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On the Inextensibility Assumption in the Stability of Elastic Rings: Overhaul of a Traditional Paradigm

Federico Guarracino[1]
Department of Structural Engineering,
University of Naples "Federico II",
80125 Napoli, Italy
e-mail: fguarrac@unina.it

Ida Mascolo[1]
Department of Structural Engineering,
University of Naples "Federico II",
80125 Napoli, Italy
e-mail: ida.mascolo@unina.it

One of the oldest and most common structural engineering issues is the elastic buckling of circular rings under external pressure, which has a fundamental importance in a number of applications in general mechanics, engineering and bio-physics, just to name a few. Levy is considered to have provided the first significant solution to this problem in 1884, and most stability textbooks make reference to this original solution, which is based on the Euler–Bernoulli beam model. Following this incipit, over the past 140 years a huge number of papers have continued to analyze many special cases and extensions. However, the majority of these studies tend to build on the a priori assumption of inextensibility of the ring centerline without investigating the real significance and extent of this condition. Here, in the framework of a general nonlinear kinematic, the problem is re-examined from its roots, and it is shown that not only the inextensibility paradigm cannot straightforwardly lead to the classic solution in a coherent energy framework but also the extensibility of the ring centerline is necessary to allow a unified and meaningful treatment of buckling and initial postbuckling behavior for a complete variety of cases. On these bases, some facts and results in the literature are rectified and discussed. [DOI: 10.1115/1.4070586]

Keywords: elastic rings, axial extensibility, buckling, post-buckling

1 Introduction

The phenomenon of buckling of thin-walled rings and curved rods under compressive loads is a rather common experience. The instability arises as a consequence of the fact that, starting from a certain level of the load, the structure finds it more favorable to change its curvature than to deform axially and thus may abruptly shift from a circular shape to an asymmetric one. Elastic buckling of thin-walled rings was first explored by Lévy [1], and a number of expressions for the critical compressive loads are given in several reviews and textbooks [2–5].

Differently from what happens in the case of Euler's strut, i.e., an initially ideal straight column, where the axial deformability can be harmlessly disregarded, the prebuckling geometry of a compressed ring has a much greater influence on its buckling and postbuckling behavior. In fact, not accounting for the axial deformability of the ring is a much more subtle matter and may lead to at least some misunderstandings, if not even misleading results [6].

The scope of the present article is to overhaul the seemingly generally accepted paradigm of axial inextensibility [3,7–9] and to present a solution that instead fully accounts for the prebuckling extensibility of the middle line of the ring. In this manner, it is shown that some hidden issues are present from an energy standpoint in all those formulations that assume, on the contrary, an a priori axial inextensibility of the circular ring.

Since the topic has been the object of a very large number of works, it must be noticed that some past approaches considered the deformability of the ring axis [4,9,10], but they are commonly limited to the determination of buckling loads and do not frame the mechanical foundation of the resulting quasi-inextensible solution in a general architecture.

The presented formulation, in contrast, aims to provide a physically meaningful energy treatment of buckling and initial postbuckling of elastic rings for a number of instances, which allows to rectify some known results and to highlight the features of some types of applied loads.

The plan of the article is the following: first, the problem is restated in full generality with respect to the fundamental kinematics and to the definition of the applied loading (Sec. 2). Then, starting from an energy approach, the equations of equilibrium and their solution are found for a variety of loads and the quasi-inextensibility condition yielded by the Castigliano–Menabrea principle is pointed out and discussed in detail (Sec. 3). A treatment of the initial postbuckling behavior of the system is then proposed (Sec. 4), highlighting the influence of the axial deformation and amending some results in the literature. Finally, on the basis of the proposed approach, some considerations are drawn on the often

[1]Corresponding authors.

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discussed behavior of the different types of applied loads with respect to rigid-body displacements (Sec. 5). In Appendix A, the implications deriving from an a priori assumption of the inextensibility of the ring axis are pointed out and illustrated in detail providing some remarks on the underlying kinematics.

2 The General Framework

2.1 Basic Definitions. The problem is that of the initial buckling of a linearly elastic thin circular ring with a mean radius R and a wall cross section of area A and inertia I, subjected to an external inward pressure p, which is taken as positive (see Fig. 1). The hypotheses are those from the Euler–Bernoulli beam theory with shear deformation considered negligible.

Initially, the middle line of the ring experiences a uniform hoop contraction, ε_0, and its radius decreases evenly. As a result, the ring shrinks while maintaining its original shape (dash-dotted line in Fig. 1). Aside from the shrinking on account of the constant axial compressive force $N_0 = -pR$, for a critical value of the external pressure, p_k, an asymmetric, adjacent or buckled status can take place (thick solid line in Fig. 1).

The expressions of the axial stretch ε (taken positive for elongation), of the curvature, χ of the ring axis and of the radial and tangential components of the displacement, w and v, respectively (see Fig. 1), can be written as follows:

$$\begin{aligned}
\varepsilon_t &= \varepsilon_0 + \varepsilon = \frac{N_0}{\mathrm{EA}} + \varepsilon,\\
\chi_t &= \chi_0 + \chi = \frac{1}{R} - \frac{1}{R + w_0} + \chi \approx \frac{w_0}{R^2} + \chi,\\
w_t &= w_0 + w = R\frac{N_0}{\mathrm{EA}} + w,\\
v_t &= v_0 + v = 0 + v
\end{aligned} \tag{1}$$

where the subscript (t) stands for any total status and the subscript (0) stands for the initial uniformly contracted status. EA is the axial stiffness of the ring wall section, and the prebuckling deformation is assumed infinitesimal.

In a Green-Lagrange framework, the axial stretch, ε, and the curvature, χ, can be related to the components of displacement by the following relationships:

$$\begin{aligned}
\varepsilon &= \sqrt{1 + \frac{2(w+v')}{R} + \frac{(w+v')^2 + (v-w')^2}{R^2}} - 1 =\\
&= \frac{w+v'}{R} + \frac{(v-w')^2}{2R^2} - \frac{(w+v')(v-w')^2}{2R^3}\\
&\quad + \frac{(w+v')^2(v-w')^2}{2R^4} - \frac{(v-w')^4}{8R^4} + \ldots,
\end{aligned} \tag{2}$$

$$\chi = \frac{1}{R}\varphi' = \frac{v' - w''}{R^2}\left[1 + \frac{1}{2}\left(\frac{v-w'}{R}\right)^2 + \ldots\right] \tag{3}$$

where the superscript ($'$) indicates differentiation with respect to the angular coordinate θ and $\varphi = \arcsin\left(\frac{v-w'}{R}\right)$ is the rotation of the cross section of the ring.

2.2 Energy Approach. The total potential energy of the system is given, as usual, by the difference between the strain energy, U, and the work done by the applied forces, W, that is,

$$\Pi = U - W \tag{4}$$

The strain energy, accounting for the axial deformability of the ring axis, can be written as follows:

$$\begin{aligned}
U &= \frac{\mathrm{EA}}{2}\int_0^{2\pi} {\varepsilon_t}^2 ds + \frac{\mathrm{EI}}{2}\int_0^{2\pi} {\chi_t}^2 ds =\\
&= \frac{\mathrm{EA}}{2}\int_0^{2\pi} ({\varepsilon_0}^2 + 2\varepsilon_0\varepsilon + \varepsilon^2) ds\\
&\quad + \frac{\mathrm{EI}}{2}\int_0^{2\pi} ({\chi_0}^2 + 2\chi_0\chi + \chi^2) ds
\end{aligned} \tag{5}$$

where EI is the flexural stiffness of the ring wall section and $ds = Rd\theta$. Equation (5) can be conveniently rearranged in the following form:

$$U = U_0 + U_1 + U_{0,1} \tag{6}$$

where U_0 is the strain energy associated with the initial uniform hoop contraction, ε_0, U_1 is the strain energy associated with the initial buckling shape, and $U_{0,1}$ is the interaction term, that is,

$$\begin{aligned}
U_0 &= \frac{\mathrm{EA}}{2}\int_0^{2\pi} \varepsilon_0^2 R d\theta + \frac{\mathrm{EI}}{2}\int_0^{2\pi} \chi_0^2 R d\theta,\\
U_1 &= \frac{\mathrm{EA}}{2}\int_0^{2\pi} \varepsilon^2 R d\theta + \frac{\mathrm{EI}}{2}\int_0^{2\pi} \chi^2 R d\theta,\\
U_{0,1} &= \frac{\mathrm{EA}}{2}\int_0^{2\pi} 2\varepsilon_0\varepsilon R d\theta + \frac{\mathrm{EI}}{2}\int_0^{2\pi} 2\chi_0\chi R d\theta
\end{aligned} \tag{7}$$

It will be shown in the following that the interaction term $U_{0,1}$ plays a fundamental role both in the buckling and postbuckling behavior of the ring.

The work done by the external load during the initial uniform hoop contraction,

$$W_0 = \int_0^{2\pi} -pw_0 R d\theta = \frac{2\pi p^2 R^3}{EA} \tag{8}$$

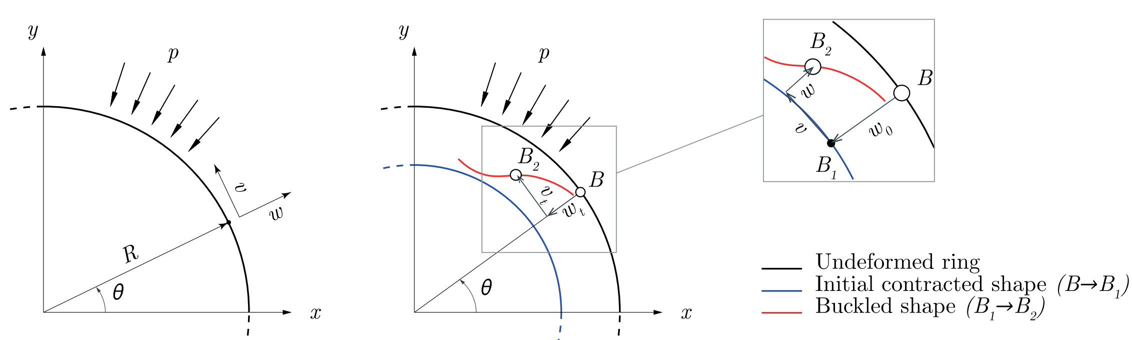

Fig. 1 A circular ring under external pressure, from left to right: initial configuration, uniform contracted status, and buckled status. The radial component of displacement is taken as positive when outward; the tangential component is taken as positive when directed as the polar angle θ, which is anticlockwise; and the pressure p is taken as positive when inward.

only depends on the radial component of the displacement and it is not affected by the particular loading type.

On the contrary, when the ring attains the buckled configuration the work done by the applied pressure, W_1, involves additional terms that depend on the type of loading under consideration.

The types of loading that are usually accounted for the problem at hand are as follows:

- *dead load* (*d*.): the load remains normal to the undeflected axis of the element;
- *hydrostatic load* (*h*.): the load remains normal to the deflected axis of the element. It is the case of the pressure exerted by liquids or gases;
- *centrally directed load* (*c*.): the load remains directed toward the center of the ring;
- *inverse-square central load* (*is*.): the load remains directed toward the center of the ring, and its magnitude obeys to an inverse-square law with respect to the distance from the center.

It is assumed that in the case of dead, centrally directed, and inverse-square central loads, the force is applied per unit initial length of the ring in accordance with the literature and follows from the fact that in the last two cases, the loading is generally conceived as produced by a very large number of closely spaced radial cables attached to the undeformed configuration, like a wheel subjected to many thin spokes under tension. This experimental arrangement was originally devised by Schmidt [11] and recently realized by Gaibotti et al. [12].

A schematic representation of the different loads behaviors is given in Fig. 2.

It is worth noticing that all types of loading taken into account can be considered conservative [13] only under certain conditions, which will be discussed more in detail in Sec. 5. In fact, as pointed out by Oran [14], certain degree of misconception can be found in the literature, likely fueled by the fact that the conservative character of loads initially normal to the surface, although conditional, is often taken for granted.

In the actual fact, the work done on the ring by the dead load (*d*.) is that of a force that is constant in direction and magnitude and depends only on the initial and final configurations in the absence of any rigid-body rotation, see Sec. 5.

The work done by the hydrostatic load (*h*.) is that of a pressure of a liquid at rest, which in the case at hand is obviously a conservative force [15]. Finally, the work done by the centrally directed loads, (*c*.) and (*is*.), is the product of the force by the change in the distance from the pole, so that it results independent of the path in the absence of any rigid-body translation, see again Sec. 5.

The work of the applied load can be thus written as follows:

$$W^i = W_0 + W_1^i, \qquad i = d,\, h,\, c,\, is \tag{9}$$

where the superscript i stands for the different loading cases.

In the loading case (*d*.), the work of the elementary force $p^d ds = pRd\theta$ only involves the radial displacement component, w (Fig. 2) and can be written as follows:

$$W_1^d = -\frac{p}{2}\int_0^{2\pi} 2Rw d\theta \tag{10}$$

The work of the hydrostatic load, W_1^h, at buckling can be expressed in terms of the product of the pressure p times the change in the area enclosed by the centroidal surface of the ring, which can be exactly calculated by means of the following formula, as shown in Ref. [4]:

$$W_1^h = -\frac{p}{2}\int_0^{2\pi} w\left(w + v'\right) + v\left(v - w'\right) d\theta \tag{11}$$

In the loading case (*c*.), the work of the applied pressure is associated with the variation in the radius length ΔR and results in

$$\begin{aligned} W_1^c &= -p\int_0^{2\pi} \Delta R R d\theta = \\ &= -p\int_0^{2\pi} \left(\sqrt{(R+w)^2 + v^2} - R\right) R d\theta = \\ &= W_1^d - \frac{p}{2}\int_0^{2\pi} \left[v^2\left(1 - \frac{w}{R} + \frac{w^2}{R^2} - \frac{v^2}{4R^2}\right) + \ldots \right] d\theta \end{aligned} \tag{12}$$

Finally, in the loading case (*is*.), the load intensity is proportional to $\frac{R^2}{(R+\Delta R)^2}$, so that, by integrating over [0, ΔR], it is $p\frac{R\Delta R}{R+\Delta R}$ and the work results in the following equation:

$$\begin{aligned} W_1^{is} &= -\int_0^{2\pi} p\frac{R\Delta R}{R+\Delta R} R d\theta = \\ &= -pR^2\int_0^{2\pi} \left(1 - \frac{R}{\sqrt{(R+w)^2 + v^2}}\right) d\theta = \\ &= W_1^d - \frac{p}{2}\int_0^{2\pi} \left[v^2\left(1 - \frac{3w}{R} + \frac{6w^2}{R^2} - \frac{3v^2}{4R^2}\right) \right. \\ &\quad \left. + 2w^2\left(-1 + \frac{w}{R} - \frac{w^2}{R^2}\right) + \ldots \right] d\theta \end{aligned} \tag{13}$$

3 Buckling Analysis

Solely for the buckling analysis, the treatment of the problem can be simplified by retaining only the terms up to the second-order of magnitude in the displacement field. Thus, the expression of the total potential energy of the system

$$\overline{\Pi}^i = \overline{U} - \overline{W}^i \tag{14}$$

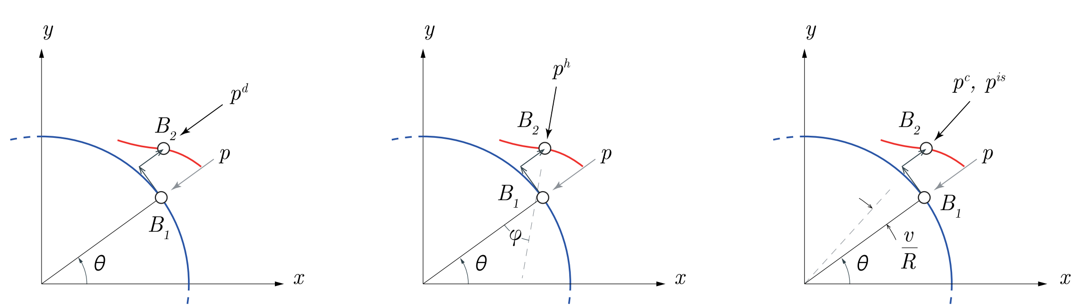

Fig. 2 Types of loading: hydrostatic load, p^h, dead load, p^d, centrally directed load, p^c, and inverse-square central load, p^{is}. The dash-dotted and thick solid lines represent the initially contracted shape and the buckled shape, respectively.

becomes

$$\overline{\Pi}^d = U_0 + W_0 + \frac{EA}{2R}\int_0^{2\pi}\left(w+v'\right)^2 d\theta + \frac{EI}{2R^3}\int_0^{2\pi}\left(v'-w''\right)^2 d\theta - \frac{p}{2}\int_0^{2\pi}\left(v-w'\right)^2 d\theta,$$
$$\overline{\Pi}^h = \overline{\Pi}^d + \frac{p}{2}\int_0^{2\pi} w\left(w+v'\right) + v\left(v-w'\right)d\theta, \tag{15}$$
$$\overline{\Pi}^c = \overline{\Pi}^d + \frac{p}{2}\int_0^{2\pi} v^2 d\theta,$$
$$\overline{\Pi}^{is} = \overline{\Pi}^d + \frac{p}{2}\int_0^{2\pi} v^2 - 2w^2 d\theta$$

where the overline denotes the second-order approximation.

By introducing the radius of gyration $\rho = \sqrt{I/A}$, $H = (\rho/R)^2$, the load coefficient $\lambda = R^3 p/\mathrm{EI}$, and setting $\kappa = 2R^3/EI$ and $\widetilde{\Pi}_0 = \kappa(U_0 + W_0) = \mathrm{const}$, the total potential energy of the system can be written in a dimensionless form $\widetilde{\Pi} = \kappa\Pi$ as follows:

$$\widetilde{\Pi}^d = \widetilde{\Pi}_0 + \frac{1}{H}\int_0^{2\pi}\left(w+v'\right)^2 d\theta + \int_0^{2\pi}\left(v'-w''\right)^2 d\theta - \lambda\int_0^{2\pi}\left(v-w'\right)^2 d\theta - 2H\lambda\int_0^{2\pi}(v'-w'')d\theta,$$
$$\widetilde{\Pi}^h = \widetilde{\Pi}^d + \lambda\int_0^{2\pi} w\left(w+v'\right) + v\left(v-w'\right)d\theta, \tag{16}$$
$$\widetilde{\Pi}^c = \widetilde{\Pi}^d + \lambda\int_0^{2\pi} v^2 d\theta,$$
$$\widetilde{\Pi}^{is} = \widetilde{\Pi}^d + \lambda\int_0^{2\pi} v^2 - 2w^2 d\theta$$

3.1 Euler–Lagrange Equations. The Euler–Lagrange equations associated with the stationary value of the energy functional $\widetilde{\Pi}^i$ result in the following equation:

$$\begin{aligned}
d.\quad & Hw^{(3)} - (1+H)v'' - (1-H\lambda)w' - H\lambda v = 0\\
& H\left(\lambda w'' - v^{(3)} + w^{(4)}\right) + (1-H\lambda)v' + w = 0\\
h.\quad & Hw^{(3)} - (1+H)v'' - w' = 0\\
& H\left(\lambda w'' - v^{(3)} + w^{(4)}\right) + v' + (1+H\lambda)w = 0\\
c.\quad & Hw^{(3)} - (1+H)v'' - (1-H\lambda)w' = 0\\
& H\left(\lambda w'' - v^{(3)} + w^{(4)}\right) + (1-H\lambda)v' + w = 0\\
is.\quad & Hw^{(3)} - (1+H)v'' - (1-H\lambda)w' = 0\\
& H\left(\lambda w'' - v^{(3)} + w^{(4)}\right) + (1-H\lambda)v' + (1-2H\lambda)w = 0
\end{aligned} \tag{17}$$

These equations can be written in the compact form, in a similar manner to Refs. [5,16], as follows:

$$\begin{cases} L_1^i w + L_2^i v = 0 \\ L_1^i v + L_3^i w = 0 \end{cases} \tag{18}$$

where the operators $\boldsymbol{L}^i = \{L_1^i, L_2^i, L_3^i\}$ are given as follows:

$$\begin{aligned}
\boldsymbol{L}^d &= \{Hf^{(3)} - (1-H\lambda)f', (1+H)f'' + H\lambda f,\\
&\quad Hf^{(4)} + H\lambda f'' + f\}\\
\boldsymbol{L}^h &= \{Hf^{(3)} - f', (1+H)f'',\\
&\quad Hf^{(4)} + H\lambda f'' + (1+H\lambda)f\}\\
\boldsymbol{L}^c &= \{Hf^{(3)} - (1-H\lambda)f', (1+H)f'',\\
&\quad Hf^{(4)} + H\lambda f'' + f\}\\
\boldsymbol{L}^{is} &= \{Hf^{(3)} - (1-H\lambda)f', (1+H)f'',\\
&\quad Hf^{(4)} + H\lambda f'' + (1-2H\lambda)f\}
\end{aligned} \tag{19}$$

By manipulating Eqs. (18), it is possible to write

$$\begin{cases} L_1^i L_3^i w + L_2^i L_3^i v = 0 \\ L_1^i L_3^i w + L_1^i L_1^i v = 0 \end{cases} \tag{20}$$

and by a simple addition, it is

$$\left(L_2^i L_3^i - {L_1^i}^2\right)v + \left(L_1^i L_3^i - L_1^i L_3^i\right)w = 0 \tag{21}$$

Since $(L_1^i L_3^i - L_1^i L_3{}^i) = 0$, the following expression

$$\left(L_2^i L_3^i - {L_1^i}^2\right)v = 0 \tag{22}$$

that involves the tangential component of displacement, v, only can be obtained. For each of the different loading cases, it results

$$\begin{aligned}
d.\quad & v^{(6)} + (2+\lambda)v^{(4)} + (1+2\lambda)v'' + \lambda v = 0,\\
h.\quad & v^{(6)} + (2+\lambda+H\lambda)v^{(4)} + (1+\lambda+H\lambda)v'' = 0,\\
c.\quad & v^{(6)} + (2+\lambda-H\lambda)v^{(4)} + (1+(2-H\lambda)\lambda)v'' = 0,\\
is.\quad & v^{(6)} + (2+\lambda-H\lambda)v^{(4)} + (1-(2+\lambda)H\lambda))v'' = 0
\end{aligned} \tag{23}$$

Finally, noticing that for thin rings the deformation of the axis, i.e., $H\lambda = Rp/EA$, can be neglected with respect to the unity at the onset of buckling, it is possible to simplify Eqs. (23) as follows:

$$\begin{aligned}
d.\quad & v^{(6)} + (2+\lambda)v^{(4)} + (1+2\lambda)v'' + \lambda v = 0,\\
h.\quad & v^{(6)} + (2+\lambda)v^{(4)} + (1+\lambda)v'' = 0,\\
c.\quad & v^{(6)} + (2+\lambda)v^{(4)} + (1+2\lambda)v'' = 0,\\
is.\quad & v^{(6)} + (2+\lambda)v^{(4)} + v'' = 0
\end{aligned} \tag{24}$$

3.2 The Principle of Castigliano–Menabrea and the Quasi-Inextensibility of the Ring Axis. The condition for the existence of two adjacent equilibrium states of the ring, the circular and an asymmetric one, under the same load intensity is that the first variation of the potential energy with respect to a small displacement is zero.

Now, by virtue of the principle of minimum strain energy by Castigliano–Menabrea [17], only the displacement fields that minimize the dimensionless functional

$$\widetilde{U} = \kappa U_0 + \frac{1}{H}\int_0^{2\pi}\left(w+v'\right)^2 d\theta + \int_0^{2\pi}\left(v'-w''\right)^2 d\theta - \lambda\int_0^{2\pi}\left(v-w'\right)^2 d\theta \tag{25}$$

are actual states of adjacent equilibrium for the ring. Given that the constant term κU_0 and the last term are immaterial to the problem, attention can be focused on the sum

$$\frac{1}{H}\int_0^{2\pi}\left(w+v'\right)^2 d\theta + \int_0^{2\pi}\left(v'-w''\right)^2 d\theta \tag{26}$$

Therefore, since

$$\frac{1}{H} >> 1 \tag{27}$$

all the displacement fields for which the integral

$$\int_0^{2\pi}\left(w+v'\right)^2 d\theta \tag{28}$$

has a value different from zero are not apt to minimize the strain energy (25).

This means that the resulting kinematic condition

$$w = -v' \tag{29}$$

that involves the linear part of Eq. (2) has to be interpreted as a "quasi-inextensibility" outcome that arises at the act of buckling as a natural consequence of the principle of minimum strain energy by Castigliano–Menabrea.

On the contrary, several works, such as Refs. [7,9,12], assume Eq. (29) as an a priori kinematic condition that implies some hidden inconsistencies (see Appendix A) and also has relevant consequences on the analysis of the initial postbuckling behavior of the ring.

3.3 Solution of the Buckling Problem. The solution of the differential equations (24) provides

$$\begin{aligned}
v^d &= C_1^d \sin\theta + C_2^d \cos\theta C_3^d \theta \sin\theta + C_4^d \theta\cos\theta \\
&\quad + C_5^d \sin\theta\sqrt{\lambda} + C_6^d \cos\theta\sqrt{\lambda} \\
v^h &= \frac{C_1^h - C_2^h}{(\lambda^h)^2}\sinh(\theta\lambda^h) + \frac{C_1^h + C_2^h}{(\lambda^h)^2}\cosh(\theta\lambda^h) \\
&\quad - C_3^h\cos(\theta) - C_4^h \sin(\theta) + C_5^h + C_6^h\theta \\
v^c &= C_1^c \sinh(\theta\lambda_1^c) + C_2^c \sinh(\theta\lambda_2^c) + C_3^c\cosh(\theta\lambda_1^c) \\
&\quad + C_4^c\cosh(\theta\lambda_2^c) + C_5^c + C_6^c\theta \\
v^{is} &= \frac{C_1^{is} - C_2^{is}}{(\lambda_2^{is})^2}\sinh(\theta\lambda_2^{is}) + \frac{C_1^{is} + C_2^{is}}{(\lambda_2^{is})^2}\cosh(\theta\lambda_2^{is}) \\
&\quad + \frac{C_3^{is} - C_4^{is}}{(\lambda_1^{is})^2}\sinh(\theta\lambda_1^{is}) + \frac{C_3^{is} + C_4^{is}}{(\lambda_1^{is})^2}\cosh(\theta\lambda_1^{is}) \\
&\quad + C_5^{is} + C_6^{is}\theta
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
\lambda^h &= \sqrt{-1-\lambda}, \\
\lambda_{1,2}^c &= \sqrt{-\left(1+\frac{\lambda}{2}\right) \pm \sqrt{\lambda\left(\frac{\lambda}{4}-1\right)}}, \\
\lambda_{1,2}^{is} &= \sqrt{-\left(1+\frac{\lambda}{2}\right) \pm \sqrt{\lambda\left(\frac{\lambda}{4}+1\right)}}
\end{aligned} \tag{31}$$

At this stage, in order to rule out from buckling modes any rigid-body translation, all linear combinations of $\sin\theta$ and $\cos\theta$ need to be null, along with any constant term that can be interpreted as a rigid-body rotation, as it will be discussed in Sec. 5. Moreover, for 2π-periodicity, any combination of the secular terms θ, $\theta\cos\theta$, and $\theta\sin\theta$ must vanish.

Solutions (30) can be thus rewritten as follows:

$$\begin{aligned}
v^d &= C_5^d \sin\theta\sqrt{\lambda} + C_6^d\cos\theta\sqrt{\lambda}, \\
v^h &= \frac{C_1^h - C_2^h}{(\lambda)^2}\sinh(\theta\lambda) + \frac{C_1^h + C_2^h}{(\lambda)^2}\cosh(\theta\lambda), \\
v^c &= C_1^c\sinh(\theta\lambda_1) + C_2^c\sinh(\theta\lambda_2) + C_3^c\cosh(\theta\lambda_1) \\
&\quad + C_4^c\cosh(\theta\lambda_2), \\
v^{is} &= \frac{C_1^{is} - C_2^{is}}{(\lambda_2^{is})^2}\sinh(\theta\lambda_2^{is}) + \frac{C_1^{is} + C_2^{is}}{(\lambda_2^{is})^2}\cosh(\theta\lambda_2^{is}) \\
&\quad + \frac{C_3^{is} - C_4^{is}}{(\lambda_1^{is})^2}\sinh(\theta\lambda_1^{is}) + \frac{C_3^{is} + C_4^{is}}{(\lambda_1^{is})^2}\cosh(\theta\lambda_1^{is})
\end{aligned} \tag{32}$$

By imposing the following general conditions of continuity

$$v[0] = v[2\pi], \quad v'[0] = v'[2\pi], \ldots, v^{(n)}[0] = v^{(n)}[2\pi] \tag{33}$$

a system of homogeneous equations is obtained from Eq. (32). The determinant of the coefficients of these equations can be made equal to zero, and a set of characteristic equations is obtained whose roots represent the eigenvalues λ_{cr} of the system as functions of the natural numbers $n = 0, 1, 2, \ldots$

$$\begin{aligned}
d. &\quad \lambda_{cr}^d = 4n^2, \quad \lambda_{cr}^d = (1+2n)^2, \\
h. &\quad \lambda_{cr}^h = 4n^2 - 1, \quad \lambda_{cr}^h = 4n(1+n), \\
c. &\quad \lambda_{cr}^c = \frac{(n^2-1)^2}{n^2-2}, \\
is. &\quad \lambda_{cr}^{is} = \frac{(n^2-1)^2}{n^2}
\end{aligned} \tag{34}$$

For dead loading (*d.*), the lowest physically admissible eigenvalue would seem to be $\lambda_{cr} = 1$, but it corresponds to an unbuckled mode and therefore does not indicate an equilibrium bifurcation point. Actually, for dead load, the lowest significant eigenvalue results $\lambda_{cr} = 4$.

The lowest physically admissible eigenvalue for the case of hydrostatic loading (*h.*) is obtained, for $n = 1$, $\lambda_{cr} = 3$. This result coincides, in full generality, with the classical solution by Lévy [1].

In the case of centrally directed loading (*c.*), the lowest physically significant eigenvalue for $n = 2$ is $\lambda_{cr} = 9/2$, and in the case of inverse-square central loading (*is.*) for $n = 2$ is $\lambda_{cr} = 9/4$.

Overall, the solutions of Eqs. (24), which actually correspond to the lowest significant bifurcation loads in the absence of any rigid-body displacement, result in the following equation:

$$\begin{aligned}
d. &\quad \lambda_{cr}^d = 4, \quad v^d = C\cos 2\theta, \quad \text{and} \quad w^d = 2C\sin 2\theta, \\
h. &\quad \lambda_{cr}^h = 3, \quad v^h = C\cos 2\theta, \quad \text{and} \quad w^h = 2C\sin 2\theta, \\
c. &\quad \lambda_{cr}^c = \frac{9}{2}, \quad v^c = C\cos 2\theta, \quad \text{and} \quad w^c = 2C\sin 2\theta, \\
is. &\quad \lambda_{cr}^{is} = \frac{9}{4}, \quad v^{is} = C\cos 2\theta, \quad \text{and} \quad w^{is} = 2C\sin 2\theta
\end{aligned} \tag{35}$$

As anticipated in Sec. 3.2, these solutions naturally satisfy Eq. (29).

4 Analysis of the Initial Postbuckling Behavior

In order to analyze the postbuckling behavior of an elastic ring under the various loading conditions examined in the previous sections, resort is here made once again to the general energy formulation given in Sec. 2.

In fact, while a classic procedure for examining the initial postbuckling behavior of a structure developed by Koiter [18] involves the solution of a set of linear equations to determine the small initial changes in the buckled form, a more straightforward procedure, as discussed in detail by Thompson [19], is to use the known linear buckling mode as the assumed form for a nonlinear Rayleigh–Ritz analysis.

Actually, it is assumed that the deformation of the elastic ring can be analyzed into mode-forms, represented by the eigenfunctions (35) of the buckling problem, the amplitudes of those will supply the generalized coordinates for the system.

Of course, the method can only be used in the context of an appropriate nonlinear energy formulation, which is here provided by Eqs. (4), (5), and (9), accounting for both axial and flexural deformability of the ring wall.

The behavior of the ring is thus studied near the first critical equilibrium state by analyzing the nonlinear solution of the equilibrium equations $\partial\Pi^i/\partial C$, which can be considered correct in the immediate vicinity of this state. The amplitude C of the critical modes (35) is the only free parameter in the analysis.

For the initial postbuckling analysis, the expression of the total potential energy of the system (4) is approximated up to the fourth order.

Table 1 presents total potential energy terms obtained by the use of critical modes (35), and the equilibrium conditions $\partial\Pi^i/\partial C$ yield the nonlinear relationships between the load coefficients and the amplitude C of the displacement reported in Table 2.

Table 1 Total potential energy terms (up to the fourth order)

Load	U_0	W_0	U_1	$U_{0,1}$	W_1
d.					0
h.					$\frac{3\pi}{2}C^2p$
c.	$\frac{\pi R^3}{EA}p^2(1+H)$	$\frac{2\pi R^3 p^2}{EA}$	$\frac{243\pi \mathrm{EA}C^2}{32R}\left(\frac{C^2}{R^2}\left(1+\frac{16}{3}H\right)+\frac{64}{27}H\right)$	$-\frac{9\pi}{2}C^2p\left(1-\frac{27}{16}\frac{C^2}{R^2}\right)$	$-\frac{\pi}{2}\left(C^2+\frac{13}{16R^2}C^4\right)p$
is.					$\frac{\pi}{2}\left(7C^2+\frac{297}{16R^2}C^4\right)p$

Table 2 Load coefficients and buckling values

Load type	λ	λ_{cr}
d.	$\left(4+\frac{C^2}{\rho^2}\left(\frac{27}{16}+9H\right)\right)\left(1-\frac{27}{16}\frac{HC^2}{\rho^2}\right)^{-1}$	4
h.	$\left(3+\frac{C^2}{\rho^2}\left(\frac{81}{64}+\frac{27}{4}H\right)\right)\left(1-\frac{81}{64}\frac{HC^2}{\rho^2}\right)^{-1}$	3
c.	$\left(\frac{9}{2}+\frac{C^2}{\rho^2}\left(\frac{243}{128}+\frac{81}{8}H\right)\right)\left(1-2\frac{HC^2}{\rho^2}\right)^{-1}$	9/2
is.	$\left(\frac{9}{4}+\frac{C^2}{\rho^2}\left(\frac{243}{256}+\frac{81}{16}H\right)\right)\left(1-\frac{39}{128}\frac{HC^2}{\rho^2}\right)^{-1}$	9/4

The values of the buckling multipliers λ_{cr} in Table 2 are obtained by taking the limit of the nonlinear relationships, $\lambda = \lambda(C)$, for $C \to 0$ and obviously coincide with those in Eqs. (35).

It is worth noticing that in the case of the dead load (*d.*) being the work of the external pressure null, the buckling phenomenon is triggered by the elastic energy stored in the ring as a consequence of the prebuckling work of the applied pressure.

Beyond the onset of buckling, the nonlinear relationships of Table 2 provide the initial postbuckling behavior for all the considered load cases that is plotted in Fig. 3. The initial postbuckling behavior results stable in all four cases, including the one of inverse-square central loading, differently from what was reported in Ref. [20] as a consequence of a long-lasting debate started by Ref. [8].

These results have been here obtained in full generality, and it is important to point out that the initial postbuckling behavior of elastic rings results mostly governed by the terms related to the axial stiffness, EA, as it happens in the von Karmann analysis of the stability of thin elastic plates [21], and it has been observed in the case of cylindrical shells [22]. This is indicated by the fact that the contribution of the flexural stiffness EI to the postbuckling carrying capacity is represented by the terms in Table 2 that are affected by the factor H, which is $<< 1$.

5 The Role of the Rigid-Body Displacements

The buckling coefficients (35) have been derived in Sec. 3 by ruling out any rigid-body displacement.

In the general energy framework proposed here, a rigid-body displacement field for the ring can be given by the superposition of a horizontal displacement, α_x, and of a vertical displacement, α_y,

$$\begin{aligned} v &= -\alpha_x \sin\theta + \alpha_y \cos\theta \\ w &= \alpha_x \cos\theta + \alpha_y \sin\theta \end{aligned} \tag{36}$$

plus a rotation about the center of the ring, β, which can be derived from basic considerations of differential geometry (see Fig. 4) by solving the following set of equations:

$$\begin{aligned} \Delta x &= v\sin\theta + w\cos\theta \\ &= -R(\cos\theta\cos\beta - \sin\theta\sin\beta - \cos\theta) \approx \\ &\approx R\left(\frac{\beta^2}{2}\cos\theta + \beta\sin\theta\right) + o(\beta^3), \\ \Delta y &= v\cos\theta - w\sin\theta \\ &= R(\sin\theta\cos\beta + \cos\theta\sin\beta - \sin\theta) \approx \\ &\approx R\left(-\frac{\beta^2}{2}\sin\theta + \beta\cos\theta\right) + o(\beta^3) \end{aligned} \tag{37}$$

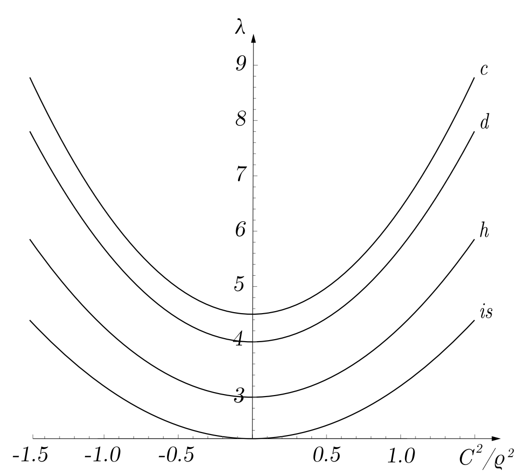

Fig. 3 Postbuckling behavior in the various loading cases

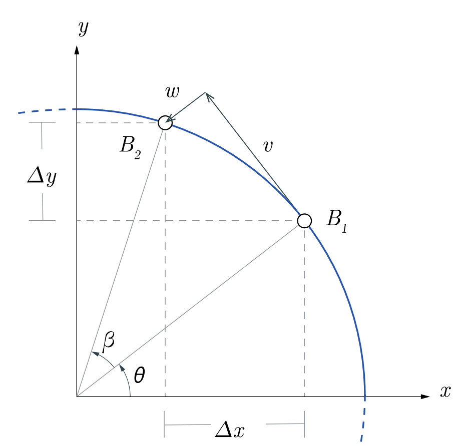

Fig. 4 Components of displacements from a rigid-body rotation

that in the vicinity of the undeformed configuration yields [7]

$$
\begin{aligned}
v &\approx R\beta, \\
w &\approx -R\frac{\beta^2}{2}
\end{aligned}
\tag{38}
$$

By taking only the linear terms in Eqs. (38), the most general solutions of the differential Eqs. (24), i.e., Eq. (30), show that the buckling mode for the dead load (d.) does not contemplate a possible rigid-body rotation and the buckling modes for the centrally directed loads (c. and is.) do not contemplate a possible rigid-body translation.

With respect to the values of the buckling pressure, these findings can be examined in the framework of the proposed energy approach. In fact, the contribution of rigid-body displacements (rd) to the terms of the total potential energy is expressed as follows:

$$
\begin{aligned}
\Pi^d_{rd} &= U^d_{0,1} - W^d_{rd} = 0, \\
\Pi^h_{rd} &= U^h_{0,1} - W^h_{rd} = 0, \\
\Pi^c_{rd} &= U^c_{0,1} - W^c_{rd} = \frac{1}{2}\pi p\left(\alpha_x^2 + \alpha_y^2\right), \\
\Pi^{is}_{rd} &= U^{is}_{0,1} - W^{is}_{rd} = -\frac{1}{2}\pi p\left(\alpha_x^2 + \alpha_y^2\right)
\end{aligned}
\tag{39}
$$

and the contribution of a rigid-body rotation (rr) is given as follows:

$$
\begin{aligned}
\Pi^d_{rr} &= U^d_{0,1} - W^d_{rr} = -\pi\beta^2 pR^2, \\
\Pi^h_{rr} &= U^h_{0,1} - W^h_{rr} = 0, \\
\Pi^c_{rr} &= U^c_{0,1} - W^c_{rr} = 0, \\
\Pi^{is}_{rr} &= U^{is}_{0,1} - W^{is}_{rr} = 0
\end{aligned}
\tag{40}
$$

Equations (39) and (40) reveal that under dead loading (d.), the total potential energy of the ring is affected only by rigid-body rotations and the total potential energy of the ring under centrally directed loads (c. and is.) is affected only by rigid-body translations.

Under the hydrostatic load (h.), whose buckling modes contemplate both rigid-body displacements and rotations, the total potential energy of the ring is not affected by any rigid-body displacement.

This clarifies the structure of the general solutions (30): the buckling modes for dead loading (d.) can accommodate rigid-body translations but not rigid-body rotations without the value of the critical pressure being affected. The buckling modes under centrally directed loads (c. and is.), on the contrary, can accommodate rigid-body rotations but not rigid-body translations without the value of the critical pressures being affected.

The buckling mode under hydrostatic pressure (h.), finally, can accommodate any kind of rigid-body displacement.

Actually, as shown in Fig. 5, differently from what happens in the case of hydrostatic pressure, p^h, where the load setup remains the same regardless of the displacement path, in the case of dead load, p^d, the work done by the load is path dependent with respect to rigid-body rotations, and the loading condition cannot be considered conservative in any case. In fact, if the ring first buckles and then rotates, the buckling loads (34) result unaffected; conversely, if the displacement path begins with a rigid-body rotation, the load arrangement changes, leading to a different structure of the buckling problem.

The same occurs in the case of a centrally-directed load, p^c or p^{is}, in the presence of a rigid-body translation, which induces a change in the position of the ring center and results again in a different loading arrangement with a consequent change in the buckling loads (see Fig. 6).

In order to make the buckling solutions compatible with the presence of external constraints, one can imagine to add the rigid-body displacements to the buckling modes (35) and write

$$
\begin{aligned}
v &= C\left(\cos 2\theta - \alpha_x \sin\theta + \alpha_y \cos\theta + R\beta\right), \\
w &= C\left(2\sin 2\theta + \alpha_x \cos\theta + \alpha_y \sin\theta - R\frac{\beta^2}{2}\right)
\end{aligned}
\tag{41}
$$

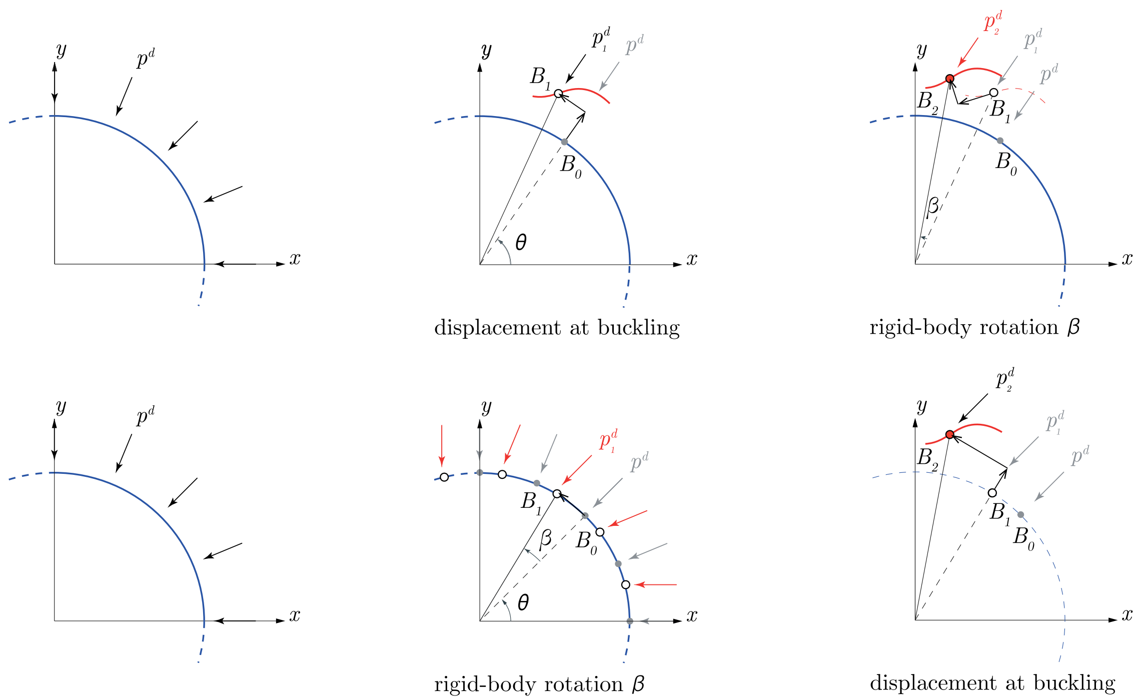

Fig. 5 Different displacement paths under dead load, p^d, in the presence of a rigid-body rotation β: the ring first buckles under the load p^d_1 and then rigidly rotates by β under the load p^d_2 (top panel), or vice versa, it first rigidly rotates by β under the load p^d_1 without altering its geometry and then buckles under the load p^d_2 (bottom panel)

By using these shape functions (41), the proposed Ritz approach would then provide critical loads that can be written as follows:

$$
\begin{aligned}
&d. \quad \lambda_{\mathrm{cr}} = 4\left(1+\frac{2}{9}\beta^2 R^2\right)^{-1},\\
&h. \quad \lambda_{\mathrm{cr}} = 3,\\
&c. \quad \lambda_{\mathrm{cr}} = \frac{9}{2}\left(1-\frac{\alpha_x^2+\alpha_y^2}{8}\right)^{-1},\\
&is. \quad \lambda_{\mathrm{cr}} = \frac{9}{4}\left(1+\frac{\alpha_x^2+\alpha_y^2}{16}\right)^{-1}
\end{aligned}
\tag{42}
$$

where the value of the rigid-body displacement coefficients α_x, α_y, and β can be determined by imposing the external constraints.

Equation (42) can be thus considered to represent an approximate solution of the problem, and it is worth noticing that they are able to return, among the others, the value provided by Chwalla and Kolbrunner [23], i.e., $\lambda_{\mathrm{cr}}^{d} = 3.265$, and discussed by Singer and Babcock in Refs. [7,24] by setting $\beta = R^{-1}$.

In fact, as said earlier, the differential equations of the problem have been derived by a variational formulation built under precise hypotheses on the behavior of the loads. In this respect, Eq. (42) are coherent with the solutions (35) only in the case of $\beta = 0$ for dead load and $\alpha_x = \alpha_y = 0$ for centrally directed loads, representing, in all other occurrences, only an energy estimate for the buckling loads [8].

Also, by imposing the continuity conditions (33) and the 2π-periodicity on the solutions of the differential equations (24), one is left only with solutions that cannot contemplate any discontinuity introduced by the presence of external constraints (as it is the case, for example, of $\lambda = 0.701$, i.e., the critical constant-directional pressure coefficient for the one-clamp ring considered by Schmidt [11]).

Overall, with the notable exception of the hydrostatic case ($h.$), great care must be paid to any external constraint arrangement, which can potentially alter the behavior of the applied load as a consequence of the allowed rigid-body displacements.

6 Conclusions

In the present work, one of the cornerstones of elastic stability, that is the elastic buckling of circular rings under external pressure, has been re-examined from its roots in the framework of an energy approach and on the basis of a general nonlinear kinematics. By overhauling the generally accepted paradigm of axial inextensibility and by presenting a formulation that fully accounts instead for the extensibility of the middle line of the ring, it has been shown that:

(1) the classic a priori inextensibility paradigm cannot straightforwardly lead to the classic solutions in a general energy framework where, on the contrary, the extensibility of the ring wall is necessary to allow a unified and meaningful treatment of buckling and initial postbuckling behavior for a complete variety of cases.

(2) A number of inconsistencies stem from the necessity of adapting the energy formulation to the a priori assumption of inextensibility, such as the paradox of a zero value of the work of the applied pressure at buckling in the case of dead load and several kinematic contradictions.

(3) The hoop deformation of a compressed ring has a significant influence on its postbuckling behavior, making it stable also in cases where it has been traditionally considered unstable.

(4) The possible influence of rigid-body displacements must be read in the framework of the correct loading hypotheses, and it is null only in those cases where the acting pressure can be considered utterly conservative, a fact that contributes to clarifying some known issues in the literature with respect to the influence of external constraints.

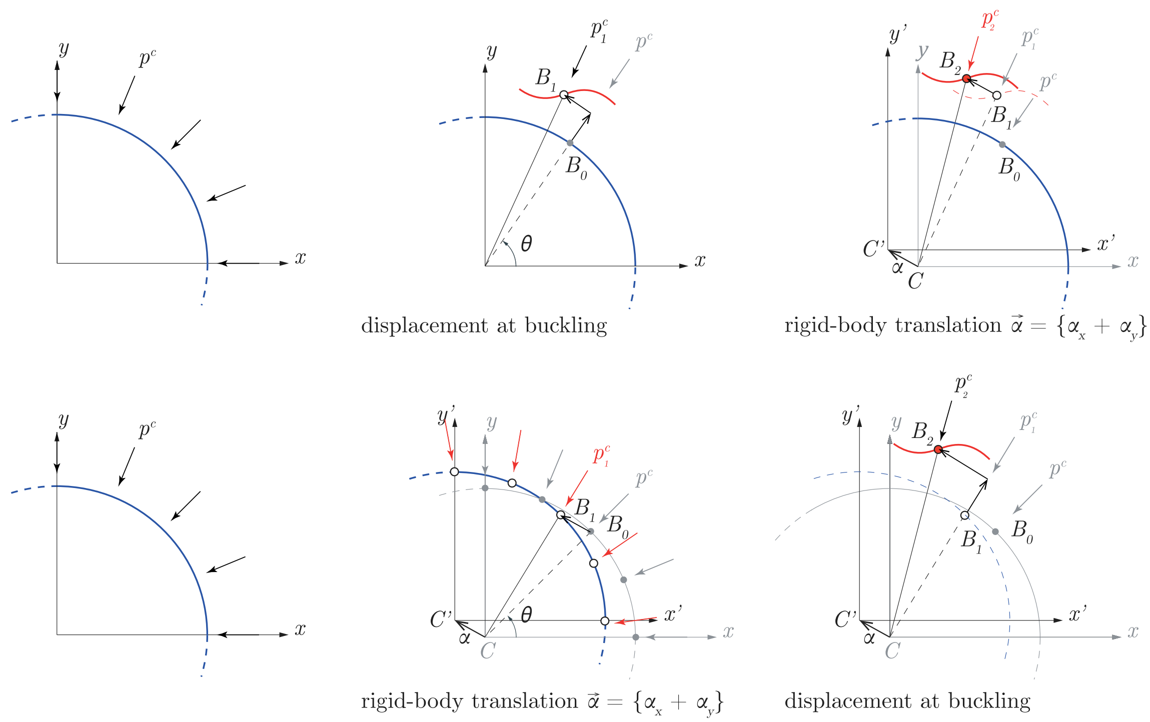

Fig. 6 Different displacement paths under a centrally directed load, p^c, in the presence of a rigid-body translation α: the ring first buckles under the load p_1^c and then rigidly translates by α under the load p_2^c (top panel), or vice versa, it first rigidly translates by α without altering its geometry under the load p_1^c and then buckles under the load p_2^c (bottom panel)

Overall, the proposed unified treatment of prebuckling, buckling, and initial postbuckling of a compressed circular ring marks the fact that its prebuckling configuration has a much greater influence on the whole phenomenon with respect to what happens in the case of an initially ideal straight column. These results could also suggest a line of investigation into problems that involve more complex behaviors of the material, beyond linear elasticity [25].

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Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The authors attest that all data for this study are included in the paper.

Appendix A: Energy Balance and the Issues Deriving From a Common A Priori Inextensibility Assumption

In Sec. 3.2, it has been discussed how the negligibility of the axial deformation, $H\lambda$, of the ring axis with respect to the unity in Eqs. (23) leads to buckling shapes that can be defined as quasi-inextensible.

As a matter of fact, a large part of the published literature over the years [7–9,26] assumes a priori that the ring is inextensible in the sense that Eq. (29) holds true. However, it is immediate to realize that in such a case the straightforward application of a Rayleigh–Ritz procedure with respect to the expression of the total potential energy (14), that is by inserting in the functional

$$\overline{\Pi}^{i}_{inext} = U_0 + \overline{U}_1 + \overline{U}_{0,1} - (W_0 + \overline{W}^{i}_1) = \overline{U}_1 - \overline{W}^{i}_1 \tag{A1}$$

the eigenmodes (35) and making it stationary, $\partial\overline{\Pi}^{i}_{\text{inext}}/\partial C = 0$, returns incorrect values for the buckling loads [6], i.e., no solution in the case (*d.*), $\lambda_{\text{cr}} = 12$ in the case (*h.*), $\lambda_{\text{cr}} = -36$ in the case (*c.*) and $\lambda_{\text{cr}} = 36/7$ in the case (*is.*), see Table 3.

The reason for this apparent paradox is not very obvious and deserves a more in-depth investigation.

In fact, since the strain energy in Eq. (A1) is evaluated only with respect to the bending deformation and by completely neglecting the axial one, the attention can be focused on the work of the applied loads $\overline{W}^{i}_1$.

It can be found that, by assuming the second-order truncation of Eq. (2) to be zero and inserting the consequent condition

$$\frac{w + v'}{R} + \frac{(v - w')^2}{2R^2} = 0 \tag{A2}$$

into the expressions (11)–(13) approximated to the second order, it is

$$\begin{aligned}
\widehat{\overline{W}}{}^{d}_1 &= -\frac{p}{2}\int_0^{2\pi} 2Rv' - (v - w')^2 d\theta = \\
&= \frac{p}{2}\int_0^{2\pi} (v - w')^2 d\theta, \\
\widehat{\overline{W}}{}^{h}_1 &= \widehat{\overline{W}}{}^{d}_1 - \frac{p}{2}\int_0^{2\pi} w\left(w + v'\right) + v\left(v - w'\right) d\theta, \\
\widehat{\overline{W}}{}^{c}_1 &= \widehat{\overline{W}}{}^{d}_1 - \frac{p}{2}\int_0^{2\pi} v^2 d\theta, \\
\widehat{\overline{W}}{}^{is}_1 &= \widehat{\overline{W}}{}^{d}_1 - \frac{p}{2}\int_0^{2\pi} v^2 - 2w^2 d\theta
\end{aligned} \tag{A3}$$

At this point, repeating the previous Rayleigh–Ritz procedure with $\widehat{\overline{W}}{}^{i}_1$ instead of $\overline{W}^{i}_1$, the buckling coefficients λ_{cr} are found to coincide with the correct values (35).

However, even if this approach, that is, for example, implicitly adopted by Budiansky [3], may appear at first sight legit, the devil is in the details and underlies an inconsistency in the kinematics of the model since the eigenmodes (35) satisfy condition (29) but not condition (A2).

More explicitly, on one hand, the condition (A2) is required to modify the expression of the work of the applied forces in order to obtain the correct values of the buckling pressure; on the other hand, the same condition (A2) is not satisfied by the eigenmodes associated with the solution. Also, since these eigenmodes verify the condition (29), should the condition (A2) also be satisfied,

Table 3 Total potential energy terms and corresponding buckling loads

Load type	U_0	W_0	$\overline{U}_1$	$\overline{U}_{0,1}$	$\overline{W}_1 - \overline{U}_{0,1}$	λ_{cr}
Present extensible formulation (Eqs. (4), (5), and (9))						
d.					$9\pi/2C^2p_k$	4
h.	$\frac{\pi R^3 p_k^2}{EA}$	$\frac{2\pi R^3 p_k^2}{EA}$	$\frac{18\pi EI}{R^3}C^2$	$-\frac{9\pi}{2}C^2 p_k$	$6\pi C^2 p_k$	3
c.					$4\pi C^2 p_k$	9/2
is.					$8\pi C^2 p_k$	9/4
A priori inextensible formulation (A1)						
Load type	$\overline{U}_0$	$\overline{W}_0$	$\overline{U}_1$	$\overline{U}_{0,1}$	$\overline{W}_1$	λ_{cr}
d.					0	no sol
h.	0	0	$\frac{18\pi EI}{R^3}C^2$	0	$\pi/2C^2p_k$	12
c.					$3\pi/2C^2p_k$	−36
is.					$7\pi/2C^2p_k$	36/7
A priori inextensible formulation (A1) amended with Eqs. (A3)						
Load type	$\widehat{\overline{U}}_0$	$\widehat{\overline{W}}_0$	$\widehat{\overline{U}}_1$	$\widehat{\overline{U}}_{0,1}$	$\widehat{\overline{W}}_1$	λ_{cr}
d.					$9\pi/2C^2p_k$	4
h.	0	0	$\frac{18\pi EI}{R^3}C^2$	0	$6\pi C^2 p_k$	3
c.					$4\pi C^2 p_k$	9/2
is.					$8\pi C^2 p_k$	9/4

this would imply

$$v - w' = 0 \tag{A4}$$

that is, $\varphi = 0$ and consequently $\chi = 0$, i.e., no possible bending deformation of the ring wall.

The thing is that in an a priori inextensible formulation, the non-occurring terms U_0 and W_0 (Eqs. (7) and (8)) in the correct expression of the total potential energy are immaterial with respect to the variation $\delta U_0/\delta C$ and can be omitted without any harm. On the contrary, the mixed term $U_{0,1}$ (Eq. (7)) cannot be overlooked and plays a hidden but crucial role.

Actually, in the proposed extensible formulation at the onset of buckling, the mixed term $U_{0,1}$ reduces to

$$\overline{U}_{0,1} = -pR\int_0^{2\pi} \frac{(w' - v)^2}{2R^2} R d\theta = -\frac{p}{2}\int_0^{2\pi} \left(w' - v\right)^2 d\theta \tag{A5}$$

and corresponds exactly to the difference $\overline{W}_1 - \widehat{\overline{W}}_1$, as shown in Table 3, a fact that leads to the correct buckling loads without giving rise to any inconsistency.

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