by all the star operators A_s, $|0\rangle$ is the completely polarized state corresponding to all the spins in the +1 eigenstate of σ^z, and $\sigma_i^z(g)$ is the z component of the spin at site i in the state $g|0\rangle$.

Let us try to understand the phases of this model. When $\beta=0$, we have the pure Kitaev toric code. Its ground state is a closed string condensed phase. An x (z) string is a collection of spins that are flipped in the σ^z (σ^x) basis. The term with the plaquette operator says that only closed strings are allowed. The term containing the star operators A_s makes instead closed strings of flipped spins to be created and fluctuate. This phase is topologically ordered, as is shown by a nonvanishing topological entropy [5–7]. We can regard the β-dependent term as a kind of tension for the z strings. As we increase β, larger loops are less favored. Indeed one can see that for small β the model is the toric code in an external magnetic field. For larger β the phase is not topologically ordered, as one can infer from the vanishing of the topological entropy [37]. This is why one can use topological entropy as an order parameter [22,37,39]. One expects that for a particular value of β, the system undergoes a QPT from the topologically ordered phase to a "magnetically" ordered phase. The authors in [37] proved that this model has a second-order phase transition at $\beta_c=(1/2)\ln(\sqrt{2}+1)$. For $\beta<\beta_c$, the system has a topologically ordered phase, with $S_{\text{topo}}=2$, and for $\beta>\beta_c$ the topological entropy vanishes: $S_{\text{topo}}=0$. It is very important to notice that despite being not topologically ordered, such a phase is not a Landau-Ginzburg phase. There is no local order parameter to characterize it [37].

We now analyze this transition using the fidelity between two ground states $|\beta\rangle$ and $|\beta+\delta\beta\rangle$ corresponding to slightly different values of the relevant parameter β. Therefore, we consider the following quantity:

$$F(\beta,\beta+\delta\beta) = \langle gs(\beta)|gs(\beta+\delta\beta)\rangle = \sum_g \frac{e^{-(\beta+1/2\,\delta\beta)\Sigma_i\sigma_i^z(g)}}{\sqrt{Z(\beta)}\sqrt{Z(\beta+\delta\beta)}}. \tag{4}$$

Expanding Eq. (4) to second order in $\delta\beta$, i.e., $F\approx 1-g_{\beta\beta}\delta\beta^2$, we obtain the following *fidelity metric* [13] $g_{\beta\beta}$:

$$g_{\beta\beta} = \frac{1}{4}\left[\frac{\Sigma_{g\in G}[\Sigma_i\sigma_i^z(g)]^2 e^{-\beta\Sigma_j\sigma_j^z(g)}}{Z(\beta)} - \left(\sum_{g\in G}\frac{\Sigma_i\sigma_i^z(g)e^{-\beta\Sigma_j\sigma_j^z(g)}}{Z(\beta)}\right)^2\right]. \tag{5}$$

Much in the spirit of the fidelity approach, near the quantum phase transition there is an enhancement in the distinguishability between the ground states $|\beta\rangle$ and $|\beta+\delta\beta\rangle$, resulting in a superextensive scaling of the singular behavior of this metric at the critical point. Indeed, this singular behavior can be captured by mapping this quantum model to a classical statistical model in the following way. Any group element $g\in G$ is the product of the star operators in some set $\mathcal{S}(g)$. Thus, $g|0\rangle$ is completely specified by the same set, modulo the product of all such operators, equal to the identity for periodic boundary conditions (i.e., for a torus of genus 1). Then, for every two configurations specified by $\{g\in G\}$ there will correspond one configuration $\{\theta\}$ of a classical Ising model with degrees of freedom θ_s on the sites, such that $\theta_s=-1$ (+1) when the corresponding star operator A_s is (is not) acting on the site s. Since a spin σ_i^z can be flipped only by its two neighboring θ spins, we have that $\sigma_i=\theta_s\theta_s'$, with s and s' the end points of the bond i. In that case, defining $E_{\text{Ising}}=J\Sigma_{\langle s,s'\rangle}\theta_s\theta_s'$, we obtain $Z_{\text{Ising}}=\Sigma_\theta e^{-\beta\Sigma_{\langle s,s'\rangle}\theta_s\theta_s'}=2\Sigma_{g\in G}e^{-\beta\Sigma_i\sigma_i^z(g)}=2Z(\beta)$, where we took $\beta=J/T$ for the Ising model. Using this equality, we can write Eq. (5) as

$$g_{\beta\beta} = \frac{1}{4\beta^2}C_v, \tag{6}$$

where C_v is the specific heat of the 2D Ising model. It is well known that C_v has a logarithmic divergence at criticality [41]. Hence mapping to the classical Ising model reveals that the fidelity metric has a logarithmic divergence,

$$g_{\beta\beta} \sim \ln|\beta_c/\beta - 1|, \tag{7}$$

at $\beta_c=\frac{1}{2}\ln(\sqrt{2}+1)$.

In [37], the authors remark that indeed the phase transition to the topologically ordered phase could be detected by the local magnetization $m(\beta)=\frac{1}{N}\Sigma_i\langle\hat{\sigma}_i^z\rangle=\frac{1}{N}E_{\text{Ising}}(\beta)$, with its first derivative equal to the specific heat, i.e., $\frac{\partial m}{\partial\beta}=-\frac{1}{N}\beta^2 C_{\text{Ising}}(\beta)$, where N is the number of sites. We see that the fidelity metric captures very naturally this divergence since it is equivalent to the specific heat, which diverges at the critical point.

TOPOLOGICAL QPT IN THE QUANTUM EIGHT-VERTEX MODEL

We now turn to analyze another model that exhibits a transition to a topological phase, and in which the mapping to a classical statistical model can be performed to analyze the scaling of the fidelity metric near the critical point. This model is the so-called quantum eight-vertex model, defined and studied in Refs. [35,40]. We proceed to review this model very briefly.

The classical eight-vertex model [41] consists of arrows placed along the bonds of a square lattice. The arrows can point in either direction along each of the bonds, subject to the constraint that an even number of arrows go into (and out of) each site. There are eight distinct configurations for the arrows around each site satisfying this constraint. Each vertex configuration is assigned an energy ϵ_i. Furthermore, by imposing toroidal boundary conditions, symmetry under rotations, and inversions of all spins (i.e., zero external electric field), one finds that there are only two independent Boltzmann weights, usually denoted by c and d, with $c=e^{-\epsilon_c/T}$ and $d=e^{-\epsilon_d/T}$. The partition function of this model has the form

$Z(c,d)=\Sigma_{\mathcal{C}} c^{n_c(\mathcal{C})} d^{n_d(\mathcal{C})}$, with $n_c(\mathcal{C})$ and $n_d(\mathcal{C})$ the number of c- and d-type vertices for the configuration $\mathcal{C}$. The total energy for a configuration $\mathcal{C}$ is given by $E=n_c(\mathcal{C})\epsilon_c+n_d(\mathcal{C})\epsilon_d$.

This classical model can be solved exactly in the thermodynamic limit by computing the free-energy density using the highest eigenvalue of the transfer matrix [41]. It exhibits ordered phases for $d>c+2$ and $d<c-2$, where the $\mathbb{Z}_2$ symmetry of flipping all the arrows is spontaneously broken, while the system is disordered for $|c-d|<2$. For $d<c-2$, there is a proliferation of c vertices, while for $d>c+2$ the d vertices dominate. These phases are called "antiferroelectric."

One unusual feature of the classical model is that the critical exponents change continuously along the critical lines $d=c+2$ and $d=c-2$, with the free-energy density having a singular behavior near $d=c-2$ of the form

$$f_{\rm sing} \sim ||d-c|-2|^{\pi/\mu}, \tag{8}$$

with $\mu=2\tan^{-1}\sqrt{cd}$. When $\pi/\mu=m$, with m an integer, this expression is changed by an additional logarithmic divergence: $f_{\rm sing}\sim ||d-c|-2|^{\pi/\mu}\ln(|d-c|-2)$. The model is also critical along the lines $c=0$, $d\leqslant 2$ and $d=0$, $c\leqslant 2$, since it reduces to the disordered phase of the six-vertex model, which has an infinite correlation length. The points $c=0$, $d=2$ and $c=2$, $d=0$ are BKT critical points. There, the exponent π/μ diverges.

The quantum eight-vertex model [35] is defined such that its Hilbert space basis $\{|\mathcal{C}\rangle\}$ is given by the configuration space of the classical eight-vertex model, with each state real and orthonormal to each other. The Hamiltonian of this model is of the form $H=\Sigma_i Q_i$, with Q_i positive operators, chosen such that H annihilates the following state:

$$|gs(c^2,d^2)\rangle = \frac{1}{\sqrt{Z_{8V}^{Q}(c^2,d^2)}}\sum_{\{\mathcal{C}\}} c^{\hat{n}_c(\mathcal{C})}d^{\hat{n}_d(\mathcal{C})}|\mathcal{C}\rangle, \tag{9}$$

with the normalization factor given by $Z_{8V}^{Q}(c^2,d^2)=\Sigma_{\{\mathcal{C}\}}c^{2\hat{n}_c(\mathcal{C})}d^{2\hat{n}_d(\mathcal{C})}$, where $\hat{n}_c(\mathcal{C})$ and $\hat{n}_d(\mathcal{C})$ are the number operators for the c- and d-type vertices, for the configuration $\mathcal{C}$ [40]. The authors in [35,40] noted that since the normalization factor above is the partition function for the classical two-dimensional eight-vertex model with weights c^2 and d^2, then the ground-state phase diagram for the quantum model is identical to the classical one, but given in terms of c^2 and d^2. The quantum model exhibits a topologically ordered phase in the region of the phase diagram that corresponds to the disordered phase of the classical model. Indeed, the topological entropy in the quantum model is given by $S_{\rm topo}=-\ln(2)$ in the topological phase $|d^2-c^2|<2$, while it is zero elsewhere. In particular, for $c^2=d^2=1$ one recovers the ground state of the Kitaev model [38].

Let us now pursue a fidelity analysis of this quantum phase transition. Again the mapping to the classical model proves useful. As we will see, the fidelity metric is equal to the fluctuations in the number of c- and d-type vertices of the classical model. This will provide us with the scaling of the metric near the phase transition.

Consider then the fidelity between two ground states for slightly different values of the parameters c^2 and d^2 and expand it to second order in c^2 and d^2: $F=\langle gs(c^2,d^2)|gs(c^2+\delta c^2,d^2+\delta d^2)\rangle \approx 1-g_{c^2c^2}(\delta c^2)^2 - g_{d^2d^2}(\delta d^2)^2 - g_{c^2d^2}(\delta c^2\delta d^2)$, where the metric elements of the 2×2 fidelity metric are given by

$$g_{c^2c^2} = \frac{1}{4c^4}(\langle n_c^2\rangle - \langle n_c\rangle^2), \tag{10}$$

$$g_{d^2d^2} = \frac{1}{4d^4}(\langle n_d^2\rangle - \langle n_d\rangle^2), \tag{11}$$

$$g_{c^2d^2} = \frac{1}{2c^2d^2}(\langle n_c n_d\rangle - \langle n_c\rangle\langle n_d\rangle), \tag{12}$$

where the averages are now taken with respect to the classical eight-vertex model.

Using this equivalence with the classical model, we can get the scaling of those metric elements near criticality by using the expression for the free-energy density $f=-T\lim_{N\to\infty}N^{-1}\ln Z(c^2,d^2)$ as a generating function for correlations, by differentiating with respect to the energies ϵ_c and ϵ_d. We obtain then that the dominant scaling near criticality of the metric elements is

$$g_{c^2c^2}, g_{d^2d^2}, g_{c^2d^2} \sim ||d^2-c^2|-2|^{\pi/\mu-2}, \tag{13}$$

and $||d^2-c^2|-2|^{\pi/\mu-2}\ln||d^2-c^2|-2|$ for π/μ an integer. Then, we have an algebraic divergence of the fidelity metric only for $\pi/\mu-2<0$, and a logarithmic divergence for $\pi/\mu-2=0$. Using the fact that near criticality $\mu=2\tan^{-1}\sqrt{c^2d^2}$, those two conditions can be written as $1<c^2d^2$ and $c^2d^2=1$, respectively. Contrary to the case analyzed before, the metric now diverges as a power law instead of logarithmically, but only for a certain region of the phase diagram.

Some remarks are now due. The eight-vertex model can be shown to be equivalent to two classical square Ising lattices, coupled with a quartic spin term [41], with both models at the same temperature. It is interesting to note that the curve $c^2d^2=1$ corresponds to the line along which the coupling between the four spins disappears, and separates the region where this coupling is ferromagnetic and antiferromagnetic. The region $1<c^2d^2$ corresponds to this last case.

Furthermore, one can consider the effects of disorder on the previous scaling results for the metric, only for $c^2d^2=1$. Along this curve in the phase diagram, the eight-vertex model maps onto two decoupled Ising lattice models, in which a random-bond disorder can be introduced. Specifically, consider that every bond has a probability p to have a coupling constant J' between nearest-neighbor spins, while a probability $1-p$ to have a coupling constant J, with $J\neq J'$. Then in the limit $p\ll 1$ and for the reduced temperature $t=(T-T_c)/T_c\to 0$, the specific heat C_v scales as $\log(\log|t|)$ for $t\ll t_p$, while $C_v\sim -\log|t|$ for $t_p\ll t\ll 1$. Here $t_p\sim\exp(-{\rm const}/p)$ is some characteristic temperature scale [42,43]. Since the metric elements (9)–(11) are proportional to the specific heat C_v for $c^2d^2=1$, we conclude that the fidelity metric elements present this novel doubly logarithmic divergence for the disordered quantum eight-vertex model as well, for $t \ll t_p$.

Finally, there are many models that are either a special case of the eight-vertex model or equivalent to it, such as the quantum *XYZ* chain model, ice model, *F* model, etc. [41]. Therefore, by analyzing the fidelity metric in the quantum eight-vertex model, we can infer its behavior in many other models and predict its divergence only for regions in the phase diagram that correspond to the condition $1 \leqslant c^2 d^2$.

CONCLUSIONS AND OUTLOOK

In this paper, we have performed a fidelity analysis of quantum phase transitions to topologically ordered phases. We have considered two different systems that can be naturally mapped onto classical statistical-mechanical models. The mapping reveals that the fidelity metric corresponds to derivatives of the free energy with respect to some parameters of the model, giving rise to correlations in the classical system.

We discovered a logarithmic divergence in the fidelity metric for the extension of the toric code model near the transition to the topologically ordered state. This may be related to the fact that in this model there is no local order parameter nor symmetry breaking, and only topological order is involved. In perspective, this is an aspect that deserves more investigation. On the other hand, the quantum eight-vertex model still has a power-law divergence at the transition to the topological phase, but exhibits this singularity for a restricted region of the phase diagram only.

A satisfactory understanding of the relation between fidelity metric singularities and the nature of different topological as well as standard orders involved in the transitions is a primary goal for future investigations.

Note added. Recently, we became aware of related results for topological phases in the Kitaev honeycomb model in [44,45].

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