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On the absolute centre and the autocommutator subgroup of a group

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Abstract

Let G be a group. The *autocentre*, or *absolute centre*, of G is the subgroup of G of all the elements fixed by every automorphism of G, while the *autocommutator subgroup* of G is the subgroup of G generated by all the elements $g^{-1}g^{\alpha}$, with $g \in G$ and $\alpha \in \operatorname{Aut} G$. In the present article we introduce some homological tools to study the properties of these subgroups and, by means of these, we extend several results for finite and infinite groups. In particular, we show that the torsion subgroup of the autocentre of G lies almost always inside the Frattini subgroup of G and give polynomial bounds for the order of the autocommutator subgroup of G, improving those previously given by Hegarty.

Keywords: Automorphisms of groups, autocentre, autocommutator.

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1 Introduction

Let G be a group. In 1994, P.V. Hegarty [12], following the same definition given in 1966 by A.P. Šapiro [25] for abelian groups, introduced the *absolute centre* or *autocentre* of G as the set

$$L(G) = \{g \in G \,|\, g^{\alpha} = g, \alpha \in \operatorname{Aut}(G)\}$$

of all the elements of G which are fixed by every automorphism of G. In the same article, Hegarty introduced the *autocommutator subgroup* of G, namely the subgroup

$$G^{*} = \langle g^{-1}g^{\alpha} \,|\, g \in G, \alpha \in \operatorname{Aut}(G)\rangle.$$

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The importance of these subgroups immediately stood out as a way, among other things, to study the holomorph H of G, i.e. the semidirect product between $\mathrm{Aut}(G)$ and G via the natural action of the former on the latter. In fact, it is immediate to see that $L(G)$ equals the centre of H and that G^* is the commutator subgroup $[G, H]$.

In recent years, numerous works have explored the properties of these subgroups and the relations among them, often in connection with finiteness conditions, the isomorphism problem, the study of $\mathrm{Aut}(G)$, and extended central series of groups. We briefly recall a few examples. Concerning finitary conditions, in [12] and [13] the author considered the case where the quotient $G/L(G)$ is finite and examined its relation to the finiteness of G^*. A stronger result, removing the assumption on $G/L(G)$, was later obtained in [6]. More generally, in [7] the authors investigated groups in which $G/L(G)$ is locally finite. Concerning the isomorphism problems, in [5] it has been proved that the isomorphism class of G is totally determined by the isomorphism classes of $L(G)$ and $G/L(G)$, while in [9] the authors found out that all cyclic groups can be the autocentre of some group. We remark here that in the upcoming article [2], it will be proved in particular that every finite abelian group can be the autocentre of a group. Following the study made in [8], different authors studied under many aspects the subgroup of automorphisms of G acting trivially on $G/L(G)$, among which we find [18, 19, 22, 23] and [4]. Moreover, in [17] the authors introduced the so-called *autocentral series* of a group, which in particular has been investigated in greater detail and generality in [7] and in [3]. Finally, we want to mention also that autocentres and autocommutators have recently been used in the context of abelian groups (see [14]), of geometric group theory (see [10]) and of the study of homomorphisms of groups (see [21]).

One of the aims of this work is to introduce some (co)homological tools which are meant to be used to improve and extend results in the different areas described above. Here we will in particular take care of the connection between the autocentre and the Frattini subgroup in first place. In particular we will give the following generalisation to infinite groups of the key theorem of [16], which shows that, essentially, the torsion subgroup of the autocentre of a group lies almost always in the Frattini subgroup of the group.

Theorem A. Let G be a group, let A be the autocentre of G and let T be the torsion subgroup of A. Then the following conditions are equivalent

(1) $G = H \times K$, where H has order 2 and K contains no central subgroup of order 2;

(2) T is not contained in $\Phi(G)$.

In second place, we will apply the tools developed to bound the order of G^* as a function of the index $|G : L(G)|$ of $L(G)$ in G, provided that the index is finite. A first attempt has already been done in [12], where the author found that, if $|G : L(G)| = n$, then $|G^*| \leq n^{n((n-1)^2+\lfloor n/2\rfloor)\lfloor \log_2 n\rfloor}$. Our proof makes use of the Lyndon-Hochschild-Serre spectral sequence as well as some basic properties of the theory of group extensions and shows that the order of $|G^*|$ is always less than or equal to n^3.

Theorem B. Let G be a group, let A be the autocentre of G and let T be the torsion subgroup of A. If G/A is finite of order n, then T is finite and

(i) if T is contained in $\Phi(G)$, then $|T| \leq n^2$ and $|G^*| \leq n^3$;

(ii) if T is not contained in $\Phi(G)$ and $2 \nmid |G^* \cap T|$, then $|T| \leq 2n^2$ and $|G^*| \leq n^3$;

(iii) if T is not contained in $\Phi(G)$ and $2 \mid |G^* \cap T|$, then $|T| \leq n^2/2$ and $|G^*| \leq n^3/2$.

Finally, we give some examples showing that our results are optimal in the sense that their main hypotheses cannot be easily relaxed. However, one of the purposes of this article is to show that the introduction of some homological tools can lead to relevant extensions of classical results and to significant improvements to the bounds already obtained in the particular case of a group with autocentre of finite index. As it is not difficult to show that such a group has finite automorphism group, this article is also meant to be a further step in the study of groups with finitely many automorphisms.

1.1 Notation

$\mathrm{Aut}(G)$: The group of all automorphisms of G.
$\Phi(G)$: the Frattini subgroup of G.
G' : the derived subgroup of G, i.e. the subgroup $[G,G]$.
G_{ab} : the *abelianization* of G, i.e. G/G'.
$\mathrm{Dr}_{i=1}^{n} G_i$: the direct product of the groups $G_1, \ldots, G_n$.
$M(G)$: the Schur Multiplier of G, i.e. the second homology group for trivial group action $H_2(G, \mathbb{Z})$.
M^Q : the set of elements of the Q-module M fixed by the action of the group Q.
φ^* : the map from $H^2(Q_1, A)$ to $H^2(Q, A)$ induced by $\varphi\colon Q \to Q_1$.
φ_* : the map from $H^2(Q, A)$ to $H^2(Q, A_1)$ induced by $\varphi\colon A \to A_1$.

We will employ the additive notation for the abelian groups. Moreover, functions will always act on the right. The rest of the group-theoretical notation is standard and can be found in [20], while for homological results we mainly refer to [24].

2 Tools and results

Here we recall a well-known lemma due to Stammbach (see [24, Proposition II.4.3]) which we will need in the following.

Lemma 2.1. *Let $E\colon A \rightarrowtail G \twoheadrightarrow Q$ and $E_1\colon A_1 \rightarrowtail G_1 \twoheadrightarrow Q_1$ be extensions with abelian kernels and let Δ and Δ_1 be the respective cohomology classes. Then there exists a homomorphism $\gamma\colon G \to G_1$ making the following diagram*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\alpha"'] & G \arrow[r, two heads] \arrow[d, dashed, "\gamma"'] & Q \arrow[d, "\beta"] \\
A_1 \arrow[r, tail] & G_1 \arrow[r, two heads] & Q_1
\end{tikzcd}$$

commute if and only if

(i) α is a homomorphism of Q-modules and

(ii) $\Delta\alpha_* = \Delta_1\beta^*$.

Notice that for central extensions, the first property is always satisfied. In the notation of this lemma, let $E = E_1$. If we now endow $H^2(Q, A)$ with the usual structure of $Aut(A) \times Aut(Q)$-module, it is immediate to rephrase the lemma when α and β are bijective maps: in this case the prescribed (automorphism) γ of G exists if and only if $\Delta(\alpha,\beta) = (\Delta\alpha_*)(\beta^{-1})^* = \Delta$. This will be the way we are going to use the result.

Let $E\colon A \rightarrowtail G \twoheadrightarrow Q$ be a group extension and let Δ be its cohomology class. We will say for brevity that Δ is *surjective* if there exists a 2-cocycle representative of Δ which is a surjective map. It is clear that if Δ is surjective, then any representative of Δ is surjective. Moreover, E will be called an *autocentral extension* if the image of A is contained in the autocentre of G.

We will also use the results in [1] about automorphisms of direct products of groups. This will be done without further reference.

Proposition 2.2. *Let $A \rightarrowtail G \twoheadrightarrow Q$ be an autocentral extension with cohomology class Δ, let C be a direct factor of A which is the direct product of periodic locally cyclic subgroups and let ε be the projection of A onto C. Then either $\Delta\varepsilon_*$ is surjective or C is a direct factor of G of order at most 2.*

Proof. Notice that if C is a direct factor of G, then it can admit only the identity automorphism since it lies in the autocentre of G by hypothesis, and so it has order at most 2. Hence we may assume that C is not a direct factor of G, so that in particular it is not trivial. Moreover, we may assume without loss of generality that C is a locally cyclic p-group for a prime p. Take into account the map $\varepsilon_*\colon H^2(Q,A) \to H^2(Q,C)$ induced by ε and assume for a contradiction that a 2-cocycle $\delta\colon Q \times Q \to C$ representing $\Delta\varepsilon_*$ is not surjective. Let I be the image of δ; we claim that I is a subgroup of C. Let z be any element of I and let u and v be elements of Q such that $(u,v)\delta = z$. Fix a positive integer n and define $\gamma_n\colon (x,y) \in Q\times Q \mapsto (x,y)\delta + (n-1)z \in C$, so that $(u,v)\gamma_n = nz$. As clearly $\gamma_n - \delta$ is a 2-coboundary, both δ and γ_n are representatives of $\Delta\varepsilon_*$, we have that the corresponding group extensions are isomorphic, which leads to the existence of two elements u_n and v_n of Q such that $(u_n,v_n)\delta = nz$. So our claim is proved. Then, I is a proper subgroup of C of order say p^m for a non-negative integer m. Since C is abelian and I has order p^m, the mapping $\varphi\colon c \in C \to (1+p^m)c \in C$ is a non-trivial automorphism of C which acts trivially on I. Let now write $A = B \times C$ and let $\alpha\colon (b,c) \in A \mapsto (b, c\varphi) \in A$. Take into account the long exact cohomology sequence associated to the short exact sequence of trivial Q-modules $B \rightarrowtail A \twoheadrightarrow C$. By naturality, we may find out what the induced action of α on $H^2(Q,A)$ is, namely we have the following commuting diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(Q,B) \arrow[r] \arrow[d, "id_{B*}"'] & H^2(Q,A) \arrow[r] \arrow[d, dashed, "\alpha_*"'] & H^2(Q,C) \arrow[r] \arrow[d, "\varphi_*"] & \cdots \\
\cdots \arrow[r] & H^2(Q,B) \arrow[r] & H^2(Q,A) \arrow[r] & H^2(Q,C) \arrow[r] & \cdots
\end{tikzcd}$$

In fact, let σ and τ be the immersions of B and C into A, respectively. Since $H^2(Q,-)$ is an additive (covariant) functor, it preserves splitting sequences and hence $H^2(Q,A) = \operatorname{Im}\sigma_* \oplus \operatorname{Im}\tau_*$. From this, we may write $\Delta = \Delta_1 + \Delta_2$, with $\Delta_1 \in \operatorname{Im}\sigma_*$ and $\Delta_2 \in \operatorname{Im}\tau_*$. However, we have already shown that the image of any representative of $\Delta\varepsilon_*$, which is Δ_2, has order p^m, so $\Delta\alpha_* = (\Delta_1+\Delta_2)\alpha_* = \Delta_1 id_{B*} + \Delta_2\varphi_* = \Delta_1 + (1+p^m)\Delta_2 = \Delta$. If we now choose β to be the identity map on Q, we immediately get that $(\Delta\alpha_*)\beta^* = \Delta$, so that Lemma 2.1 gives an automorphism of G acting non-trivially on C, a contradiction. □

A first interesting corollary is the following

Corollary 2.3. *Let G be a group and let T be the torsion subgroup of the autocentre A of G. If T has cardinality $\aleph$, then G/A has cardinality at least $\aleph$.*

We now give an example showing that the hypothesis on the torsion part of the autocentre of G cannot be relaxed in the two previous results, even if we take into account an infinite cyclic subgroups of the autocentre. In fact, one can construct a group G with an infinite, countable autocentre A, such that G/A is finite.

Example 2.4. There exists a group whose autocentre is an infinite cyclic group which is not contained in the Frattini subgroup of the group.

Proof. Let p be an odd prime and let q be a prime such that $p^2|q-1$. Let Q be the semidirect product of a cyclic group of order p^2 acting faithfully on a cyclic group of order q. One can easily prove by calculation or using Lemma 2.1 that $\operatorname{Aut} Q$ acts trivially on Q_{ab}. Let now A be an infinite cyclic group, let Δ be an element of order p of

$$\operatorname{Ext}(Q_{ab}, A) \simeq A/p^2A$$

and let G be the central extension of A by Q determined by Δ. As $\operatorname{Aut} Q$ acts trivially on Q_{ab}, it also centralizes Δ, while the only non-trivial automorphism of A does not fix Δ. Hence, Lemma 2.1 shows that A coincides with the autocentre of G. On the other hand, if we let $\tau : Q \longrightarrow G$ be a transversal function, by the definition of Δ we get that $\langle Q^\tau \rangle$ contains pA and not A, so that it is a maximal subgroup of G. This shows in particular that A is not contained in the Frattini subgroup of G. □

Next example shows that one cannot dismiss the locally cyclic assumption in Proposition 2.2. The group in question will be constructed with the aid of a well-known example due to Prüfer of an infinite reduced abelian group which is not the direct sum of cyclic groups.

Example 2.5. There exists a periodic autocentral extension $A \rightarrowtail G \twoheadrightarrow Q$ of cohomology class Δ, such that Δ is not surjective.

Proof. Let p be a prime number and let A be the group generated by the elements $a_0, a_1, \ldots, a_n, \ldots$ and subject to the defining relations

$$pa_0 = 0\,,\ p^i a_i = a_0 \text{ and } a_i + a_j = a_j + a_i \text{ for every } i, j \in \mathbb{N}.$$

For every $i \geq 1$, put $b_i = a_i - pa_{i+1}$ and let B be the subgroup generated by all the b_i. Then B is a basic subgroup of A and A/B is isomorphic to a Prüfer p-group. We now use Dirichlet's theorem on arithmetic progressions to find, for every positive integer i and for $j \in \{1, 2\}$, a couple of primes ${}^jq_i$ such that ${}^jl_ip^i = {}^jq_i - 1$ for a positive integer ${}^jl_i$ and that ${}^mq_i = {}^nq_j$ if and only if $i = j$ and $m = n$. Let ${}^jQ_i$ be the semidirect product of a cyclic group $\langle {}^jx_i \rangle$ of order p^i acting faithfully on a cyclic group $\langle {}^jy_i \rangle$ of order ${}^jq_i$ for $j \in \{1, 2\}$ and a positive integer i, let $Q_i = {}^1Q_i \times {}^2Q_i$, let Q be the direct product of all the Q_i and let P be the subgroup generated by all the ${}^jx_i$.

Let $M(Q)$ be the Schur Multiplier of Q. We want to show that $M(Q)$ is a p-group, which is equivalent to require that its ${}^jq_i$-component is trivial for any positive integer i and $j \in \{1,2\}$. To this aim, put $r = {}^jq_i$, $R = {}^jQ_i$ and let M_r be the r-component of $M(Q)$. Since R' is a normal Sylow r-subgroup of Q, it follows that M_r is isomorphic to the Q-stable subgroup $M(R')^Q$ of $M(R')$, i.e. the subgroup of $M(R')$ consisting of every element which is fixed by the map induced by conjugation of Q on $M(R')$. However, since the action of Q on R' is exactly that of R on it, we have that $M(R')^Q = M(R')^R$ and this is the trivial group because of the fixed-point-free action of R on R'. This shows that $M(Q)$ is a p-group. If we now let $H_n = \mathrm{Dr}_{i=1}^n Q_i$ and let $P_n = P \cap H_n$, which clearly is a Sylow p-subgroup of H_n, it follows from a well-known result by Swan for finite groups (see for instance [15, Theorem 4.3.5]), that $M(H_n)_p \simeq M(N_{H_n}(P_n))_p = M(P_n)_p$, and so, since homology commutes with direct limits, we have that

$$M(Q) = M(Q)_p \simeq M(P)_p = M(P).$$

We are now going to define a bilinear map from the direct product $P \times P$ to A. To this aim, take two elements of P, $x = {}^1x_1{}^{{}^1u_1}\,{}^2x_1{}^{{}^2u_1}\,{}^1x_2{}^{{}^1u_2}\,{}^2x_2{}^{{}^2u_2}\dots{}^1x_n{}^{{}^1u_n}\,{}^2x_n{}^{{}^2u_n}$ and $y = {}^1x_1{}^{{}^1v_1}\,{}^2x_1{}^{{}^2v_1}\,{}^1x_2{}^{{}^1v_2}\,{}^2x_2{}^{{}^2v_2}\dots{}^1x_n{}^{{}^1v_n}\,{}^2x_n{}^{{}^2v_n}$ and let $f\colon P \times P \to A$ be the map defined by

$$(x,y)f = \sum_{i=1}^{n}({}^1u_i\,{}^2v_i - {}^2u_i\,{}^1v_i)b_i.$$

It is easy to check that f is bilinear and hence it can be extended to a bilinear homomorphism $\overline{f}\colon P \otimes P \to A$. On the other hand, it is clear that the subgroup $\langle x \otimes x \,|\, x \in P\rangle$ of $P \otimes P$ lies in the kernel of $\overline{f}$, so that it induces a non-zero homomorphism $\varphi : P \wedge P \to A$, where $P \wedge P$ is the so-called exterior square of P. As $P \wedge P \simeq M(P) \simeq M(Q)$, regarding A as a trivial Q-module we may find an element Δ of $H^2(Q,A)$ corresponding to φ. Hence, let G be the central extension of A by Q relative to Δ. Let ε be the identity automorphism of A and let δ be a representative of $\Delta\varepsilon_* = \Delta$. Since no element of $A \setminus B$ is in the image of δ, δ is not surjective. We only need to show that (the image of) A is indeed the autocentre of G. Assume that $A \leq G$. Since, for each positive integer i and $j \in \{1,2\}$, $\langle {}^jy_i\rangle$ is characteristic in Q and $\langle {}^jx_i\rangle\langle {}^jy_i\rangle$ is the centre of $Q/\langle {}^jy_i\rangle$, it follows that every ${}^jQ_i$ is characteristic in Q. On the other hand, $\mathrm{Aut}({}^jQ_i)$ acts trivially on $({}^jQ_i)_{ab}$, so that $\mathrm{Aut}(Q)$ centralizes $M(Q)$ and hence also $\mathrm{Hom}(M(Q),A)$. From Lemma 2.1 it follows that the only isomorphisms of A which extend to G must induce automorphisms of $H^2(Q,A)$ centralizing Δ. Let α be one of these. Then, by the definition of Δ, α has to centralize the whole B. However, if α moves any a_i, it has to move some element of B by the construction of the latter, so α can only be the trivial automorphism and this shows that A is the autocentre of G. □

Proposition 2.2 already allows us to give an independent proof of the key Theorem of [16]. However, we give here the following generalised version of that result. For the essential properties of the Frattini subgroup we refer to [20], while for the properties of pure subgroups and basic subgroups of abelian groups we refer to [11].

Proof of Theorem A. Assume first that G satisfies (1). Then K is maximal in G, so that H is not contained in $\Phi(G)$. On the other hand, since K contains no central subgroup of order 2, H is a characteristic subgroup of $H \times Z(K) = Z(G)$, so that it is characteristic in G. From this and from the fact that H has only two elements, it follows that H is contained in T.

Assume now that G does not satisfy (1) and let Δ be the cohomology class of

$$A \rightarrowtail G \twoheadrightarrow Q = G/A.$$

If a subgroup H of A is a direct factor of G, then it can admit only the identity automorphism because it lies in the autocentre of G, and so it has order at most 2. If H had order 2, then, by hypothesis, we would find a subgroup of K of order 2 which is also central in G, and hence H would not be contained in the autocentre of G. Then no non-trivial subgroup of A is a direct factor of G. Let now B be a basic subgroup of T and let $C = \langle c\rangle$ be a cyclic direct factor of B. As C is pure in B and purity is a transitive relation, C is a pure subgroup of A and hence it is even a direct factor of A. Let ε be the projection of A onto C and let δ be a representative of $\Delta\varepsilon_*$. It follows from Proposition 2.2, that δ is surjective, so that, by the definition of a 2-cocycle, we can find two non-trivial elements x and y of Q such that $(x, y)\delta = c$. Hence, for a transversal function $\tau\colon Q \to G$, we get that $c = ((xy)^\tau)^{-1}x^\tau y^\tau$. Let now X be a subgroup of G such that $G = \langle c, X\rangle$. Since A is central in G, we may assume that $(xy)^\tau$, x^τ and y^τ belong to X, so that c itself belongs to X and we have that $G = X$. This shows that c is a non-generator of G and hence $C = \langle c\rangle$ is a subgroup of $\Phi(G)$. By the generality of C, we have that the whole B, which is a direct product of cyclic subgroups, is contained in $\Phi(G)$. Since, in general, every divisible subgroup of a group is always contained in the Frattini subgroup of the same group, we have that the divisible group T/B is contained in the Frattini subgroup of G/B and this implies that $T \leq \Phi(G)$. □

On the other hand, Example 2.4 shows that non-periodic subgroups of the autocentre behave quite badly regarding the embedding into the Frattini subgroup.

Recall here that a class $\mathfrak{F}$ of groups is called a *formation* if it is closed under homomorphic images and if, for each group G and normal subgroups H and K of G, from $G/H \in \mathfrak{F}$ and $G/K \in \mathfrak{F}$ it follows that $G/H\cap K \in \mathfrak{F}$. A formation is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$ for any group G.

Next result about saturated formations is also an improvement of the analogous result for finite groups in [16].

Corollary 2.6. *Let $\mathfrak{F}$ be a saturated formation containing a cyclic group of order 2. Let G be a group and let T be the torsion subgroup of the autocentre of G. Then G is in $\mathfrak{F}$ if and only if G/T is in $\mathfrak{F}$.*

Proof. Since a formation is closed under taking homomorphic images, we have only to prove the sufficiency part. To this aim, assume that G/T belongs to $\mathfrak{F}$. Clearly, if T is contained in $\Phi(G)$, then $G \in \mathfrak{F}$, because $\mathfrak{F}$ is saturated, so we may suppose this is not the case, so that, by Theorem A we have that $G = H \times K$ where H has order 2 and K contains no central subgroup of order 2. In particular, the torsion subgroup of the autocentre of K, which is $T \cap K$, is contained in $\Phi(K)$ again by Theorem A. Since $K/T \cap K$ is isomorphic to G/T and the latter belongs to $\mathfrak{F}$, we have that $K/\Phi(K) \in \mathfrak{F}$. This implies that the saturated formation $\mathfrak{F}$ contains K, so that also $G/H \simeq K$ is in $\mathfrak{F}$. On the other hand, $G/K \simeq H \in \mathfrak{F}$ by hypothesis and hence G itself belongs to $\mathfrak{F}$. □

Just as in [16], we remark here that the consideration of the saturated formation of all finite groups of odd order makes it impossible for us to get rid of the hypothesis about the cyclic group of order 2.

As one last application of Proposition 2.2, we now improve the bound given in [12] for the autocommutator subgroup of a group with autocentre of finite index.

Proof of Theorem B. First, since A is abelian, it is well-known that the transfer homomorphism from G to A is given by the mapping $\tau\colon g \in G \mapsto g^n \in A$. If α is any automorphism of G and g is any element of G, we have that $(g^{-1}g^{\alpha})^n = (g^{-1}g^{\alpha})^{\tau} = (g^{-n}(g^n)^{\alpha}) = 1$, so that G^* has finite exponent dividing n. Moreover, this shows that $G^* \cap A$ is contained in T and that, in particular, a bound for $|G^*|$ will derive from a bound on T.

Let $Q = G/A$ and let Δ be the cohomology class of the autocentral extension

$$\begin{tikzcd} A \arrow[r, tail] & G \arrow[r, two heads] & Q. \end{tikzcd}$$

Let C be any locally cyclic direct factor of T. It follows from Proposition 2.2 that there is a surjective function from $Q \times Q$ to C, so that C cannot be infinite. In particular, T has no non-trivial divisible direct factor and hence is reduced. Moreover, again from Propostion 2.2 we have that T is finite, because otherwise it would contain infinitely many cyclic direct factors (see for instance [11]). Therefore, T is a direct factor of A and we may take into account the canonical projection ε of A onto T.

Assume now that T is contained in the Frattini subgroup of G, so that G has no cyclic direct factor of order 2 by Proposition 2.2. Then, a further application of Proposition 2.2 yields that $\Delta\varepsilon_*$ is surjective and hence we get that $|T| \leq n^2$. Since we clearly have that $|G^*/G^* \cap T| \leq |Q| = n$, we obtain that $|G^*| \leq n^3$.

Assume now that T is not contained in $\Phi(G)$. Then, Proposition 2.2 gives that $G = H \times K$, where $|H| = 2$ and K contains no central subgroup of order 2. Let L be the torsion subgroup of $K \cap A$, which is easily seen to be the autocentre of K (see for instance [1]). If H is not contained in G^*, then $G^* = K^*$ and hence, by the first part of the proof applied on K, we have that $|G^*| \leq n^3$. To complete the proof, we assume that H is a subgroup of G^*, which is equivalent to assuming the existence of a subgroup R of Q of index 2. Consider now the following extension

$$\begin{tikzcd} R \arrow[r, tail] & Q \arrow[r, two heads] & S = Q/R \end{tikzcd}$$

and take into account the Lyndon-Hochschild-Serre spectral sequence for cohomology associated with the above extension and the trivial module L. Now, since L contains no elements of order 2, we have $E_2^{2,0} \simeq H^2(S, L) = 0$ and, for the same reason, we also get that

$$E_2^{1,1} \simeq H^1(S, H^1(R, L)) \simeq \mathrm{Hom}(S, \mathrm{Hom}(R, L)) = 0.$$

This shows that $H^2(Q, L)$ is isomorphic to $H^2(R, L) \simeq H^2(R, L)^S$, so that any representative of $\Delta\varepsilon_*$ can be thought of as a (surjective) function from $R \times R$ to L. Since $|T| = 2|L|$, we finally get that $|T| \leq 2(n/2)^2 = n^2/2$ and also that $|G^*| \leq n^3/2$. ☐

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