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A characterization of subgame perfect equilibria in single-leader multi-follower games

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Abstract
Single-leader multi-follower games model the sequential strategic interactions between one player who chooses first, the leader, and other players, the followers, who respond simultaneously after observing the choice of the leader. In this paper we focus on the concept of subgame perfect equilibrium (SPE for short) in single-leader multi-follower games when the followers are involved in a generalized game and have a non-unique joint best response for each choice of the leader. We first show the connections between the concept of an SPE and those of a pessimistic and optimistic equilibrium, which are usually adopted and extensively studied in this framework. We then prove an existence result for SPEs and a characterization of the set of SPE-outcomes. Specifically, an action profile is shown to be an SPE-outcome if and only if the joint response of the followers to the leader is rational and the leader's payoff lies between the payoffs achievable in a pessimistic equilibrium and in an optimistic equilibrium. Finally, moving to the broader framework of multi-leader-follower games, we show the existence of SPEs and we identify a subset of SPE-outcomes for a class of multi-leader-follower games where the game played by the leaders involves a generalized weighted potential structure.

Keywords Single-leader multi-follower Stackelberg game · Subgame perfect equilibrium · Pessimistic and optimistic equilibrium · Generalized Nash equilibrium · Multi-leader-follower game · Weighted potential game

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1 Introduction

Since their first application, in the seminal article by Sherali et al. (1983), to an oligopolistic market model with a first-mover quantity-setting firm and a fringe of second-mover quantity-setting competitors, single-leader multi-follower games have played a significant role in the analysis of many real-world situations, especially in the fields of Economics and Operations Research. To mention a few, some sub-fields of application include supply chains, energy markets, cybersecurity, resource management, public policy analysis on environmental issues, and transportation networks. Single-leader multi-follower games are non-cooperative $(1 + M)$-player sequential games in which one player, *the leader*, acts in the initial stage and makes her decision first, while the other $M \geq 2$ players, *the followers*, involved in a simultaneous-move game in the subsequent stage, observe the leader's decision and respond to it by simultaneously making a choice. This interaction structure subsumes the standard two-player Stackelberg game (or single-leader single-follower game, introduced in Stackelberg (1934) to describe the interaction between two firms in a sequential duopoly) and the standard bilevel optimization model, topics of study in which Jacqueline Morgan is recognized as a pioneer and a leading figure for her in-depth analysis of the existence of solutions as well as their approximation, well-posedness and stability, as witnessed by the intense contribution to the literature and by papers such as Morgan (1989); Loridan and Morgan (1989a, b); Lignola and Morgan (1995); Loridan and Morgan (1996); Lignola and Morgan (1997), that are milestones in the field: see Caruso et al. (2020) for a comprehensive overview on approximation and regularization issues in Stackelberg games and bilevel optimization.

In single-leader multi-follower games, Morgan's investigations on equilibrium concepts and related properties mainly focused on the relevant issues listed below.

- *Approximation of pessimistic equilibria:* Analysis of the approximate solutions in the cases of a zero-sum second-stage game (Morgan and Raucci 1997; Mallozzi and Morgan 2001) and of a more general two-player second-stage game (Morgan and Raucci 1999; Mallozzi and Morgan 2006) as well as of the viscosity solutions (Lignola and Morgan 2014a); see also Lignola and Morgan (2014b) for the wider framework where the second stage involves quasi-variational inequalities.
- *Existence and approximation of optimistic equilibria:* Analysis of approximate solutions in the case of a zero-sum second-stage game (Morgan and Raucci 1997), of the existence of optimistic equilibria in the broader framework of generalized bilevel programming problems (Lignola and Morgan 1998) and in a Banach space setting (Lignola and Morgan 2002), of the approximate equilibria via mixed strategies (Mallozzi and Morgan 2006); see also Lignola and Morgan (2001, 2012, 2015) for the case of variational or quasi-variational inequalities in the second stage.

- *Intermediate hierarchical equilibria:* Analysis of the existence and approximation of solutions in Mallozzi and Morgan (1996) as well as in Mallozzi and Morgan (2005) for the special case of a game of Cournot competition at the second stage.
- *Well-posedness:* Analysis of the optimistic solutions in optimization problems with Nash equilibrium constraints Lignola and Morgan (2006); see also Lignola and Morgan (2000); Del Prete et al. (2003) for the case of a variational inequality in the second stage.
- *Existence in the case of a unique equilibrium in the second-stage game:* Analysis of the existence of solutions when the followers are involved in a two-stage game of Cournot competition Flåm et al. (2002); Mallozzi and Morgan (2005).
- *Regularization of the second-stage game:* Analysis of the sequence of equilibria of single-leader two-follower games perturbed via the Tikhonov regularization Morgan and Patrone (2006).

In this paper, we consider single-leader multi-follower games where the set of feasible choices for each follower depends on the leader's decisions and, possibly, on the actions of the rival followers, meaning that the followers are involved in a generalized game in the second stage and so respond to the leader by choosing a social Nash equilibrium (or generalized Nash equilibrium, see Arrow and Debreu (1954) and also Facchinei and Kanzow (2010) for a survey paper), and we focus on the subgame perfect equilibrium concept, as introduced in Selten (1965) (henceforth SPE, for short), a traditional refinement of the Nash equilibrium notion which is largely employed in sequential games and, more generally, in dynamic games (see, for example, Fudenberg and Tirole (1991); Maschler et al. (2020) for games with finite action sets and Haurie et al. (2012) for dynamic contexts).

The literature on the SPE concept in single-leader multi-follower games has focused mainly on the special case of a single follower, as summarized in Table 1. Moreover, in the case of multiple followers and of a unique Nash equilibrium at the second-stage game, the results on existence, uniqueness, and approximation of the outcome corresponding to an SPE can be deduced from the literature related to the *Stackelberg-Nash equilibrium* concept in that the followers' strategy profile of an SPE is uniquely determined (see, for example, Sherali et al. (1983); Tobin (1992); Raucci (1998); Flåm et al. (2002); Mallozzi and Morgan (2005)). Therefore, with the notable exception of Morgan and Patrone (2006, Section 4), where an SPE is shown to be generated by the limit of sequences of equilibria of Tikhonov-perturbed single-leader two-follower games, the investigation around the concept of SPE in single-leader multi-follower games is quite limited.

The aim of this paper is to study the existence and characterization of an SPE in single-leader multi-follower games in the case of a generalized second-stage game and non single-valuedness of the followers' best reply correspondence. We start our analysis recalling the notions of pessimistic equilibrium and optimistic equilibrium, concepts widely considered in the literature when the followers have a non-unique optimal response to each choice of the leader and motivated by the extreme beliefs that the leader has about how the followers make their choices out of the set of Nash equilibria of the second-stage game; these notions are the generalization of the traditional pessimistic and optimistic (or weak and strong Stackelberg) equilibria defined in the case

Table 1 Literature results on SPEs in single-leader single/multi-follower games.

	Game features	Achievements on SPEs
Loridan and Morgan (1989b)	Single follower and single-valued best reply correspondence	Variational stability analysis under perturbation
Morgan and Patrone (2006)	Single follower and two followers	Construction of an SPE from the limit of equilibria of games perturbed via the Tikhonov regularization
Caruso et al. (2019)	Single follower	Learning approach to select an SPE relying on the costs faced by players to deviate from current actions
Caruso et al. (2024a)	Single follower and multi-valued best reply correspondence	Asymptotic behavior and variational stability for SPEs under perturbation
Caruso et al. (2024b)	Single follower	General constructive method to generate an SPE via sequences of SPEs of well-behaved perturbed games

of a single follower: see, for example, Caruso et al. (2020, Definitions 4.2.2 and 4.2.3) for single-leader single-follower games as well as Caruso et al. (2020, Section 4.4.3.2) or Aussel and Svensson (2020, Definitions 3.3.1 and 3.3.2) for single-leader multi-follower games.

First, we present the relations between the SPE concept and the notions of pessimistic and optimistic equilibria. More precisely, we show that it is possible to identify the SPEs associated to pessimistic and optimistic equilibria, which reflect the pessimistic or optimistic attitude of the leader and, consequently, that the sets of pessimistic and optimistic equilibria are subsets of the set of SPE-outcomes of the game (namely, of the set of action profiles actually played according to an SPE). In the second place, under some assumptions that guarantee the existence of an SPE, we show the following characterization of the set of the SPE-outcomes: the action profiles played in an SPE coincide with the action profiles in which the followers respond optimally to the leader by choosing a generalized Nash equilibrium of the second-stage game and the leader's payoff is in between the payoffs corresponding to a pessimistic equilibrium and an optimistic equilibrium. In particular, when there is only one follower, such a characterizing property coincides with the definition of a *lower Stackelberg equilibrium*, a concept introduced in Loridan and Morgan (1986) in the analysis of approximate solutions to pessimistic bilevel optimization problems (see also Loridan and Morgan (1992) for results in the framework of regularization of pessimistic bilevel optimization problems and Caruso et al. (2025) for variational stability issues). Finally, we investigate the possibility of expanding the analysis of the existence and characterization of SPEs to the framework of games with many leaders and many followers. In particular, we show a result on the existence of SPEs and identify a subset of SPE-outcomes of a class of multi-leader-follower games whose first-stage games involve a weighted potential (see Monderer and Shapley (1996) for definition and Facchini et al. (1997); Caruso et al. (2018) for characterization of weighted potential games) in the same spirit of Lignola and Morgan (2024), who study the pessimistic solutions and define suitable approximate and viscosity solutions in multi-leader common-follower games.

The paper is organized as follows. Section 2 presents the context of single-leader multi-follower games, related solution concepts, and interconnections. More precisely, in Section 2.1 an overview of the equilibrium notions associated to a single-leader multi-follower game is provided: the notions of a Stackelberg-Nash equilibrium, pessimistic equilibrium, optimistic equilibrium, intermediate hierarchical Nash equilibrium, and SPE are recalled and discussed, while in Section 2.2 the connections between the SPE and the other mentioned equilibrium concepts are proved and representative examples are illustrated. Section 3 is devoted to the core of the investigation, that is, the characterization of the SPEs: after listing the assumptions for the analysis, we show a preliminary result on the properties of the followers' best reply correspondence, the existence of an SPE, and the characterization and the closure of the set of SPE-outcomes. Section 4 extends the analysis on SPEs to the class of multi-leader-follower games: we discuss the issue of the existence of SPEs through an example and introduce the class of multi-leader-follower for which the existence of SPEs is established and the identification of a subset of SPE-outcomes is shown. Finally, Section 5 provides conclusions and some directions for future research.

2 Single-leader multi-follower games

We start this section by presenting the mathematical framework associated to single-leader multi-follower games. Then, we recall the main solution concepts associated to such a class of games and show the connections among them through illustrative examples.

Let Γ be a single-leader multi-follower game, that is, a non-cooperative sequential two-stage game where the interactions among players are described as follows.

1. At the first stage: one player, henceforth called the *leader*, makes a choice x out of her set of available actions X.
2. At the second stage: $M \geq 2$ players, henceforth called *followers*, observe the choice made by the leader in the first stage and simultaneously choose an action; more precisely, each follower $j \in J = \{1, \dots, M\}$ chooses an action y_j in his own set of available actions $\mathcal{Y}_j(x, \boldsymbol{y}_{-j}) \subseteq Y_j$, which depends on the leader's action x and on the action profile $\boldsymbol{y}_{-j}:=(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_M)$ of the followers other than j.

Once the leader's action x and the followers' action profile $\boldsymbol{y} = (y_1, \dots, y_M)$ are played, the two-stage interaction ends and the players' payoffs associated to the outcome $(x, \boldsymbol{y})$ are $l(x, \boldsymbol{y}) \in \mathbb{R}$ for the leader and $f_j(x, \boldsymbol{y}) \in \mathbb{R}$ for the follower j, for each $j \in J$.

Putting $\boldsymbol{Y}_{-j}:=Y_1 \times \dots \times Y_{j-1} \times Y_{j+1} \times \dots \times Y_M$ and $\boldsymbol{Y} = Y_1 \times \dots \times Y_M$, the set of actions available to the follower j is described via the set-valued map $\mathcal{Y}_j \colon X \times \boldsymbol{Y}_{-j} \rightrightarrows Y_j$, and the payoff functions of the leader and the follower j are $l \colon X \times \boldsymbol{Y} \to \mathbb{R}$ and $f_j \colon X \times \boldsymbol{Y} \to \mathbb{R}$, respectively. As standard in the literature on single-leader multi-follower games, we assume that the followers adopt a Nash equilibrium action profile after observing the choice of the leader at the first stage. More precisely, given the action x chosen by the leader, the followers are involved in a simultaneous-move game $\Gamma(x)$ and, assuming that players pursue to maximize their payoffs, they react to her by choosing an action profile $\bar{\boldsymbol{y}}$ in the set

$$\begin{aligned}\mathcal{N}(x) &= \left\{ (\bar{y}_1, \ldots, \bar{y}_M) \in \boldsymbol{Y} \mid \bar{y}_j \in \operatorname*{arg\,max}_{z_j \in \mathcal{Y}_j(x, \bar{\boldsymbol{y}}_{-j})} f_j(x, z_j, \bar{\boldsymbol{y}}_{-j}) \text{ for each } j \in J \right\} \\ &= \left\{ (\bar{y}_1, \ldots, \bar{y}_M) \in \boldsymbol{Y} \mid f_j(x, \bar{y}_j, \bar{\boldsymbol{y}}_{-j}) = \max_{z_j \in \mathcal{Y}_j(x, \bar{\boldsymbol{y}}_{-j})} f_j(x, z_j, \bar{\boldsymbol{y}}_{-j}) \text{ for each } j \in J \right\},\end{aligned}$$

that is a generalized (or social) Nash equilibrium of the game $\Gamma(x) = \langle J, (\mathcal{Y}_j(x, \cdot))_{j \in J}, (f_j(x, \cdot))_{j \in J} \rangle$. We refer to the set-valued map $\mathcal{N} \colon X \rightrightarrows \boldsymbol{Y}$ as the *followers' best reply (or Nash equilibrium) correspondence*. In light of the notation above, we denote the entire single-leader multi-follower game by $\Gamma = \langle \{1\}, J, X, (\mathcal{Y}_j)_{j \in J}, l, (f_j)_{j \in J} \rangle$.

2.1 Equilibrium concepts

Depending on the structure of the followers' best reply correspondence and on the attitude of the followers while reacting to the decision of the leader, various solution concepts have been proposed for single-leader multi-follower games.

In the situation where the followers have a unique joint best reaction to each action of the leader, she can unambiguously anticipate the action profile chosen by the followers and take her corresponding optimal choice. The associated outcome is often referred to as *Stackelberg-Nash equilibrium*.

Definition 1 An action profile $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ is a Stackelberg-Nash equilibrium of Γ if $\bar{x} \in \arg\max_{x \in X} l(x, \bar{\boldsymbol{n}}(x))$, where $\{\bar{\boldsymbol{n}}(x)\} = \mathcal{N}(x)$ for each $x \in X$, and $\bar{\boldsymbol{y}} = \bar{\boldsymbol{n}}(\bar{x})$.

Remark 1 This sequential interaction structure has been first addressed in Sherali et al. (1983) when the followers are involved in a Cournot oligopoly. Subsequently, the existence of Stackelberg-Nash equilibria when the followers are involved in a Cournot oligopoly (also named *Stackelberg-Nash-Cournot equilibria*) has been investigated in Flåm et al. (2002) in the case of inverse demand set-valued map and in Mallozzi and Morgan (2005) in the case of possibly discontinuous demand function, whereas uniqueness and algorithms for the Stackelberg-Nash-Cournot equilibrium have been analyzed in Tobin (1992). For existence and approximation results of the Stackelberg-Nash equilibria when the followers' best reply correspondence is assumed to be single-valued, see Raucci (1998).

It is worth remarking that since the followers are involved in a generalized game after the choice of the leader, the Nash equilibrium correspondence $\mathcal{N}$ generally fails to be single-valued, even under strict concavity of the followers' payoff functions (see, for example, Facchinei and Kanzow (2010, Section 4.3)). In the papers cited in Remark 1, the followers are involved in a traditional (non-generalized) game, meaning that the set of available actions to each follower are independent of the choice of the rival followers, for which uniqueness of Nash equilibrium can be actually ensured.[1]

When followers do not have a unique best reaction to each choice of the leader, which is "frequent" when the followers are involved in a generalized game, ambiguities arise for the leader who cannot exactly anticipate the action profile the followers will choose. In this framework, two approaches are mainly considered, which correspond to the generalization of the pessimistic and optimistic approaches widely investigated in the literature on single-leader single-follower games (see, for example, Caruso et al. (2020, Section 4.2) for a comprehensive look).

In the pessimistic case, a pessimistic leader believes that the followers will react by choosing the worst Nash equilibrium for her among those in $\mathcal{N}(x)$, meaning the one that minimizes her payoff. The corresponding solution concept is recalled below.

Definition 2 An action profile $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ is a pessimistic equilibrium of Γ if $\bar{x} \in \arg\max_{x\in X} \min_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y})$ and $\bar{\boldsymbol{y}} \in \arg\min_{\boldsymbol{y}\in\mathcal{N}(\bar{x})} l(\bar{x}, \boldsymbol{y})$.

Remark 2 Originally introduced in Leitmann (1978), we point out that a slightly different definition for the pessimistic equilibrium solution concept requires just $\bar{\boldsymbol{y}} \in \mathcal{N}(\bar{x})$ instead of $\bar{\boldsymbol{y}} \in \arg\min_{\boldsymbol{y}\in\mathcal{N}(\bar{x})} l(\bar{x}, \boldsymbol{y})$, see Başar and Olsder (1998, Definition 3.35), and it is often used.[2] Regardless of the definition taken into account, the existence of a pessimistic equilibrium is not guaranteed in general, even when the action sets are compact, the payoff functions are continuous and there is a single follower (see, for example, Loridan and Morgan (1986, Remark 2.1) and Başar and Olsder (1998, Section 3.6)). To circumvent this drawback, relevant notions of surrogate solutions have been proposed and investigated: approximate solutions (see Morgan and Raucci (1997, Section 5) for zero-sum games and Morgan and Raucci (1999, Section 2) for non-zero-sum games), approximate mixed solutions (see Mallozzi and Morgan (2001) for zero-sum games and Mallozzi and Morgan (2006) for antipotential games), and viscosity solutions (see Lignola and Morgan (2014a)).

In the opposite case, an optimistic leader thinks that the followers will choose the best Nash equilibrium for her in $\mathcal{N}(x)$, or she is able to force the followers to choose the action profile in $\mathcal{N}(x)$ that maximizes her payoff. This circumstance leads to the following equilibrium concept.

Definition 3 An action profile $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ is an optimistic equilibrium of Γ if $\bar{x} \in \arg\max_{x\in X} \max_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y})$ and $\bar{\boldsymbol{y}} \in \arg\max_{\boldsymbol{y}\in\mathcal{N}(\bar{x})} l(\bar{x}, \boldsymbol{y})$.

Remark 3 Unlike the pessimistic case, the optimistic case is more tractable and has been much more investigated, especially in the optimization literature, where the

[1] We recall that results on the uniqueness of Nash equilibrium have been provided in Rosen (1965); Karamardian (1969); Gabay and Moulin (1980); Ceparano and Quartieri (2017); Scalzo (2020) for N-player games, in Sherali et al. (1983); Tobin (1992); Mouche and Quartieri (2013, 2018); Caruso et al. (2026) for Cournot games, and in Caruso et al. (2018) for two-player weighted potential games.

[2] In terms of payoffs, the difference is that the leader's payoff at a pessimistic equilibrium as defined in Definition 2 and in Leitmann (1978, Definition 3.2) is exactly $\max_{x\in X} \min_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y})$, while at a pessimistic equilibrium as defined in Başar and Olsder (1998, Definition 3.35) is at least $\max_{x\in X} \min_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y})$. These different notions in literature reflect the analogous duality appearing in the case of a single follower, see Leitmann (1978, Definition 2.2) and Başar and Olsder (1998, Definition 3.28).

problem of finding an optimistic equilibrium is often referred to as the optimization problem with Nash equilibrium constraints (see Lignola and Morgan (2002)). The existence of an optimistic equilibrium has been successfully addressed in Morgan and Raucci (1997, Proposition 2.1) when followers are involved in a zero-sum game, Lignola and Morgan (1998, Corollary 4.4) in a possibly discontinuous setting, Lignola and Morgan (2002, Section 3) when action sets are Banach spaces, and Aussel and Svensson (2020, Corollary 3.3.6) in a differentiable setting. In addition, approximate solutions have been investigated in Morgan and Raucci (1997, Section 3) and Mallozzi and Morgan (2006), and well-posedness in Lignola and Morgan (2006). Concerning numerical approaches, most of the algorithms to approximate optimistic equilibria rely on reformulations of the corresponding optimization problem with Nash equilibrium constraints (see the survey Aussel and Svensson (2020, Section 3.3.3).[3]

In addition to the pessimistic and optimistic approaches, an intermediate situation in which the leader has information on how followers will choose their action profiles in $\mathcal{N}(x)$ has been investigated in Mallozzi and Morgan (1996). This information allows the leader to attribute a probability distribution (reflecting her beliefs) over the set $\mathcal{N}(x)$ for each of her decisions $x \in X$ and leads to the definition of the *intermediate hierarchical Nash equilibrium* concept Mallozzi and Morgan (1996), Definition 2.1). In order to simplify, assume that $\mathcal{N}(x)$ has a finite number of elements for each $x \in X$, namely $\mathcal{N}(x) = \{\boldsymbol{y}^1(x), \ldots, \boldsymbol{y}^{\sigma_x}(x)\}$. Let $\pi^k(x) \in [0, 1]$ be the probability that the followers jointly choose $\boldsymbol{y}^k(x)$ in $\mathcal{N}(x)$, and put $D(x) := (\pi^1(x), \ldots, \pi^{\sigma_x}(x))$ and $D := \{D(x) \mid x \in X\}$.

Definition 4 An action profile $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ is an intermediate hierarchical Nash equilibrium of Γ with respect to D if $\bar{x} \in \arg\max_{x \in X} \sum_{k=1}^{\sigma_x} \pi^k(x) l(x, \boldsymbol{y}^k(x))$ and $\bar{\boldsymbol{y}} \in \mathcal{N}(\bar{x})$.

Remark 4 The intermediate hierarchical Nash equilibrium concept extends the notion of *intermediate Stackelberg equilibrium* introduced for single-leader single-follower games (see Mallozzi and Morgan (1995, Definitions 2.1 and 3.1) and, depending on the specific probability distribution D, one can recover both the pessimistic and the optimistic equilibrium concepts. In the case the followers are involved in a non-generalized game at the second stage, the existence of an intermediate hierarchical Nash equilibrium and its regularized version have been examined in Mallozzi and Morgan (1996, Theorems 3.1 and 3.2) when $\mathcal{N}(x)$ is a measurable set, and in Mallozzi and Morgan (2005) when $\mathcal{N}(x)$ is the set of equilibria of a Cournot oligopoly. Note that the leader's payoff at an intermediate hierarchical Nash equilibrium is always in between the corresponding ones at a pessimistic equilibrium and at an optimistic equilibrium, provided that the last equilibria exist, see Mallozzi and Morgan (1996, Remark 2.1).

Finally, let us present the notion of *subgame perfect equilibrium* for a single-leader multi-follower game, which is the main solution concept of our investigation in this

[3] Reformulations involve the wider class of generalized bilevel programming problems or mathematical programs with equilibrium constraints; see, for example, Luo et al. (1996); Ye et al. (1997); Lignola and Morgan (1998); Outrata et al. (1998); Colson et al. (2007).

paper. First, we recall that a *strategy for the follower* j is a function $\vartheta_j\colon X \to Y_j$ that assigns a follower j's action to each action of the leader (that is, to each point the follower j is required to take a decision; see, for example, Fudenberg and Tirole (1991, Section 3.4.1) and Maschler et al. (2020, Definition 3.9) for the definition of players' strategy). Hence, a strategy ϑ_j describes the instructions on how the follower j plays for each possible choice $x \in X$ of the leader, including the circumstances in which the follower j may not be required to choose an action; in fact, the leader will choose only one action in the set X. Denoted by Y_j^X and $\boldsymbol{Y}^X$ the set of strategies of the follower j and the set of followers' strategy profiles, respectively, defined by

$$Y_j^X := \{\vartheta_j\colon X \to Y_j\} \quad \text{and} \quad \boldsymbol{Y}^X := \prod_{j\in J} Y_j^X = \{\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_M)\colon X \to \boldsymbol{Y}\},$$

the definition of subgame perfect equilibrium (henceforth SPE) is provided below.

Definition 5 A strategy profile $(\bar{x}, \bar{\boldsymbol{\vartheta}}) \in X \times \boldsymbol{Y}^X$ is a subgame perfect equilibrium of Γ if the following conditions are satisfied:

(SPE1) $\bar{\boldsymbol{\vartheta}}(x) \in \mathcal{N}(x)$ for each $x \in X$;
(SPE2) $\bar{x} \in \arg\max_{x\in X} l(x, \bar{\boldsymbol{\vartheta}}(x))$.

An action profile $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ is a subgame perfect equilibrium outcome of Γ if there exists a followers' strategy profile $\bar{\boldsymbol{\vartheta}} \in \boldsymbol{Y}^X$ such that $(\bar{x}, \bar{\boldsymbol{\vartheta}})$ is a subgame perfect equilibrium of Γ and $\bar{\boldsymbol{\vartheta}}(\bar{x}) = \bar{\boldsymbol{y}}$.

Definition 5 represents the explicit specification of the notion of SPE originally introduced in Selten (1965) (see also, for example, Fudenberg and Tirole (1991, Section 3.5) and Maschler et al. (2020, Section 7.1) as a refinement of the Nash equilibrium concept for extensive-for games, in the class of single-leader multi-follower games.[4] The idea behind the SPE concept is to rule out irrational behaviors of the players even in the points that are not actually reached during the play of the game. Hence, players act according to the so-called *principle of sequential rationality*: they must play optimally from every point of the game onwards, even regarding to off-equilibrium path actions (see Maschler et al. (2020, Chapter 7) for further discussion). In single-leader multi-follower games, this principle is expressed in Definition 5 by condition (SPE1), where the followers are required to play a generalized Nash equilibrium for each subgame $\Gamma(x)$ they may be involved in when varying the choice x of the leader, and by condition (SPE2), where the leader is required to play optimally knowing the strategy profile that the followers will implement. Moreover, to each SPE corresponds the related SPE-outcome, which is just the action profile actually played according to the SPE strategy profile, namely the realization of the SPE (see Maschler et al. (2020, Sections 3.3 and 3.6) for further discussion on the notion of an outcome of a game).

Remark 5 The SPE notion has been a subject of investigation in two-stage games, especially in single-leader single-follower games. In fact, Loridan and Morgan (1989b) presents an approximation scheme in the case of single-valuedness of the follower's

[4] The concept of subgame perfect equilibrium is largely employed as a solution concept even in the wider framework of dynamic games; see, for example, Başar and Olsder (1998); Haurie et al. (2012).

best reply correspondence, Raucci (1998) study the existence and approximation of SPEs in single-leader two-follower games with a single-valued Nash equilibrium correspondence, Morgan and Patrone (2006) shows that an SPE can be generated from the limit points of games perturbed via the Tikhonov regularization both in the case of one follower and two followers, Caruso et al. (2019) proposes a learning procedure to approach an SPE motivated according to the costs that the leader and the follower face when deviating from their current actions, Caruso et al. (2024a) and Caruso et al. (2024b) examine the issues of asymptotic behavior under perturbation and of construction of SPEs in single-leader single-follower games, respectively, and Caruso et al. (2025) shows the role played by SPEs in the issue of variational stability of bilevel optimization problems.

2.2 Connections among the equilibrium concepts

The solution concepts presented in the previous section are closely related. In particular, we show the connections between the SPE-outcomes and the pessimistic equilibria, optimistic equilibria, and intermediate hierarchical Nash equilibria. Throughout this section, we consider a single-leader multi-follower game Γ and we denote by $\mathcal{E}_P(\Gamma)$, $\mathcal{E}_O(\Gamma)$, $\mathcal{E}_I^D(\Gamma)$, and $\mathcal{E}_{SP}(\Gamma)$ the set of pessimistic equilibria, optimistic equilibria, intermediate hierarchical Nash equilibria with respect to D, and SPE-outcomes of Γ, respectively.

First, when the followers have a unique joint best reply for each choice of the leader, the set of SPE-outcomes coincides with the sets of pessimistic equilibria and of optimistic equilibria.

Proposition 1 *Assume that the Nash equilibrium correspondence $\mathcal{N}$ is single-valued. Then $\mathcal{E}_{SP}(\Gamma) = \mathcal{E}_P(\Gamma) = \mathcal{E}_O(\Gamma)$.*

Proof Since $\mathcal{N}$ is single-valued, suppose $\mathcal{N}(x) = \{\bar{\boldsymbol{n}}(x)\}$ for each $x \in X$ where $\bar{\boldsymbol{n}}\colon X \to \boldsymbol{Y}$ is the function that assigns to each leader's action x the unique Nash equilibrium of the subgame $\Gamma(x)$. On the one hand, the equality $\mathcal{E}_P(\Gamma) = \mathcal{E}_O(\Gamma)$ follows immediately from the fact that $\min_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y}) = \max_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y}) = l(x, \bar{\boldsymbol{n}}(x))$ for each $x \in X$. On the other hand, by Definition 5, the action profile $(\bar{x}, \bar{\boldsymbol{y}})$ is an SPE-outcome if and only if $\bar{x} \in \arg\max_{x\in X} l(x, \bar{\boldsymbol{n}}(x))$ and $\bar{\boldsymbol{y}} = \bar{\boldsymbol{n}}(\bar{x})$. Hence $\mathcal{E}_{SP}(\Gamma) = \mathcal{E}_P(\Gamma) = \mathcal{E}_O(\Gamma)$. □

In general, regardless of the structure of the Nash equilibrium correspondence, pessimistic and optimistic approaches recalled in Section 2.1 lead to the description of particular SPEs.

Theorem 2 *Each strategy profile $(\bar{x}, \bar{\boldsymbol{\vartheta}})$ satisfying*

$$\bar{x} \in \arg\max_{x \in X} \min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) \quad \textit{and} \quad \bar{\boldsymbol{\vartheta}}(x) \in \arg\min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) \quad \textit{for any } x \in X, \tag{1}$$

is an SPE of Γ.

Proof Let $(\bar{x}, \bar{\boldsymbol{\vartheta}}) \in X \times \boldsymbol{Y}^X$ be such that (1) holds, and let us show that it satisfies conditions (SPE1) and (SPE2) in Definition 5.

The requirement (SPE1) is clearly satisfied. To prove (SPE2), first note that the action profile $(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x}))$ is a pessimistic equilibrium of Γ, by Definition 2. So, on the one hand,

$$l(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x})) = \min_{\boldsymbol{y} \in \mathcal{N}(\bar{x})} l(\bar{x}, \boldsymbol{y}) = \max_{x \in X} \min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}).$$

On the other hand, since $\bar{\boldsymbol{\vartheta}}(x) \in \arg\min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y})$ for each $x \in X$, we have $l(x, \bar{\boldsymbol{\vartheta}}(x)) = \min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y})$ for each $x \in X$, which implies

$$\sup_{x \in X} l(x, \bar{\boldsymbol{\vartheta}}(x)) = \sup_{x \in X} \min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}).$$

From the above equalities, it follows that

$$l(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x})) = \max_{x \in X} \min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) = \sup_{x \in X} l(x, \bar{\boldsymbol{\vartheta}}(x)),$$

hence, (SPE2) is satisfied. □

Theorem 3 *Each strategy profile* $(\bar{x}, \bar{\boldsymbol{\vartheta}})$ *satisfying*

$$\bar{x} \in \arg\max_{x \in X} \max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) \quad \textit{and} \quad \bar{\boldsymbol{\vartheta}}(x) \in \arg\max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) \quad \textit{for any } x \in X, \tag{2}$$

is an SPE of Γ.

Proof The proof is analogous to the one of Theorem 2.

Let $(\bar{x}, \bar{\boldsymbol{\vartheta}}) \in X \times \boldsymbol{Y}^X$ be such that (2) holds. The condition (SPE1) in Definition 5 is obviously satisfied. To show (SPE2), it is worth noting preliminarily that, in light of Definition 3, the action profile $(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x}))$ is an optimistic equilibrium of Γ, so

$$l(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x})) = \max_{\boldsymbol{y} \in \mathcal{N}(\bar{x})} l(\bar{x}, \boldsymbol{y}) = \max_{x \in X} \max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}).$$

Moreover, from $\bar{\boldsymbol{\vartheta}}(x) \in \arg\max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y})$ for each $x \in X$, it follows that $l(x, \bar{\boldsymbol{\vartheta}}(x)) = \max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y})$ for each $x \in X$, and then

$$\sup_{x \in X} l(x, \bar{\boldsymbol{\vartheta}}(x)) = \sup_{x \in X} \max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}).$$

The above equalities imply

$$l(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x})) = \max_{x \in X} \max_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) = \sup_{x \in X} l(x, \bar{\boldsymbol{\vartheta}}(x)),$$

therefore, (SPE2) is proved. □

Theorems 2 and 3 show that pessimistic and optimistic approaches bring to specific refinements of the SPE, motivated by the leader's corresponding pessimistic or optimistic beliefs about the behavior of the followers when they choose their actions in $\mathcal{N}(x)$. One can refer to the strategy profiles satisfying (1) and (2), respectively, as *pessimistic SPEs* and *optimistic SPEs* of Γ. Additionally, since the outcomes of the strategy profiles defined in (1) and (2) are pessimistic equilibria and optimistic equilibria, respectively, we can immediately deduce from Theorems 2 and 3 the connections between the sets of these equilibria and the set of SPE-outcomes.

Proposition 4 *The following statements hold:*

i) if $\arg\min_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y}) \neq \emptyset$ *for each* $x \in X$*, then* $\mathcal{E}_P(\Gamma) \subseteq \mathcal{E}_{SP}(\Gamma)$*;*
ii) if $\arg\max_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y}) \neq \emptyset$ *for each* $x \in X$*, then* $\mathcal{E}_O(\Gamma) \subseteq \mathcal{E}_{SP}(\Gamma)$.

Proof To show $i)$, let $(\bar{x}, \bar{\boldsymbol{y}}) \in \mathcal{E}_P(\Gamma)$. Consider a strategy profile for the followers $\widetilde{\boldsymbol{\vartheta}} : X \to \boldsymbol{Y}$ that satisfies

$$\widetilde{\boldsymbol{\vartheta}}(\bar{x}) = \bar{\boldsymbol{y}} \quad \text{and} \quad \widetilde{\boldsymbol{\vartheta}}(x) \in \arg\min_{\boldsymbol{y}\in\mathcal{N}(x)} l(x, \boldsymbol{y}) \quad \text{for each } x \neq \bar{x},$$

which is well-defined by the hypothesis. In light of Theorem 2, the strategy profile $(\bar{x}, \widetilde{\boldsymbol{\vartheta}})$ is an SPE and its related outcome is $(\bar{x}, \bar{\boldsymbol{y}})$. Hence, $(\bar{x}, \bar{\boldsymbol{y}}) \in \mathcal{E}_{SP}(\Gamma)$. The proof of $ii)$ is analogous. □

Remark 6 The hypotheses in $i)$ and $ii)$ of Proposition 4 are technical assumptions just to ensure that the followers' strategy profiles $\bar{\boldsymbol{\vartheta}}$ satisfying (1) and (2) are well-defined.[5] Note that although an SPE-outcome is an action profile as well as for pessimistic and optimistic equilibria, the definition of the SPE-outcome depends on the existence of the corresponding SPE and so, in this case, on the existence of a function $\bar{\boldsymbol{\vartheta}}$ for which (1) and (2) hold.

To better illustrate the concepts of pessimistic and optimistic equilibrium, SPE and SPE-outcome defined in Section 2.1 and the related connections, we provide an example of single-leader multi-follower game where players have finite action sets.

Example 1 Consider the single-leader two-follower game $\Gamma = \langle\{1\}, \{1, 2\}, X, \mathcal{Y}_1, \mathcal{Y}_2, l, f_1, f_2\rangle$, where each player has two available actions and the set of actions available for each follower is independent of the actions of the rival follower. More precisely $X = \{P, Q\}, Y_1 = \{T, B\}, Y_2 = \{L, R\}, \mathcal{Y}_1(x, y_2) = \{T, B\}$ and $\mathcal{Y}_2(x, y_1) = \{L, R\}$ for each x, y_1, y_2, and the payoff functions are defined over $\{P, Q\} \times \{T, B\} \times \{L, R\}$ by

$$l(P, T, L) = 0, \qquad l(P, T, R) = 1, \qquad l(P, B, L) = 3, \qquad l(P, B, R) = 2$$

[5] Such hypotheses are both satisfied if, for example, the function $l(x, \cdot)$ is continuous over $\boldsymbol{Y}$ for each $x \in X$ and the set-valued map $\mathcal{N}$ has non-empty compact values. This last condition will be examined in the next section.

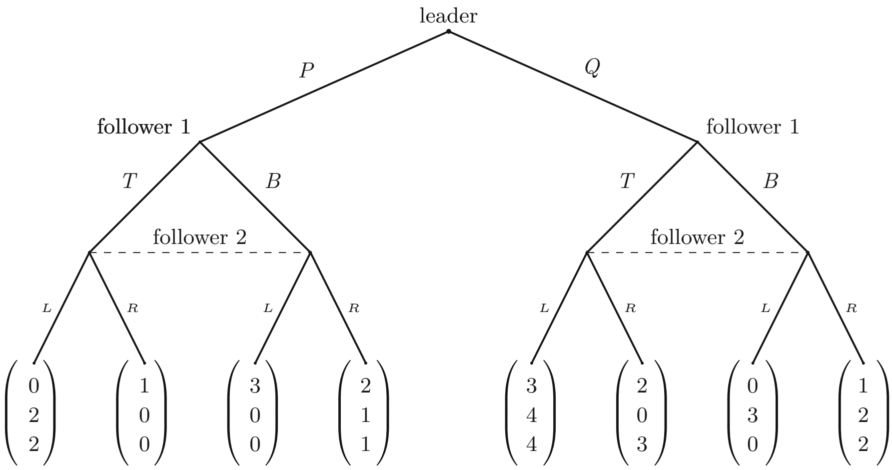

Fig. 1 Extensive-form description of $\Gamma = \langle\{1\}, \{1, 2\}, \{P, Q\}, \{T, B\}, \{L, R\}, l, f_1, f_2\rangle$.

$$\begin{array}{llll} l(Q,T,L)=3, & l(Q,T,R)=2, & l(Q,B,L)=0, & l(Q,B,R)=1; \\ f_1(P,T,L)=2, & f_1(P,T,R)=0, & f_1(P,B,L)=0, & f_1(P,B,R)=1 \\ f_1(Q,T,L)=4, & f_1(Q,T,R)=0, & f_1(Q,B,L)=3, & f_1(Q,B,R)=2; \\ f_2(P,T,L)=2, & f_2(P,T,R)=0, & f_2(P,B,L)=0, & f_2(P,B,R)=1 \\ f_2(Q,T,L)=4, & f_2(Q,T,R)=3, & f_2(Q,B,L)=0, & f_2(Q,B,R)=2. \end{array}$$

The game Γ can be described via the extensive-form representation in Figure 1.

For each choice of the leader $x \in \{P, Q\}$, the subgames $\Gamma(P)$ and $\Gamma(Q)$ played by the followers have two Nash equilibria, namely (T, L) and (B, R). So, the Nash equilibrium correspondence $\mathcal{N}\colon \{P, Q\} \rightrightarrows \{T, B\} \times \{L, R\}$ is always multi-valued and defined by

$$\mathcal{N}(x) = \{(T, L), (B, R)\} \quad \text{for each } x \in \{P, Q\}.$$

On the one hand, by computing the marginal values

$$\min_{(y_1,y_2)\in\mathcal{N}(P)} l(P, y_1, y_2) = 0 \quad \text{and} \quad \min_{(y_1,y_2)\in\mathcal{N}(Q)} l(Q, y_1, y_2) = 1,$$
$$\max_{(y_1,y_2)\in\mathcal{N}(P)} l(P, y_1, y_2) = 2 \quad \text{and} \quad \max_{(y_1,y_2)\in\mathcal{N}(Q)} l(Q, y_1, y_2) = 3,$$

we obtain

$$\arg\max_{x\in\{P,Q\}} \min_{(y_1,y_2)\in\mathcal{N}(x)} l(x, y_1, y_2) = \{Q\} \quad \text{and} \quad \arg\min_{(y_1,y_2)\in\mathcal{N}(Q)} l(Q, y_1, y_2) = \{(B, R)\},$$
$$\arg\max_{x\in\{P,Q\}} \max_{(y_1,y_2)\in\mathcal{N}(x)} l(x, y_1, y_2) = \{Q\} \quad \text{and} \quad \arg\max_{(y_1,y_2)\in\mathcal{N}(Q)} l(Q, y_1, y_2) = \{(T, L)\}.$$

Hence, Γ has one pessimistic equilibrium and one optimistic equilibrium, namely the action profiles (Q, B, R) and (Q, T, L), respectively. On the other hand, there are four SPEs in Γ, defined as follows:

$$(Q, \bar{\boldsymbol{\vartheta}}_1(\cdot)), \quad \text{where} \quad \bar{\boldsymbol{\vartheta}}_1(x) = \begin{cases} (T, L), & \text{if } x = P \\ (T, L), & \text{if } x = Q, \end{cases}$$

$$(Q, \bar{\boldsymbol{\vartheta}}_2(\cdot)), \quad \text{where} \quad \bar{\boldsymbol{\vartheta}}_2(x) = \begin{cases} (T, L), & \text{if } x = P \\ (B, R), & \text{if } x = Q, \end{cases}$$

$$(Q, \bar{\boldsymbol{\vartheta}}_3(\cdot)), \quad \text{where} \quad \bar{\boldsymbol{\vartheta}}_3(x) = \begin{cases} (B, R), & \text{if } x = P \\ (T, L), & \text{if } x = Q, \end{cases}$$

$$(P, \bar{\boldsymbol{\vartheta}}_4(\cdot)), \quad \text{where} \quad \bar{\boldsymbol{\vartheta}}_4(x) = \begin{cases} (B, R), & \text{if } x = P \\ (B, R), & \text{if } x = Q. \end{cases}$$

The SPE-outcomes of Γ are the action profiles (Q, T, L), (Q, B, R), and (P, B, R). In particular, the outcome (Q, T, L) corresponds to the SPEs $(Q, \bar{\boldsymbol{\vartheta}}_1)$ and $(Q, \bar{\boldsymbol{\vartheta}}_3)$, while (Q, B, R) corresponds to the SPE $(Q, \bar{\boldsymbol{\vartheta}}_2)$, and (P, B, R) corresponds to the SPE $(P, \bar{\boldsymbol{\vartheta}}_4)$. Finally, note that the SPE-outcomes (Q, B, R) and (Q, T, L) coincide with the pessimistic and optimistic equilibrium of Γ, respectively. Instead, the SPE-outcome (P, B, R) is neither a pessimistic nor an optimistic equilibrium.

Finally, in order to analyze the connections between the SPNE-outcomes and the intermediate hierarchical Nash equilibria, it is important to point out that the latter concept presupposes the probability distribution $D(x)$ on the set of generalized Nash equilibria of the second-stage game $\mathcal{N}(x)$ to be part of the primitives of the game for each $x \in X$; instead, this is not needed for the definition of an SPNE-outcome. Given this premise, an SPNE-outcome can be considered as an intermediate hierarchical Nash equilibrium with respect to an appropriate family of probability distributions D, as proved in the next result where we assume that $\mathcal{N}(x)$ is a finite set for each $x \in X$, consistently with Definition 4.

Proposition 5 *The following inclusion holds:*

$$\mathcal{E}_{SP}(\Gamma) \subseteq \bigcup_{D \in \mathfrak{D}} \mathcal{E}_I^D(\Gamma),$$

where $\mathfrak{D}$ is the set of all possible probability distributions over $\mathcal{N}(x)$ when varying $x \in X$.

Proof Put $\mathcal{N}(x) = \{\boldsymbol{y}^1(x), \dots, \boldsymbol{y}^{\sigma_x}(x)\}$ for each $x \in X$. Let $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ be an SPE-outcome of Γ and $\bar{\boldsymbol{\vartheta}} \in \boldsymbol{Y}^X$ be the corresponding followers' strategy whereby $(\bar{x}, \bar{\boldsymbol{\vartheta}})$ is an SPE and $\bar{\boldsymbol{\vartheta}}(\bar{x}) = \bar{\boldsymbol{y}}$. Fixed $x \in X$, by Definition 5 we know that $\bar{\boldsymbol{\vartheta}}(x) \in \mathcal{N}(x)$, so there is $\bar{k}(x) \in \{1, \dots, \sigma_x\}$ such that $\bar{\boldsymbol{\vartheta}}(x) = \boldsymbol{y}^{\bar{k}(x)}(x)$. Consider $\widetilde{D} = \{\widetilde{D}(x) \mid x \in X\}$, where $\widetilde{D}(x) = (\widetilde{\pi}^1(x), \dots, \widetilde{\pi}^{\sigma_x}(x))$ is defined as follows:

$$\widetilde{\pi}^k(x) = \begin{cases} 1, & \text{if } k = \bar{k}(x) \\ 0, & \text{if } k \in \{1, \dots \sigma_x\} \setminus \{\bar{k}(x)\}. \end{cases}$$

Consequently,

$$l(x, \bar{\boldsymbol{\vartheta}}(x)) = \sum_{k=1}^{\sigma_x} \widetilde{\pi}^k(x)\, l(x, \mathbf{y}^k(x)),$$

and, since $(\bar{x}, \bar{\boldsymbol{\vartheta}})$ is an SPE, we have that

$$\bar{x} \in \arg\max_{x \in X} l(x, \bar{\boldsymbol{\vartheta}}(x)) = \arg\max_{x \in X} \sum_{k=1}^{\sigma_x} \widetilde{\pi}^k(x)\, l(x, \mathbf{y}^k(x)).$$

Therefore, $(\bar{x}, \bar{\mathbf{y}})$ is an intermediate hierarchical Nash equilibrium with respect to $\widetilde{D}$, that is $(\bar{x}, \bar{\mathbf{y}}) \in \mathcal{E}_I^{\widetilde{D}}(\Gamma)$. □

The following example shows that, even in the simpler framework of single-leader single-follower games, the inclusion proved in Proposition 5 cannot be reversed, which means that it is possible to find a family of probability distributions D for which an intermediate hierarchical Nash equilibrium with respect to D is not an SPE-outcome.

Example 2 (Example 2.1 in Mallozzi and Morgan (1996)) Consider the single-leader single-follower game $\Gamma = \langle \{1\}, \{1\}, X, \mathcal{Y}, l, f \rangle$, where $X = Y = [-1, 1]$, $\mathcal{Y}(x) = [-1, 1]$ for each x, and the payoff functions are defined over $[-1, 1] \times [-1, 1]$ by

$$l(x, y) = -x - y \quad \text{and} \quad f(x, y) = -|x^4 - y^2|.$$

The follower's best reply correspondence $\mathcal{N}$ assigns to each $x \in [-1, 1]$ the set

$$\mathcal{N}(x) = \{-x^2, x^2\} \subseteq [-1, 1].$$

For each $x \in X$, let $\pi^1(x) = \frac{1}{8}$ and $\pi^2(x) = \frac{7}{8}$ be the probability that the follower chooses $-x^2$ and x^2 as the best response to the leader, respectively; so $D(x) = (\frac{1}{8}, \frac{7}{8})$ for each $x \in X$.

On the one hand, since

$$\arg\max_{x \in [-1,1]} [\pi^1(x)\, l(x, -x^2) + \pi^2(x)\, l(x, x^2)] = \left\{-\frac{2}{3}\right\},$$

the intermediate hierarchical Nash equilibria of Γ with respect to D are $(-\frac{2}{3}, \frac{4}{9})$ and $(-\frac{2}{3}, -\frac{4}{9})$. On the other hand, Γ has two SPEs, namely

$$(-1, \bar{\boldsymbol{\vartheta}}_1(\cdot)), \text{ where } \bar{\boldsymbol{\vartheta}}_1(x) = -x^2 \quad \text{and} \quad \left(-\frac{1}{2}, \bar{\boldsymbol{\vartheta}}_2(\cdot)\right), \text{ where } \bar{\boldsymbol{\vartheta}}_2(x) = x^2.$$

Therefore, the SPE-outcomes of Γ are $(-1, -1)$ and $(-\frac{1}{2}, \frac{1}{4})$.

3 Characterization of subgame perfect equilibria

In this section, we focus on the SPE concept and we will show an existence result for SPEs and a characterization of the set SPE-outcomes in single-leader multi-follower games. Preliminarily, let us present the assumptions on the game $\Gamma = \langle \{1\}, J, X, (\mathcal{Y}_j)_{j\in J}, l, (f_j)_{j\in J}\rangle$ (see Border (1985, Chapter 11) and Aubin and Frankowska (2009, Chapter 1) for basic notions and continuity properties on correspondences) and a preparatory result on the followers' best reply correspondence $\mathcal{N}$ that will be used in the sequel.

- $(\mathcal{A}_1)$ X is a non-empty compact subset of $\mathbb{R}^n$ and Y_j is a non-empty compact and convex subset of $\mathbb{R}^{m_j}$ for each $j \in J$;
- $(\mathcal{A}_2)$ the set-valued map $\mathcal{Y}_j(x,\cdot)\colon \boldsymbol{Y}_{-j} \rightrightarrows Y_j$ has non-empty convex values for each $x \in X$, for each $j \in J$;
- $(\mathcal{A}_3)$ $\mathcal{Y}_j$ is a closed set-valued map for each $j \in J$, that is the set

$$Gr(\mathcal{Y}_j){:=}\{(x, \boldsymbol{y}_{-j}, y_j) \in X \times \boldsymbol{Y}_{-j} \times Y_j \mid y_j \in \mathcal{Y}_j(x, \boldsymbol{y}_{-j})\} \subseteq X \times \boldsymbol{Y}$$

  is closed;
- $(\mathcal{A}_4)$ $\mathcal{Y}_j$ is a lower semicontinuous set-valued map for each $j \in J$, that is[6] for each $(x, \boldsymbol{y}_{-j}, y_j) \in Gr(\mathcal{Y}_j)$ and each sequence $(x^n, \boldsymbol{y}^n_{-j})_n \subseteq X \times \boldsymbol{Y}_{-j}$ converging to $(x, \boldsymbol{y}_{-j})$, there exists a sequence $(y^n_j)_n \subseteq Y_j$ converging to y_j such that $y^n_j \in \mathcal{Y}_j(x^n, \boldsymbol{y}^n_{-j})$ for each $n \in \mathbb{N}$;
- $(\mathcal{A}_5)$ the function l is continuous;
- $(\mathcal{A}_6)$ the function f_j is continuous and $f_j(x, \cdot, \boldsymbol{y}_{-j})$ is quasi-concave over Y_j for each $(x, \boldsymbol{y}_{-j}) \in X \times \boldsymbol{Y}_{-j}$, for each $j \in J$.

Lemma 6 *Assume that $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ and $(\mathcal{A}_6)$ hold. Then:*

i) the set-valued map $\mathcal{N}$ has non-empty and compact values;
ii) the graph of $\mathcal{N}$, namely

$$Gr(\mathcal{N}) = \{(x, \boldsymbol{y}) \in X \times \boldsymbol{Y} \mid \boldsymbol{y} \in \mathcal{N}(x)\},$$

is a compact subset of $X \times \boldsymbol{Y}$.

Proof For the sake of readability, we divide the proof in steps.

Claim 1: $\mathcal{N}$ has non-empty values.

Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ and $(\mathcal{A}_6)$ ensure that the game $\Gamma(x) = \langle J, (\mathcal{Y}_j(x,\cdot))_{j\in J}, (f_j(x,\cdot))_{j\in J}\rangle$ satisfies the hypotheses of the Arrow-Debreu-Nash theorem for the existence of a generalized Nash equilibrium (see Ichiishi (1983) or also, for example, Aubin (2007, Theorem 3 of Section 9.3.2)), and so $\mathcal{N}(x)$ is not empty for each $x \in X$. In fact, fixed $x \in X$ and $j \in J$, note that: the follower j's action set Y_j is compact and convex for each $j \in J$, by $(\mathcal{A}_1)$; the set-valued map $\mathcal{Y}_j(x,\cdot)$ has non-empty convex

[6] Equivalently, $\mathcal{Y}_j(x, \boldsymbol{y}_{-j}) \subseteq \operatorname{Lim\,inf}_{n\to+\infty} \mathcal{Y}_j(x^n, \boldsymbol{y}^n_{-j})$ for each sequence $(x^n, \boldsymbol{y}^n_{-j})_n \subseteq X \times \boldsymbol{Y}_{-j}$ converging to $(x, \boldsymbol{y}_{-j})$, where $\operatorname{Lim\,inf}_{n\to+\infty} \mathcal{Y}_j(x^n, \boldsymbol{y}^n_{-j})$ denotes the lower limit of the sequence of sets $(\mathcal{Y}_j(x^n, \boldsymbol{y}^n_{-j}))_n$ in the sense of Painlevé-Kuratowsi, see Kuratowski (1966, Section 29).

values, by $(\mathcal{A}_2)$, and it has closed values, as $\mathcal{Y}_j(x,\cdot)$ is closed, by $(\mathcal{A}_3)$, and this implies that $\mathcal{Y}_j(x,\cdot)$ is also closed-valued; $\mathcal{Y}_j(x,\cdot)$ is both lower semicontinuous, by $(\mathcal{A}_4)$, and upper semicontinuous, as $\mathcal{Y}_j(x,\cdot)$ is closed and Y_j is compact, by $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$ (see, for example, Border (1985, Proposition 11.9(b))); the follower j's payoff function $f_j(x,\cdot)$ is continuous over $\boldsymbol{Y}$ and quasi-concave over Y_j, by $(\mathcal{A}_6)$.

Claim 2: $\mathcal{N}$ is closed.

The finding can be proved by extending the result in Lignola and Morgan (1998, Corollary 3.4), obtained in the framework of generalized bilevel programming problems, to the case of $M > 2$ players. For the sake of completeness, we provide here a direct proof.

First, consider the marginal function $v_j : X \times \boldsymbol{Y}_{-j} \to \mathbb{R}$ of the follower j, defined by

$$v_j(x, \boldsymbol{y}_{-j}) = \sup_{y_j \in \mathcal{Y}_j(x, \boldsymbol{y}_{-j})} f_j(x, y_j, \boldsymbol{y}_{-j}).$$

Since $\mathcal{Y}_j$ is lower semicontinuous and f_j is continuous, the Berge maximum theorem (see, for example, Aubin and Frankowska (2009, Theorem 1.4.16)) implies that v_j is lower semicontinuous.

Now, to show that $\mathcal{N}$ is closed, consider $(x, \boldsymbol{y}) \in X \times \boldsymbol{Y}$ and a sequence $(x^n, \boldsymbol{y}^n)_n$ converging to $(x, \boldsymbol{y})$ such that $\boldsymbol{y}^n \in \mathcal{N}(x^n)$ for each $n \in \mathbb{N}$. We need to prove that $\boldsymbol{y} \in \mathcal{N}(x)$; that is, $y_j \in \mathcal{Y}_j(x, \boldsymbol{y}_{-j})$ and $f_j(x, y_j, \boldsymbol{y}_{-j}) \geq f_j(x, z_j, \boldsymbol{y}_{-j})$ for each $z_j \in \mathcal{Y}(x, \boldsymbol{y}_{-j})$, for each $j \in \boldsymbol{J}$. So, fix $j \in \boldsymbol{J}$. On the one hand, $y_j \in \mathcal{Y}_j(x, \boldsymbol{y}_{-j})$ because $y_j^n \in \mathcal{Y}_j(x^n, \boldsymbol{y}_{-j}^n)$ for each $n \in \mathbb{N}$ and $\mathcal{Y}_j$ is closed, by $(\mathcal{A}_3)$. On the other hand, pick $z_j \in \mathcal{Y}(x, \boldsymbol{y}_{-j})$. In light of the lower semicontinuity of the marginal function v_j and $(\mathcal{A}_6)$ (in particular, the upper semicontinuity of f_j), we see that

$$\begin{aligned} f_j(x, z_j, \boldsymbol{y}_{-j}) \leq v_j(x, \boldsymbol{y}_{-j}) &\leq \liminf_{n\to+\infty} v_j(x^n, \boldsymbol{y}_{-j}^n) \\ &= \liminf_{n\to+\infty} f_j(x^n, y_j^n, \boldsymbol{y}_{-j}^n) \\ &\leq \limsup_{n\to+\infty} f_j(x^n, y_j^n, \boldsymbol{y}_{-j}^n) \leq f_j(x, y_j, \boldsymbol{y}_{-j}), \end{aligned}$$

where the equality follows from the fact that $v_j(x^n, \boldsymbol{y}_{-j}^n) = f_j(x^n, y_j^n, \boldsymbol{y}_{-j}^n)$, as $(y_j^n, \boldsymbol{y}_{-j}^n) = \boldsymbol{y}^n \in \mathcal{N}(x^n)$. Therefore, $\boldsymbol{y} = (y_j, \boldsymbol{y}_{-j}) \in \mathcal{N}(x)$ and so $\mathcal{N}$ is closed.

Claim 3: $\mathcal{N}$ has compact-values.

By *Claim 2* and assumption $(\mathcal{A}_1)$, $\mathcal{N}(x)$ is a closed subset of the compact set $\boldsymbol{Y}$ for each $x \in X$. Hence, $\mathcal{N}(x)$ is compact for each $x \in X$.

Claim 4: $Gr(\mathcal{N})$ is compact.

Again, *Claim 2* and the assumption $(\mathcal{A}_1)$ imply also that $Gr(\mathcal{N})$ is a closed subset of the compact set $X \times \boldsymbol{Y}$. So $Gr(\mathcal{N})$ is compact. □

The preparatory Lemma 6 allows us to show the following result on the existence of SPEs.

Proposition 7 *Assume that $(\mathcal{A}_1)$–$(\mathcal{A}_6)$ hold. Then, there exists an SPE of* Γ.

Proof In light of $ii)$ in Lemma 6 and $(\mathcal{A}_5)$, we have

$$\operatorname*{arg\,max}_{(x,\boldsymbol{y})\in Gr(\mathcal{N})} l(x,\boldsymbol{y}) \neq \emptyset.$$

Pick $(\bar{x},\bar{\boldsymbol{y}}) \in \operatorname{arg\,max}_{(x,\boldsymbol{y})\in Gr(\mathcal{N})} l(x,\boldsymbol{y})$ and consider a function $\bar{\boldsymbol{\vartheta}}\colon X \to \boldsymbol{Y}$ such that

$$\bar{\boldsymbol{\vartheta}}(\bar{x}) = \bar{\boldsymbol{y}} \quad \text{and} \quad \bar{\boldsymbol{\vartheta}}(x) \in \mathcal{N}(x) \quad \text{for each } x \in X \setminus \{\bar{x}\},$$

which is well-defined as $\mathcal{N}$ is non-empty valued (by $i)$ in Lemma 6). The strategy profile $(\bar{x},\bar{\boldsymbol{\vartheta}})$ is actually an SPE of Γ. In fact, by the definition of $\bar{\boldsymbol{\vartheta}}$, it clearly follows that $\bar{\boldsymbol{\vartheta}}(x) \in \mathcal{N}(x)$ for each $x \in X$, so (SPE1) in Definition 5 holds true. Moreover, by the definition of $(\bar{x},\bar{\boldsymbol{y}})$ and $\bar{\boldsymbol{\vartheta}}(\bar{x})$, we have $l(\bar{x},\bar{\boldsymbol{\vartheta}}(\bar{x})) = l(\bar{x},\bar{\boldsymbol{y}}) \geq l(x,\boldsymbol{y})$ for each $(x,\boldsymbol{y}) \in Gr(\mathcal{N})$. Since $\bar{\boldsymbol{\vartheta}}(x) \in \mathcal{N}(x)$ for each $x \in X$, the inequality $l(\bar{x},\bar{\boldsymbol{\vartheta}}(\bar{x})) \geq l(x,\bar{\boldsymbol{\vartheta}}(x))$ holds also for each $x \in X$. Hence, even (SPE2) in Definition 5 is satisfied. □

In the next result, a characterization of the SPE-outcomes of Γ is presented. To avoid cumbersome notation, let us denote v^- and v^+, respectively

$$v^- := \sup_{x\in X} \inf_{\boldsymbol{y}\in\mathcal{N}(x)} l(x,\boldsymbol{y}) \quad \text{and} \quad v^+ := \sup_{x\in X} \sup_{\boldsymbol{y}\in\mathcal{N}(x)} l(x,\boldsymbol{y}).$$

Theorem 8 *Assume that $(\mathcal{A}_1)$–$(\mathcal{A}_6)$ hold. Then*

$$\mathcal{E}_{SP}(\Gamma) = \left\{(\bar{x},\bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y} \mid \bar{\boldsymbol{y}} \in \mathcal{N}(\bar{x}) \textit{ and } l(\bar{x},\bar{\boldsymbol{y}}) \in [v^-,v^+]\right\}.$$

Proof First note that the interval $[v^-,v^+]$ is well-defined. Indeed, $\inf_{\boldsymbol{y}\in\mathcal{N}(x)} l(x,\boldsymbol{y}) \in \mathbb{R}$ for each $x \in X$ since $\mathcal{N}$ has compact values (by $i)$ in Lemma 6) and l is continuous; therefore, $v^- > -\infty$ as X is non-empty. Moreover, in light of $ii)$ in Lemma 6 and the continuity of l, from the equality $v^+ = \sup_{(x,\boldsymbol{y})\in Gr(\mathcal{N})} l(x,\boldsymbol{y})$ it follows that $v^+ \in \mathbb{R}$. Consequently, $-\infty < v^- \leq v^+ < +\infty$. Now, let us continue the proof showing the two inclusions separately.

Inclusion $\subseteq$. Let $(\bar{x},\bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ be an SPE-outcome of Γ, and let $\bar{\boldsymbol{\vartheta}}\colon X \to \boldsymbol{Y}$ be a followers' strategy profile such that $(\bar{x},\bar{\boldsymbol{\vartheta}})$ is an SPE and $\bar{\boldsymbol{\vartheta}}(\bar{x}) = \bar{\boldsymbol{y}}$, whose existence is guaranteed by Definition 5. Clearly, $\bar{\boldsymbol{y}} \in \mathcal{N}(\bar{x})$ in light of (SPE1) in Definition 5.

On the one hand, since $(\bar{x},\bar{\boldsymbol{y}}) \in Gr(\mathcal{N})$, we have

$$l(\bar{x},\bar{\boldsymbol{y}}) \leq \sup_{(x,\boldsymbol{y})\in Gr(\mathcal{N})} l(x,\boldsymbol{y}) = v^+.$$

On the other hand, from (SPE2) in Definition 5, it follows that

$$l(\bar{x},\bar{\boldsymbol{y}}) = l(\bar{x},\bar{\boldsymbol{\vartheta}}(\bar{x})) \geq l(x,\bar{\boldsymbol{\vartheta}}(x)) \geq \inf_{\boldsymbol{y}\in\mathcal{N}(x)} l(x,\boldsymbol{y}) \quad \text{for each } x \in X,$$

so $l(\bar{x},\bar{\boldsymbol{y}}) \geq v^-$. Therefore, $l(\bar{x},\bar{\boldsymbol{y}}) \in [v^-,v^+]$.

Inclusion $\supseteq$. Let $(\bar{x}, \bar{\boldsymbol{y}}) \in X \times \boldsymbol{Y}$ be an action profile that satisfies $\bar{y} \in \mathcal{N}(\bar{x})$ and $l(\bar{x}, \bar{\boldsymbol{y}}) \in [v^-, v^+]$. Consider a followers' strategy profile $\bar{\boldsymbol{\vartheta}} \colon X \to \boldsymbol{Y}$ such that

$$\bar{\boldsymbol{\vartheta}}(\bar{x}) = \bar{\boldsymbol{y}} \quad \text{and} \quad \bar{\boldsymbol{\vartheta}}(x) \in \arg\min_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}) \quad \text{for each } x \in X \setminus \{\bar{x}\},$$

which is well-defined because $\mathcal{N}$ is non-empty and compact valued (by i) in Lemma 6) and l is continuous. Clearly, $(\bar{x}, \bar{\boldsymbol{y}})$ is the outcome of the strategy profile $(\bar{x}, \bar{\boldsymbol{\vartheta}})$. We need to show that $(\bar{x}, \bar{\boldsymbol{\vartheta}})$ is an SPE.

In light of the definition of $\bar{\boldsymbol{\vartheta}}$ and the fact that $\bar{y} \in \mathcal{N}(\bar{x})$, we have $\bar{\boldsymbol{\vartheta}}(x) \in \mathcal{N}(x)$ for each $x \in X$. So, condition (SPE1) in Definition 5 is satisfied.

To prove (SPE2) in Definition 5, it suffices to show that $l(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x})) \geq l(x, \bar{\boldsymbol{\vartheta}}(x))$ for each $x \neq \bar{x}$. Pick $x \in X \setminus \{\bar{x}\}$. The definition of $\bar{\boldsymbol{\vartheta}}$ implies that $l(x, \bar{\boldsymbol{\vartheta}}(x)) \leq l(x, \boldsymbol{y})$ for each $\boldsymbol{y} \in \mathcal{N}(x)$, so

$$l(x, \bar{\boldsymbol{\vartheta}}(x)) \leq \inf_{\boldsymbol{y} \in \mathcal{N}(x)} l(x, \boldsymbol{y}).$$

Since $l(\bar{x}, \bar{\boldsymbol{y}}) \geq v^-$ and using the above inequality, we get

$$\begin{aligned} l(\bar{x}, \bar{\boldsymbol{\vartheta}}(\bar{x})) = l(\bar{x}, \bar{\boldsymbol{y}}) \geq v^- & \\ & \geq \sup_{w \in X \setminus \{\bar{x}\}} \inf_{\boldsymbol{y} \in \mathcal{N}(w)} l(w, \boldsymbol{y}) \\ & \geq \sup_{w \in X \setminus \{\bar{x}\}} l(w, \bar{\boldsymbol{\vartheta}}(w)) \geq l(x, \bar{\boldsymbol{\vartheta}}(x)). \end{aligned}$$

This shows that (SPE2) in Definition 5 is valid. Therefore, $(\bar{x}, \bar{\boldsymbol{y}})$ is an SPE-outcome of Γ. ◻

Remark 7 Showing that the set-valued map $\mathcal{N}$ is closed in Lemma 6 is a crucial argument in the proofs of Proposition 7 and Theorem 8. Such an issue has been already investigated in the literature in the case of two players: beyond Lignola and Morgan (1998, Corollary 3.4), we recall Lignola and Morgan (2002, Theorem 3.2) when the action sets of the players are subsets of reflexive Banach spaces, Morgan and Scalzo (2004, Corollary 3.1 and Theorem 3.2) and Morgan and Scalzo (2008, Theorem 1) when the players' payoff functions are sequentially pseudocontinuous, and Morgan and Scalzo (2007, Theorem 1) for finite economies and sequentially pseudocontinuous utility functions.[7] These findings allow to address the existence of SPEs and the characterization of SPE-outcomes also in possibly infinite-dimensional or discontinuous frameworks. Moreover, we point out that only the upper semicontinuity of l would be sufficient to prove Proposition 7 and only the lower semicontinuity of $l(x, \cdot)$ over $\boldsymbol{Y}$ would suffice to prove Theorem 8. The overall stronger assumption $(\mathcal{A}_5)$ is stated just for the sake of exposition. Finally, in the case where the followers are involved in a non-generalized game at the second stage, note that the *Inclusion* $\subseteq$ of Theorem 8 can be obtained also by applying Proposition 5 and Mallozzi and Morgan (1996, Remark 2.1).

[7] For the broader context of stability of Nash equilibria with respect to perturbations of the characteristics of players, we recall the results in Scalzo (2009, 2012, 2019) on the upper semicontinuity of the Nash equilibrium correspondence in discontinuous settings.

Remark 8 Theorem 8 extends the characterization of SPE-outcomes provided for single-leader single-follower games in Caruso et al. (2025, Theorem 4.3), where the set of SPE-outcomes is shown to coincide with the set of *lower Stackelberg equilibrium pairs* of the associated bilevel optimization problems (see Loridan and Morgan (1986, Remark 5.3) for the original definition of the lower Stackelberg equilibrium concept and also Loridan and Morgan (1992) as well as Dempe (2002, Section 7.2) for results connected to the asymptotic behavior of pessimistic and optimistic bilevel optimization problems under perturbation, respectively).

The presented characterization of SPE-outcomes describes also an additional connection between SPEs and pessimistic and optimistic equilibria. In fact, Theorem 8 tells us that in a single-leader multi-follower game the leader's payoff at an SPE is always in between her payoffs at a pessimistic equilibrium and at an optimistic equilibrium, provided that the last equilibria exist. Moreover, on the contrary, whenever an outcome of a single-leader multi-follower game requires the rationality of the followers' play (meaning that they choose a Nash equilibrium after observing the action of the leader) and the leader's payoff is in between the worst and the best she can obtain when the followers play rationally, that outcome is necessarily generated by an SPE.

Remark 9 Let us see how the characterization provided in Theorem 8 is revealed in Example 1. We recall that $\mathcal{E}_P(\Gamma) = \{(Q, B, R)\}, \mathcal{E}_O(\Gamma) = \{(Q, T, L)\}$ and $\mathcal{E}_{SP}(\Gamma) = \{(Q, B, R), (Q, T, L), (P, B, R)\}$, so we have $v^- = l(Q, B, R) = 1$ and $v^+ = l(Q, T, L) = 3$. Since $l(P, B, R) = 2$, then actually $l(x, y_1, y_2) \in [v^-, v^+]$ for each $(x, y_1, y_2) \in \mathcal{E}_{SP}(\Gamma)$.

Finally, we point out that Theorem 8 immediately implies also that the set of SPE-outcomes is closed, as stated in the following result.

Proposition 9 *Assume that $(\mathcal{A}_1)$–$(\mathcal{A}_6)$ hold. Then $\mathcal{E}_{SP}(\Gamma)$ is a closed subset of $X \times \boldsymbol{Y}$.*

Proof In light of Theorem 8, we know that $\mathcal{E}_{SP}(\Gamma) = Gr(\mathcal{N}) \cap l^{-1}([v^-, v^+])$. Since $\mathcal{N}$ is a closed set-valued map (by *Claim 2* in the proof of Lemma 6) and l is continuous (by assumption $(\mathcal{A}_5)$), then both $Gr(\mathcal{N})$ and $l^{-1}([v^-, v^+])$ are closed sets. Therefore, $\mathcal{E}_{SP}(\Gamma) = Gr(\mathcal{N}) \cap l^{-1}([v^-, v^+])$ is also a closed set. □

4 Extension to multi-leader-follower games

The aim of this section is to discuss whether and how the investigation on SPEs presented in Sections 2 and 3 can be extended to the situation of more than one leader.

Unlike single-leader multi-follower games, in the first stage of a multi-leader-follower game Λ there are $N \geq 2$ players (the *leaders*) who make a decision simultaneously; then the followers make their choices after the decision of the leaders. More precisely, each leader $i \in I = \{1, \dots, N\}$ chooses an action x_i in her own set of available actions $\mathcal{X}_i(\boldsymbol{x}_{-i}) \subseteq X_i$, which depends on the action profile $\boldsymbol{x}_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ of the leaders other than i. Putting $\boldsymbol{X}_{-i} := X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_N$ and $\boldsymbol{X} := X_1 \times \dots \times X_N$, the sets of available actions for the leader i and the follower j are represented, respectively, by the set-valued maps

$\mathcal{X}_i\colon \boldsymbol{X}_{-i} \rightrightarrows X_i$ and $\mathcal{Y}_j\colon \boldsymbol{X} \times \boldsymbol{Y}_{-j} \rightrightarrows Y_j$, and the corresponding payoff functions are $l_i\colon \boldsymbol{X} \times \boldsymbol{Y} \to \mathbb{R}$ and $f_j\colon \boldsymbol{X} \times \boldsymbol{Y} \to \mathbb{R}$, respectively.

Given the notation above, we denote the multi-leader-follower game by $\Lambda = \langle I, J, (\mathcal{X}_i)_{i\in I}, (\mathcal{Y}_j)_{j\in J}, (l_i)_{i\in I}, (f_j)_{j\in J}\rangle$ and the subgame in which the followers are involved after the leaders choose the action profile $\boldsymbol{x} = (x_1, \dots, x_N)$, by $\Lambda(\boldsymbol{x}) = \langle J, (\mathcal{Y}_j(\boldsymbol{x}, \cdot))_{j\in J}, (f_j(\boldsymbol{x}, \cdot))_{j\in J}\rangle$.

Before providing the definition of SPE in this framework, let us define the set of action profiles available to the leaders as

$$\boldsymbol{\mathcal{X}} := \{\boldsymbol{x} = (x_1, \dots, x_N) \in \boldsymbol{X} \mid x_i \in \mathcal{X}_i(\boldsymbol{x}_{-i}) \text{ for each } i \in I\}. \tag{3}$$

Hence, the best reply (or Nash equilibrium) correspondence of the followers $\mathcal{N}$ is defined on $\boldsymbol{\mathcal{X}}$ by

$$\mathcal{N}(\boldsymbol{x}) = \left\{(\bar{y}_1, \dots, \bar{y}_M) \in \boldsymbol{Y} \mid \bar{y}_j \in \operatorname*{arg\,max}_{z_j \in \mathcal{Y}_j(\boldsymbol{x}, \bar{\boldsymbol{y}}_{-j})} f_j(\boldsymbol{x}, z_j, \bar{\boldsymbol{y}}_{-j}) \text{ for each } j \in J\right\} \subseteq \boldsymbol{Y};$$

while, the set of follower j's strategies and the set of followers' strategy profiles are defined, respectively, by

$$Y_j^{\boldsymbol{\mathcal{X}}} := \{\psi_j\colon \boldsymbol{\mathcal{X}} \to Y_j\} \quad \text{and} \quad \boldsymbol{Y}^{\boldsymbol{\mathcal{X}}} := \prod_{j\in J} Y_j^{\boldsymbol{\mathcal{X}}} = \{\boldsymbol{\psi} = (\psi_1, \dots, \psi_M)\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{Y}\}.$$

The definition of SPE in a multi-leader-follower game is presented below.

Definition 6 A strategy profile $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{Y}^X$ is a subgame perfect equilibrium of Λ if the following conditions are satisfied:

(SPE1) $\bar{\boldsymbol{\psi}}(\boldsymbol{x}) \in \mathcal{N}(\boldsymbol{x})$ for each $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$;

(SPE2) $\bar{x}_i \in \arg\max_{x_i \in \mathcal{X}_i(\bar{\boldsymbol{x}}_{-i})} l(x_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(x_i, \bar{\boldsymbol{x}}_{-i}))$ for each $i \in I$.

An action profile $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{Y}$ is a subgame perfect equilibrium outcome of Λ if there exists a followers' strategy profile $\bar{\boldsymbol{\psi}} \in \boldsymbol{Y}^{\boldsymbol{\mathcal{X}}}$ such that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}})$ is a subgame perfect equilibrium of Λ and $\bar{\boldsymbol{\psi}}(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{y}}$.

In particular, Definition 6 posits that, according to an SPE $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}})$, the action profile $\bar{\boldsymbol{\psi}}(\bar{x}) = (\bar{\psi}_1(\bar{\boldsymbol{x}}), \dots, \bar{\psi}_M(\bar{\boldsymbol{x}})) \in \boldsymbol{Y}$ played by the followers is a generalized Nash equilibrium of the game $\Lambda(\bar{\boldsymbol{x}}) = \langle J, (\mathcal{Y}_j(\bar{\boldsymbol{x}}, \cdot))_{j\in J}, (f_j(\bar{\boldsymbol{x}}, \cdot))_{j\in J}\rangle$ and the action profile $\bar{\boldsymbol{x}} = (\bar{x}_1, \dots, \bar{x}_N) \in \boldsymbol{\mathcal{X}}$ played by the leaders is a generalized Nash equilibrium of the game $\Upsilon(\bar{\boldsymbol{\psi}}) = \langle I, (\mathcal{X}_i)_{i\in I}, (l_i(\cdot, \bar{\boldsymbol{\psi}}(\cdot)))_{i\in I}\rangle$.

The existence of SPEs for multi-leader-follower games is generally more difficult to ensure than for single-leader multi-follower games. Let us show below an example of a game having two leaders and one follower (a so-called multi-leader common-follower game, which is a special type of multi-leader-follower game), compact action sets, linear payoff functions, and single-valued best reply correspondence, that has no SPEs.[8]

[8] Such an example is a slight modification of Pang and Fukushima (2005, Example 4) used to describe a game that has no *L/F Nash equilibrium*, a solution concept for multi-leader-follower games different from the SPE.

Example 3 (Example 4 in Pang and Fukushima (2005)) Consider the two-leader single-follower game $\Lambda = \langle \{1,2\}, \{1\}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, l_1, l_2, f \rangle$, where the sets of available actions for each player are $\mathcal{X}_1(x_2) = \mathcal{X}_2(x_1) = \mathcal{Y}(x_1, x_2) = [0,1]$ for each $x_1, x_2 \in [0,1]$ and the payoff functions are defined over $[0,1]^3$ by

$$\begin{aligned} &l_1(x_1, x_2, y) = -\frac{1}{2}x_1 - y, \quad l_2(x_1, x_2, y) = \frac{1}{2}x_2 + y, \\ &f(x_1, x_2, y) = -\frac{1}{2}y^2 - y(x_1 + x_2 - 1). \end{aligned}$$

The follower's best reply correspondence $\mathcal{N} \colon [0,1]^2 \rightrightarrows [0,1]$ is defined as follows:

$$\mathcal{N}(x_1, x_2) = \arg\max_{y \in [0,1]} f(x_1, x_2, y) = \begin{cases} \{0\}, & \text{if} \quad x_1 + x_2 > 1 \\ \{1 - x_1 - x_2\} & \text{if} \quad x_1 + x_2 \leq 1. \end{cases}$$

So $\mathcal{N}$ is single-valued and the follower has a unique optimal strategy, namely $\psi \colon [0,1]^2 \to [0,1]$ defined by $\psi(x_1, x_2) = \max\{0, 1 - x_1 - x_2\}$.

The leaders, able to anticipate the optimal strategy of the follower, are involved in the game $\Upsilon(\psi) = \langle \{1,2\}, [0,1], [0,1], l_1(\cdot,\cdot,\psi(\cdot,\cdot)), l_2(\cdot,\cdot,\psi(\cdot,\cdot)) \rangle$. Since

$$\begin{aligned} &\arg\max_{x_1 \in [0,1]} l_1(x_1, x_2, \max\{0, 1 - x_1 - x_2\}) = \{1 - x_2\} \\ &\arg\max_{x_2 \in [0,1]} l_2(x_2, x_2, \max\{0, 1 - x_1 - x_2\}) = \begin{cases} \{0\}, & \text{if } x_1 < 1/2 \\ \{0,1\}, & \text{if } x_1 = 1/2 \\ \{1\}, & \text{if } x_1 > 1/2, \end{cases} \end{aligned}$$

the game $\Upsilon(\psi)$ has no Nash equilibria. Hence, the two-leader single-follower game Λ does not have SPEs.

However, starting from the seminal paper Sherali (1984), the existence of SPEs (and in some cases, even uniqueness and computation) has been shown in specific classes of multi-leader-follower games under assumptions that guarantee the single-valuedness of the Nash equilibrium correspondence; see, for example, Su (2007); Demiguel and Xu (2009); Hu and Fukushima (2013); Mallozzi and Messalli (2017); Julien (2017); Herty et al. (2022); Caruso et al. (2024) and also Aussel and Svensson (2020, Section 3.4.1) for further discussion. Instead, when the Nash equilibrium correspondence is not single-valued, Ceparano and Morgan (2017, Section 3) and Mallozzi and Messalli (2017, Section 5) investigated the existence of particular selections of SPEs based on *passive beliefs* of the followers and on an *optimistic view* of the leaders, respectively.

Remark 10 As well as for single-leader multi-follower games, even in multi-leader-follower games the pessimistic and optimistic beliefs of the leaders lead to define suitable notions of pessimistic and optimistic equilibrium. We recall, for example, Başar and Olsder (1998); Lignola and Morgan (2024) for the multi-leader common-follower version of the pessimistic equilibrium and Pang and Fukushima (2005); Kulkarni and Shanbhag (2014, 2015) for multi-leader-follower versions of the optimistic equilibrium. However, these pessimistic and optimistic solution concepts generally have no connections with the SPE concept, differently from what has been shown in Theorems 2 and 3 and Proposition 4 for single-leader multi-follower games (see discussion in Kulkarni and Shanbhag (2015, Section 1), Aussel and Svensson (2018), Aussel and Svensson (2020, Section 3.4) and Caruso et al. (2024, Remark 1)).

We examine the SPEs in the case of a possible multi-valuedness of the Nash equilibrium correspondence for the specific class of multi-leader-follower games defined below.

Definition 7 A multi-leader-follower game $\Lambda = \langle I, J, (\mathcal{X}_i)_{i\in I}, (\mathcal{Y}_j)_{j\in J}, (l_i)_{i\in I}, (f_j)_{j\in J}\rangle$ is said to be a *generalized first-stage weighted potential MLF game* if there exist $p\colon \boldsymbol{X} \times \boldsymbol{Y} \to \mathbb{R}$, $h_i\colon \boldsymbol{X}_{-i} \to \mathbb{R}$ for each $i \in I$ and $w_i \in \mathbb{R}^{++}$ for each $i \in I$ such that

$$l_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) = w_i\, p(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) + h_i(\boldsymbol{x}_{-i}),$$

for each $(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) \in X_i \times \boldsymbol{X}_{-i} \times \boldsymbol{Y}$. The function p is called the potential of Λ.

The payoff function of the leader i displayed in Definition 7 is additively separable with respect to the action profile of the rival leaders and to a strategic contribution common to all leaders weighted by w_i. Hence, in particular, for each fixed $\boldsymbol{y} \in \boldsymbol{Y}$ the N-player game $\langle I, (\mathcal{X}_i)_i, (l_i(\cdot, \boldsymbol{y}))_{i\in I}\rangle$ belongs to the class of generalized *weighted potential game*; a family of simultaneous-move games introduced in Monderer and Shapley (1996) and deeply studied in the literature.[9] The class of generalized first-stage weighted potential MLF games is an immediate extension of the classes of *weighted potential CF games* and of *weighted MF-potential games* introduced in Lignola and Morgan (2024, 2026) to investigate pessimistic solutions and the corresponding approximate and viscosity solutions in non-generalized multi-leader common-follower games and in non-generalized multi-leader multi-follower games. More precisely, when $\boldsymbol{\mathcal{X}} = \boldsymbol{X}$ and $M = 1$ ($\mathcal{Y}_j : X \rightrightarrows Y_j$ for any $j \in J$, respectively), from Definition 7 we recover the characterization of the weighted potential CF games in Lignola and Morgan (2024, Proposition 3.6) (of the weighted MF-potential games in Lignola and Morgan (2026, Proposition 2.1), respectively).

Let us show a simple characterization of the generalized first-stage weighted potential MLF games (which extends Caruso et al. (2018, Proposition 2)) and a preparatory result on the SPEs of such a class of games that will be used to study the existence of SPEs.

Proposition 10 *A game Λ is a generalized first-stage weighted potential MLF game if and only if for there exist $c\colon \boldsymbol{X} \times \boldsymbol{Y} \to \mathbb{R}$, $r_i\colon X_i \to \mathbb{R}$, $s_i\colon \boldsymbol{X}_{-i} \to \mathbb{R}$ and $w_i \in \mathbb{R}^{++}$ for each $i \in I$ such that*

$$l_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) = r_i(x_i) + s_i(\boldsymbol{x}_{-i}) + w_i\, c(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}),$$

for each $(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) \in X_i \times \boldsymbol{X}_{-i} \times \boldsymbol{Y}$.

[9] See Facchini et al. (1997) for a characterization of potential games, Neyman (1997); Caruso et al. (2018) for results on the uniqueness of Nash equilibrium, Facchinei et al. (2011) for generalized potential games, and Pusillo (2008) for further discussion and references.

Proof The sufficiency is straightforward by setting $r_i \equiv 0$, $s_i \equiv h_i$, and $c \equiv p$. Fixed $i \in I$, the converse implication comes by defining p and h_i as follows:

$$p(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) = c(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) + \sum_{k \in I} \frac{r_k(x_i)}{w_k} \quad \text{and} \quad h_i(\boldsymbol{x}_{-i}) = s_i(\boldsymbol{x}_{-i}) - w_i \sum_{k \in I \setminus \{i\}} \frac{r_k(x_i)}{w_k},$$

for each $(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) \in X_i \times \boldsymbol{X}_{-i} \times \boldsymbol{Y}$. $\square$

Lemma 11 *Let Λ be a generalized first-stage weighted potential MLF game with potential p. If $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{Y}^{\boldsymbol{\mathcal{X}}}$ is an SPE of the single-leader multi-follower game $\Gamma_p = \langle \{1\}, J, \boldsymbol{\mathcal{X}}, (\mathcal{Y}_j)_{j \in J}, p, (f_j)_{j \in J} \rangle$, then $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}})$ is an SPE of Λ.*

Proof First, by Definition 7, the leader i's payoff function has the following representation

$$l_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) = w_i\, p(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) + h_i(\boldsymbol{x}_{-i}), \tag{4}$$

where $p \colon \boldsymbol{X} \to \mathbb{R}$ and $h_i \colon \boldsymbol{X}_{-i} \to \mathbb{R}$.

Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{Y}^{\boldsymbol{\mathcal{X}}}$ be an SPE of the single-leader multi-follower game $\Gamma_p = \langle \{1\}, J, \boldsymbol{\mathcal{X}}, (\mathcal{Y}_j)_{j \in J}, p, (f_j)_{j \in J} \rangle$, that is, where the leader's payoff function is p. From (SPE1) in Definition 5, we have $\bar{\boldsymbol{\psi}}(\boldsymbol{x}) \in \mathcal{N}(\boldsymbol{x})$ for each $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, so (SPE1) in Definition 6 holds. Moreover, (SPE2) in Definition 5 implies that

$$p(\bar{x}, \bar{\boldsymbol{\psi}}(\bar{\boldsymbol{x}})) \geq p(\boldsymbol{x}, \bar{\boldsymbol{\psi}}(\boldsymbol{x})) \quad \text{for each } \boldsymbol{x} \in \boldsymbol{\mathcal{X}}.$$

Fix $i \in I$. By the above inequality and the definition of $\boldsymbol{\mathcal{X}}$, it follows in particular that

$$p(\bar{x}_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(\bar{x}_i, \bar{\boldsymbol{x}}_{-i})) \geq p(x_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(x_i, \bar{\boldsymbol{x}}_{-i})) \quad \text{for each } x_i \in \mathcal{X}_i(\bar{\boldsymbol{x}}_{-i}).$$

So, in light of (4) and since $w_i > 0$, we have

$$\begin{aligned} l_i(\bar{x}_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(\bar{x}_i, \bar{\boldsymbol{x}}_{-i})) &= w_i\, p(\bar{x}_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(\bar{x}_i, \bar{\boldsymbol{x}}_{-i})) + h_i(\bar{\boldsymbol{x}}_{-i}) \\ &\geq w_i\, p(x_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(x_i, \bar{\boldsymbol{x}}_{-i})) + h_i(\bar{\boldsymbol{x}}_{-i}) \\ &= l_i(x_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(x_i, \bar{\boldsymbol{x}}_{-i})) \end{aligned}$$

for each $x_i \in \mathcal{X}_i(\bar{\boldsymbol{x}}_{-i})$. Hence, $\bar{x}_i \in \arg\max_{x_i \in \mathcal{X}_i(\bar{\boldsymbol{x}}_{-i})} l(x_i, \bar{\boldsymbol{x}}_{-i}, \bar{\boldsymbol{\psi}}(x_i, \bar{\boldsymbol{x}}_{-i}))$, that is, (SPE2) in Definition 6 holds. Therefore, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\psi}})$ is an SPE of the multi-leader-follower game Λ. $\square$

The next result provides the existence of SPEs and identifies some SPE-outcomes for the class of generalized first-stage weighted potential MLF games.

Theorem 12 *Let Λ be a generalized first-stage weighted potential MLF game with potential p. Assume that:*

$(\mathcal{H}_1)$ X_i is a non-empty compact subset of $\mathbb{R}^{n_i}$ for each $i \in I$;
$(\mathcal{H}_2)$ Y_j is a non-empty compact and convex subset of $\mathbb{R}^{m_j}$ for each $j \in J$;
$(\mathcal{H}_3)$ $\mathcal{X}_i$ is a closed set-valued map for each $i \in I$;

($\mathcal{H}_4$) the set-valued map $\mathcal{Y}_j(\boldsymbol{x}, \cdot): \boldsymbol{Y}_{-j} \rightrightarrows Y_j$ has non-empty convex values for each $\boldsymbol{x} \in \boldsymbol{X}$, for each $j \in J$;
($\mathcal{H}_5$) $\mathcal{Y}_j$ is a closed set-valued map for each $j \in J$;
($\mathcal{H}_6$) $\mathcal{Y}_j$ is a lower semicontinuous set-valued map for each $j \in J$;
($\mathcal{H}_7$) the potential function p is continuous;
($\mathcal{H}_8$) the function f_j is continuous and $f_j(\boldsymbol{x}, \cdot, \boldsymbol{y}_{-j})$ is quasi-concave over Y_j for each $(\boldsymbol{x}, \boldsymbol{y}_{-j}) \in \boldsymbol{X} \times \boldsymbol{Y}_{-j}$, for each $j \in J$.

Then, there exists an SPE of Λ. Moreover,

$$\left\{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{Y} \mid \bar{\boldsymbol{y}} \in \mathcal{N}(\bar{\boldsymbol{x}}) \text{ and } p(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [\tau^-, \tau^+]\right\} \subseteq \mathcal{E}_{SP}(\Lambda),$$

where $\tau^- = \sup_{\boldsymbol{x} \in \boldsymbol{\mathcal{X}}} \inf_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} p(\boldsymbol{x}, \boldsymbol{y})$ and $\tau^+ = \sup_{\boldsymbol{x} \in \boldsymbol{\mathcal{X}}} \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} p(\boldsymbol{x}, \boldsymbol{y})$.

Proof Assumption ($\mathcal{H}_3$) implies that the set $\boldsymbol{\mathcal{X}}$ defined in (3) is closed and, being a subset of $\boldsymbol{X}$, by ($\mathcal{H}_1$) we have that $\boldsymbol{\mathcal{X}}$ is compact. Moreover, in light of ($\mathcal{H}_2$) and ($\mathcal{H}_4$)–($\mathcal{H}_8$), the single-leader multi-follower game $\Gamma_p = \langle \{1\}, J, \boldsymbol{\mathcal{X}}, (\mathcal{Y}_j)_{j \in J}, p, (f_j)_{j \in J} \rangle$ satisfies the assumptions of Proposition 7. Hence, there exists an SPE of Γ_p, which is also an SPE of Λ by Lemma 11. So, there exists an SPE of Λ. To show the last part of the theorem, it suffices to note that, in light of Theorem 8, we have

$$\mathcal{E}_{SP}(\Gamma_p) = \left\{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \boldsymbol{\mathcal{X}} \times \boldsymbol{Y} \mid \bar{\boldsymbol{y}} \in \mathcal{N}(\bar{\boldsymbol{x}}) \text{ and } p(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in [\tau^-, \tau^+]\right\},$$

where $\tau^- = \sup_{\boldsymbol{x} \in \boldsymbol{\mathcal{X}}} \inf_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} p(\boldsymbol{x}, \boldsymbol{y})$ and $\tau^+ = \sup_{\boldsymbol{x} \in \boldsymbol{\mathcal{X}}} \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} p(\boldsymbol{x}, \boldsymbol{y})$, and that $\mathcal{E}_{SP}(\Gamma_p) \subseteq \mathcal{E}_{SP}(\Lambda)$ by Lemma 11. □

Remark 11 In addition to Lignola and Morgan (2024), multi-leader-follower games involving potential game structures have also been analyzed in Kulkarni and Shanbhag (2014, 2015); Steffensen (2021); Caruso et al. (2024). In particular, the class of games presented in Definition 7 contains the class of *quasi-potential multi-leader multi-follower games* introduced in Kulkarni and Shanbhag (2015, Definition 3.2) and, when $w_i = 1$ for each $i \in I$, it is included in the class of *multi-leader multi-follower potential games* proposed in Kulkarni and Shanbhag (2014, Definition 2.2); in both cases, equilibrium concepts connected to the optimistic approach are considered.

5 Conclusions

This paper has investigated the concept of an SPE in single-leader multi-follower games where the followers play a generalized game at the second stage and where the Nash equilibrium correspondence is not necessarily single-valued. The investigation has shown a close connection to the notions of pessimistic and optimistic equilibrium, which have been widely employed in both theoretical and applied analyzes involving the mentioned class of games. In particular, it has been shown that some SPEs can be identified by the definitions of pessimistic and optimistic equilibrium (Theorems 2 and 3) and, as a consequence, that the set of SPE-outcomes contains those of both the pessimistic and optimistic equilibria (Proposition 4). Moreover, after providing a result on the existence of SPEs (Proposition 7), the set of SPE-outcomes has been proved to coincide with the set of action profiles in which the followers react rationally to the leader and the leader's payoff is in between what she would achieve in a pessimistic and in an optimistic equilibrium (Theorem 8). This characterization has been used to establish the closedness of the set of SPE-outcomes (Proposition 9). Finally, the investigation has been extended to the framework of multi-leader-follower games and the related issues of existence of SPEs, of the identification of SPE-outcomes and connections to other solution concepts of the literature (Theorem 12). In fact, most of the findings of the paper (particularly those concerning the characterization of SPE-outcomes and the connections of SPEs with pessimistic and optimistic equilibria) are independent of the nature of the players' action sets and continue to hold true when, for instance, the set of players' feasible action sets is finite, as in Example 1.

In conclusion, it is worth mentioning two research directions that could be pursued in the future. First, one might extend the characterization of the set of SPE-outcomes to multi-stage games in the framework of continuous games of perfect (and hopefully imperfect) information, where the existence and characterization of SPEs was a relevant topic of investigation (see, for example, Harris (1985); Hellwig et al. (1990)). Second, one might investigate the issue of subgame perfection when the followers adhere to equilibrium concepts that take into account the other-regarding behavior of players. This too has been a subject of extensive analysis in non-cooperative games by Jacqueline Morgan and some of her former students: see, for instance, De Marco and Morgan (2008a, b); Mallozzi and Tijs (2009); De Marco and Morgan (2011); De Marco et al. (2022); Scalzo (2023); De Marco et al. (2024); Scalzo (2026).

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest. The first two authors of the present article are members of the National Group for Mathematical Analysis, Probability and their Applications (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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