



Free boundary cluster with Robin condition on the transmission interface

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Abstract

We formulate and study a variational two-phase free boundary problem with Robin condition on the interface between the two phases, and we prove existence and regularity of solutions in dimension two.

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1 Introduction

Free boundary problems with two and more phases are often used to describe models in different areas of Physics, Engineering and Life Sciences, for instance in Fluid Dynamics (Bernoulli free boundary problems), Dynamics of Populations (optimal partitions problems), Mechanics and Phase Transition (obstacle problems). The different phases are called *segregated* if they occupy different space regions; segregation occurs for instance in the two-phase Bernoulli problem, the two-phase obstacle problem and optimal partitions problems.

In all these cases the interaction between the different phases is supposed to be *competitive*, in particular, the interfaces are not formed because it is convenient energetically, but due to the lack of space. For instance, if we have two disjoint one-phase solutions of the variational Bernoulli (or obstacle) problem, then the couple they form is a minimizer to the corresponding two-phase problem, and even if the two phases are very close to each other, an interface is not formed (we briefly discuss this phenomenon in Sect. 1.1).

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In this paper, we consider a two-phase problem, in which the phases are still *segregated*, but the interaction along the free interface is *collaborative*. In this case, if two or more disjoint one-phase solutions are sufficiently close, then it is energetically convenient for them to create a free interface, that is, the formation of clusters is incentivized.

We introduce the functional in Sect. 1.2, while in Sect. 1.3 we state the variational problem and the main results of the paper.

1.1 The classical one-phase and two-phase Bernoulli free boundary problems

Let D be a smooth bounded open set in \mathbb{R}^d . Let $g : \partial D \rightarrow \mathbb{R}$ be a given nonnegative function and $\lambda > 0$ a given constant. The classical one-phase Bernoulli problem can be stated as follows. Find a domain $\Omega \subset D$ and a function $u : D \rightarrow \mathbb{R}$ such that $u = g$ on ∂D and

$$\Delta u = 0 \text{ in } \Omega, \quad u = 0 \text{ and } |\nabla u| = \lambda \text{ on } \partial\Omega \cap D.$$

In the seminal paper [1] Alt and Caffarelli showed that the existence of such a couple (u, Ω) can be obtained by minimizing the functional

$$J_\lambda(u) = \int_D |\nabla u|^2 + \lambda^2 |\{u > 0\} \cap D|,$$

among all functions in $H^1(D)$ such that $u = g$ on ∂D , and then taking $\Omega := \{u > 0\}$.

In the two-phase problem, we are given two constants $\lambda_1 > 0$ and $\lambda_2 > 0$ and two nonnegative functions $g_1 : \partial D \rightarrow \mathbb{R}$ and $g_2 : \partial D \rightarrow \mathbb{R}$ with disjoint supports. Then, the two-phase free boundary problem is the following. Find two *disjoint* sets Ω_1 and Ω_2 in D and two functions $u_1 : D \rightarrow \mathbb{R}$ and $u_2 : D \rightarrow \mathbb{R}$ such that $u_1 = g_1$ and $u_2 = g_2$ on ∂D and

$$\begin{cases} \Delta u_1 = 0 \text{ in } \Omega_1 & \text{and} & \Delta u_2 = 0 \text{ in } \Omega_2; \\ u_1 = 0 \text{ and } |\nabla u_1| = \lambda_1 & \text{on } D \cap \partial\Omega_1 \setminus \partial\Omega_2; \\ u_2 = 0 \text{ and } |\nabla u_2| = \lambda_2 & \text{on } D \cap \partial\Omega_2 \setminus \partial\Omega_1; \\ u_1 = u_2 = 0 \text{ and } |\nabla u_1|^2 - |\nabla u_2|^2 = \lambda_1^2 - \lambda_2^2 & \text{on } D \cap \partial\Omega_1 \cap \partial\Omega_2. \end{cases} \tag{1.1}$$

The existence of a solution can be obtained by minimizing the functional

$$J_{\lambda_1, \lambda_2}(u) = \int_D |\nabla u|^2 + \lambda_1^2 |\{u > 0\} \cap D| + \lambda_2^2 |\{u < 0\} \cap D|,$$

among all functions in $H^1(D)$ such that $u = g_1 - g_2$ on ∂D , and then taking $\Omega_1 = \max\{u, 0\}$, $\Omega_2 = \max\{-u, 0\}$, $\Omega_1 = \{u_1 > 0\}$ and $\Omega_2 = \{u_2 > 0\}$ (see [2], [14] and [6]).

In the two-phase problem, the two-phase interface $\partial\Omega_1 \cap \partial\Omega_2$ is formed when the two sets Ω_1 and Ω_2 act as geometric obstacles to each other; if Ω_1 and Ω_2 are disjoint one-phase solutions, then the two-phase interface is simply not formed. In other words, if u_1 and u_2 are minimizers of the one-phase functionals J_{λ_1} and J_{λ_2} such that $u_1 u_2 \equiv 0$, then it is immediate to check that $u = u_1 - u_2$ is a minimizer of the two-phase functional J_{λ_1, λ_2} . In fact, if $v \in H^1(D)$ is such that $v = u$ on ∂D , then $v_+ = u_1$ and $v_- = u_2$ on ∂D and so, by the optimality of u_1 and u_2 , we get

$$J_{\lambda_1, \lambda_2}(v) = J_{\lambda_1}(v_+) + J_{\lambda_2}(v_-) \geq J_{\lambda_1}(u_1) + J_{\lambda_2}(u_2) = J_{\lambda_1, \lambda_2}(u).$$

1.2 A two-phase problem with Robin condition on the free interface

In this paper we study a different type of two-phase problem in which the two state functions u_1 and u_2 might not vanish on the interface $\partial\Omega_1 \cap \partial\Omega_2$. Precisely, given $\beta > 0$, $\Lambda > 0$ and a fixed set D , we consider the functional

$$J_{\beta,\Lambda}(u, \Omega_1, \Omega_2) = \int_D |\nabla u|^2 dx + \beta \int_{\partial\Omega_1 \cap \partial\Omega_2} u^2 d\mathcal{H}^{d-1} + \Lambda |\{u > 0\} \cap D|,$$

defined for couples of disjoint domains Ω_1, Ω_2 in D and functions $u \in H^1(D)$ with $u = 0$ on $D \setminus (\Omega_1 \cup \Omega_2)$. We will then show that if (Ω_1, Ω_2, u) locally minimize $J_{\beta,\Lambda}$ in D , then the functions

$$u_1 = u \mathbb{1}_{\Omega_1} \quad \text{and} \quad u_2 = u \mathbb{1}_{\Omega_2}$$

are solutions to the problem

$$\begin{cases} \Delta u_1 = 0 \text{ in } \Omega_1 & \text{and} & \Delta u_2 = 0 \text{ in } \Omega_2 \\ u_1 = 0 \text{ and } |\nabla u_1| = \sqrt{\Lambda} & \text{on } D \cap \partial\Omega_1 \setminus \partial\Omega_2 \\ u_2 = 0 \text{ and } |\nabla u_2| = \sqrt{\Lambda} & \text{on } D \cap \partial\Omega_2 \setminus \partial\Omega_1 \\ u_1 = u_2 \text{ and } |\nu \cdot \nabla u_1| + |\nu \cdot \nabla u_2| = \beta(u_1 + u_2) & \text{on } D \cap \partial\Omega_1 \cap \partial\Omega_2, \end{cases} \tag{1.2}$$

ν being the normal to $\partial\Omega_1 \cap \partial\Omega_2$. Moreover, the interface $\partial\Omega_1 \cap \partial\Omega_2$ is a critical point of the functional $J_{\beta,\Lambda}$ in the sense that

$$\frac{d}{dt} \Big|_{t=0} J_{\beta,\Lambda}(u, \Phi_t(\Omega_1), \Phi_t(\Omega_2)) = 0,$$

for every diffeomorphism Φ_t of the form $\Phi_t(x) = x + t\xi(x)$ generated by a smooth vector field ξ compactly supported in D . This condition can be explicitly written in terms of u_1, u_2 and the mean curvature of the interface (see [9]).

Notice that if (u_1, Ω_1) and (u_2, Ω_2) are two minimizers of the one-phase Bernoulli functional $J_{\sqrt{\Lambda}}$ with disjoint supports ($\Omega_1 \cap \Omega_2 = \emptyset$), the triple $(\Omega_1, \Omega_2, u = u_1 + u_2)$ might not be a minimizer of $J_{\beta,\Lambda}$, even if the Hausdorff distance between Ω_1 and Ω_2 is strictly positive. In fact, it might be convenient to enlarge the domains Ω_1 and Ω_2 in order to obtain a non-empty interface $\partial\Omega_1 \cap \partial\Omega_2$ that will allow competitors which are not vanishing identically on the entire free boundaries $\partial\Omega_1$ and $\partial\Omega_2$. This is illustrated by the following one-dimensional example.

Example 1.1 (Formation of an interface in 1D) Let $\varepsilon > 0$ and $\beta > 0$ be fixed. We consider the interval $D = [-1 - \varepsilon, 1 + \varepsilon]$ and the boundary data $g_1, g_2 : \partial D \rightarrow \mathbb{R}$ given by

$$g_1(-1 - \varepsilon) = 1, \quad g_1(1 + \varepsilon) = 0, \quad g_2(-1 - \varepsilon) = 0, \quad g_2(1 + \varepsilon) = 1.$$

The minimizers of the one-phase function

$$J_1 = \int_D |u'(x)|^2 dx + |\{u > 0\} \cap D|$$

with boundary conditions g_1 and g_2 are respectively the functions

$$u_1(x) = (-x - \varepsilon)_+ \quad \text{and} \quad u_2(x) = (x - \varepsilon)_+.$$

If we consider the sets $\Omega_1 = (-1 - \varepsilon, -\varepsilon)$ and $\Omega_2 = (\varepsilon, 1 + \varepsilon)$, then we have that

$$J_{\beta,1}(u_1 + u_2, \Omega_1, \Omega_2) = J_1(u_1) + J_1(u_2) = 4.$$

On the other hand, by taking

$$\tilde{\Omega}_1 = [-1 - \varepsilon, 0] \quad \tilde{\Omega}_2 = [0, 1 + \varepsilon] \quad u(x) = \begin{cases} 1 & \text{if } x = -1 - \varepsilon, \\ \ell & \text{if } x = 0, \\ 1 & \text{if } x = 1 + \varepsilon, \end{cases}$$

and extending u linearly on the intervals $[-1 - \varepsilon, 0]$ and $[0, 1 + \varepsilon]$ we obtain that

$$J_{\beta,1}(u, \tilde{\Omega}_1, \tilde{\Omega}_2) = 2 \frac{(1 - \ell)^2}{1 + \varepsilon} + \beta \ell^2 + 2 + 2\varepsilon.$$

Setting the parameter ℓ to be the optimal one, $\ell = \frac{2}{2 + \beta + \varepsilon\beta}$, we get that

$$J_{\beta,1}(u, \tilde{\Omega}_1, \tilde{\Omega}_2) = 2 \frac{(1 + \varepsilon)\beta^2}{(2 + \beta + \varepsilon\beta)^2} + \beta \left(\frac{2}{2 + \beta + \varepsilon\beta} \right)^2 + 2 + 2\varepsilon.$$

When $\varepsilon = 0$, we get

$$J_{\beta,1}(u, \tilde{\Omega}_1, \tilde{\Omega}_2) = 2 \frac{\beta^2}{(2 + \beta)^2} + \beta \left(\frac{2}{2 + \beta} \right)^2 + 2 = 2 \frac{\beta^2 + 2\beta}{\beta^2 + 4\beta + 4} + 2 < 4.$$

In conclusion, if we fix $\beta > 0$ we can find $\varepsilon_0 > 0$ such that

$$J_{\beta,1}(u, \tilde{\Omega}_1, \tilde{\Omega}_2) < 4 \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

which means that for those choices of β and ε , the combination of the two one-phase solutions is not optimal.

1.3 Setting of the problem and main theorem

We will define the variational problem for the functional $J_{\beta,\Lambda}$ in the class of sets of finite perimeter and Sobolev functions. Then, we will prove an existence theorem in this class and we will show that the minimizers are regular. We fix the boundary data for Ω_1, Ω_2 and g . Precisely, let

- E_1 and E_2 be two smooth, bounded and disjoint sets of positive distance in \mathbb{R}^d ;
- $D := \mathbb{R}^d \setminus (\overline{E_1} \cup \overline{E_2})$;
- $\Omega_i = E_i$ in $\mathbb{R}^d \setminus D$;
- $g \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a non-negative function such that

$$g \equiv 1 \quad \text{on } E_1 \cup E_2.$$

We define the following admissible set of functions

$$\mathcal{V} = \left\{ u \in H^1(\mathbb{R}^d) : u \geq 0 \text{ in } \mathbb{R}^d \text{ and } u - g \in H_0^1(D) \right\}.$$

Then, fixed $u \in \mathcal{V}$, we define the admissible set $\mathcal{A}(u)$ as the set of all couples (Ω_1, Ω_2) of Lebesgue measurable sets such that:

- $\Omega_1 \cap \Omega_2 = \emptyset, E_1 \subset \Omega_1$ and $E_2 \subset \Omega_2$ Lebesgue almost-everywhere;
- Ω_1 and Ω_2 have finite perimeter (as subsets of \mathbb{R}^d);
- $\{u > 0\} \subset \Omega_1 \cup \Omega_2$ Lebesgue almost-everywhere.

For every $\beta > 0$ and $\Lambda > 0$, we consider the functional $J_{\beta,\Lambda}$, defined for functions $u \in \mathcal{V}$ and couples of sets $(\Omega_1, \Omega_2) \in \mathcal{A}(u)$, as

$$J_{\beta,\Lambda}(u, \Omega_1, \Omega_2) := \int_D |\nabla u|^2 dx + \beta \int_{\partial^* \Omega_1 \cap \partial^* \Omega_2} u^2 d\mathcal{H}^{d-1} + \Lambda |\{u > 0\} \cap D|,$$

where $\partial^* \Omega_j$ is the reduced boundary of Ω_j ; we recall that since u is a bounded Sobolev function, the second integral is well-defined (see Sect. 2).

In this paper we consider the variational problem

$$\min \left\{ J_{\beta,\Lambda}(u, \Omega_1, \Omega_2) : u \in \mathcal{V}, (\Omega_1, \Omega_2) \in \mathcal{A}(u) \right\}. \tag{1.3}$$

The main result is the following.

Theorem 1.2 *Let D be a smooth bounded open set in \mathbb{R}^2 . Given sets E_1 and E_2 , and a function g as above, there are a function $u \in \mathcal{V}$ and sets $(\Omega_1, \Omega_2) \in \mathcal{A}(u)$ that solve the variational problem (1.3). Conversely, if (u, Ω_1, Ω_2) is a solution to (1.3), then also $(u, \tilde{\Omega}_1, \tilde{\Omega}_2)$ is a solution to (1.3), where*

$$\tilde{\Omega}_1 = \{u > 0\} \cap \Omega_1 \quad \text{and} \quad \tilde{\Omega}_2 = \{u > 0\} \cap \Omega_2.$$

Moreover,

- (i) *The boundary $\partial\{u > 0\}$ is $C^{1,\alpha}$ -regular in D ;*
- (ii) *The interface $\partial\Omega_1 \cap \partial\Omega_2$ is C^∞ in the open set $D \cap \{u > 0\}$ and is C^1 regular up to the boundary $D \cap \partial\{u > 0\}$. Moreover, $\partial\Omega_1 \cap \partial\Omega_2$ reaches $\partial\{u > 0\}$ orthogonally.*

Sketch of the proof and plan of the paper

In order to prove Theorem 1.2, we first introduce a family of approximating problems in Sect. 4. Then, passing to the limit, we obtain a function $u \in \mathcal{V}$ and a couple of disjoint sets Ω_1 and Ω_2 . We cannot obtain immediately that (u, Ω_1, Ω_2) is a solution to (1.3), since there is not a uniform bound on the perimeter of the approximating sets, so we do not a priori have that Ω_1 and Ω_2 are sets of locally finite perimeter in D . Instead, we are able to prove that u satisfies an almost-minimality condition involving the one-phase Alt-Caffarelli functional, which allows to prove that the set $\{u > 0\}$ is regular (Theorem 9.1). This solves the problem only in part because we only have that

$$\{u > 0\} = \Omega_1 \cup \Omega_2 \quad \text{in } D.$$

We then show that the sets Ω_1 and Ω_2 are almost-minimizers of the perimeter in $\{u > 0\} \cap D$, which implies that (in low dimension) the free interface $\partial\Omega_1 \cup \partial\Omega_2$ is smooth in $D \cap \{u > 0\}$. Thus, in order to prove that Ω_1 and Ω_2 have finite perimeter it is sufficient to study the behavior of the interface $\partial\Omega_1 \cup \partial\Omega_2$ close to the free boundary $\partial\{u > 0\}$ (see Theorem 10.1). We show that Ω_1 and Ω_2 are minimizers in $\{u > 0\}$ of a weighted perimeter functional, the weight being precisely the function u^2 , which is $C^{0,\alpha}$ and positive in $\{u > 0\}$, but as it approaches the free boundary $\partial\{u > 0\}$ we have that

$$u^2(x) \sim \text{dist}^2(x, \partial\{u > 0\}).$$

In order to deal with this degenerate weight, we perform a 2D conformal change of coordinates, which flattens $\partial\{u > 0\}$ to a line; then we rotate Ω_1 around this line in order to obtain an almost-minimizer of the perimeter in \mathbb{R}^4 . This allows to conclude that $\partial\Omega_1 \cup \partial\Omega_2$ is the

union of C^1 curves that meet $\partial\{u > 0\}$ orthogonally in a (locally) finite number of points. This concludes the proof of Theorem 10.1 and also shows that Ω_1 and Ω_2 have locally finite perimeter. In order to show that Ω_1 and Ω_2 have finite perimeter, in Proposition 8.1 we prove that $\{u > 0\}$ contains strictly both \bar{E}_1 and \bar{E}_2 . Then, in Sect. 11 we show that Ω_1 and Ω_2 are actually minimizers of (1.3) and we complete the proof of Theorem 1.2.

2 Sets of finite perimeter and Sobolev functions

2.1 Caccioppoli sets

For any Lebesgue measurable set $\Omega \subset \mathbb{R}^d$, we define

$$\text{Per}(\Omega) := \sup \left\{ \int_{\Omega} \text{div } \xi(x) \, dx \mid \xi \in C_c^1(\mathbb{R}^d), \|\xi\|_{L^\infty(\mathbb{R}^d)} \leq 1 \right\},$$

and we say that Ω is of finite perimeter (Caccioppoli set) if

$$\text{Per}(\Omega) < +\infty.$$

Given $\alpha \in [0, 1]$, we say that the set Ω has a Lebesgue density α at $x \in \mathbb{R}^d$ if

$$\lim_{r \rightarrow 0} \frac{|\Omega \cap B_r(x)|}{|B_r|} = \alpha.$$

We define the set $\Omega^{(\alpha)}$ as

$$\Omega^{(\alpha)} := \left\{ x \in \mathbb{R}^d \mid \lim_{r \rightarrow 0} \frac{|\Omega \cap B_r(x)|}{|B_r|} = \alpha \right\}.$$

Given a set of finite perimeter $\Omega \subset \mathbb{R}^d$ we will denote by $\partial^*\Omega$ its reduced boundary and by ν_Ω the generalized exterior normal. We recall that

$$\text{Per}(\Omega) = \mathcal{H}^{d-1}(\partial^*\Omega),$$

and that for any $\xi \in C_c^1(\mathbb{R}^d)$

$$\int_{\Omega} \text{div } \xi(x) \, dx = \int_{\partial^*\Omega} \xi \cdot \nu_\Omega \, d\mathcal{H}^{d-1},$$

where \mathcal{H}^{d-1} denotes the $(d - 1)$ -dimensional Hausdorff measure in \mathbb{R}^d . Moreover, we recall that at every point of the reduced boundary, $\partial^*\Omega$ has Lebesgue density $1/2$, that is,

$$\partial^*\Omega \subset \Omega^{(1/2)}.$$

We also recall the following well-known result by Federer

$$\mathcal{H}^{d-1}(\mathbb{R}^d \setminus \{\Omega^{(0)} \cup \Omega^{(1)} \cup \partial^*\Omega\}) = 0,$$

which can also be stated as in the lemma below.

Lemma 2.1 *If Ω is a set of finite perimeter in \mathbb{R}^d , then up to a set of zero \mathcal{H}^{d-1} measure*

$$\Omega^{(1/2)} = \partial^*\Omega \quad \text{and} \quad \Omega^{(0)} \cup \Omega^{(1/2)} \cup \Omega^{(1)} = \mathbb{R}^d.$$

Finally, we conclude this section with the following proposition

Proposition 2.2 *Let A and B be two disjoint sets of finite perimeter in \mathbb{R}^d . Then,*

$$\partial^* A = \left(\partial^* A \cap \partial^* B\right) \cup \left(\partial^* A \setminus \partial^* B\right) \quad \text{and} \quad \partial^* B = \left(\partial^* A \cap \partial^* B\right) \cup \left(\partial^* B \setminus \partial^* A\right),$$

the set $A \cup B$ is a set of finite perimeter and, up to a set of zero \mathcal{H}^{d-1} -measure, and

$$\partial^*(A \cup B) = \left(\partial^* A \setminus \partial^* B\right) \cup \left(\partial^* B \setminus \partial^* A\right). \tag{2.1}$$

In particular,

$$\text{Per}(A) + \text{Per}(B) = \text{Per}(A \cup B) + 2 \mathcal{H}^{d-1}(\partial^* A \cap \partial^* B). \tag{2.2}$$

Proof Up to a set of zero \mathcal{H}^{d-1} measure, we have that

$$\partial^* A \setminus \partial^* B = A^{(1/2)} \cap \left(B^{(0)} \cup B^{(1)}\right) = A^{(1/2)} \cap B^{(0)}.$$

Analogously, $\partial^* B \setminus \partial^* A = B^{(1/2)} \cap A^{(0)}$. On the other hand

$$\partial^*(A \cup B) = (A \cup B)^{(1/2)} = \left(A^{(1/2)} \cap B^{(0)}\right) \cup \left(B^{(1/2)} \cap A^{(0)}\right),$$

which proves (2.1). Finally, (2.2) follows since the sets

$$\partial^* A \setminus \partial^* B \quad \partial^* B \setminus \partial^* A \quad \text{and} \quad \partial^* A \cap \partial^* B$$

are disjoint. □

As a consequence of Lemma 2.1, one can obtain the following decomposition.

Proposition 2.3 *Let A and B be two sets of finite perimeter in \mathbb{R}^d . Then, also $A \setminus B$ and $B \setminus A$ have finite perimeter and we have the following decompositions (up to sets of zero \mathcal{H}^{d-1} measure) of $\partial^* A$, $\partial^* B$, $\partial^*(A \setminus B)$ and $\partial^*(B \setminus A)$ into disjoint sets:*

$$\begin{aligned} \partial^* A &= \left(A^{(1/2)} \cap B^{(0)}\right) \cup \left(A^{(1/2)} \cap B^{(1)}\right) \cup \left(A^{(1/2)} \cap B^{(1/2)}\right) \\ \partial^* B &= \left(B^{(1/2)} \cap A^{(0)}\right) \cup \left(B^{(1/2)} \cap A^{(1)}\right) \cup \left(B^{(1/2)} \cap A^{(1/2)}\right) \\ \partial^*(A \setminus B) &= \left(A^{(1)} \cap B^{(1/2)}\right) \cup \left(A^{(1/2)} \cap B^{(0)}\right) \cup \left(A^{(1/2)} \cap B^{(1/2)} \cap (A \cup B)^{(1)}\right) \\ \partial^*(B \setminus A) &= \left(B^{(1)} \cap A^{(1/2)}\right) \cup \left(B^{(1/2)} \cap A^{(0)}\right) \cup \left(B^{(1/2)} \cap A^{(1/2)} \cap (A \cup B)^{(1)}\right). \end{aligned}$$

2.2 Sobolev functions and capacity

Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. We recall that u is a Sobolev function ($u \in H^1(\mathbb{R}^d)$), if $u \in L^2(\mathbb{R}^d)$ and $\nabla u \in L^2(\mathbb{R}^d; \mathbb{R}^d)$, where ∇u is the distributional gradient of u . Given a measurable set $\Omega \subset \mathbb{R}^d$ and a Sobolev function $u \in H^1(\mathbb{R}^d)$, we say that $u \in \tilde{H}_0^1(\Omega)$ if

$$u = 0 \quad \text{almost everywhere on} \quad \mathbb{R}^d \setminus \Omega.$$

If Ω is an open set, we can also define the space $H_0^1(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the Sobolev norm

$$\|u\|_{H^1} := \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{1/2}.$$

It is well-known that both $H_0^1(\Omega)$ and $\tilde{H}_0^1(\Omega)$ are closed (with respect to both the strong and the weak H^1 -convergence) linear subspaces of $H^1(\mathbb{R}^d)$ and that, for any open set Ω , $H_0^1(\Omega) \subset \tilde{H}_0^1(\Omega)$, while the converse inclusion is in general false.

Given any set $A \subset \mathbb{R}^d$ and any ball $B_{2R}(x_0)$, we define

$$\text{cap}\left(A; B_{2R}(x_0)\right) := \inf \left\{ \int |\nabla \varphi|^2 dx : \varphi \in H_0^1(B_{2R}(x_0)), \right. \\ \left. \varphi \geq 1 \text{ in a neighborhood of } B_R(x_0) \cap A \right\}.$$

We say that a set A has zero capacity if

$$\text{cap}\left(A; B_{2R}(x_0)\right) = 0 \quad \text{for every ball } B_{2R}(x_0) \subset \mathbb{R}^d.$$

We recall the following properties of the capacity.

- If a set $A \subset \mathbb{R}^d$ has zero capacity, then $|A| = 0$ and $\mathcal{H}^{d-1}(A) = 0$.
- Given $u \in H^1(\mathbb{R}^d)$, there exists a set of zero capacity \mathcal{N}_u such that

$$\lim_{r \rightarrow 0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x) dx \text{ exists for every } x_0 \in \mathbb{R}^d \setminus \mathcal{N}_u.$$

In particular, to every $u \in H^1(\mathbb{R}^d)$, we can associate a representative

$$\tilde{u} : \mathbb{R}^d \rightarrow \mathbb{R}$$

defined pointwise everywhere as follows:

$$\tilde{u}(x_0) := \lim_{r \rightarrow 0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x) dx \text{ if } x_0 \in \mathbb{R}^d \setminus \mathcal{N}_u$$

while $\tilde{u}(x_0) = 0$ if $x_0 \in \mathcal{N}_u$.

- Suppose that a sequence $u_n \in H^1(\mathbb{R}^d)$ converges strongly in H^1 to $u \in H^1(\mathbb{R}^d)$. Let \tilde{u}_n and \tilde{u} be the representatives defined above and let \mathcal{N}_{u_n} and \mathcal{N}_u be the corresponding sets of zero capacity. Then, there are a subsequence u_{n_k} and a set of zero capacity \mathcal{N} such that

$$\mathcal{N}_u \subset \mathcal{N} \quad \text{and} \quad \bigcup_{n \geq 1} \mathcal{N}_{u_n} \subset \mathcal{N},$$

and

$$\lim_{k \rightarrow \infty} \tilde{u}_{n_k}(x) = \tilde{u}(x) \text{ for every } x \in \mathbb{R}^d \setminus \mathcal{N}. \tag{2.3}$$

For simplicity, we will identify any function $u \in H^1(\mathbb{R}^d)$ with its representative \tilde{u} and if a sequence $u_{n_k} \in H^1(\mathbb{R}^d)$ satisfies (2.3), then we will say that it converges quasi-everywhere to $u \in H^1(\mathbb{R}^d)$.

2.3 Traces of Sobolev functions on the boundary of sets of finite perimeter

Let Ω be a set of finite perimeter in \mathbb{R}^d and let $u \in H^1(\mathbb{R}^d)$. Let $\tilde{u} : \mathbb{R}^d \rightarrow \mathbb{R}$ be the representative of u defined for every point x_0 outside a set of zero capacity \mathcal{N}_u (and defined as zero on \mathcal{N}_u). Then, \tilde{u} is defined at every point of $\partial^* \Omega \setminus \mathcal{N}_u$. Since \mathcal{N}_u has zero \mathcal{H}^{d-1} -measure, we have that \tilde{u} is defined \mathcal{H}^{d-1} -almost everywhere on $\partial^* \Omega$. We also notice that

$$\tilde{u} : \partial^* \Omega \rightarrow \mathbb{R}$$

is a \mathcal{H}^{d-1} measurable function. Indeed, since $u \in H^1(\mathbb{R}^d)$ is a strong H^1 limit of a sequence of C^∞ functions, we have that $\tilde{u} : \partial^*\Omega \rightarrow \mathbb{R}$ is a pointwise limit of smooth functions. From now on, we will write u instead of \tilde{u} .

The next two propositions allow to write the functional $J_{\beta,\Lambda}$ in an equivalent way.

Proposition 2.4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded quasi-open set of finite perimeter and let $\partial^*\Omega$ be its reduced boundary. Let $u \in \tilde{H}_0^1(\Omega)$ and let $\tilde{u} : \mathbb{R}^d \rightarrow \mathbb{R}$ be a representative of u defined up to a set of zero capacity. Then,*

$$\tilde{u} = 0 \quad \mathcal{H}^{d-1}\text{-almost everywhere on } \Omega^{(1/2)}.$$

In particular,

$$\tilde{u} = 0 \quad \mathcal{H}^{d-1}\text{-almost everywhere on } \partial^*\Omega.$$

Proof Without loss of generality, we can suppose that

$$0 \leq u \leq 1.$$

For every $n \geq 1$, we consider the functional

$$F_n : H_0^1(\Omega) \rightarrow \mathbb{R} \quad F_n(v) = \int_{\Omega} |\nabla v|^2 dx + n \int_{\Omega} |v - u|^2 dx.$$

The functional F_n admits a unique minimizer in $H_0^1(\Omega)$ that we denote by u_n . By construction, testing the optimality of u_n with $v = u$, we get

$$\int_{\Omega} |\nabla u_n|^2 dx + n \int_{\Omega} |u_n - u|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx.$$

In particular, the sequence u_n converges strongly $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to the function u . Moreover, u_n solves the PDE

$$-\Delta u_n = n(u - u_n) \quad \text{in } \Omega \quad u \in H_0^1(\Omega). \tag{2.4}$$

We notice that since u_n minimizes F_n and since $0 \leq u \leq 1$, then also

$$0 \leq u_n \leq 1.$$

Thus, the right-hand side $n(u - u_n)$ of (2.4) is bounded. Let now $x_0 \in \Omega^{(1/2)}$, that is,

$$\lim_{r \rightarrow 0} \frac{|B_r(x_0) \cap \Omega|}{|B_r(x_0)|} = \frac{1}{2}.$$

By [7, Proposition 4.6] we have that for $r > 0$ small enough

$$\|u_n\|_{L^\infty(B_r(x_0))} \leq r^\beta C_n,$$

for some constant C_n depending on u_n and some dimensional constant $\beta > 0$. In particular,

$$u_n = 0 \quad \text{on } \Omega^{(1/2)}.$$

Now, since

$$\int_{\Omega} |\nabla u_n|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx \quad \text{for every } n \geq 1$$

we get that the convergence of u_n to u is strong in $H_0^1(\Omega)$. It is well-known that there is a subsequence of u_n converging pointwise quasi-everywhere to u . In particular, the same subsequence converges pointwise \mathcal{H}^{d-1} -almost everywhere on $\partial^*\Omega$. Thus, (the representative of) u vanishes \mathcal{H}^{d-1} -almost everywhere on both $\Omega^{(1/2)}$ and $\partial^*\Omega$. \square

Proposition 2.5 *Let Ω_1 and Ω_2 be two sets of finite perimeter in \mathbb{R}^d such that*

$$|\Omega_1 \cap \Omega_2| = 0.$$

Then, for any function $u \in \tilde{H}_0^1(\Omega_1 \cup \Omega_2)$, we have that

$$\int_{\partial^* \Omega_1 \cap \partial^* \Omega_2} u^2 d\mathcal{H}^{d-1} = \frac{1}{2} \left(\int_{\partial^* \Omega_1} u^2 d\mathcal{H}^{d-1} + \int_{\partial^* \Omega_2} u^2 d\mathcal{H}^{d-1} \right). \tag{2.5}$$

Proof We first notice that the reduced boundaries $\partial^* \Omega_1$ and $\partial^* \Omega_2$ can be decomposed as

$$\begin{aligned} \partial^* \Omega_1 &= \left(\partial^* \Omega_1 \cap \partial^* \Omega_2 \right) \cup \left(\partial^* \Omega_1 \setminus \partial^* \Omega_2 \right) \quad \text{and} \\ \partial^* \Omega_2 &= \left(\partial^* \Omega_1 \cap \partial^* \Omega_2 \right) \cup \left(\partial^* \Omega_2 \setminus \partial^* \Omega_1 \right). \end{aligned}$$

Thus, in order to prove (2.5), it is sufficient to prove that

$$u = 0 \quad \mathcal{H}^{d-1}\text{-almost everywhere on} \quad \left(\partial^* \Omega_1 \setminus \partial^* \Omega_2 \right) \cup \left(\partial^* \Omega_2 \setminus \partial^* \Omega_1 \right). \tag{2.6}$$

Let $x_0 \in \partial^* \Omega_1 \setminus \partial^* \Omega_2$. Since

$$\mathcal{H}^{d-1} \left(\mathbb{R}^d \setminus \left(\Omega_2^{(0)} \cup \Omega_2^{(1)} \cup \partial^* \Omega_2 \right) \right) = 0,$$

we can suppose that $x_0 \in \Omega_2^{(0)} \cup \Omega_2^{(1)}$, but since $x_0 \in \Omega_1^{(1/2)}$ and $\Omega_1 \cap \Omega_2 = 0$, we get that necessarily $x_0 \in \Omega_2^{(0)}$. But then,

$$x_0 \in (\Omega_1 \cup \Omega_2)^{(1/2)}.$$

Thus, by Proposition 2.4, we get that $u(x_0) = 0$. This proves (2.6) and (2.5). □

2.4 A semicontinuity lemma

In the proof of the main theorem we will repeatedly use the following lemma, which is a restatement of a lemma from [9].

Lemma 2.6 ([9]) *Let $A \subset \mathbb{R}^d$ be a bounded open set. Let $u_n \in H^1(A)$ be a sequence of functions converging to $u_\infty \in H^1(A)$ weakly in $H^1(A)$, strongly in $L^2(A)$ and pointwise almost-everywhere. Let $\Omega_n \subset A$ be a sequence of sets of locally finite perimeter in A converging almost-everywhere (in A) to the set of locally finite perimeter $\Omega_\infty \subset A$. Then,*

$$\int_{A \cap \partial^* \Omega_\infty} u_\infty^2 d\mathcal{H}^{d-1} \leq \liminf_{n \rightarrow \infty} \int_{A \cap \partial^* \Omega_n} u_n^2 d\mathcal{H}^{d-1}. \tag{2.7}$$

Proof The proof is precisely the one from [9, Lemma 2.4]. We report it here for the sake of completeness. The key observation is that given $u \in H^1(A)$ and a set of locally finite perimeter $\Omega \subset A$, we have

$$\int_{A \cap \partial^* \Omega} u^2 d\mathcal{H}^{d-1} = \sup \left\{ \int_{A \cap \Omega} \operatorname{div} (u^2 \xi) dx \mid \xi \in C_c^1(A; \mathbb{R}^d), |\xi| \leq 1 \right\}.$$

We now fix a vector field $\xi \in C_c^1(A; \mathbb{R}^d)$, $|\xi| \leq 1$ and we compute

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{A \cap \partial^* \Omega_n} u_n^2 d\mathcal{H}^{d-1} &\geq \liminf_{n \rightarrow \infty} \int_{A \cap \Omega_n} \operatorname{div} (u_n^2 \xi) dx \\ &= \liminf_{n \rightarrow \infty} \int_A \left(2(u_n \xi) \cdot (\mathbb{1}_{\Omega_n} \nabla u_n) + (u_n \mathbb{1}_{\Omega_n}) (u_n \operatorname{div} \xi) \right) dx \end{aligned}$$

Now, since $\mathbb{1}_{\Omega_n} \nabla u_n$ converges weakly in L^2 to $\mathbb{1}_{\Omega_\infty} \nabla u_\infty$, we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{A \cap \partial^* \Omega_n} u_n^2 d\mathcal{H}^{d-1} &\geq \int_A \left(2(u_\infty \xi) \cdot (\mathbb{1}_{\Omega_\infty} \nabla u_\infty) + (u_\infty \mathbb{1}_{\Omega_\infty}) (u_\infty \operatorname{div} \xi) \right) dx \\ &= \int_{A \cap \Omega_\infty} \operatorname{div} (u_\infty^2 \xi) dx. \end{aligned}$$

Taking the supremum over ξ , we get (2.7). □

3 Almost-minimality and Hölder estimates

In this section, we prove two general technical results on the continuity of subharmonic functions which are almost-minimizers of the Dirichlet energy in a suitable sense. We will use these estimates in Sects. 4 and 9.

Lemma 3.1 (A growth estimate) *Let D be a bounded open set in \mathbb{R}^d , $x_0 \in D$ and let $u \in H^1(D)$ be a function such that*

- (a) u is non-negative and subharmonic in D ;
- (b) There are constants $\alpha \in [0, 1]$, $K > 0$ and $r_0 > 0$ such that $B_{r_0}(x_0) \subset D$ and

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \int_{B_r(x_0)} |\nabla(u + \varphi)|^2 dx + Kr^{d-1+\alpha},$$

for every $r \in (0, r_0)$ and every $\varphi \in H_0^1(B_r(x_0))$ with $\varphi \geq 0$ in $B_r(x_0)$.

Then, for every $r \leq r_0/4$, we have

$$\int_{B_r(x_0)} u(x) dx - u(x_0) \leq \frac{C_d \sqrt{K}}{\alpha + 1} r^{\frac{1+\alpha}{2}}.$$

In particular, if $u(x_0) = 0$, then

$$\|u\|_{L^\infty(B_r(x_0))} \leq \frac{C_d \sqrt{K}}{\alpha + 1} r^{\frac{1+\alpha}{2}} \text{ for every } r \leq \frac{r_0}{8}.$$

Proof Without loss of generality, we can suppose that $x_0 = 0$ and $D = B_{r_0}$. Then, for every $r \in (0, r_0)$, we have

$$0 \leq \int_{\partial B_r} u d\mathcal{H}^{d-1} - u(0) \leq \frac{1}{d\omega_d} \int_0^r s^{1-d} \Delta u(B_s) ds.$$

Thus, we only need to estimate $\Delta u(B_r)$. In order to do so, we test the optimality of u with $u + t\varphi$, where $\varphi(x) := \frac{1}{r}(2r - |x|)_+$.

$$-2 \int_{B_r} \nabla u \cdot \nabla \varphi dx \leq t \int_{B_r} |\nabla \varphi|^2 + \frac{K}{t} r^{d-1+\alpha} \leq t\omega_d r^{d-2} + \frac{K}{t} r^{d-1+\alpha}.$$

Taking

$$t := \sqrt{\frac{K}{\omega_d}} r^{\frac{1+\alpha}{2}},$$

we obtain

$$-2 \int_{B_r} \nabla u \cdot \nabla \varphi \, dx \leq \sqrt{K\omega_d} r^{d-\frac{3}{2}+\frac{\alpha}{2}},$$

and so, for every $r \leq \frac{r_0}{2}$, we get

$$\Delta u(B_{r/2}) \leq \int_{B_r} \varphi(x) \Delta u(x) \, dx \leq - \int_{B_r} \nabla \varphi \cdot \nabla u \, dx \leq \frac{1}{2} \sqrt{K\omega_d} r^{d-\frac{3}{2}+\frac{\alpha}{2}}.$$

Then, for every $r \leq \frac{r_0}{4}$, we can estimate

$$\int_{\partial B_r} u \, d\mathcal{H}^{d-1} - u(0) \leq \frac{2^{d-1} \sqrt{K\omega_d}}{d\omega_d} \int_0^r s^{1-d} s^{d-\frac{3}{2}+\frac{\alpha}{2}} \, ds \leq \frac{2^d \sqrt{K\omega_d}}{d\omega_d} \frac{r^{\frac{1+\alpha}{2}}}{\alpha+1}.$$

Now, using the non-negativity and the subharmonicity of u , we get the claim. □

Lemma 3.2 (Hölder continuity) *Let D be a bounded open set in \mathbb{R}^d let $u \in H^1(D)$ be a function which is subharmonic in D and such that*

$$0 \leq u \leq M \text{ in } D,$$

for some constant $M > 0$. Given $\delta \in (0, 1)$, we define the set

$$D_\delta := \{x \in D \mid \text{dist}(x, \partial D) > \delta\}, \tag{3.1}$$

and we suppose that there are constants $\alpha \in [0, 1]$ and $K > 0$ such that

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq \int_{B_r(x_0)} |\nabla(u + \varphi)|^2 \, dx + Kr^{d-1+\alpha},$$

for every $x_0 \in D_\delta$, every $r \in (0, \delta)$ and every $\varphi \in H_0^1(B_r(x_0))$ with $\varphi \geq 0$ in $B_r(x_0)$. Then,

$$|u(x) - u(y)| \leq C_d \left(M + \frac{\sqrt{K}}{\alpha+1} \right) |x - y|^{\frac{1+\alpha}{3+\alpha}},$$

for every $x, y \in D_\delta$ such that $|x - y| < (\frac{\delta}{16})^2$

Proof We apply the previous lemma, to $x_0 = y$ and $r = |x - y|^\gamma$. Then,

$$\begin{aligned} u(x) - u(y) &\leq \int_{B_r(x)} u - u(y) \leq \frac{(r + |x - y|)^d}{r^d} \int_{B_{r+|x-y|}(y)} u - u(y) \\ &\leq \frac{(r + |x - y|)^d - r^d}{r^d} M + \int_{B_{r+|x-y|}(y)} u - u(y) \\ &\leq \frac{|x - y|}{r} M d \frac{(r + |x - y|)^{d-1}}{r^{d-1}} + \frac{C_d \sqrt{K}}{\alpha+1} (r + |x - y|)^{\frac{1+\alpha}{2}}. \end{aligned}$$

Now, since $|x - y| < 1$ and $r = |x - y|^\gamma$, with $\gamma \in (0, 1)$, we get

$$u(x) - u(y) \leq d2^d M |x - y|^{1-\gamma} + \frac{2C_d \sqrt{K}}{\alpha+1} |x - y|^\gamma \frac{1+\alpha}{2}.$$

Choosing $\gamma = \frac{2}{3+\alpha}$, we get that

$$1 - \gamma = \gamma \frac{1 + \alpha}{2} = \frac{\alpha + 1}{\alpha + 3},$$

and so we get the claim. Finally, we notice that we should have the inequality $r + |x - y| < \frac{\delta}{8}$, which is satisfied for instance when $|x - y|^\gamma < \frac{\delta}{16}$. □

4 Non-degenerate approximating problems

In this section, we define a sequence of non-degenerate problems, approximating (1.3), in which the competitors u are a priori bounded from below by a fixed positive constant. We consider the family of approximating problems

$$\min \left\{ J_\varepsilon(u, \Omega_1, \Omega_2) : u \in \mathcal{V}, (\Omega_1, \Omega_2) \in \mathcal{A}(u) \right\}, \tag{4.1}$$

where the functional J_ε is defined as

$$J_\varepsilon(u, \Omega_1, \Omega_2) := \int_D |\nabla u|^2 dx + \Lambda |\{u > 0\} \cap D| + \frac{\beta}{2} \left(\int_{\partial^* \Omega_1} u^2 d\mathcal{H}^{d-1} + \int_{\partial^* \Omega_2} u^2 d\mathcal{H}^{d-1} \right) + \varepsilon (\text{Per}(\Omega_1) + \text{Per}(\Omega_2)).$$

Proposition 4.1 [Solutions to the approximating problem] *Let D be a bounded open set in \mathbb{R}^d . Then, for any $\varepsilon > 0$ there are $u_\varepsilon \in \mathcal{V}$ and $(\Omega_\varepsilon^1, \Omega_\varepsilon^2) \in \mathcal{A}(u_\varepsilon)$ such that:*

- (i) $(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$ is a solution to (4.1);
- (ii) The function u_ε is Hölder continuous in D and for every $\delta > 0$ there is a constant $C_\delta > 0$, depending on $\delta, d, \beta, \Lambda$ and $\|g\|_{L^\infty}$, such that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_\delta |x - y|^{1/3} \quad \text{for every } x, y \in D_\delta, \tag{4.2}$$

where D_δ is the set defined in (3.1).

- (iii) There is a constant $\rho > 0$, depending only on d, Λ and $\|g\|_{L^\infty}$, such that

$$\Omega_\varepsilon^1 \cup \Omega_\varepsilon^2 \subset (E_1 \cup E_2) + B_\rho. \tag{4.3}$$

Proof Let $\varepsilon > 0$ be fixed. We divide the proof into several steps.

Existence. Let $\{(u_{\varepsilon,i}, \Omega_{\varepsilon,i}^1, \Omega_{\varepsilon,i}^2)\}_{i \in \mathbb{N}}$ be a minimizing sequence for (4.1). Since we can use (g, E_1, E_2) as a competitor against $(u_{\varepsilon,i}, \Omega_{\varepsilon,i}^1, \Omega_{\varepsilon,i}^2)$, we have that

$$\int_D |\nabla u_{\varepsilon,i}|^2 dx + \varepsilon (\text{Per}(\Omega_{\varepsilon,i}^1) + \text{Per}(\Omega_{\varepsilon,i}^2)) \leq J_\varepsilon(g, E_1, E_2) \leq J_1(g, E_1, E_2). \tag{4.4}$$

Moreover, since $u_{\varepsilon,i} - g \in H_0^1(D)$, by the Poincaré inequality with constant C_D , we have

$$\begin{aligned} \|u_{\varepsilon,i}\|_{L^2(D)} &\leq \|u_{\varepsilon,i} - g\|_{L^2(D)} + \|g\|_{L^2(D)} \\ &\leq C_D \|\nabla(u_{\varepsilon,i} - g)\|_{L^2(D)} + \|g\|_{L^2(D)}. \end{aligned}$$

Thus, the sequence $\{u_{\varepsilon,i}\}_{i \in \mathbb{N}}$ is bounded in $H^1(D)$. Hence, up to a subsequence, there exists a function $u_\varepsilon \in H_0^1(\mathbb{R}^d)$ such that $\nabla u_{\varepsilon,i} \rightarrow \nabla u_\varepsilon$ weakly in $L^2(\mathbb{R}^d)$ and $u_{\varepsilon,i} \rightarrow u_\varepsilon$ strongly in $L^2(\mathbb{R}^d)$ and pointwise almost-everywhere. In particular, the almost-everywhere convergence gives that

$$\mathbb{1}_{\{u_\varepsilon > 0\}} \leq \liminf_{i \rightarrow \infty} \mathbb{1}_{\{u_{\varepsilon,i} > 0\}}. \tag{4.5}$$

Using again (4.4), we obtain

$$\text{Per}(\Omega_{\varepsilon,i}^1) + \text{Per}(\Omega_{\varepsilon,i}^2) \leq \frac{1}{\varepsilon} J_1(g, E_1, E_2).$$

Hence, there are sets of finite perimeter Ω_ε^1 and Ω_ε^2 such that

$$\Omega_\varepsilon^1 \cap \Omega_\varepsilon^2 = \emptyset \quad E_1 \subset \Omega_\varepsilon^1 \quad \text{and} \quad E_2 \subset \Omega_\varepsilon^2,$$

and such that (up to a subsequence)

$$\mathbb{1}_{\Omega_{\varepsilon,i}^1} \rightarrow \mathbb{1}_{\Omega_\varepsilon^1} \quad \text{and} \quad \mathbb{1}_{\Omega_{\varepsilon,i}^2} \rightarrow \mathbb{1}_{\Omega_\varepsilon^2},$$

strongly in $L^1(\mathbb{R}^d)$ and pointwise almost-everywhere. Together with (4.5), this implies that

$$\{u_\varepsilon > 0\} \subset \Omega_\varepsilon^1 \cup \Omega_\varepsilon^2$$

so $(\Omega_\varepsilon^1, \Omega_\varepsilon^2) \in \mathcal{A}(u_\varepsilon)$. Finally, the semicontinuity of J_ε (see Lemma 2.6) gives that $(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$ is a solution to the variational problem (4.1).

Subharmonicity of u_ε . Let v be a function in $H^1(D)$ such that

$$v \leq u_\varepsilon \quad \text{in } D \quad \text{and} \quad v = u_\varepsilon \quad \text{on } \partial D.$$

Then, testing the optimality of $(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$ with $(v_+, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$ and using the fact that

$$v_+ \leq u_\varepsilon \quad \text{in } D,$$

we get

$$\int_D |\nabla v|^2 dx \geq \int_D |\nabla v_+|^2 dx \geq \int_D |\nabla u_\varepsilon|^2 dx.$$

In particular, if φ is a nonnegative function compactly supported in D , then we can apply the above inequality to $v := u - t\varphi$ for some $t > 0$. Then, by sending t to zero, we get that

$$- \int_D \nabla \varphi \cdot \nabla u_\varepsilon dx \geq 0,$$

which means that the distributional Laplacian Δu_ε is a positive Radon measure in D .

Hölder continuity of u_ε . Let $\delta \in (0, 1)$ and let D_δ be given by (3.1). Let $x_0 \in D_\delta$ and $0 < R < \delta$. Let $\varphi \in H_0^1(B_R(x_0))$ be such that $\varphi \geq 0$ in $B_R(x_0)$. We set

$$\tilde{\Omega}_\varepsilon^1 := \Omega_\varepsilon^1 \cup B_R(x_0) \quad \text{and} \quad \tilde{\Omega}_\varepsilon^2 := \Omega_\varepsilon^2 \setminus B_R(x_0),$$

so that $(\tilde{\Omega}_\varepsilon^1, \tilde{\Omega}_\varepsilon^2) \in \mathcal{A}(u_\varepsilon)$. Using $(u_\varepsilon + \varphi, \tilde{\Omega}_\varepsilon^1, \tilde{\Omega}_\varepsilon^2)$ to test the optimality of $(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$, we get

$$\begin{aligned} & \int_{B_R(x_0)} |\nabla u_\varepsilon|^2 dx + \Lambda |B_R(x_0) \cap \{u_\varepsilon > 0\}| + \frac{\beta}{2} \int_{\partial^* \Omega_\varepsilon^1} u_\varepsilon^2 d\mathcal{H}^{d-1} + \frac{\beta}{2} \int_{\partial^* \Omega_\varepsilon^2} u_\varepsilon^2 d\mathcal{H}^{d-1} \\ & \leq \int_{B_R(x_0)} |\nabla(u_\varepsilon + \varphi)|^2 dx + \Lambda |B_R| + \frac{\beta}{2} \int_{\partial^* \tilde{\Omega}_\varepsilon^1} u_\varepsilon^2 d\mathcal{H}^{d-1} \\ & \quad + \frac{\beta}{2} \int_{\partial^* \tilde{\Omega}_\varepsilon^2} u_\varepsilon^2 d\mathcal{H}^{d-1} + \varepsilon \mathcal{H}^{d-1}(\partial B_R). \end{aligned}$$

Moreover, the definition of $(u + \varphi, \tilde{\Omega}_\varepsilon^1, \tilde{\Omega}_\varepsilon^2)$ gives that, for $j = 1, 2$,

$$\begin{aligned} \int_{\partial^* \tilde{\Omega}_\varepsilon^j} u_\varepsilon^2 d\mathcal{H}^{d-1} &\leq \int_{\partial^* \Omega_\varepsilon^j} u_\varepsilon^2 d\mathcal{H}^{d-1} + \int_{\partial B_R(x_0)} u_\varepsilon^2 d\mathcal{H}^{d-1} \\ &\leq \int_{\partial^* \Omega_\varepsilon^j} u_\varepsilon^2 d\mathcal{H}^{d-1} + d\omega_d R^{d-1} \|g\|_{L^\infty(D)}^2. \end{aligned} \tag{4.6}$$

Combining these two estimates, we obtain

$$\begin{aligned} \int_{B_R(x_0)} |\nabla u_\varepsilon|^2 dx &\leq \int_{B_R(x_0)} |\nabla(u_\varepsilon + \varphi)|^2 dx + \Lambda |B_R| + (\varepsilon + \beta \|g\|_{L^\infty(D)}^2) d\omega_d R^{d-1} \\ &\leq \int_{B_R(x_0)} |\nabla(u_\varepsilon + \varphi)|^2 dx + \left(\omega_d \Lambda \delta + (1 + \beta \|g\|_{L^\infty(D)}^2) d\omega_d \right) R^{d-1}. \end{aligned}$$

By Lemma 3.2 we get that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C|x - y|^{1/\beta} \quad \text{for every } x, y \in D_\delta, \tag{4.7}$$

such that $|x - y| \leq \frac{\delta^2}{256}$, where the constant C depends only on d, Λ, β and $\|g\|_{L^\infty}$.

Boundedness of Ω_ε^1 and Ω_ε^2 . We first notice that u_ε is a subsolution of the Alt-Caffarelli functional, that is,

$$\int_{\mathbb{R}^d} |\nabla u_\varepsilon|^2 dx + \Lambda |\{u_\varepsilon > 0\}| \leq \int_{\mathbb{R}^d} |\nabla v|^2 dx + \Lambda |\{v > 0\}|,$$

for every $v \in H^1(\mathbb{R}^d)$ such that $u - v \in H_0^1(D)$ and $0 \leq v \leq u_\varepsilon$ on \mathbb{R}^d . In particular, this implies (see for instance [15]) that the set $\{u_\varepsilon > 0\}$ lies in a sufficiently large ball B_ρ . Now, since outside $\{u_\varepsilon > 0\}$ the functional J_ε only accounts for the perimeter of Ω_ε^1 and Ω_ε^2 , we have that these sets should be contained in the convex envelope of $\{u_\varepsilon > 0\}$, which in particular gives (4.3). □

5 The limit of the non-degenerate solutions

In this section we define the function u (Sect. 5.1) and the sets (Ω_1, Ω_2) (Sect. 5.2), which we will prove to be solutions to the initial problem (1.3). Throughout this section, for any $\varepsilon > 0$, we fix a solution $(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$ of the approximating problem (4.1) for J_ε .

5.1 The limit function

It is immediate to check that, there is a function

$$u \in H^1(\mathbb{R}^d),$$

and a sequence $\varepsilon_n \rightarrow 0$ such that

- for every fixed $\delta > 0, u_{\varepsilon_n} \rightarrow u$ uniformly in D_δ as $n \rightarrow \infty$;
- $u_{\varepsilon_n} \rightarrow u$ strongly in $L^2(\mathbb{R}^d)$ and pointwise almost-everywhere in \mathbb{R}^d ;
- $\nabla u_{\varepsilon_n} \rightarrow \nabla u$ weakly in $L^2(\mathbb{R}^d)$.

By construction, we have $u - g \in H_0^1(D)$, while Proposition 4.1 gives that

$$u \in H_0^1((E_1 \cup E_2) + B_\rho) \quad \text{and} \quad u \in C^{0,1/\beta}(\overline{D}_\delta) \quad \text{for every } \delta > 0.$$

Moreover,

$$0 \leq u \leq \|g\|_{L^\infty(\mathbb{R}^d)} \quad \text{and} \quad \Delta u \geq 0 \quad \text{in } D.$$

5.2 The limit sets

We next construct the sets Ω_1 and Ω_2 . Choose a ball

$$B_R(x_0) \subset\subset D \cap \{u > 0\}.$$

Then, there are $t > 0$ and $\delta > 0$ such that

$$\overline{B}_R(x_0) \subset D_\delta \cap \{u \geq t\},$$

where D_δ is given by (3.1). By the uniform convergence of u_{ε_n} to u on D_δ , we can find $n_0 \geq 1$ such that, the following inequality holds for every $n \geq n_0$:

$$u_{\varepsilon_n} \geq \frac{t}{2} \quad \text{in } \overline{B}_R(x_0).$$

Using this inequality and the optimality of $(u_{\varepsilon_n}, \Omega_{\varepsilon_n}^1, \Omega_{\varepsilon_n}^2)$, we can estimate

$$\begin{aligned} \frac{2}{\beta} J_\varepsilon(g, E_1, E_2) &\geq \frac{2}{\beta} J_\varepsilon(u_{\varepsilon_n}, \Omega_{\varepsilon_n}^1, \Omega_{\varepsilon_n}^2) \geq \int_{B_R(x_0) \cap \partial^* \Omega_{\varepsilon_n}^1} u_{\varepsilon_n}^2 d\mathcal{H}^{d-1} + \int_{B_R(x_0) \cap \partial^* \Omega_{\varepsilon_n}^2} u_{\varepsilon_n}^2 d\mathcal{H}^{d-1} \\ &\geq \frac{t^2}{4} \left(\text{Per}(\Omega_{\varepsilon_n}^1; B_R(x_0)) + \text{Per}(\Omega_{\varepsilon_n}^2; B_R(x_0)) \right). \end{aligned}$$

Thus, the sets

$$\Omega_{\varepsilon_n}^1 \cap B_R(x_0) \quad \text{and} \quad \Omega_{\varepsilon_n}^2 \cap B_R(x_0)$$

have uniformly bounded perimeter. In particular, up to a subsequence there are sets

$$\Omega_{R,x_0}^1 \cap B_R(x_0) \quad \text{and} \quad \Omega_{R,x_0}^2 \cap B_R(x_0),$$

of finite perimeter and such that, as $n \rightarrow \infty$,

$$\mathbb{1}_{\Omega_{\varepsilon_n}^1 \cap B_R(x_0)} \rightarrow \mathbb{1}_{\Omega_{R,x_0}^1} \quad \text{and} \quad \mathbb{1}_{\Omega_{\varepsilon_n}^2 \cap B_R(x_0)} \rightarrow \mathbb{1}_{\Omega_{R,x_0}^2},$$

pointwise almost-everywhere and strongly in $L^1(\mathbb{R}^d)$. Thus, by a diagonal sequence argument, we can define the sets Ω_1 and Ω_2 as the union of Ω_{R,x_0}^1 and Ω_{R,x_0}^2 over all balls

$$B_R(x_0) \subset\subset D \cap \{u > 0\},$$

of radius $R \in \mathbb{Q}$ and center with rational coordinates $x_0 \in \mathbb{Q}^d$,

$$\Omega_i := E_i \cup \bigcup_{R,x_0} \Omega_{R,x_0}^i \quad \text{for } i = 1, 2.$$

By construction, Ω_1 and Ω_2 have locally finite perimeter in $D \cap \{u > 0\}$ and satisfy

$$E_i \subset \Omega_i \subset \left((E_1 \cup E_2) \cap B_\rho \right),$$

where we recall that $D := \mathbb{R}^d \setminus (\overline{E}_1 \cup \overline{E}_2)$. Moreover, we still have the pointwise convergence of the corresponding characteristic functions, that is, for $i = 1, 2$,

$$\mathbb{1}_{\Omega_i} = \lim_{n \rightarrow \infty} \mathbb{1}_{\Omega_{\varepsilon_n}^i \cap \{u > 0\}} \quad \text{in } \mathbb{R}^d,$$

which implies that, for almost every $x \in \mathbb{R}^d$,

$$\mathbb{1}_{\Omega_1 \cap \Omega_2}(x) = \mathbb{1}_{\Omega_1}(x) \cdot \mathbb{1}_{\Omega_2}(x) = \lim_{n \rightarrow \infty} \left(\mathbb{1}_{\Omega_{\varepsilon_n}^1}(x) \cdot \mathbb{1}_{\Omega_{\varepsilon_n}^2}(x) \cdot \mathbb{1}_{\{u > 0\}} \right) = 0,$$

the sets Ω_1 and Ω_2 are disjoint, $|\Omega_1 \cap \Omega_2| = 0$.

Remark 5.1 Notice that we do not have a priori that Ω_1 and Ω_2 have finite perimeter in \mathbb{R}^d , so at this stage they might not be in the admissible class $\mathcal{A}(u)$ defined in Sect. 1.3.

6 Almost-minimality and Lipschitz estimates of u

In this section we will show that u is an almost-minimizer (in some suitable sense) of the classical one-phase functional of Alt and Caffarelli. From this we deduce the Lipschitz growth of u on the boundary, which we will use in Sect. 9.1 in order to deduce the convergence of the blow-up sequences of u .

Lemma 6.1 *Let $u \in H^1(\mathbb{R}^d)$ be the function from Sect. 5.1 and let $\overline{B}_r(x_0) \subset \mathbb{R}^d \setminus \overline{E}_2$. Suppose that $v \in H^1(\mathbb{R}^d)$ is a function such that $v - u \in H_0^1(B_r(x_0))$. Then,*

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 dx + \Lambda |B_r(x_0) \cap \{u > 0\}| \\ \leq \int_{B_r(x_0)} |\nabla v|^2 dx + \Lambda |B_r(x_0) \cap \{v > 0\}| + \beta \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{d-1}. \end{aligned}$$

Proof We can suppose that $v \geq 0$. Then, testing the minimality of $(u_{\varepsilon_n}, \Omega_{\varepsilon_n}^1, \Omega_{\varepsilon_n}^2)$, with the function v (which we can do since $v - g \in H_0^1(D)$) and the sets

$$\Omega_{\varepsilon_n}^1 \cup B_r(x_0) \quad \text{and} \quad \Omega_{\varepsilon_n}^2 \setminus B_r(x_0),$$

we obtain

$$\begin{aligned} \int_D |\nabla u_{\varepsilon_n}|^2 dx + \Lambda |D \cap \{u_{\varepsilon_n} > 0\}| \\ \leq \int_D |\nabla v|^2 dx + \Lambda |D \cap \{v > 0\}| + \beta \int_{\partial B_r(x_0)} u_{\varepsilon_n}^2 d\mathcal{H}^{d-1}. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we get the claim. □

Lemma 6.2 *Let $u \in H^1(\mathbb{R}^d)$ be the function defined in Sect. 5.1. For every $\delta \in (0, 1)$ and every $x_0 \in D_\delta \cap \{u = 0\}$, we have*

$$\|u\|_{L^\infty(B_r(x_0))} \leq Cr \quad \text{for every} \quad r < \frac{\delta}{256},$$

where C is a constant depending only on d, β, Λ and $\|g\|_{L^\infty}$.

Proof We set for simplicity $M := \|g\|_{L^\infty}$. By Lemma 6.1, we have that for every $r \in (0, \delta)$

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 dx &\leq \int_{B_r(x_0)} |\nabla(u + \varphi)|^2 dx + \Lambda |B_r| + \beta \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{d-1} \\ &\leq \int_{B_r(x_0)} |\nabla(u + \varphi)|^2 dx + \left(\Lambda \omega_d \delta + \beta d \omega_d M^2 \right) r^{d-1}, \end{aligned}$$

for every $\varphi \in H_0^1(B_r(x_0))$ such that $\varphi \geq 0$ in $B_r(x_0)$. Applying Lemma 3.1, we obtain

$$\|u\|_{L^\infty(B_r(x_0))} \leq C_d(\Lambda + \beta M^2)^{1/2} r^{1/2} \quad \text{for every } r \leq \frac{\delta}{8}.$$

Using this estimate in Lemma 6.1, we get that, for every $r \in (0, \delta/8)$,

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 dx &\leq \int_{B_r(x_0)} |\nabla(u + \varphi)|^2 dx + \Lambda|B_r| + \beta \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{d-1} \\ &\leq \int_{B_r(x_0)} |\nabla(u + \varphi)|^2 dx + C_d(\Lambda + \beta(\Lambda + \beta M^2))r^d, \end{aligned}$$

for every nonnegative $\varphi \in H_0^1(B_r(x_0))$. Applying again Lemma 3.1, we get the claim. \square

7 Non-degeneracy of u

In this section we show that u is a subsolution of the Alt-Caffarelli functional. From this information, we can immediately deduce that $\{u > 0\}$ has finite perimeter in D_δ , for every $\delta > 0$. Moreover, the suboptimality of u implies that it is non-degenerate, which assures that the blow-up limits of u are not identically zero.

Lemma 7.1 *Let $u \in H^1(\mathbb{R}^d)$ be the function from Sect. 5.1. Then,*

$$\int_D |\nabla u|^2 dx + \Lambda|\{u > 0\} \cap D| \leq \int_D |\nabla v|^2 dx + \Lambda|\{v > 0\} \cap D|,$$

for any $v \in H^1(\mathbb{R}^d)$ such that

$$v - u \in H_0^1(D) \quad \text{and} \quad 0 \leq v \leq u \quad \text{in } D.$$

Proof Testing the optimality of $(u_{\varepsilon_n}, \Omega_{\varepsilon_n}^1, \Omega_{\varepsilon_n}^2)$ with $(v, \Omega_{\varepsilon_n}^1, \Omega_{\varepsilon_n}^2)$, we obtain

$$\int_D |\nabla u_{\varepsilon_n}|^2 dx + \Lambda|\{u_{\varepsilon_n} > 0\} \cap D| \leq \int_D |\nabla v|^2 dx + \Lambda|\{v > 0\} \cap D|.$$

Passing to the limit as $n \rightarrow \infty$, we get the claim. \square

As an immediate consequence, we have

Corollary 7.2 *Let $u \in H^1(\mathbb{R}^d)$ be the function defined in Sect. 5.1. Then:*

- (i) *the set $\{u > 0\}$ has locally finite perimeter in D ;*
- (ii) *there is a constant $\eta > 0$ such that for every $x_0 \in D \cap \overline{\{u > 0\}}$*

$$\|u\|_{L^\infty(B_r(x_0))} \geq \eta r \quad \text{for every } B_{2r}(x_0) \subset D.$$

Proof See [1] or [15]. \square

8 Density estimate and its consequences

In this section, we will show that the free boundary $\partial\{u > 0\}$ is not touching ∂E_1 and ∂E_2 . This is a crucial step in proving that Ω_1 and Ω_2 have finite perimeter in \mathbb{R}^d . The main result is the following.

Proposition 8.1 (Non-collapsing) *Let $u \in H^1(\mathbb{R}^d)$ be the function from Sect. 5.1. Then, there is a positive constant $t > 0$ such that $u \geq t$ in a neighborhood of $\bar{E}_1 \cup \bar{E}_2$.*

The proof of Proposition 8.1 is based on the following lemma.

Lemma 8.2 (Density estimate) *Let $u \in H^1(\mathbb{R}^d)$ be the function from Sect. 5.1. There is a constant $c > 0$ and $R_0 > 0$, depending only on d, β, Λ and $\|g\|_{L^\infty}$, such that*

$$|B_R(x_0) \cap \{u = 0\}| \geq c|B_R| \text{ for every } B_R(x_0) \subset D \text{ with } u(x_0) = 0.$$

Proof We notice that by Lemma 6.2, we have that

$$\|u\|_{L^\infty(B_\rho(x_0))} \leq C\rho \text{ for every } \rho < \frac{R}{256}. \tag{8.1}$$

Now, we consider the competitor

$$\tilde{\Omega}_1 := \Omega_1 \cup B_{R/256}(x_0) \quad \tilde{\Omega}_2 := \Omega_2 \setminus B_{R/256}(x_0) \quad \tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \mathbb{R}^d \setminus B_{R/256}(x_0) \\ h(x) & \text{if } x \in B_{R/256}(x_0) \end{cases},$$

h being the harmonic extension of u in $B_{R/256}(x_0)$. Setting for simplicity $r = \frac{R}{256}$ and $x_0 = 0$, by Lemma 6.1, we have that

$$\int_{B_r} |\nabla u|^2 dx + \Lambda|B_r \cap \{u > 0\}| \leq \int_{B_r} |\nabla h|^2 dx + \Lambda|B_r \cap \{h > 0\}| + \beta d \omega_d r^{d-1} C^2 r^2.$$

The rest of the proof follows the analogous lemma from [1]. Using the fact that h is harmonic and strictly positive in B_r , we have

$$\int_{B_r} |\nabla(u - h)|^2 dx \leq \Lambda|B_r \cap \{u = 0\}| + \beta C^2 d \omega_d r^{d+1}.$$

By the Poincaré inequality, there is a dimensional constant C_d such that

$$\left(\frac{1}{|B_r|} \int_{B_r} (h - u) dx \right)^2 \leq \frac{1}{|B_r|} \int_{B_r} (h - u)^2 dx \leq \frac{C_d r^2}{|B_r|} \left(\Lambda|B_r \cap \{u = 0\}| + \beta C^2 r^{d+1} \right).$$

On the other hand, the combination of Corollary 7.2 and the subharmonicity of u in B_R , gives that for any $\kappa \in (0, 1)$

$$\frac{1}{|B_{\kappa r}|} \int_{B_{\kappa r}} h(x) dx = \int_{\partial B_r} u \geq 2^{-d} cr.$$

On the other hand, (8.1) implies that

$$\frac{1}{|B_{\kappa r}|} \int_{B_{\kappa r}} u(x) dx \leq C\kappa r.$$

Thus,

$$(c - C2^d \kappa)^2 \leq \frac{C_d}{|B_r|} \left(\Lambda|B_r \cap \{u = 0\}| + \beta C^2 r^{d+1} \right).$$

Choosing κ and $R > 0$ small enough, we get the claim. □

Proof of Proposition 8.1 Suppose that $x_0 \in D$ is such that

$$u(x_0) = 0 \quad \text{and} \quad \delta := \text{dist}(x_0, \partial E_1) < \delta_0,$$

where the constant $\delta_0 > 0$ will be chosen later. Let y_0 be the projection of x_0 on E_1 and let $R := 2|x_0 - y_0|$. We consider the competitor

$$\tilde{\Omega}_1 := \Omega_1 \cup B_R(y_0) \quad \tilde{\Omega}_2 := \Omega_2 \setminus B_R(y_0) \quad \tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \mathbb{R}^d \setminus (B_R(y_0) \cap D) \\ h(x) & \text{if } x \in B_R(y_0) \cap D \end{cases},$$

h being the harmonic extension of u in $B_R(y_0) \cap D$. We notice that \tilde{u} is the solution to

$$\min \left\{ \int_{B_R(y_0)} |\nabla v|^2 dx \mid u - v \in H_0^1(B_R(y_0)) \right\}.$$

Thus, by [15, Lemma 3.7] and the fact that $u = g$ on E_1 , we have that

$$\frac{1}{R^2} |\{u = 0\} \cap B_R(y_0)| \left(\int_{\partial B_R(y_0)} u d\mathcal{H}^{d-1} \right)^2 \leq C_d \int_{B_R(y_0)} |\nabla(u - h)|^2 dx. \quad (8.2)$$

On the other hand, using Lemma 6.1, we get

$$\int_{B_R(y_0)} |\nabla(u - h)|^2 dx \leq \Lambda |B_R(y_0) \cap \{u = 0\}| + \beta \int_{\partial B_R(y_0)} u^2 d\mathcal{H}^{d-1}. \quad (8.3)$$

Now, since $u \equiv 1$ on E_1 and $u \leq 1$ in \mathbb{R}^d , we get that

$$\frac{1}{C_1} \leq \int_{\partial B_R(y_0)} u d\mathcal{H}^{d-1} \quad \text{and} \quad \int_{\partial B_R(y_0)} u^2 d\mathcal{H}^{d-1} \leq d\omega_d R^{d-1},$$

where $C_1 > 0$ is a constant depending only on E_1 (notice that since E_1 is regular, for $\delta > 0$ small, we can choose $C_1 \simeq 2$). Thus, combining (8.2) and (8.3), we get

$$\frac{1}{R^2} |\{u = 0\} \cap B_R(y_0)| \leq C_1^2 C_d \left(\Lambda + \frac{\beta}{R} \right) |B_R|,$$

which by the density estimate Lemma 8.2, gives a contradiction when R is small enough. \square

9 Regularity of the free boundary $\partial\{u > 0\}$

In this section we prove the following.

Theorem 9.1 *Let $u \in H^1(\mathbb{R}^2)$ be the function defined in Sect. 5.1. Then $\partial\{u > 0\} \cap D$ is a $C^{1,\alpha}$ curve.*

The proof is based on the fact that u satisfies an almost-minimality condition in D . Precisely, by Lemma 3.1 and Lemma 6.1, we have that, given a compact set $K \subset D$, there are constants $C > 0$ and $r_0 > 0$ for which

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 dx + \Lambda |B_r(x_0) \cap \{u > 0\}| \\ \leq \int_{B_r(x_0)} |\nabla v|^2 dx + \Lambda |B_r(x_0) \cap \{v > 0\}| + Cr^{d+1}, \end{aligned} \quad (9.1)$$

for every $r < r_0$, $x_0 \in \partial\{u > 0\} \cap D$ and $v \in H^1(B_r(x_0))$ such that $u - v \in H_0^1(B_r(x_0))$. In Sect. 9.1, we use the almost-minimality to show that when $d \leq 4$ every blow-up u is a half-plane solution (that is, solution of the form (9.2)) of the classical Alt-Caffarelli functional. Then, in Sect. 9.2, we show that in dimension $d = 2$, we can use the epiperimetric inequality from [14] to conclude the proof of Theorem 9.1.

Remark 9.2 We notice that the function u might not be smooth in the open set $\{u > 0\}$. In fact, u is not even C^1 as the gradient is not continuous across $\partial\Omega_1 \cap \partial\Omega_2$. We stress that we can still use the 2D epiperimetric inequality from [14] together with the almost-minimality condition (9.1) to prove the $C^{1,\alpha}$ regularity of the free boundary, but we cannot improve this regularity to C^∞ .

Remark 9.3 We expect Theorem 9.1 to hold in every dimension $2 \leq d \leq 4$, as there are several epsilon-regularity results for functions u satisfying almost-minimality conditions similar to (9.1) (see for instance [4, 5, 8, 13]), but we stress that none of these results directly apply to (9.1). In fact, the almost-minimality of u only holds around points at the boundary $\partial\{u > 0\}$ (and in our case u is not even $C^{1,\alpha}$ in $\{u > 0\}$), which essentially requires [4, 5, 8, 13] to be revisited in order to be used in our context. We choose the approach from [13] which limits Theorem 9.1 to the case $d = 2$, but on the other hand is based on the epiperimetric inequality from [14], which works without any modifications in our case.

9.1 Blow-up sequences and blow-up limits

Let $x_0 \in \partial\{u > 0\} \cap D$. We define

$$u_r(x) := \frac{1}{r}u(x_0 + rx).$$

Let r_n be an infinitesimal sequence. Then, for n large enough, the sequence u_{r_n} is uniformly bounded in L^∞ in every ball $B_{2R} \subset \mathbb{R}^d$. Moreover, by Lemma 3.2, u_{r_n} is uniformly bounded also in $C^{0,1/2}(B_R)$. Thus, up to a (non-relabelled) subsequence u_{r_n} it converges uniformly in B_R . By a diagonal sequence argument, there is a continuous function

$$u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$$

and a subsequence u_{r_n} such that u_{r_n} converges to u_0 uniformly on every ball $B_R \subset \mathbb{R}^d$. We will say that u_0 is a blow-up limit of u at x_0 .

Proposition 9.4 *Let $2 \leq d \leq 4$ and let u be the function from Sect. 5.1. Then every blow-up limit $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ of u at a point $x_0 \in \partial\{u > 0\} \cap D$ is of the form*

$$u_0(x) = \sqrt{\Lambda} (x \cdot v)_+ \tag{9.2}$$

for some unit vector $v \in \mathbb{R}^d$.

Proof Let u_{r_n} be a blow-up sequence converging to u_0 . We notice that, by Corollary 7.2, u_0 is non-trivial. Moreover, using the almost-minimality condition (9.1) we get that u_0 is a local minimizer of the Alt-Caffarelli functional (see for instance [1] or [15]). Precisely,

$$\int_{B_R} |\nabla u|^2 dx + \Lambda|B_R \cap \{u > 0\}| \leq \int_{B_R} |\nabla v|^2 dx + \Lambda|B_R \cap \{v > 0\}|,$$

for every $B_R \subset \mathbb{R}^d$ and every $v \in H^1(B_R)$ such that $u - v \in H_0^1(B_R)$. Moreover, the almost-minimality condition (9.1) implies that every blow-up limit u_0 is 1-homogeneous

(see [13]). When $d \leq 4$, using [3] and [10], this gives that every blow-up limit u_0 is of the form (9.2). □

9.2 Epiperimetric inequality and regularity of $\partial\{u > 0\}$

For any $\varphi \in H^1(B_1)$ we consider the Weiss' boundary adjusted energy introduced in [16]

$$W(\varphi) := \int_{B_1} |\nabla\varphi|^2 dx + \Lambda|\{\varphi > 0\} \cap B_1| - \int_{\partial B_1} \varphi^2 d\mathcal{H}^{d-1}.$$

Let K be a compact set contained in D and let $C > 0$ and r_0 be the constants from the almost-minimality condition (9.1). Let $x_0 \in \partial\{u > 0\} \cap K$ be fixed and let $u_r(x) := \frac{1}{r}u(x_0 + rx)$. Then, the derivative of $W(u_r)$ is given by

$$\frac{\partial}{\partial r} W(u_r) = \frac{d}{r}(W(z_r) - W(u_r)) + \frac{1}{r} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^{d-1}(x),$$

where $z_r : B_1 \rightarrow \mathbb{R}$, $z_r(x) := |x|u_r(\frac{x}{|x|})$, is the 1-homogeneous extension of u_r in B_1 . Now, by the 2D epiperimetric inequality of [14], we have that there is a constant $\varepsilon \in (0, 1)$ such that for every $r > 0$, there exists a function $h_r : B_1 \rightarrow \mathbb{R}$ with $h_r = u_r = z_r$ on ∂B_1 and

$$W(h_r) - \Theta \leq (1 - \varepsilon)(W(z_r) - \Theta), \quad \text{where } \Theta := \Lambda \frac{|B_1|}{2}.$$

Now, using the almost minimality (9.1) of u , we have that for every $r \leq r_0$

$$\begin{aligned} \frac{\partial}{\partial r} W(u_r) &= \frac{2}{r}(W(z_r) - W(u_r)) + \frac{1}{r} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^1(x) \\ &= \frac{2}{r} \left((W(z_r) - \Theta) - (W(u_r) - \Theta) \right) + \frac{1}{r} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^1(x) \\ &\geq \frac{2}{r} \left(\frac{1}{1 - \varepsilon} (W(h_r) - \Theta) - (W(u_r) - \Theta) \right) + \frac{1}{r} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^1(x) \\ &\geq \frac{2}{r} \left(\frac{1}{1 - \varepsilon} (W(u_r) - \Theta) - \frac{Cr}{1 - \varepsilon} - (W(u_r) - \Theta) \right) + \frac{1}{r} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^1(x) \\ &\geq \frac{2}{r} \frac{\varepsilon}{1 - \varepsilon} (W(u_r) - \Theta) - \frac{2C}{1 - \varepsilon} + \frac{1}{r} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^1(x). \end{aligned}$$

Taking $\gamma = \frac{2\varepsilon}{1 - \varepsilon}$, we get that

$$\frac{\partial}{\partial r} \left(\frac{W(u_r)}{r^\gamma} + \frac{2Cr^{1-\gamma}}{1 - 3\varepsilon} \right) \geq \frac{1}{r^{1+\gamma}} \int_{\partial B_1} |x \cdot \nabla u - u|^2 d\mathcal{H}^1(x),$$

which implies that

$$W(u_r) \leq C_0 r^\gamma \quad \text{where} \quad C_0 := \frac{W(u_{r_0})}{r_0^\gamma} + \frac{2Cr_0^{1-\gamma}}{1 - 3\varepsilon}.$$

By a standard argument (see for instance [14], [13] or [15]), this implies the uniqueness of the blow-up limit at x_0 and the $C^{1,\alpha}$ -regularity of $\partial\{u > 0\}$ in K , which concludes the proof of Theorem 9.1. □

10 Regularity of the free interface

In this section we prove the following theorem.

Theorem 10.1 *Let u be the function from Sect. 5.1 and let Ω_1 and Ω_2 be the sets constructed in Sect. 5.2. Then,*

- (i) *In any dimension $d \geq 2$, the free interface $\partial^* \Omega_1 \cap (D \cap \{u > 0\})$ is a C^∞ manifold up to a closed singular set of Hausdorff dimension at most $d - 8$;*
- (ii) *In dimension $d = 2$, the contact set $\partial \Omega_1 \cap \partial \Omega_2 \cap (D \cap \partial \{u > 0\})$ is discrete in D ;*
- (iii) *In dimension $d = 2$, in a neighborhood of every point $x \in \partial \Omega_1 \cap \partial \Omega_2 \cap (D \cap \partial \{u > 0\})$ the boundary $\partial \Omega_1 \cap \partial \Omega_2 \cap \{u > 0\}$ is a C^∞ curve, C^1 regular up to the endpoint x , and attaches orthogonally to $\partial \{u > 0\}$ at x .*

10.1 Minimality and regularity of the free interface in $D \cap \{u > 0\}$

Proposition 10.2 (Minimality of the limit sets) *In any $d \geq 2$, let Ω_1 and Ω_2 be the sets from Sect. 5.2 and u be the function from Sect. 5.1. Then, for every open set $A \subset\subset D \cap \{u > 0\}$,*

$$\int_{A \cap \partial^* \Omega_1} u^2 d\mathcal{H}^{d-1} \leq \int_{A \cap \partial^* \tilde{\Omega}_1} u^2 d\mathcal{H}^{d-1}, \tag{10.1}$$

for every set $\tilde{\Omega}_1$ of locally finite perimeter in $D \cap \{u > 0\}$ such that $\tilde{\Omega}_1 \Delta \Omega_1 \subset\subset A$.

In particular, $\partial^ \Omega_1 \cap (D \cap \{u > 0\})$ is a $C^{1,\alpha}$ -regular manifold up to a closed singular set of Hausdorff dimension at most $d - 8$.*

Proof Without loss of generality we can suppose that A is a finite union of balls

$$A = \bigcup_{j=1}^N B_{r_j}(x_j).$$

Let $(u_n, \Omega_n^1, \Omega_n^2) := (u_{\varepsilon_n}, \Omega_{\varepsilon_n}^1, \Omega_{\varepsilon_n}^2)$ be the sequence of minimizers from Sect. 5.1 and Sect. 5.2 converging to (u, Ω_1, Ω_2) . Then, by construction

$$\Omega_n^1 \cap \Omega_n^2 = \emptyset \quad \text{and} \quad (\Omega_n^1 \cap A) \cup (\Omega_n^2 \cap A) = A,$$

and the same holds for the limit sets Ω_1 and Ω_2 . Let now $\tilde{\Omega}_1$ be such that $\tilde{\Omega}_1 \Delta \Omega_1 \subset\subset A$. Notice that we can find a family of balls $B_{\rho_j}(x_j)$, $j = 1, \dots, N$, such that

$$B_{\rho_j}(x_j) \subset B_{r_j}(x_j) \quad \text{for every } j = 1, \dots, N;$$

$$\tilde{\Omega}_1 \Delta \Omega_1 \subset\subset \bigcup_{j=1}^N B_{\rho_j}(x_j) \subset\subset \bigcup_{j=1}^N B_{r_j}(x_j),$$

and such that for every $j = 1, \dots, N$, we have

$$\mathcal{H}^{d-1}(\partial^* \Omega_1 \cap \partial B_{\rho_j}(x_j)) = 0 \quad \text{and} \quad \mathcal{H}^{d-1}(\partial^* \Omega_n^1 \cap \partial B_{\rho_j}(x_j)) = 0 \quad \text{for every } n \geq 1.$$

We define

$$B := \bigcup_{j=1}^N B_{\rho_j}(x_j),$$

and consider the sets

$$\tilde{\Omega}_n^1 := (\tilde{\Omega}_1 \cap B) \cup (\Omega_n^1 \setminus B) \quad \text{and} \quad \tilde{\Omega}_n^2 := (B \setminus \tilde{\Omega}_1) \cup (\Omega_n^2 \setminus B).$$

Testing the optimality of $(u_n, \Omega_n^1, \Omega_n^2)$ with $(u_n, \tilde{\Omega}_n^1, \tilde{\Omega}_n^2)$, we obtain

$$\int_{B \cap \partial^* \tilde{\Omega}_1} u_n^2 + \int_{\partial^* \Omega_n^1 \setminus B} u_n^2 = \int_{\partial^* \tilde{\Omega}_n^1} u_n^2 \geq \int_{\partial^* \Omega_n^1} u_n^2 = \int_{B \cap \partial^* \Omega_n^1} u_n^2 + \int_{\partial^* \Omega_n^1 \setminus B} u_n^2$$

which we can write simply as

$$\int_{B \cap \partial^* \tilde{\Omega}_1} u_n^2 \geq \int_{B \cap \partial^* \Omega_n^1} u_n^2.$$

Using Lemma 2.6 and the uniform convergence of u_n to u , we get

$$\int_{B \cap \partial^* \tilde{\Omega}_1} u^2 = \lim_{n \rightarrow \infty} \int_{B \cap \partial^* \tilde{\Omega}_1} u_n^2 \quad \text{and} \quad \int_{B \cap \partial^* \Omega_1} u^2 \leq \liminf_{n \rightarrow \infty} \int_{B \cap \partial^* \Omega_n^1} u_n^2,$$

which concludes the proof of (10.1). The regularity of $\partial\Omega_1$ in A then follows (as in [9]) from the fact that $u \in C^{0,\alpha}(A)$ and $u \geq t > 0$ in A , for some constant $t > 0$. □

10.2 Minimality of the free interface up to the boundary of $\{u > 0\}$

Proposition 10.3 (Minimality of the limit sets) *In any $d \geq 2$, let Ω_1 and Ω_2 be the sets from Sect. 5.2 and u be the function from Sect. 5.1. Then, for every ball $B_r(x_0) \subset\subset D$,*

$$\int_{B_r(x_0) \cap \partial^* \Omega_1} u^2 d\mathcal{H}^{d-1} \leq \int_{B_r(x_0) \cap \partial^* \tilde{\Omega}_1} u^2 d\mathcal{H}^{d-1}, \tag{10.2}$$

for every set $\tilde{\Omega}_1$ such that $\tilde{\Omega}_1 \Delta \Omega_1 \subset\subset B_r(x_0)$ and which is of finite perimeter in $B_r(x_0)$.

Proof Let $\tilde{\Omega}_1$ be such that $\tilde{\Omega}_1 \Delta \Omega_1 \subset\subset B_r(x_0)$. Since u satisfies the minimality condition from Lemma 7.1, we have that there are a sequence $\delta_n \rightarrow 0$ and a constant $C > 0$ such that

$$\mathcal{H}^{d-1}(B_r(x_0) \cap \partial^* \{u > \delta_n\}) \leq C \quad \text{for every } n \geq 1.$$

Moreover, we can suppose that

$$\mathcal{H}^{d-1}(\partial^* \{u > \delta_n\} \cap \partial^* \Omega_1) = \mathcal{H}^{d-1}(\partial^* \{u > \delta_n\} \cap \partial^* \tilde{\Omega}_1) = 0.$$

Now, consider the sets

$$A_n := B_r(x_0) \cap \{u > \delta_n\} \quad \text{and} \quad \Omega_n := (A_n \cap \tilde{\Omega}_1) \cup (\Omega_1 \setminus A_n).$$

Since $\Omega_n \Delta \Omega_1 \subset\subset B_r(x_0) \cap \{u > 0\}$, by Proposition 10.2, we have that

$$\int_{B_r(x_0) \cap \partial^* \Omega_1} u^2 \leq \int_{B_r(x_0) \cap \partial^* \Omega_n} u^2,$$

which we write as

$$\begin{aligned} & \int_{(B_r(x_0) \cap \partial^* \Omega_1) \cap A_n} u^2 + \int_{(B_r(x_0) \cap \partial^* \Omega_1) \setminus A_n} u^2 \\ & \leq \int_{(B_r(x_0) \cap \partial^* \tilde{\Omega}_1) \cap A_n} u^2 + \int_{(B_r(x_0) \cap \partial^* \Omega_1) \setminus A_n} u^2 + 2 \int_{B_r(x_0) \cap \partial^* A_n} u^2 \\ & \leq \int_{(B_r(x_0) \cap \partial^* \tilde{\Omega}_1) \cap A_n} u^2 + \int_{(B_r(x_0) \cap \partial^* \Omega_1) \setminus A_n} u^2 + 2\delta_n^2 C, \end{aligned}$$

which gives

$$\int_{(B_r(x_0) \cap \partial^* \Omega_1) \cap A_n} u^2 \leq \int_{(B_r(x_0) \cap \partial^* \tilde{\Omega}_1) \cap A_n} u^2 + 2\delta_n^2 C.$$

Passing to the limit as $n \rightarrow \infty$, we get (10.2). □

10.3 Regularity of the free interface up to the boundary $\partial\{u > 0\}$

In this section we will need the $C^{1,\alpha}$ regularity of the free boundary $\partial\{u > 0\}$ in D , so in order to have Theorem 9.1 we assume that $d = 2$.

Let $x_0 \in D \cap \{u > 0\}$ and $B_r(x_0)$ be a (small) ball contained in D . Without loss of generality, we suppose that $x_0 = (0, 0)$. We define the function $h : B_r \rightarrow \mathbb{R}$ as

$$\Delta h = 0 \text{ in } B_r \cap \{u > 0\}, \quad h = u \text{ in } \partial B_r, \quad h = 0 \text{ in } B_r \cap \partial\{u > 0\}.$$

Then, h is $C^{1,\alpha}$ regular in $B_r \cap \{u > 0\}$ up to the boundary $B_r \cap \partial\{u > 0\}$ and moreover, there is a $C^{0,\alpha}$ -regular strictly positive function

$$a : B_r \cap \overline{\{u > 0\}} \rightarrow \mathbb{R}$$

such that

$$a(x) = \frac{h(x)}{u(x)} \text{ for } x \in B_r \cap \{u > 0\}; \quad a(x) = \frac{|\nabla h|(x)}{\sqrt{\Lambda}} \text{ for } x \in B_r \cap \partial\{u > 0\}.$$

Moreover, choosing $r > 0$ small enough the set $B_r \cap \{u > 0\}$ is simply connected, so we can find a function $w : B_r \cap \{u > 0\} \rightarrow \mathbb{R}$ such that

$$\partial_x w = -\partial_y h \quad \text{and} \quad \partial_y w = \partial_x h \quad \text{in } B_r \cap \{u > 0\},$$

w being defined as

$$w(x, y) = \int_{\gamma_{x,y}} (\partial_y h \, dx - \partial_x h \, dy),$$

where $\gamma_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is any C^1 curve connecting $(0, 0)$ to (x, y) in $B_r \cap \{u > 0\}$. Then, the map

$$\Phi : B_r \cap \overline{\{u > 0\}} \rightarrow \mathbb{R} \times \mathbb{R}, \quad \Phi(x, y) := (w(x, y), h(x, y)),$$

is $C^{1,\alpha}$ smooth in $B_r \cap \overline{\{u > 0\}}$ (up to the boundary $B_r \cap \partial\{u > 0\}$) and the set

$$A := \Phi(B_r \cap \overline{\{u > 0\}})$$

is a relatively open subset of the upper half-plane $\{(w, h) \in \mathbb{R}^2 \mid h \geq 0\}$. We notice that for r small enough the function Φ is invertible. Then, we define

$$\varphi : A \cap \{(w, h) \in \mathbb{R}^2 \mid h \geq 0\} \rightarrow \mathbb{R}, \quad \varphi := \frac{1}{|\nabla h|a^2} \circ \Phi^{-1}.$$

Then, φ is $C^{0,\alpha}$ and is bounded from below by a positive constant. We will show that in the new coordinates the set $\Omega := \Phi(\Omega_1)$ locally minimizes the functional

$$\mathcal{F}_2(\Omega) := \int_{\{h \geq 0\} \cap \partial^* \Omega} h^2 \varphi(w, h) \, d\mathcal{H}^1(w, h).$$

In fact, since $\partial^* \Omega_1$ is a C^1 curve, it is sufficient to check that for any

$$\gamma : [0, 1] \rightarrow B_r \cap \overline{\{u > 0\}}, \quad \gamma(t) = (x(t), y(t)),$$

we have

$$\begin{aligned} \int_{\Phi(\gamma)} h^2 \varphi(w, h) &:= \int_0^1 h^2(\gamma(t)) \varphi(\Phi(\gamma(t))) \sqrt{(x' \partial_x w + y' \partial_y w)^2 + (x' \partial_x h + y' \partial_y h)^2} \, dt \\ &= \int_0^1 h^2(\gamma(t)) \varphi(\Phi(\gamma(t))) \sqrt{(x' \partial_y h - y' \partial_x h)^2 + (x' \partial_x h + y' \partial_y h)^2} \, dt \\ &= \int_0^1 h^2(\gamma(t)) \varphi(\Phi(\gamma(t))) \sqrt{(\partial_x h)^2 + (\partial_y h)^2} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \\ &= \int_0^1 u^2(\gamma(t)) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt, \end{aligned}$$

which concludes the proof. In order to conclude the proof of the C^1 regularity of $\partial \Omega_1$, it is sufficient to prove that at the point $(w_0, h_0) = (0, 0)$ the set $\Omega := \Phi(\Omega_1)$ has a unique blow-up limit given by

$$\Omega_0 := \{(w, h) \in \mathbb{R}^2 \mid h \geq 0, w > 0\}.$$

In order to do so, we consider the set

$$\mathcal{R} := \left\{ (w, X) \in \mathbb{R} \times \mathbb{R}^3 \mid (w, |X|) \in \Omega \right\}.$$

Then, \mathcal{R} is a local minimizer of the functional

$$\mathcal{F}_4(\mathcal{R}) := \int_{\partial^* \mathcal{R}} \varphi(w, |X|) \, d\mathcal{H}^3(w, X),$$

among all sets with the same symmetries as \mathcal{R} , that is, all sets of the form

$$\tilde{\mathcal{R}} := \left\{ (w, X) \in \mathbb{R} \times \mathbb{R}^3 \mid (w, |X|) \in \tilde{\Omega} \right\},$$

for some $\tilde{\Omega} \subset \{(w, h) \in \mathbb{R}^2 \mid h \geq 0\}$. Now, by the monotonicity formula for the local minimizers of the area (see for instance [11]), we have that any blow-up limit \mathcal{R}_0 of \mathcal{R} is a cone in \mathbb{R}^4 , which is area-minimizing with respect to perturbations that preserve the symmetries of \mathcal{R} . But then, since the dimension of $\partial \mathcal{R}_0$ is less than 7, we have that $\partial \mathcal{R}_0$ is necessarily a plane (with the same symmetries as \mathcal{R}). Thus, $\partial \mathcal{R}_0$ is orthogonal to the line $\{0, 0, 0\} \times \mathbb{R}$, which concludes the proof of the uniqueness of the blow-up, which implies points (ii) and (iii) of Theorem 10.1.

11 Proof of Theorem 1.2

In order to prove the existence of a solution to (1.3), we observe that as a consequence of an almost-minimality condition involving the one-phase Alt-Caffarelli functional and Theorem 10.1, we have that the sets Ω_1 and Ω_2 constructed in Sect. 5.2 have locally finite perimeter in D . It remains to prove that

- (i) Ω_1 and Ω_2 are sets of finite perimeter in \mathbb{R}^2 ;
- (ii) (u, Ω_1, Ω_2) is a solution to (1.3).

First, for $j = 1, 2$ and $\delta > 0$, we define

$$E_j^\delta := \left\{ x \in \mathbb{R}^2 \setminus E_j \mid \text{dist}(x, E_j) > \delta \right\}.$$

Then, $E_1^\delta \cap \Omega_1$ and $E_2^\delta \cap \Omega_2$ are sets of finite perimeter and, by Proposition 8.1,

$$\text{Per}(E_1^\delta \cap \Omega_1) + \text{Per}(E_2^\delta \cap \Omega_2) < C,$$

where C is a constant that does not depend on δ . Thus, passing to the limit as $\delta \rightarrow 0$, we get that Ω_1 and Ω_2 have finite perimeter. Next, in order to prove (ii), we consider $\tilde{u} \in \mathcal{V}$ and $(\tilde{\Omega}_1, \tilde{\Omega}_2) \in \mathcal{A}(u)$. Testing the optimality of $(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2)$ we get that

$$J_\varepsilon(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2) \leq J_\varepsilon(\tilde{u}, \tilde{\Omega}_1, \tilde{\Omega}_2).$$

Now, the semicontinuity lemma (Lemma 2.6) gives that

$$J_{\beta, \Lambda}(u, \Omega_1, \Omega_2) = \liminf_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon, \Omega_\varepsilon^1, \Omega_\varepsilon^2) \leq \lim_{\varepsilon \rightarrow 0} J_\varepsilon(\tilde{u}, \tilde{\Omega}_1, \tilde{\Omega}_2) = J_{\beta, \Lambda}(\tilde{u}, \tilde{\Omega}_1, \tilde{\Omega}_2),$$

which concludes the proof of the existence.

Moreover, if (u, Ω_1, Ω_2) is any solution to (1.3), then it satisfies the minimality conditions from Lemma 6.1, Lemma 7.1, Proposition 10.2 and Proposition 10.3, so by Theorem 9.1 and Theorem 10.1, the claims (i) and (ii) of Theorem 1.2 follow.

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