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# On the thermal stresses in chiral porous elastic beams

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**Abstract** This paper is concerned with the strain gradient theory of porous thermoelastic solids. We study the deformation of isotropic chiral cylinders subjected to a temperature field that is linear in the axial coordinate. It is shown that the solution can be reduced to the study of two-dimensional problems. The results are used to investigate the deformation of a circular cylinder subjected to a uniform temperature variation. In contrast to the case of achiral materials, the thermal field in chiral cylinders produces torsional effects.

**Keywords** Strain gradient elasticity · Porous solids · Chiral materials · Thermal effects in beams · Circular cylinders

## 1 Introduction

The behavior of chiral elastic bodies is a topic of current interest both from a theoretical and an experimental point of view. The chiral elastic solid was used as model for various crystalline materials, carbon nanotubes, bones, honeycomb structures, as well as composites with inclusions. Eringen [1] introduced the theory of microstretch continua as a generalization of Cosserat theory. In this theory, the material particles undergo a uniform microdilatation and a rigid microrotation. The intended applications of the theory are to porous media filled with gas, composite materials reinforced with chopped elastic fibers, bones and other materials with microstructure. In the case of a microstretch continuum, the change of microdeformation tensor, from the constant value in the natural state, has the form $\varphi\delta_{ij} + \varepsilon_{jik}\psi_k$, where $\varphi$ is the microdilatation function, $\psi_k$ is the microrotation vector, $\delta_{ij}$ is the Kronecker delta and $\varepsilon_{jik}$ is the alternating symbol. In the absence of microrotations, the equations of the linear theory of microstretch elastic solids coincide with the equations of the elastic materials with voids introduced by Cowin and Nunziato [2] to study the deformation of porous solids. In the theory of elastic materials with voids, the volume fraction field is a function strictly positive and less than 1. Considering the multiple possibilities of using the solid microstretch model, in this paper we will use the theory of microstretch continua to investigate the deformation of porous bodies. In what follows, the subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate. The function $\varphi$ is dimensionless, and the functions $\varphi_{,j}$ and $u_{i,rs}$ have the same dimensions. If the functions $\varphi$ and $\varphi_{,j}$ are considered as independent constituent variables, then the second-order partial derivatives of the displacement vector have to be included in the set of independent constituent variables. Thus, the introduction

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of the microdilatation function requires a strain gradient theory. Toupin [3,4] and Mindlin [5] have introduced the strain gradient theory of elasticity. Interest in the study of this theory is stimulated by the fact that it is suitable to investigate problems related to the effects of size and nanotechnology. The deformation of chiral elastic materials cannot be described within classical elasticity [6]. In various papers, the authors have studied the behavior of chiral materials by using the linear theory of gradient elasticity (see, e.g., [7–9] and references therein). The strain gradient elasticity has been used to investigate the behavior of carbon nanotubes (see, e.g., [10,11]).

This paper is concerned with a theory of thermoelasticity for isotropic microstretch continua, without microrotations, where the second-order displacement gradient is added to the classical set of independent constitutive variables. We have considered this theory since the chirality behavior in strain gradient theory is controlled by a single material parameter, in contrast to the three additional material parameters required in Cosserat theory. We study the equilibrium problem for a cylinder which, in the absence of body forces and lateral loading, is subjected to prescribed surface tractions on bases and to a thermal field that is linear in the axial coordinates. The origin of this problem goes back to the work of Boley and Weiner [12] that is devoted to classical thermoelasticity. The deformation of achiral porous elastic solids has been investigated in various papers (see, e.g., [2,13–18]). In this paper, we focus our attention to the case of homogeneous and isotropic chiral porous elastic materials. Since the cancellous bone is considered as a porous body [19] as well as a chiral material [20], it seems that the linear theory of gradient elasticity of porous solids is adequate to describe the mechanical behavior of bones. This paper is concerned with uncoupled system in the sense that the temperature field can be found by solving the heat flow problem associated with the heat conduction and energy equation. We shall treat the temperature field as having already been determined [21]. The temperature is a prescribed function that is independent of time.

The paper is structured as follows. First, we present the basic equations of chiral porous thermoelastic solids and formulate the problem of deformation of the right cylinders. We have introduced mechanical loads on the ends in order to compare the effects of thermal field with those produced by the resultants of the tractions that act on the bases. Then, we investigate the generalized plane strain problem. In the following section, we establish the solution of the problem when the temperature distribution is independent of the axial coordinate. It is shown that this thermal field produces extension, bending and torsion. The next section deals with the deformation of cylinders subjected to a temperature field that is linear in the axial coordinate. We present a method to reduce the three-dimensional problem to the study of plane problems. The results are used to study the deformation of a circular cylinder subjected to a uniform temperature variation. In contrast to the case of centrosymmetric materials, the thermal field in chiral cylinders produces torsional effects.

## 2 Preliminaries

In this section, we present the equations of equilibrium in the context of the strain gradient theory of porous thermoelastic materials. Mindlin [5] presented three forms of the linear theory of gradient elasticity. In what follows, we use the strain measures introduced in the first form of the theory. We consider a body that in the undeformed state occupies the bounded region $B$ with Lispchitz boundary $\partial B$. The boundary $\partial B$ consists of the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). We denote by $C$ the union of the edges. Throughout this paper, a rectangular Cartesian coordinate system $Ox_j$, $(j = 1, 2, 3)$, is used. We shall employ the usual summation and differentiation conventions.

We assume that $B$ is occupied by a homogeneous and isotropic chiral thermoelastic solid. Let $u_i$ be the components of the displacement vector, and let $\varphi$ be the microstretch function (microdilatation function). The strain tensors are defined by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}. \tag{1}$$

In the case of chiral porous thermoelastic bodies, the constitutive equations are given by (see, e.g., [1,8,22])

$$\begin{aligned}
\tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \mathrm{d}\varphi\delta_{ij} + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}) - bT\delta_{ij},\\
\mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik})\\
&\quad + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} + \beta_1\delta_{ij}\varphi_{,k} + \beta_2(\delta_{ik}\varphi_{,j} + \delta_{jk}\varphi_{,i})\\
&\quad + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}),\\
\sigma_i &= \beta_1\kappa_{rri} + 2\beta_2\kappa_{irr} + a_0\varphi_{,i}, \quad g = de_{rr} + \xi\varphi - \beta T,
\end{aligned} \tag{2}$$

where $\tau_{ij}$ is the stress tensor, $\mu_{ijk}$ is the dipolar stress tensor, $\sigma_i$ is the microstretch stress vector, $g$ is the intrinsic body force, $T$ is the temperature measured from the constant absolute temperature of reference state, $\delta_{ij}$ is Kronecker delta, $\varepsilon_{ijk}$ is the alternating symbol, and $\lambda, \mu, \alpha_s (s = 1, 2, \ldots, 5), \beta_1, \beta_2, \beta, b, d, a_0, \xi$ and $f$ are constitutive constants. In the case of achiral materials, the coefficient $f$ is equal to zero. In what follows, we assume that the elastic potential is a positive definite quadratic form in the variables $e_{ij}, \kappa_{ijk}, \varphi$ and $\varphi_{,k}$. The equations of equilibrium, in the absence of body loads, are given by

$$\tau_{ji,j} - \mu_{kji,kj} = 0, \quad \sigma_{j,j} - g = 0. \tag{3}$$

The equilibrium theory of linear elastic heat conductors has been studied in various books (see., e.g., [13,22,23]). We investigate the effects of temperature variation of the deformation of cylinders.

Toupin [3,4] introduced the functions $P_i$, $R_i$ and $Q_i$ defined by

$$\begin{aligned}
P_i &= (\tau_{ki} - \mu_{ski,s})n_k - D_j(n_r\mu_{rji}) + n_s n_p \mu_{spi}(D_k n_k),\\
R_i &= \mu_{rsi}n_r n_s,\ Q_i = < \mu_{pji}n_p n_q > \varepsilon_{jrq}s_r \ \text{ on } \partial B,
\end{aligned} \tag{4}$$

where $n_j$ are the components of the outward unit normal of $\partial B$, $D_i$ are the components of the surface gradient, $D_i = (\delta_{ik} - n_i n_k)\partial/\partial x_k$, $s_i$ are the components of the unit vector tangent to $C$ and $< g >$ denotes the difference of limits of $g$ from both sides of $C$.

In the case of traction problem, the boundary conditions are [1,24]

$$P_i = \widetilde{P}_i,\ R_i = \widetilde{R}_i,\ \sigma_i n_i = \widetilde{\sigma} \ \text{ on } \partial B\backslash C,\ Q_i = \widetilde{Q}_i \ \text{ on } C, \tag{5}$$

where $\widetilde{P}_i$, $\widetilde{R}_i$, $\widetilde{\sigma}$ and $\widetilde{Q}_i$ are prescribed functions.

We assume that the region $B$ from here on refers to the interior of a right cylinder of length $h$ with the cross section $\Sigma$ and the lateral boundary $\Pi$. Let $\Gamma$ be the boundary of $\Sigma$. The coordinate system consists of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the origin $O$. We choose the system $Ox_j$ such that $x_3$-axis is parallel to the generator of the cylinder and the $x_1Ox_2$ plane contains one of the terminal cross sections. We denote by $\Sigma_1$ and $\Sigma_2$, respectively, the cross section located at $x_3 = 0$ and $x_3 = h$. Let $\Gamma_\alpha$ be the boundary of $\Sigma_\alpha$, $(\alpha = 1, 2)$. Throughout this paper, the Greek subscripts range over the integers $(1, 2)$.

In what follows, we assume that the thermal field is linear in the axial coordinate,

$$T = T_0(x_1, x_2) + x_3 T_1(x_1, x_2), \tag{6}$$

where the functions $T_0$ and $T_1$ are prescribed. We suppose that the lateral surface is smooth. This assumption implies that the functions $Q_i$ are equal to zero on $\Pi$. The cylinder is supposed to be free of lateral loading and subjected to appropriate stress resultants over its ends. On the lateral surface of the cylinder, we have the conditions

$$P_i = 0,\ R_i = 0,\ \sigma_\alpha n_\alpha = 0 \ \text{ on } \Pi. \tag{7}$$

Let $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{M} = (M_1, M_2, M_3)$ be prescribed vectors representing the resultant force and resultant moment about $O$ of the tractions acting on $\Sigma_1$. On $\Sigma_2$, there are tractions applied so as to satisfy the equilibrium conditions of the cylinder. We have introduced the mechanical loads $\mathbf{F}$ and $\mathbf{M}$ in order to compare the effects of thermal field with those produced by the resultants **F** and **M**. On the end $\Sigma_1$, we have the boundary conditions

$$\int_{\Sigma_1} P_\alpha \mathrm{d}a + \int_{\Gamma_1} Q_\alpha \mathrm{d}s = F_\alpha, \tag{8}$$

$$\int_{\Sigma_1} P_3 \mathrm{d}a + \int_{\Gamma_1} Q_3 \mathrm{d}s = F_3, \tag{9}$$

$$\int_{\Sigma_1} (x_\alpha P_3 + R_\alpha) \mathrm{d}a + \int_{\Gamma_1} x_\alpha Q_3 \mathrm{d}s = \varepsilon_{\beta\alpha 3} M_\beta, \tag{10}$$

$$\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta \mathrm{d}a + \int_{\Gamma_1} \varepsilon_{a\beta 3} x_\alpha Q_\beta \mathrm{d}s = M_3. \tag{11}$$

From (4), we find that

$$\begin{aligned} &P_i = -\tau_{3i} + 2\mu_{\alpha 3i,\alpha} + \mu_{33i,3},\ R_i = \mu_{33i} \ \text{ on } \Sigma_1, \\ &Q_i = -2\mu_{\alpha 3i} n_\alpha \ \text{ on } \Gamma_1, \end{aligned} \tag{12}$$

where $(n_1, n_2, 0)$ are the direction cosines of the exterior normal to $\Pi$.

The problem consists in finding the functions $u_i$ and $\varphi$ satisfying the equations (1)–(3) on $B$, the conditions (7) on $\Pi$ and the conditions (8)–(11) on the ends, when the temperature $T$, the constitutive coefficients and the constants $F_j$ and $M_j$ are prescribed.

## 3 Two-dimensional problems

In what follows, we will reduce the problem formulated in Sect. 2 to the study of some two-dimensional problems. In this section, we present some results concerning the generalized plane strain of a chiral thermoelastic cylinder. We assume now that a body force $\mathscr{F}_j$ and a microstretch force density $l$ are prescribed on $B$. Let us suppose that $\mathscr{F}_j$, $l$, $\widetilde{P}_i$, $\widetilde{R}_i$ and $\widetilde{\sigma}$ are independent of the axial coordinate $x_3$. We define the state of generalized plane strain of the cylinder $B$ to be that state in which the displacement vector, the microdilatation function and the temperature are independent of the axial coordinate,

$$u_j = u_j(x_1, x_2),\ \varphi = \varphi(x_1, x_2),\ T = T(x_1, x_2),\ (x_1, x_2) \in \Sigma_1. \tag{13}$$

From (1), we obtain $e_{33} = 0$, $\kappa_{j3i} = 0$ and

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}),\ 2e_{\alpha 3} = u_{3,\alpha},\ \kappa_{\alpha\beta j} = u_{j,\alpha\beta}. \tag{14}$$

The constitutive equations (2) reduce to

$$\begin{aligned} \tau_{\alpha\beta} &= \lambda e_{\rho\rho}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta} + \mathrm{d}\varphi\delta_{\alpha\beta} + f(\varepsilon_{\alpha\rho 3}\kappa_{\beta\rho 3} + \varepsilon_{\beta\rho 3}\kappa_{\alpha\rho 3}) - bT\delta_{\alpha\beta}, \\ \tau_{\alpha 3} &= 2\mu e_{\alpha 3} + f\varepsilon_{\rho\beta 3}\kappa_{\alpha\rho\beta}, \\ \mu_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\kappa_{\rho\rho\alpha}\delta_{\beta\gamma} + 2\kappa_{\gamma\rho\rho}\delta_{\alpha\beta} + \kappa_{\rho\rho\beta}\delta_{\alpha\gamma}) + \alpha_2(\kappa_{\alpha\rho\rho}\delta_{\beta\gamma} + \kappa_{\beta\rho\rho}\delta_{\alpha\gamma}) \\ &\quad + 2\alpha_3\kappa_{\rho\rho\gamma}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta\gamma} + \alpha_5(\kappa_{\gamma\beta\alpha} + \kappa_{\gamma\alpha\beta}) + \beta_1\delta_{\alpha\beta}\varphi_{,\gamma} \\ &\quad + 2\beta_2(\delta_{\alpha\gamma}\varphi_{,\beta} + \delta_{\beta\gamma}\varphi_{,\alpha}) + f(\varepsilon_{\alpha\gamma 3}e_{\beta 3} + \varepsilon_{\beta\gamma 3}e_{\alpha 3}), \\ \mu_{\alpha\beta 3} &= 2\alpha_3\kappa_{\rho\rho 3}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta 3} + f(\varepsilon_{\rho\alpha 3}e_{\beta\rho} + \varepsilon_{\rho\beta 3}e_{\alpha\rho}), \\ \sigma_\alpha &= \beta_1\kappa_{\rho\rho\alpha} + 2\beta_2\kappa_{\alpha\rho\rho} + a_0\varphi_{,\alpha}, \quad g = de_{\rho\rho} + \xi\varphi - \beta T, \end{aligned} \tag{15}$$

and

$$\begin{aligned}
\tau_{33} &= \lambda e_{\rho\rho} + \mathrm{d}\varphi - bT,\\
\mu_{3\alpha\beta} &= \frac{1}{2}\alpha_1\kappa_{\rho\rho 3}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3} + f\varepsilon_{\beta\rho 3}e_{\alpha\rho},\\
\mu_{3\alpha 3} &= \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha} + \alpha_2\kappa_{\alpha\rho\rho} + f\varepsilon_{\rho\alpha 3}e_{3\rho} + \beta_2\varphi_{,\alpha},\\
\mu_{33\alpha} &= \alpha_1\kappa_{\alpha\rho\rho} + 2\alpha_3\kappa_{\rho\rho\alpha} + \beta_1\varphi_{,\alpha} + 2f\varepsilon_{3\alpha\rho}e_{3\rho},\\
\mu_{333} &= (\alpha_1 + 2\alpha_3)\kappa_{\rho\rho 3},\ \sigma_3 = \beta_1\kappa_{\rho\rho 3}.
\end{aligned} \tag{16}$$

The equations of equilibrium in the presence of body loads can be written as

$$\tau_{\alpha i,\alpha} - \mu_{\alpha\beta i,\alpha\beta} + \mathscr{F}_i = 0,\ \sigma_{\alpha,\alpha} - g + l = 0 \ \text{ on } \Sigma_1. \tag{17}$$

It follows from (4) that in the plane problems the functions $P_i$ and $R_i$ have the following form on the lateral surface

$$\begin{aligned}
P_i &= (\tau_{\beta i} - \mu_{\rho\beta i,\rho})n_\beta - D_\rho(n_\beta\mu_{\beta\rho i}) + n_\beta n_\alpha\mu_{\beta\alpha i}(D_\rho n_\rho),\\
R_i &= \mu_{\rho\alpha i}n_\rho n_\alpha.
\end{aligned} \tag{18}$$

The conditions on the surface $\Pi$ become

$$P_i = \widetilde{P}_i,\ R_i = \widetilde{R}_i,\ \sigma_\alpha n_\alpha = \widetilde{\sigma} \ \text{ on } \Gamma_1. \tag{19}$$

The two-dimensional problem of thermoelasticity consists in finding the functions $u_j$ and $\varphi$ satisfying the Eqs. (14), (15) and (17) on $\Sigma_1$, and the boundary conditions (19) on $\Gamma_1$. We denote by $(\mathscr{H})$ the problem characterized by the Eqs. (14), (15), (17) and the boundary conditions (19). Using the results established by Hlavacek and Hlavacek [25], we can easily deduce the following proposition.

**Theorem 1** *The necessary and sufficient conditions for the existence of a solution to the problem* $(\mathscr{H})$ *are given by*

$$\begin{aligned}
\int_{\Sigma_1}\mathscr{F}_j\mathrm{d}a + \int_{\Gamma_1}\widetilde{P}_j\mathrm{d}s &= 0,\\
\int_{\Sigma_1}\varepsilon_{3\alpha\beta}x_\alpha\mathscr{F}_\beta\mathrm{d}a + \int_{\Gamma_1}\varepsilon_{3\alpha\beta}(x_\alpha\widetilde{P}_\beta + n_\alpha\widetilde{R}_\beta)\mathrm{d}s &= 0.
\end{aligned} \tag{20}$$

*In view of* (14) *and* (15)*, the equations of equilibrium can be written in the form*

$$\begin{aligned}
&\mu\Delta u_\alpha + (\lambda+\mu)u_{\beta,\beta\alpha} - 2(\alpha_3+\alpha_4)\Delta\Delta u_\alpha - 2(\alpha_1+\alpha_2+\alpha_5)\Delta u_{\beta,\beta\alpha}\\
&\quad + 2f\varepsilon_{\alpha\beta 3}\Delta u_{3,\beta} + \mathrm{d}\varphi_{,\alpha} - (\beta_1+2\beta_2)\Delta\varphi_{,\alpha} - bT_{,\alpha} + \mathscr{F}_\alpha = 0,\\
&\mu\Delta u_3 - 2(\alpha_3+\alpha_4)\Delta\Delta u_3 + 2f\varepsilon_{\rho\nu 3}\Delta u_{\nu,\rho} + \mathscr{F}_3 = 0,\\
&(\beta_1+2\beta_2)\Delta u_{\rho,\rho} + a_0\Delta\varphi - du_{\rho,\rho} - \xi\varphi + \beta T + l = 0.
\end{aligned} \tag{21}$$

*The functions* $\tau_{33}$, $\mu_{3\alpha\beta}$, $\mu_{3\alpha 3}$, $\mu_{33\alpha}$, $\mu_{333}$ *and* $\sigma_3$ *can be calculated after the determination of displacements and microdilatation.*

## 4 Plane temperature field

In this section, we present the solution of the problem under the following assumptions

$$T = T_0(x_1, x_2),\quad \mathbf{F} = F_3\mathbf{e}_3,\quad \mathbf{M} = M_j\mathbf{e}_j. \tag{22}$$

It is known that in the classical thermoelasticity, a uniform thermal field produces bending, extension and a plane deformation. We try to solve the problem by combining a solution of Saint-Venant's type with a solution of the plane problem. We seek the solution in the form

$$\begin{aligned}
u_\alpha &= -\frac{1}{2}a_\alpha x_3^2 + \varepsilon_{3\beta\alpha}a_4 x_\beta x_3 + \sum_{k=1}^{4} a_k u_\alpha^{(k)} + w_\alpha(x_1, x_2),\\
u_3 &= (a_1x_1 + a_2x_2 + a_3)x_3 + \sum_{k=1}^{4} a_k u_3^{(k)} + w_3(x_1, x_2),\\
\varphi &= \sum_{k=1}^{4} a_k \varphi^{(k)} + \psi,
\end{aligned} \tag{23}$$

where $u_j^{(k)}$, $\varphi^{(k)}$, $w_j$ and $\psi$ are unknown functions which are independent of $x_3$, and $a_k$ are unknown constants. We denote by $A^{(k)}$ $(k = 1, 2, 3, 4)$, the isothermal $(T = 0)$ plane strain problems characterized by the displacement vector $u_j^{(k)}$ and microdilatation function $\varphi^{(k)}$. The body loads and the boundary data associated with the problems $A^{(k)}$ will be precised in what follows. We introduce the notations

$$e_{\alpha\beta}^{(k)} = \frac{1}{2}(u_{\alpha,\beta}^{(k)} + u_{\beta,\alpha}^{(k)}), 2e_{\alpha3}^{(k)} = u_{3,\alpha}^{(k)}, \kappa_{\alpha\beta j}^{(k)} = u_{j,\alpha\beta}^{(k)}. \tag{24}$$

It follows from (15) that the constitutive equations imply

$$\begin{aligned}
\tau_{\alpha\beta}^{(k)} &= \lambda e_{\rho\rho}^{(k)}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}^{(k)} + \mathrm{d}\varphi^{(k)}\delta_{\alpha\beta} + f(\varepsilon_{\alpha\rho3}\kappa_{\beta\rho3}^{(k)} + \varepsilon_{\beta\rho3}\kappa_{\alpha\rho3}^{(k)}),\\
\tau_{\alpha3}^{(k)} &= 2\mu e_{\alpha3}^{(k)} + f\varepsilon_{\rho\beta3}\kappa_{\alpha\rho\beta}^{(k)},\\
\mu_{\alpha\beta\gamma}^{(k)} &= \frac{1}{2}\alpha_1(\kappa_{\rho\rho\alpha}^{(k)}\delta_{\beta\gamma} + 2\kappa_{\gamma\rho\rho}^{(k)}\delta_{\alpha\beta} + \kappa_{\rho\rho\beta}^{(k)}\delta_{\alpha\gamma})\\
&\quad + \alpha_2(\kappa_{\alpha\rho\rho}^{(k)}\delta_{\beta\gamma} + \kappa_{\beta\rho\rho}^{(k)}\delta_{\alpha\gamma}) + 2\alpha_3\kappa_{\rho\rho\gamma}^{(k)}\delta_{\alpha\beta}\\
&\quad + 2\alpha_4\kappa_{\alpha\beta\gamma}^{(k)} + \alpha_5(\kappa_{\gamma\beta\alpha}^{(k)} + \kappa_{\gamma\alpha\beta}^{(k)}) + \beta_1\delta_{\alpha\beta}\varphi_{,\gamma}^{(k)}\\
&\quad + \beta_2(\delta_{\alpha\gamma}\varphi_{,\beta}^{(k)} + \delta_{\beta\gamma}\varphi_{,\alpha}^{(k)}) + f(\varepsilon_{\alpha\gamma3}e_{\beta3}^{(k)} + \varepsilon_{\beta\gamma3}e_{\alpha3}^{(k)}),\\
\mu_{\alpha\beta3}^{(k)} &= 2\alpha_3\kappa_{\rho\rho3}^{(k)}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta3}^{(k)} + f(\varepsilon_{\rho\alpha3}e_{\beta\rho}^{(k)} + \varepsilon_{\rho\beta3}e_{\alpha\rho}^{(k)}),\\
\sigma_\alpha^{(k)} &= \beta_1\kappa_{\rho\rho\alpha}^{(k)} + 2\beta_2\kappa_{\alpha\rho\rho}^{(k)} + a_0\varphi_{,\alpha}^{(k)}, \quad g^{(k)} = de_{\rho\rho}^{(k)} + \xi\varphi^{(k)}.
\end{aligned} \tag{25}$$

We shall use the notations

$$\begin{aligned}
\tau_{33}^{(k)} &= \lambda e_{\rho\rho}^{(k)} + \mathrm{d}\varphi^{(k)}, \quad \mu_{3\alpha\beta}^{(k)} = \frac{1}{2}\alpha_1\kappa_{\rho\rho3}^{(k)}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha3}^{(k)} + f\varepsilon_{\beta\rho3}e_{\alpha\rho}^{(k)},\\
\mu_{3\alpha3}^{(k)} &= \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha}^{(k)} + \alpha_2\kappa_{\alpha\rho\rho}^{(k)} + f\varepsilon_{\rho\alpha3}e_{3\rho}^{(k)} + \beta_2\varphi_{,\alpha}^{(k)},\\
\mu_{33\alpha}^{(k)} &= \alpha_1\kappa_{\alpha\rho\rho}^{(k)} + 2\alpha_3\kappa_{\rho\rho\alpha}^{(k)} + \beta_1\varphi_{,\alpha}^{(k)} + 2f\varepsilon_{3\alpha\rho}e_{3\rho}^{(k)},\\
\mu_{333}^{(k)} &= (\alpha_1 + 2\alpha_3)\kappa_{\rho\rho3}^{(k)}, \quad \sigma_3^{(k)} = \beta_1\kappa_{\rho\rho3}^{(k)}.
\end{aligned} \tag{26}$$

We denote by $(\mathcal{T})$ the plane thermoelastic problem which corresponds to the temperature $T_0$ and is characterized by the displacement vector $w_j$ and the microdilatation $\psi$. Thus, the constitutive equations in the problem $(\mathcal{T})$

are

$$\begin{aligned}
t_{\alpha\beta} &= \lambda\eta_{\rho\rho}\delta_{\alpha\beta} + 2\mu\eta_{\alpha\beta} + \mathrm{d}\psi\delta_{\alpha\beta} + f(\varepsilon_{\alpha\rho3}\xi_{\beta\rho3} + \varepsilon_{\beta\rho3}\xi_{\alpha\rho3}) - bT_0\delta_{\alpha\beta},\\
t_{\alpha3} &= 2\mu\eta_{\alpha3} + f\varepsilon_{\rho\beta3}\xi_{\alpha\rho\beta},\\
m_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\xi_{\rho\rho\alpha}\delta_{\beta\gamma} + 2\xi_{\gamma\rho\rho}\delta_{\alpha\beta} + \xi_{\rho\rho\beta}\delta_{\alpha\gamma}) + \alpha_2(\xi_{\alpha\rho\rho}\delta_{\beta\gamma} + \xi_{\beta\rho\rho}\delta_{\alpha\gamma})\\
&\quad + 2\alpha_3\xi_{\rho\rho\gamma}\delta_{\alpha\beta} + 2\alpha_4\xi_{\alpha\beta\gamma} + \alpha_5(\xi_{\gamma\beta\alpha} + \xi_{\gamma\alpha\beta}) + \beta_1\delta_{\alpha\beta}\psi_{,\gamma},\\
&\quad + 2\beta_2(\delta_{\alpha\gamma}\psi_{,\beta} + \delta_{\beta\gamma}\psi_{,\alpha}) + f(\varepsilon_{\alpha\gamma3}\eta_{\beta3} + \varepsilon_{\beta\gamma3}\eta_{\alpha3}),\\
m_{\alpha\beta3} &= 2\alpha_3\xi_{\rho\rho3}\delta_{\alpha\beta} + 2\alpha_4\xi_{\alpha\beta3} + f(\varepsilon_{\rho\alpha3}\eta_{\beta\rho} + \varepsilon_{\rho\beta3}\eta_{\alpha\rho}),\\
\pi_\alpha &= \beta_1\xi_{\rho\rho\alpha} + 2\beta_2\xi_{\alpha\rho\rho} + a_0\psi_{,\alpha},\quad \gamma = \mathrm{d}\eta_{\rho\rho} + \xi\psi - \beta T_0,
\end{aligned}\tag{27}$$

where

$$\eta_{\alpha\beta} = \frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha}),\, 2\eta_{\alpha3} = w_{3,\alpha},\, \xi_{\alpha\beta j} = w_{j,\alpha\beta}.\tag{28}$$

We shall use the notations

$$\begin{aligned}
t_{33} &= \lambda\eta_{\rho\rho} + \mathrm{d}\psi - bT_0,\\
m_{3\alpha\beta} &= \frac{1}{2}\alpha_1\xi_{\rho\rho3}\delta_{\alpha\beta} + \alpha_5\xi_{\beta\alpha3} + f\xi_{\beta\rho3}\eta_{\alpha\rho},\\
m_{3\alpha3} &= \frac{1}{2}\alpha_1\xi_{\rho\rho\alpha} + \alpha_2\xi_{\alpha\rho\rho} + f\xi_{\rho\alpha3}\eta_{3\rho} + \beta_2\psi_{,\alpha},\\
m_{33\alpha} &= \alpha_1\xi_{\alpha\rho\rho} + 2\alpha_3\xi_{\rho\rho\alpha} + \beta_1\psi_{,\alpha} + 2f\varepsilon_{3\alpha\rho}\eta_{3\rho},\\
m_{333} &= (\alpha_1 + 2\alpha_3)\xi_{\rho\rho3}\xi_{\rho\rho3},\, \pi_3 = \beta_1\xi_{\rho\rho3}.
\end{aligned}\tag{29}$$

From (1) and (23), we obtain

$$\begin{aligned}
e_{\alpha\beta} &= \eta_{\alpha\beta} + \sum_{k=1}^{4} a_k e_{\alpha\beta}^{(k)},\, e_{\alpha3} = \frac{1}{2}\varepsilon_{3\beta\alpha}x_\beta a_4\\
&\qquad + \eta_{\alpha3} + \sum_{k=1}^{4} a_k e_{\alpha3}^{(k)},\, e_{33} = a_1x_1 + a_2x_2 + a_3,\\
\kappa_{\alpha\beta\gamma} &= \xi_{\alpha\beta\gamma} + \sum_{k=1}^{4} a_k\kappa_{\alpha\beta\gamma}^{(k)},\, \kappa_{\alpha\beta3} = \xi_{\alpha\beta3} + \sum_{k=1}^{4} a_k\kappa_{\alpha\beta3}^{(k)},\\
\kappa_{\beta3\alpha} &= \varepsilon_{3\beta\alpha}a_4,\, \kappa_{\alpha33} = -\kappa_{33\alpha} = a_4,\, \kappa_{333} = 0.
\end{aligned}\tag{30}$$

In view of (2) and (29), we find that the functions $\tau_{ij}$, $\mu_{ijk}$, $\sigma_i$ and $g$ are given by

$$\begin{aligned}
\tau_{\alpha\beta} &= [(a_1x_1 + a_2x_2 + a_3) - 2fa_4]\delta_{\alpha\beta} + \sum_{k=1}^{4} a_k\tau_{\alpha\beta}^{(k)} + t_{\alpha\beta},\\
\tau_{\alpha3} &= \mu a_4\varepsilon_{3\beta\alpha}x_\beta + 2f\varepsilon_{\alpha\rho3}a_\rho + \sum_{k=1}^{4} a_k\tau_{\alpha3}^{(k)} + t_{\alpha3},\\
\tau_{33} &= (\lambda + 2\mu)(a_1x_1 + a_2x_2 + a_3) + 4fa_4 + t_{33} + \sum_{k=1}^{4} a_k\tau_{33}^{(k)},\\
\mu_{111} &= 2(\alpha_2 - \alpha_3)a_1 + \sum_{k=1}^{4} a_k\mu_{111}^{(k)} + m_{111},
\end{aligned}$$

$$
\begin{aligned}
\mu_{222} &= 2(\alpha_2-\alpha_3)a_2+\sum_{k=1}^{4}a_k\mu_{222}^{(k)}+m_{222},\\
\mu_{221} &= (\alpha_1-2\alpha_3)a_1-fa_4x_1+\sum_{k=1}^{4}a_k\mu_{221}^{(k)}+m_{221},\\
\mu_{112} &= (\alpha_1-2\alpha_3)a_2-fa_4x_2+\sum_{k=1}^{4}a_k\mu_{112}^{(k)}+m_{112},\\
\mu_{121} &= \frac{1}{2}(2\alpha_2-\alpha_3)a_2+\frac{1}{2}fa_4x_2+\sum_{k=1}^{4}a_k\mu_{121}^{(k)}+m_{121},\\
\mu_{122} &= \frac{1}{2}(2\alpha_2-\alpha_1)a_1+\frac{1}{2}fa_4x_1+\sum_{k=1}^{4}a_k\mu_{122}^{(k)}+m_{122},\\
\mu_{\alpha 33} &= \frac{1}{2}(2\alpha_2-\alpha_1+4\alpha_4)a_4-\frac{1}{2}fa_4x_\alpha+\sum_{k=1}^{4}a_k\mu_{\alpha 33}^{(k)}+m_{\alpha 33},\\
\mu_{33\alpha} &= (\alpha_1-2\alpha_3-2\alpha_4+2\alpha_5)a_\alpha+fa_4x_\alpha+\sum_{k=1}^{4}a_k\mu_{33\alpha}^{(k)}+m_{33\alpha},\\
\mu_{\alpha 3\beta} &= \varepsilon_{3\alpha\beta}f(a_1x_1+a_2x_2+a_3)+\varepsilon_{3\alpha\beta}(2\alpha_4-\alpha_5)a_4\\
&\quad+\sum_{k=1}^{4}a_k\mu_{\alpha 3\beta}^{(k)}+m_{\alpha 3\beta},\\
\mu_{\alpha\beta 3} &= \sum_{k=1}^{4}a_k\mu_{\alpha\beta 3}^{(k)}+m_{\alpha\beta 3},\ \mu_{333}=\sum_{k=1}^{4}a_k\mu_{333}^{(k)}+m_{333},\\
\sigma_\alpha &= (2\beta_2-\beta_1)a_\alpha+\sum_{k=1}^{4}a_k\sigma_\alpha^{(k)}+\pi_\alpha,\ \sigma_3=\sum_{k=1}^{4}a_k\sigma_3^{(k)}+\pi_3,\\
g &= d(a_1x_1+a_2x_2+a_3)+\sum_{k=1}^{4}a_kg^{(k)}+\gamma.
\end{aligned}
\tag{31}
$$

Here we have used the relations (25)–(27).

Let us impose that the equations of equilibrium (3) and the boundary conditions (7) be satisfied by the functions (31). If we require that the coefficients of the constants $a_k$ that appear in the equilibrium equations be equal to zero, then we find that the functions $\tau_{\alpha j}^{(k)}, \mu_{\alpha\beta i}^{(k)}, \sigma_\alpha^{(k)}$ and $g^{(k)}$ $(k=1,2,3,4)$ satisfy the following equations

$$
\tau_{\alpha j,\alpha}^{(k)}-\mu_{\alpha\beta j,\alpha\beta}^{(k)}+\mathscr{F}_j^{(k)}=0,\quad \sigma_{\alpha,\alpha}^{(k)}-g^{(\kappa)}+l^{(k)}=0\ \text{ on } \Sigma_1, \tag{32}
$$

where we have used the notations

$$
\mathscr{F}_j^{(\rho)}=\lambda\delta_{j\rho},\quad \mathscr{F}_j^{(3)}=0,\quad \mathscr{F}_j^{(4)}=0,\quad l^{(\rho)}=-\mathrm{d}x_\rho,\quad l^{(3)}=-d,\ l^{(4)}=0. \tag{33}
$$

The equilibrium equations (3) reduce to

$$
t_{\alpha i,\alpha}-m_{\alpha\beta i,\alpha}=0,\quad \pi_{\alpha,\alpha}-\gamma=0\ \text{ on }\ \Sigma_1. \tag{34}
$$

Let us introduce the functions

$$
\begin{aligned}
P_i^{(k)} &= (\tau_{\beta i}^{(k)}-\mu_{\rho\beta i,\rho}^{(k)})n_\beta-D_\rho(n_\beta\mu_{\beta\rho i}^{(k)})+n_\beta n_\alpha\mu_{\beta\alpha i}^{(k)}(D_\rho n_\rho),\\
R_i^{(k)} &= \mu_{\rho\alpha i}^{(k)}n_\rho n_\alpha,
\end{aligned}
$$

$$\begin{aligned}&P_i^* = (t_{\beta i} - m_{\rho\beta i,\rho})n_\beta - D_\rho(n_\beta m_{\beta\rho i}) + n_\beta n_\alpha m_{\beta\alpha i}(D_\rho n_\rho),\\&R_i^* = m_{\rho\alpha i}n_\rho n_\alpha, \quad \text{on } \Gamma_1.\end{aligned} \tag{35}$$

We require that the boundary conditions (7) be satisfied for any constants $a_k$. Thus, we find that the functions $\tau_{\alpha j}^{(k)}$, $\mu_{\alpha\beta i}^{(k)}$ and $\sigma^{(k)}$ have to satisfy the following boundary conditions

$$P_i^{(k)} = \widetilde{P}_i^{(k)}, \quad R_i^{(k)} = \widetilde{R}_i^{(k)}, \quad \sigma_\alpha^{(k)}n_\alpha = \widetilde{\sigma}^{(k)} \text{ on } \Gamma_1, \tag{36}$$

where

$$\begin{aligned}
\widetilde{P}_1^{(1)} &= -\lambda x_1 n_1 + (\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1 n_2)_{,\nu}n_\alpha,\\
\widetilde{P}_2^{(1)} &= -\lambda x_1 n_2 + \frac{1}{2}(\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\rho}(n_1^2 - n_2^2)_{,\alpha}n_\rho,\ P_3^{(1)} = 2fn_2,\\
\widetilde{R}_1^{(1)} &= 2\alpha_3 - \alpha_1 + (\alpha_1 - 2\alpha_2)n_1^2,\ \widetilde{R}_2^{(1)} = (\alpha_1 - 2\alpha_2)n_1 n_2,\\
\widetilde{R}_3^{(1)} &= 0,\ \widetilde{\sigma}^{(1)} = (\beta_1 - 2\beta_2)n_1,\\
\widetilde{P}_1^{(2)} &= -\lambda x_2 n_1 + \frac{1}{2}(\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1^2 - n_2^2)_{,\alpha}n_\nu,\\
\widetilde{P}_2^{(2)} &= -\lambda x_2 n_2 + (\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1 n_2)_{,\nu}n_\alpha,\ \widetilde{P}_3^{(2)} = -2fn_1,\\
\widetilde{R}_1^{(2)} &= (\alpha_1 - 2\alpha_2)n_1 n_2,\ \widetilde{R}_2^{(2)} = 2\alpha_3 - \alpha_1 + (\alpha_1 - 2\alpha_2)n_2^2,\\
\widetilde{R}_3^{(2)} &= 0,\ \widetilde{\sigma}^{(2)} = (\beta_1 - 2\beta_2)n_2,\\
\widetilde{P}_\alpha^{(3)} &= -\lambda n_\alpha,\ \widetilde{P}_3^{(3)} = 0,\ \widetilde{R}_j^{(3)} = 0,\ \widetilde{\sigma}^{(3)} = 0,\\
\widetilde{P}_1^{(4)} &= \frac{1}{2}f[5n_1 + D_1(x_2 n_2) + D_2(x_2 n_1 - 2x_1 n_2) - 2(x_2 n_1 n_2 - x_1 n_2^2)(D_\rho n_\rho)],\\
\widetilde{P}_2^{(4)} &= \frac{1}{2}f[5n_2 + D_1(x_1 n_2 - 2x_2 n_1) + D_2(x_1 n_1) - 2(x_1 n_1 n_2 - x_2 n_1^2)(D_\rho n_\rho)],\\
\widetilde{P}_3^{(4)} &= \mu\varepsilon_{3\beta\nu}x_\nu n_\beta,\ \widetilde{R}_1^{(4)} = f(x_1 n_2^2 - x_2 n_1 n_2),\\
\widetilde{R}_2^{(4)} &= f(x_2 n_1^2 - x_1 n_1 n_2),\ \widetilde{R}_3^{(4)} = 0,\ \widetilde{\sigma}^{(4)} = 0.
\end{aligned} \tag{37}$$

We conclude that the problem $A^{(k)}$ consists in finding the functions $u_j^{(k)}$ and $\varphi^{(k)}$ satisfying the Eqs. (24), (25) and (32) on $\Sigma_1$, and the boundary conditions (36). The necessary and sufficient conditions for the existence of the solution are satisfied for each problem $A^{(k)}$. The solutions of these problems depend only on the cross section $\Sigma$ and the constitutive constants. The boundary conditions (7) reduce to

$$P_i^* = 0, \quad R_i^* = 0, \quad \pi_\alpha n_\alpha = 0 \ \text{ on } \Gamma_1. \tag{38}$$

The problem $(\mathcal{T})$ consists in finding the functions $w_j$ and $\psi$ which satisfy the Eqs. (27), (28) and (34) on $\Sigma_1$, and the boundary conditions (38) on $\Gamma_1$.

Let us determine now the constants $a_j$, $(j = 1, 2, 3, 4)$. By using the divergence theorem and the relations (3), (5) and (12), we obtain

$$\begin{aligned}
\int_{\Sigma_1} P_\alpha \mathrm{d}a + \int_{\Gamma_1} Q_\alpha \mathrm{d}s &= -\int_{\Gamma_1}(x_\alpha \widetilde{P}_3 + n_\alpha \widetilde{R}_3)\mathrm{d}s\\
&\quad - \int_{\Sigma_1}[x_\alpha(\tau_{33,3} - \mu_{333,33}) + 2\mu_{\alpha 33,3} - \mu_{33\alpha,3}]\mathrm{d}s,\\
\int_{\Sigma_1} P_3 \mathrm{d}a + \int_{\Gamma_1} Q_3 \mathrm{d}s &= -\int_{\Sigma_1}(\tau_{33} - \mu_{333,33})\mathrm{d}a,\\
\int_{\Sigma_1}(x_\alpha P_3 + R_\alpha)\mathrm{d}a + \int_{\Gamma_1} x_\alpha Q_3 \mathrm{d}s &= -\int_{\Sigma_1}[x_\alpha(\tau_{33} - \mu_{333,3}) + 2\mu_{\alpha 33} - \mu_{33\alpha}]\mathrm{d}a,\\
\int_{\Sigma_1} \varepsilon_{\alpha\beta 3}x_\alpha P_\beta \mathrm{d}s + \int_{\Gamma_1} \varepsilon_{\alpha\beta 3}x_\alpha Q_\beta \mathrm{d}s &= \int_{\Sigma_1}[\varepsilon_{\alpha\beta 3}x_\alpha(\mu_{33\beta,3} - \tau_{3\beta}) - 2\varepsilon_{\alpha\beta 3}\mu_{\alpha 3\beta}]\mathrm{d}a.
\end{aligned} \tag{39}$$

In view of (7) and (31), we find from (39) that

$$\int_{\Sigma_1} P_\alpha \mathrm{d}a + \int_{\Gamma_1} Q_\alpha \mathrm{d}s = 0.$$

Thus, the conditions (8), with $F_\alpha = 0$, are identically satisfied. It follows from (31) and (39) that the conditions (9)–(11) reduce to the following system, for the constants $a_k$

$$\begin{aligned}
&\sum_{k=1}^{4} D_{\alpha k} a_k = \varepsilon_{3\alpha\beta}(M_\beta + M_\beta^*),\\
&\sum_{k=1}^{4} D_{3k} a_k = -F_3 - F_3^*,\\
&\sum_{k=1}^{4} D_{4k} a_k = -M_3 - M_3^*.
\end{aligned} \tag{40}$$

In (40), we have used the notations

$$\begin{aligned}
D_{\alpha k} &= \int_{\Sigma_1} (x_\alpha S_{33}^{(k)} + 2N_{\alpha33}^{(k)} - N_{33\alpha}^{(k)})\mathrm{d}a,\\
D_{3k} &= \int_{\Sigma_1} S_{33}^{(k)}\mathrm{d}a,\; D_{4k} = \int_{\Sigma_1} \varepsilon_{3\alpha\beta}(x_\alpha S_{3\beta}^{(k)} + 2N_{\alpha3\beta}^{(k)})\mathrm{d}a,\\
F_3^* &= \int_{\Sigma_1} t_{33}\mathrm{d}a,\; M_\alpha^* = \varepsilon_{3\alpha\beta}\int_{\Sigma_1} (x_\beta t_{33} + 2m_{\beta33} - m_{33\beta})\mathrm{d}a,\\
M_3^* &= \int_{\Sigma_1} \varepsilon_{3\alpha\beta}(x_\alpha t_{3\beta} + 2m_{\alpha3\beta})\mathrm{d}a,
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
S_{33}^{(\rho)} &= (\lambda + 2\mu)x_\rho + \tau_{33}^{(\rho)},\; S_{33}^{(3)} = \lambda + 2\mu + \tau_{33}^{(3)},\\
S_{33}^{(4)} &= 4f + \tau_{33}^{(4)},\; S_{3\alpha}^{(\rho)} = 2f\varepsilon_{\alpha\rho3} + \tau_{\alpha3}^{(\rho)},\\
S_{3\alpha}^{(3)} &= \tau_{\alpha3}^{(3)},\; S_{3\alpha}^{(4)} = \mu\varepsilon_{3\beta\alpha}x_\beta + \tau_{\alpha3}^{(4)},\\
N_{\alpha33}^{(i)} &= \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)\delta_{i\alpha} + \mu_{\alpha33}^{(i)},\; (i = 1, 2, 3),\\
N_{\alpha33}^{(4)} &= -\frac{1}{2}fx_\alpha + \mu_{\alpha33}^{(4)},\; N_{33\alpha}^{(i)} = (\alpha_1 - 2\alpha_3 - 2\alpha_4 + \alpha_5)\delta_{i\alpha}\\
&\quad + \mu_{33\alpha}^{(i)},\quad N_{33\alpha}^{(4)} = fx_\alpha + \mu_{33\alpha}^{(4)},\; N_{\alpha3\beta}^{(\rho)} = \varepsilon_{3\alpha\beta}fx_\rho + \mu_{\alpha3\beta}^{(\rho)},\\
N_{\alpha3\beta}^{(3)} &= \varepsilon_{3\alpha\beta}f + \mu_{\alpha3\beta}^{(3)},\; N_{\alpha3\beta}^{(4)} = \varepsilon_{3\alpha\beta}(2\alpha_4 - \alpha_5) + \mu_{\alpha3\beta}^{(4)}.
\end{aligned} \tag{42}$$

The constants $D_{mn}$ $(m, n = 1, 2, 3, 4)$ can be calculated after the solving of the problems $A^{(k)}$ $(k = 1, 2, 3, 4)$. As in classical elasticity, the positive definiteness of the potential energy implies that [17,26].

$$\det(D_{mn}) \neq 0. \tag{43}$$

Thus, the constants $a_k$ are determined by the system (40). By using the reciprocal theorem, we find that

$$D_{mn} = D_{nm}.$$

The functions $t_{3j}$ and $m_{j33}$ that appear in $F_3^*$ and $M_3^*$ depend on the temperature $T_0$. We conclude that a plane temperature field produces axial extension, bending and torsion.

## 5 Deformation due to a temperature that is linear in the axial coordinate

This section is concerned with the deformation of the beam subjected to the following external data

$$T = x_3 T_1(x_1, x_2), \quad \mathbf{F} = F_\alpha \mathbf{e}_\alpha, \quad \mathbf{M} = \mathbf{0}, \tag{44}$$

where $T_1$ and $F_\alpha$ are prescribed.

We try to solve the problem assuming that

$$\begin{aligned}
u_\alpha &= -\frac{1}{2}c_\alpha x_3^2 - \frac{1}{6}b_\alpha x_3^3 + \varepsilon_{3\beta\alpha}(c_4 x_3 + \frac{1}{2}b_4 x_3^2)x_\beta \\
&\quad + \sum_{k=1}^{4}(c_k + b_k x_3)u_\alpha^{(k)} + x_3 U_\alpha + v_\alpha, \\
u_3 &= (c_1 x_1 + c_2 x_2 + c_3)x_3 + \frac{1}{2}(b_1 x_1 + b_2 x_2 + b_3)x_3^2 \\
&\quad + \sum_{k=1}^{4}(c_k + b_k x_3)u_3^{(k)} + x_3 U_3 + v_3, \\
\varphi &= \sum_{k=1}^{4}(c_k + b_k x_3)\varphi^{(k)} + x_3 \Phi + \chi,
\end{aligned} \tag{45}$$

where $(u_j^{(k)}, \varphi^{(k)})$ is the solution of problem $A^{(k)} (k = 1, 2, 3, 4)$, $U_j$, $\Phi$, $v_j$ and $\chi$ are unknown functions of $x_1$ and $x_2$, and $c_k$ $(k = 1, 2, 3, 4)$ are unknown constants. Let us consider a plane strain in which the components of the displacement vector are $v_j$ and the microdilatation function is $\chi$. The strain measures in this problem are defined by

$$\gamma_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), 2\gamma_{\alpha 3} = v_{3,\alpha}, \zeta_{\alpha\beta j} = v_{j,\alpha\beta}. \tag{46}$$

We denote by $E_{\alpha j}$ and $K_{\alpha\beta j}$ the strain measures in the plane strain problem associated with the displacement vector $U_j$ and microdilatation $\Phi$,

$$E_{\alpha\beta} = \frac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha}), 2E_{\alpha 3} = U_{3,\alpha}, K_{\alpha\beta j} = U_{j,\alpha\beta}. \tag{47}$$

In view of (1) and (45)–(47), we obtain

$$\begin{aligned}
e_{\alpha\beta} &= \sum_{k=1}^{4}(c_k + b_k x_3)e_{\alpha\beta}^{(k)} + x_3 E_{\alpha\beta} + \gamma_{\alpha\beta}, \\
e_{\alpha 3} &= \frac{1}{2}\varepsilon_{3\beta\alpha}(c_4 + b_4 x_3)x_\beta + \sum_{k=1}^{4}(c_k + b_k x_3)e_{\alpha 3}^{(k)} + \gamma_{\alpha 3} + x_3 E_{\alpha 3} \\
&\quad + \frac{1}{2}\sum_{k=1}^{4} b_k u_\alpha^{(k)} + \frac{1}{2}U_\alpha, \\
e_{33} &= c_1 x_1 + c_2 x_2 + c_3 + (b_1 x_1 + b_2 x_2 + b_3) + \sum_{k=1}^{4} b_k u_3^{(k)} + U_3, \\
\kappa_{\alpha\beta\gamma} &= \sum_{k=1}^{4}(c_k + b_k x_3)\kappa_{\alpha\beta\gamma}^{(k)} + \zeta_{\alpha\beta\gamma} + x_3 K_{\alpha\beta\gamma}, \\
\kappa_{\alpha\beta 3} &= \sum_{k=1}^{4}(c_k + b_k x_3)\kappa_{\alpha\beta 3}^{(k)} + \zeta_{\alpha\beta 3} + x_3 K_{\alpha\beta 3},
\end{aligned}$$

$$
\begin{aligned}
\kappa_{\beta 3\alpha} &= \varepsilon_{3\beta\alpha}(c_4 + b_4 x_3) + \sum_{k=1}^{4} b_k u^{(k)}_{\alpha,\beta} + U_{\alpha,\beta},\\
\kappa_{\alpha 33} &= c_\alpha + b_\alpha x_3 + \frac{1}{2}\sum_{k=1}^{4} b_k e^{(k)}_{3\alpha} + \frac{1}{2}E_{\alpha 3},\\
\kappa_{33\alpha} &= -c_k - b_\alpha x_3 + \varepsilon_{3\beta\alpha} b_4 x_\beta, \quad \kappa_{333} = b_1 x_1 + b_2 x_2 + b_3.
\end{aligned}
\tag{48}
$$

Let us introduce the notations

$$
\begin{aligned}
s_{\alpha\beta} &= \lambda\gamma_{\rho\rho}\delta_{\alpha\beta} + 2\mu\gamma_{\alpha\beta} + f(\varepsilon_{\alpha\rho 3}\zeta_{\beta\rho 3} + \varepsilon_{\beta\rho 3}\zeta_{\alpha\rho 3}) + \mathrm{d}\chi\delta_{\alpha\beta},\\
s_{\alpha 3} &= 2\mu\gamma_{\alpha 3} + f\varepsilon_{\rho\beta 3}\zeta_{\alpha\rho 3},\ s_{33} = \lambda\gamma_{\rho\rho} + \mathrm{d}\chi,\\
\nu_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\zeta_{\rho\rho\alpha}\delta_{\beta\gamma} + 2\zeta_{\gamma\rho\rho}\delta_{\alpha\beta} + \zeta_{\rho\rho\beta}\delta_{\alpha\gamma}) + \alpha_2(\zeta_{\alpha\rho\rho}\delta_{\beta\gamma} + \zeta_{\beta\rho\rho}\delta_{\alpha\gamma})\\
&\quad + 2\alpha_3\zeta_{\rho\rho\gamma}\delta_{\alpha\beta} + 2\alpha_4\zeta_{\alpha\beta\gamma} + \alpha_5(\zeta_{\gamma\beta\alpha} + \zeta_{\gamma\alpha\beta}) + f(\varepsilon_{\alpha\gamma 3}\gamma_{\beta 3} + \varepsilon_{\beta\gamma 3}\gamma_{\alpha 3})\\
&\quad + \beta_1\delta_{\alpha\beta}\chi_{,\gamma} + 2\beta_2(\delta_{\alpha\gamma}\chi_{,\beta} + \delta_{\beta\gamma}\chi_{,\alpha}),\\
\nu_{\alpha\beta 3} &= 2\alpha_3\zeta_{\rho\rho 3}\delta_{\alpha\beta} + 2\alpha_4\zeta_{\alpha\beta 3} + f(\varepsilon_{\rho\alpha 3}\gamma_{\beta\rho} + \varepsilon_{\rho\beta 3}\gamma_{\alpha\rho}),\\
\nu_{3\alpha\beta} &= \frac{1}{2}\alpha_1\zeta_{\rho\rho 3}\delta_{\alpha\beta} + \alpha_5\zeta_{\beta\alpha 3} + f\varepsilon_{\beta\rho 3}\gamma_{\alpha\rho},\\
\nu_{3\alpha 3} &= \frac{1}{2}\alpha_1\zeta_{\rho\rho\alpha} + \alpha_2\zeta_{\alpha\rho\rho} + f\varepsilon_{\rho\alpha 3}\gamma_{3\rho} + \beta_2\chi_{,\alpha},\\
\nu_{33\alpha} &= \alpha_1\zeta_{\alpha\rho\rho} + 2\alpha_3\zeta_{\rho\rho\alpha} + 2f\varepsilon_{3\alpha\rho}\gamma_{3\rho} + \beta_1\chi_{,\alpha},\\
\nu_{333} &= (\alpha_1 + 2\alpha_3)\zeta_{\rho\rho 3},\ h_\alpha = \beta_1\zeta_{\rho\rho\alpha} + 2\beta_2\zeta_{\alpha\rho\rho} + a_0\chi_{,\alpha},\\
h_3 &= \beta_1\zeta_{\rho\rho 3},\ p = \mathrm{d}\gamma_{\rho\rho} + \xi\chi.
\end{aligned}
\tag{49}
$$

Clearly, $s_{ij}$ is the stress tensor, $\nu_{ijk}$ is the dipolar stress tensor, $\pi_j$ is the microstretch stress vector and $p$ is the intrinsic body force in an isothermal plane problem corresponding to the strain measures $\gamma_{ij}$ and $\zeta_{ijk}$. We now consider a thermoelastic plane problem associated with the thermal field $T_1$, displacement vector $U_j$ and microdilatation function $\Phi$. In this problem, we denote the stress tensor, the dipolar stress tensor, the microstretch stress vector and the intrinsic body force by $T_{ij}$, $M_{ijk}$, $H_\alpha$ and $L$, respectively. Thus, we have

$$
\begin{aligned}
T_{\alpha\beta} &= \lambda E_{\rho\rho}\delta_{\alpha\beta} + 2\mu E_{\alpha\beta} + \mathrm{d}\Phi\delta_{\alpha\beta} + f(\varepsilon_{\alpha\rho 3}K_{\beta\rho 3} + \varepsilon_{\beta\rho 3}K_{\alpha\rho 3}) - bT_1\delta_{\alpha\beta},\\
T_{\alpha 3} &= 2\mu E_{\alpha 3} + f\varepsilon_{\rho\beta 3}K_{\alpha\rho\beta},\\
M_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(K_{\rho\rho\alpha}\delta_{\beta\gamma} + 2K_{\gamma\rho\rho}\delta_{\alpha\beta} + K_{\rho\rho\beta}\delta_{\alpha\gamma})\\
&\quad + \alpha_2(K_{\alpha\rho\rho}\delta_{\beta\gamma} + K_{\beta\rho\rho}\delta_{\alpha\gamma}) + 2\alpha_3 K_{\rho\rho\gamma}\delta_{\alpha\beta} + 2\alpha_4 K_{\alpha\beta\gamma} + \alpha_5(K_{\gamma\beta\alpha} + K_{\gamma\alpha\beta})\\
&\quad + \beta_1\delta_{\alpha\beta}\Phi_{,\gamma} + 2\beta_2(\delta_{\alpha\gamma}\Phi_{,\beta} + \delta_{\beta\gamma}\Phi_{,\alpha}) + f(\varepsilon_{\alpha\gamma 3}E_{\beta 3} + \varepsilon_{\beta\gamma 3}E_{\alpha 3}),\\
M_{\alpha\beta 3} &= 2\alpha_3 K_{\rho\rho 3}\delta_{\alpha\beta} + 2\alpha_4 K_{\alpha\beta 3} + f(\varepsilon_{\rho\alpha 3}E_{\beta\rho} + \varepsilon_{\rho\beta 3}E_{\alpha 3}),\\
H_\alpha &= \beta_1 K_{\rho\rho\alpha} + 2\beta_2 K_{\alpha\rho\rho} + a_0\Phi_{,\alpha},\ H_3 = \beta_1 K_{\rho\rho 3},\\
L &= dE_{\rho\rho} + \xi\Phi - \beta T_1,\ T_{33} = \lambda E_{\rho\rho} + \mathrm{d}\Phi - bT_1,\\
M_{3\alpha\beta} &= \frac{1}{2}\alpha_1 K_{\rho\rho 3}\delta_{\alpha\beta} + \alpha_5 K_{\beta\alpha 3} + f\varepsilon_{\beta\rho 3}E_{\alpha\rho},\\
M_{3\alpha 3} &= \frac{1}{2}\alpha_1 K_{\rho\rho\alpha} + \alpha_2 K_{\alpha\rho\rho} + f\varepsilon_{\rho\alpha 3}E_{3\rho} + \beta_2\Phi_{,\alpha}\\
M_{33\alpha} &= \alpha_1 K_{\alpha\rho\rho} + 2\alpha_3 K_{\rho\rho\alpha} + \beta_1\Phi_{,\alpha} + 2f\varepsilon_{3\alpha\rho}E_{3\rho}, \quad M_{333} = (\alpha_1 + 2\alpha_3)K_{\rho\rho 3}.
\end{aligned}
\tag{50}
$$

From the constitutive equations (2) and the relations (48)–(50), we find that the stress tensor $\tau_{ij}$ is given by

$$
\begin{aligned}
\tau_{\alpha\beta} &= s_{\alpha\beta} + x_3 T_{\alpha\beta} + \{\lambda[c_1 x_1 + c_2 x_2 + c_3 + (b_1 x_1 + b_2 x_2 + b_3)x_3]\\
&\quad - 2f(c_4 + b_4 x_3)\}\delta_{\alpha\beta} + \sum_{k=1}^{4}(c_k + b_k x_3)\tau^{(k)}_{\alpha\beta} + G_{\alpha\beta},
\end{aligned}
$$

$$\begin{aligned}\tau_{\alpha 3} &= s_{\alpha 3} + x_3 T_{\alpha 3} + 2f\varepsilon_{\alpha\rho 3}(c_\rho + b_\rho x_3) + \mu\varepsilon_{3\beta\alpha}(c_4 + b_4 x_3)x_\beta \\ &\quad + \sum_{k=1}^{4}(c_k + b_k x_3)\tau_{\alpha 3}^{(k)} + G_{\alpha 3}, \\ \tau_{33} &= s_{33} + x_3 T_{33} + (\lambda + 2\mu)[c_1 x_1 + c_2 x_2 + c_3 + (b_1 x_1 + b_2 x_2 + b_3)x_3] \\ &\quad + 4f(c_4 + b_4 x_3) + \sum_{k=1}^{4}(c_k + b_k x_3)\tau_{33}^{(k)} + G_{33},\end{aligned} \tag{51}$$

where

$$\begin{aligned} G_{\alpha\beta} &= f[\varepsilon_{3\rho\alpha}U_{\rho,\beta} + \varepsilon_{3\rho\beta}U_{\rho,\alpha} + \sum_{k=1}^{4} b_k(\varepsilon_{3\rho\alpha}u_{\rho,\beta}^{(k)} + \varepsilon_{3\rho\beta}u_{\rho,\alpha}^{(k)})], \\ G_{\alpha 3} &= \mu(U_\alpha + \sum_{k=1}^{4} b_k u_\alpha^{(k)}) + f\varepsilon_{\alpha\rho 3}(U_{3,\rho} + \sum_{k=1}^{4} b_k u_{3,\rho}^{(k)}) - f b_4 x_\alpha, \\ G_{33} &= (\lambda + 2\mu)(U_3 + \sum_{k=1}^{4} b_k u_3^{(k)}) + 2f\varepsilon_{3\alpha\beta}(U_{\alpha,\beta} + \sum_{k=1}^{4} b_k u_{\alpha,\beta}^{(k)}).\end{aligned} \tag{52}$$

The functions $\mu_{ijk}$, $\sigma_j$ and $g$ can be expressed as

$$\begin{aligned}\mu_{111} &= \nu_{111} + x_3 M_{111} + 2(\alpha_2 - \alpha_3)(c_1 + b_1 x_3) + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{111}^{(k)} + J_{111}, \\ \mu_{222} &= \nu_{222} + x_3 M_{222} + 2(\alpha_2 - \alpha_3)(c_2 + b_2 x_3) + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{222}^{(k)} + J_{222}, \\ \mu_{221} &= \nu_{221} + x_3 M_{221} + (\alpha_1 - 2\alpha_3)(c_1 + b_1 x_3) - f(c_4 + b_4 x_3)x_1 \\ &\quad + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{221}^{(k)} + J_{221}, \\ \mu_{112} &= \nu_{112} + x_3 M_{112} + (\alpha_1 - 2\alpha_3)(c_2 + b_2 x_3) - f(c_4 + b_4 x_3)x_2 \\ &\quad + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{112}^{(k)} + J_{112}, \\ \mu_{121} &= \nu_{121} + x_3 M_{121} + \frac{1}{2}(2\alpha_2 - \alpha_1)(c_2 + b_2 x_3) + \frac{1}{2}f(c_4 + b_4 x_3)x_2 \\ &\quad + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{121}^{(k)} + J_{121}, \\ \mu_{122} &= \nu_{122} + x_3 M_{122} + \frac{1}{2}(2\alpha_2 - \alpha_1)(c_1 + b_1 x_3) + \frac{1}{2}f(c_4 + b_4 x_3)x_1 \\ &\quad + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{122}^{(k)} + J_{122}, \\ \mu_{\rho 33} &= \nu_{\rho 33} + x_3 M_{\rho 33} + \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)(c_\rho + b_\rho x_3) \\ &\quad - \frac{1}{2}f(c_4 + b_4 x_3)x_\rho + \sum_{k=1}^{4}(c_k + b_k x_3)\mu_{\rho 33}^{(k)} + J_{\rho 33}, \\ \mu_{33\rho} &= \nu_{33\rho} + x_3 M_{33\rho} + (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5)(c_\rho + b_\rho x_3)\end{aligned}$$

$$\begin{aligned}
&\quad + f(c_4 + b_4 x_3) x_\rho + \sum_{k=1}^{4} (c_k + b_k x_3) \mu^{(k)}_{33\rho} + J_{33\rho},\\
\mu_{\alpha 3\beta} &= \nu_{\alpha 3\beta} + x_3 M_{\alpha 3\beta} + (2\alpha_4 - \alpha_5)(c_4 + b_4 x_3)\varepsilon_{\alpha\beta 3}\\
&\quad + f[c_1 x_1 + c_2 x_2 + c_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3]\varepsilon_{\alpha\beta 3}\\
&\quad + \sum_{k=1}^{4} (c_k + b_k x_3) \mu^{(k)}_{\alpha 3\beta} + J_{\alpha 3\beta},\\
\mu_{\alpha\beta 3} &= \nu_{\alpha\beta 3} + x_3 M_{\alpha\beta 3} + \sum_{k=1}^{4} (c_k + b_k x_3) \mu^{(k)}_{\alpha\beta 3} + J_{\alpha\beta 3},\\
\mu_{333} &= \nu_{333} + x_3 M_{333} + (\alpha_1 + 2\alpha_3) \sum_{k=1}^{4} (c_k + b_k x_3) \kappa^{(k)}_{\alpha\alpha 3} + J_{333},\\
\sigma_\alpha &= h_\alpha + x_3 H_\alpha + \sum_{k=1}^{4} (c_k + b_k x_3) \sigma^{(k)}_\alpha + (2\beta_2 - \beta_1)(c_\alpha + b_\alpha x_3)\\
&\quad + \beta_1 \varepsilon_{3\beta\alpha} b_4 x_\beta + \beta_2 (\sum_{k=1}^{4} b_k e^{(k)}_{\alpha 3} + E_{\alpha 3}),\\
\sigma_3 &= h_3 + x_3 H_3 + \sum_{k=1}^{4} (c_k + b_k x_3) \sigma^{(k)}_3 + (b_1 + 2\beta_2)(b_1 x_1 + b_2 x_2 + b_3)\\
&\quad + 2\beta_2 (\sum_{k=1}^{4} b_k u^{(k)}_{\rho,\rho} + U_{\rho,\rho}) + a_0 (\sum_{k=1}^{4} b_k \varphi^{(k)} + \Phi),\\
g &= p + x_3 L + \sum_{k=1}^{4} (c_k + b_k x_3) g^{(k)} + d[c_1 x_1 + c_2 x_2 + c_3\\
&\quad + (b_1 x_1 + b_2 x_2 + b_3) x_3 + \sum_{k=1}^{4} b_k u^{(k)}_3 + U].
\end{aligned} \tag{53}$$

In these relations, we have used the following notations

$$\begin{aligned}
J_{111} &= -(\alpha_1 + 2\alpha_3) b_4 x_2 + (\alpha_1 + 2\alpha_2)(U_{3,1} + \sum_{k=1}^{4} b_k u^{(k)}_{3,1}),\\
J_{222} &= (\alpha_1 + 2\alpha_3) b_4 x_1 + (\alpha_1 + 2\alpha_2)(U_{3,2} + \sum_{k=1}^{4} b_k u^{(k)}_{3,2}),\\
J_{221} &= -2\alpha_3 b_4 x_2 + \alpha_1 (U_{3,1} + \sum_{k=1}^{4} b_k u^{(k)}_{3,1}) - f(U_2 + \sum_{k=1}^{4} b_k u^{(k)}_2),\\
J_{112} &= \alpha_1 (U_{3,2} + \sum_{k=1}^{4} b_k u^{(k)}_{3,2}) + 2\alpha_3 (b_4 x_1 + \sum_{k=1}^{4} b_k u^{(k)}_1)\\
&\quad + f(U_1 + \sum_{k=1}^{4} b_k u^{(k)}_1),\\
J_{121} &= \frac{1}{2}\alpha_1 b_4 x_1 + \alpha_2 (U_{3,2} + \sum_{k=1}^{4} b_k u^{(k)}_{3,2}) - \frac{1}{2} f(U_1 + \sum_{k=1}^{4} b_k u^{(k)}_1),
\end{aligned}$$

$$
\begin{aligned}
J_{122} &= -\frac{1}{2}\alpha_1 b_4 x_2 + \alpha_2(U_{3,1} + \sum_{k=1}^{4} b_k u_{3,1}^{(k)}) + \frac{1}{2} f(U_2 + \sum_{k=1}^{4} b_k u_2^{(k)}),\\
J_{\rho 33} &= -\frac{1}{2}(\alpha_1 + 2\alpha_5)\varepsilon_{3\rho\beta} b_4 x_\beta + (\alpha_2 + 2\alpha_4 + \alpha_5)(U_{3,\rho}\\
&\quad + \sum_{k=1}^{4} b_k u_{3,\rho}^{(k)}) + \frac{1}{2} f\varepsilon_{3\rho\beta}(U_\beta + \sum_{k=1}^{4} b_k u_\beta^{(k)}),\\
J_{33\rho} &= -2(\alpha_3 + \alpha_4)\varepsilon_{3\rho\beta} b_4 x_\beta + (\alpha_1 + 2\alpha_5)(U_{3,\rho} + \sum_{k=1}^{4} b_k u_{3,\rho}^{(k)})\\
&\quad + f\varepsilon_{3\rho\beta}(U_\beta + \sum_{k=1}^{4} b_k u_\beta^{(k)}),\\
J_{\alpha 3\beta} &= 2\alpha_4(U_{\beta,\alpha} + \sum_{k=1}^{4} b_k u_{\beta,\alpha}^{(k)}) + \alpha_5(U_{\alpha,\beta} + \sum_{k=1}^{4} b_k u_{\alpha,\beta}^{(k)})\\
&\quad + \delta_{\alpha\beta}[\frac{1}{2}(\alpha_1 + 2\alpha_2)(b_1 x_1 + b_2 x_2 + b_3) + \alpha_2(U_{\rho,\rho} + \sum_{k=1}^{4} b_k u_{\rho,\rho}^{(k)})]\\
&\quad + f(U_3 + \sum_{k=1}^{4} b_k u_3^{(k)})\varepsilon_{\alpha\beta 3},\\
J_{\alpha\beta 3} &= \delta_{\alpha\beta}[(\alpha_1 + 2\alpha_3)(b_1 x_1 + b_2 x_2 + b_3) + \alpha_1(U_{\rho,\rho} + \sum_{k=1}^{4} b_k u_{\rho,\rho}^{(k)})\\
&\quad + \beta_1(\Phi + \sum_{k=1}^{4} b_k \varphi^{(k)})] + 2\alpha_5(E_{\alpha\beta} + \sum_{k=1}^{4} b_k e_{\alpha\beta}^{(k)}),\\
J_{333} &= 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)(b_1 x_1 + b_2 x_2 + b_3) + (\alpha_1 + 2\alpha_2)U_{\rho,\rho}\\
&\quad + (\beta_1 + 2\beta_2)(\Phi + \sum_{k=1}^{4} b_k \varphi^{(k)}).
\end{aligned}
\tag{54}
$$

If we take into account (32), (33), (51) and (53), then the equilibrium equations (3) reduce to the equations

$$
s_{\beta j,\beta} - \nu_{\alpha\beta j,\alpha\beta} + \Psi_j = 0, \quad h_{\alpha,\alpha} - p + \mathcal{H} = 0 \ \text{ on } \Sigma_1, \tag{55}
$$

and

$$
T_{\alpha j,\alpha} - M_{\alpha\beta j,\alpha\beta} = 0, \quad H_{\alpha,\alpha} - L = 0 \ \text{ on } \Sigma_1. \tag{56}
$$

In equations (55) and (56), we have used the notations

$$
\begin{aligned}
\Psi_\alpha &= G_{\beta\alpha,\beta} + T_{\alpha 3} - 2M_{3\rho\alpha,\rho} - \sum_{k=1}^{4}(2\mu_{3\rho\alpha,\rho}^{(k)} - \tau_{\alpha 3}^{(k)}) b_k\\
&\quad + 4f\varepsilon_{3\alpha\beta} b_\beta + \mu\varepsilon_{3\rho\alpha} b_4 x_\rho - J_{\rho\eta\alpha,\rho\eta},\\
\Psi_3 &= G_{\alpha 3,\alpha} + T_{33} - 2M_{\alpha 33,\alpha} - J_{\alpha\beta 3,\alpha\beta}\\
&\quad + (\lambda + 2\mu)(b_1 x_1 + b_2 x_2 + b_3) + 6f b_4 - \sum_{k=1}^{4}(2\mu_{\alpha 33,\alpha}^{(k)} - \tau_{33}^{(k)}) b_k,\\
\mathcal{H} &= H_3 + \beta_2(E_{\alpha 3,\alpha} + \sum_{k=1}^{4} b_k e_{\alpha 3,\alpha}^{(k)}) - d(U_3 + \sum_{k=1}^{4} b_k u_3^{(k)}) + \sum_{k=1}^{4} b_k \sigma_3^{(k)}.
\end{aligned}
\tag{57}
$$

Following (4) and (18), we define the functions

$$\begin{aligned}
\Pi_j^{(1)} &= (s_{\beta j} - \nu_{\rho\beta j,\rho})n_\beta - D_\beta(n_\rho \nu_{\rho\beta j}) + (D_\alpha n_\alpha)n_\rho n_\eta \nu_{\rho\eta j},\\
\Lambda_k^{(1)} &= \nu_{\rho\beta j}n_\rho n_\beta,\ \Lambda_j^{(2)} = M_{\rho\beta j}n_\rho n_\beta,\\
\Pi_j^{(2)} &= (T_{\beta j} - M_{\rho\beta j,\rho})n_\beta - D_\beta(n_\rho M_{\rho\beta j}) + (D_\nu n_\nu)n_\rho n_\eta M_{\rho\eta j}.
\end{aligned} \tag{58}$$

With the help of relations (36), (37), (51), (53) and (58), we see that the conditions on the lateral surface (7) reduce to

$$\Pi_j^{(1)} = B_j,\ \Lambda_j^{(1)} = C_j,\ h_\alpha n_\alpha = h \ \text{ on } \Gamma_1, \tag{59}$$

and

$$\Pi_j^{(2)} = 0,\ \Lambda_j^{(2)} = 0,\ H_\alpha n_\alpha = 0 \text{ on } \Gamma_1, \tag{60}$$

where

$$\begin{aligned}
B_\alpha &= 2n_\rho\Big(M_{3\rho\alpha} + \sum_{k=1}^{4} b_k \mu_{3\rho\alpha}^{(k)}\Big) - 2\varepsilon_{\alpha\rho3}n_\rho[f(b_1x_1 + b_2x_2 + b_3)\\
&\quad + (2\alpha_4 - \alpha_5)b_4] - B_\alpha^*,\\
B_3 &= 2n_\rho\Big[M_{\rho33} + \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)b_\rho - \frac{1}{2}fb_4x_\rho\Big]\\
&\quad + \sum_{k=1}^{4} b_k\mu_{\rho33}^{(k)} - B_3^*,\ \ C_j = -J_{\alpha\beta j}n_\alpha n_\beta,\\
h &= \Big[\beta_1\varepsilon_{3\alpha\beta}b_4x_\beta - \beta_2\Big(E_{\alpha3} + \sum_{k=1}^{4} b_k e_{\alpha3}^{(k)}\Big)\Big]n_\alpha,\\
B_j^* &= (G_{\beta j} - J_{\rho\beta j,\rho})n_\beta - D_\nu(n_\rho J_{\rho\nu j}) + (D_\alpha n_\alpha)n_\rho n_\nu J_{\rho\nu j}.
\end{aligned} \tag{61}$$

We denote by $(\mathcal{P}_1)$ the isothermal plane problem which consist in finding the functions $v_j$ and $\chi$ that satisfy the geometrical equations (46), the constitutive equations (49), the equilibrium equations (55) and the boundary conditions (59). Let us denote by $(\mathcal{P}_2)$ the thermoelastic plane problem associated with the temperature $T_1$ and characterized by the geometrical equations (47), the constitutive equations (50), the equilibrium equations (56) and the boundary conditions (60). Clearly, the necessary and sufficient conditions to solve the problem $(\mathcal{P}_2)$ are satisfied for any thermal field. The necessary and sufficient conditions to solve the problem $(\mathcal{P}_1)$ are

$$\begin{aligned}
&\int_{\Sigma_1} \Psi_j \mathrm{d}a + \int_{\Gamma_1} B_j \mathrm{d}s = 0,\\
&\int_{\Sigma_1} \varepsilon_{3\alpha\beta}x_\alpha\Psi_\beta \mathrm{d}a + \int_{\Gamma_1} \varepsilon_{3\alpha\beta}(x_\alpha B_\beta + n_\alpha C_\beta)\mathrm{d}s = 0.
\end{aligned} \tag{62}$$

By using the divergence theorem, we find that

$$\begin{aligned}
&\int_{\Sigma_1} (G_{\beta j,\beta} - J_{\rho\eta j,\rho\eta})\mathrm{d}a + \int_{\Gamma_1} B_j^* \mathrm{d}s = 0,\\
&\int_{\Sigma_1} \varepsilon_{3\alpha\beta}x_\alpha(G_{\rho\beta,\rho} - J_{\nu\rho\beta,\nu\rho})\mathrm{d}a + \int_{\Gamma_1} \varepsilon_{3\alpha\beta}(x_\alpha B_\beta^* + n_\alpha C_\beta)\mathrm{d}s = 0.
\end{aligned} \tag{63}$$

It follows from (51), (57), (61) and (63) that

$$\int_{\Sigma_1} \Psi_\alpha \mathrm{d}a + \int_{\Gamma_1} B_\alpha \mathrm{d}s = \int_{\Sigma_1} \tau_{\alpha3,3}\mathrm{d}a. \tag{64}$$

By using the equilibrium equations, we find

$$\begin{aligned}\tau_{\alpha 3} = \tau_{\alpha 3} + x_\alpha(\tau_{j3,j} - \mu_{rs3,rs}) = [x_\alpha(\tau_{\beta 3} - \mu_{\beta\nu 3,\nu})]_{,\beta}\\ + x_\alpha(\tau_{33,3} - 2\mu_{3\beta 3,3\beta} - \mu_{333,33}) + \mu_{\alpha\nu 3,\nu}.\end{aligned} \tag{65}$$

The condition $P_3 = \widetilde{P}_3$ on the lateral boundary can be expressed in the form

$$(\tau_{\beta 3} - \mu_{\beta\nu 3,\nu})n_\beta = \widetilde{P}_3 + [2\mu_{3\beta 3,3} - (\mu_{\alpha\beta 3}n_\alpha n_\gamma - \mu_{\rho\gamma 3}n_\rho n_\beta)_{,\gamma}]n_\beta, \tag{66}$$

so that the relation (65) implies that

$$\begin{aligned}\int_{\Sigma_1} \tau_{\alpha 3}\mathrm{d}a = \int_{\Gamma_1} [\mu_{\alpha\nu 3,\nu} + x_\alpha(\tau_{33,3} - 2\mu_{3\beta 3,3\beta} - \mu_{333,33})]\mathrm{d}a\\ + \int_{\Gamma_1} x_\alpha\{\widetilde{P}_3 + (\mu_{\rho\gamma 3}n_\rho n_\beta - \mu_{\alpha\beta 3}n_\alpha n_\gamma)_{,\gamma}n_\beta + 2\mu_{3\beta 3,3}n_\beta\}\mathrm{d}s.\end{aligned} \tag{67}$$

Let us note the identity

$$\int_{\Sigma_1} x_\alpha(\mu_{\rho\gamma 3}n_\rho n_\beta - \mu_{\alpha\beta 3}n_\alpha n_\gamma)_{,\gamma}n_\beta\mathrm{d}s = \int_{\Gamma_1} n_\alpha\widetilde{R}_3\mathrm{d}s - \int_{\Sigma_1} \mu_{\alpha\nu 3,\nu}\mathrm{d}a. \tag{68}$$

From (65) and (68), we get

$$\int_{\Sigma_1} \tau_{\alpha 3}\mathrm{d}a = \int_{\Sigma_1} [x_\alpha(\tau_{33,3} - \mu_{333,33}) + 2\mu_{\alpha 33,3}]\mathrm{d}a + \int_{\Gamma_1} (x_\alpha\widetilde{P}_3 + n_\alpha\widetilde{R}_3)\mathrm{d}s. \tag{69}$$

In view of (7), (51), (53) and (69), we find that

$$\int_{\Sigma_1} \tau_{\alpha 3,\alpha}\mathrm{d}a = 0,$$

so that the first two conditions from (61) are satisfied. With the help of (40), (57) and (60), we obtain

$$\begin{aligned}&\int_{\Sigma_1} \Psi_3\mathrm{d}a + \int_{\Gamma_1} B_3\mathrm{d}s = \int_{\Sigma_1} T_{33}\mathrm{d}a + \sum_{k=1}^{4} D_{3k}b_k,\\ &\int_{\Sigma_1} \varepsilon_{3\alpha\beta}x_\alpha\Psi_\beta\mathrm{d}a + \int_{\Gamma_1} \varepsilon_{3\alpha\beta}(x_\alpha B_\beta + n_\alpha C_\beta)\mathrm{d}s\\ &= \int_{\Sigma_1} \varepsilon_{3\alpha\beta}x_\alpha T_{\beta 3}\mathrm{d}a + \sum_{k=1}^{4} D_{4k}b_k.\end{aligned}$$

The last two conditions from (62) reduce to

$$\sum_{k=1}^{4} D_{3k}b_k = -\int_{\Sigma_1} T_{33}\mathrm{d}a, \sum_{k=1}^{4} D_{4k}b_k = -\int_{\Sigma_1} \varepsilon_{3\alpha\beta}x_\alpha T_{\beta 3}\mathrm{d}a. \tag{70}$$

Let us impose the conditions (8). In view of (12), we get

$$\int_{\Sigma_1} P_\alpha\mathrm{d}a + \int_{\Gamma_1} Q_s\mathrm{d}s = -\int_{\Sigma_1} (\tau_{3\alpha} - \mu_{33\alpha,3})\mathrm{d}a.$$

By using (67), we obtain

$$\begin{aligned}\int_{\Sigma_1} P_\alpha\mathrm{d}a + \int_{\Gamma_1} Q_\alpha\mathrm{d}s = -\int_{\Gamma_1} (x_\alpha\widetilde{P}_3 + n_\alpha\widetilde{R}_3)\mathrm{d}s - \int_{\Sigma_1} [x_\alpha(\tau_{33,3} - \mu_{333,33})\\ + 2\mu_{\alpha 33,3} - \mu_{33\alpha,3}]\mathrm{d}a.\end{aligned} \tag{71}$$

With the help of (7), (40), (51), (53) and (71), we see that the conditions (8) reduce to

$$\sum_{k=1}^{4} D_{\alpha k} b_k = -F_\alpha - \int_{\Sigma_1} (x_\alpha T_{33} + 2M_{\alpha 33} - M_{33\alpha})\mathrm{d}a. \tag{72}$$

The system (70), (72) uniquely determines the constants $b_k (k = 1, 2, 3, 4)$.

If we take into account the relations (38), (40), (51) and (53), we find that the conditions (9)–(11) take the form

$$\begin{aligned} &\sum_{k=1}^{4} D_{\alpha k} c_k = -\varepsilon_{3\alpha\beta}(M_\beta + M_\beta^0), \\ &\sum_{k=1}^{4} D_{3k} c_k = -F_3 - F_3^0, \sum_{k=1}^{4} D_{4k} c_k = -M_3 - M_3^0, \end{aligned} \tag{73}$$

where we have used the notations

$$\begin{aligned} F_3^0 &= \int_{\Sigma_1} [s_{33} + G_{33} - M_{333} - (\alpha_1 + 2\alpha_3) \sum_{k=1}^{4} b_k \kappa_{333}^{(k)}]\mathrm{d}a, \\ M_\alpha^0 &= \varepsilon_{3\alpha\beta} \int_{\Sigma_1} \{x_\beta [s_{33} + G_{33} - M_{333} - (\alpha_1 + 2\alpha_3) \sum_{k=1}^{4} b_k \kappa_{333}^{(k)}] \\ &\quad + 2\nu_{\beta 33} + 2J_{\beta 33} - \nu_{33\beta} - J_{33\beta}\}\mathrm{d}a, \\ M_3^0 &= \int_{\Sigma_1} \{\varepsilon_{\alpha\beta 3}[s_{\beta 3} + G_{\beta 3} - M_{33\beta} - (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5) b_\beta \\ &\quad - f b_4 x_\beta - \sum_{k=1}^{4} b_k \mu_{33\beta}^{(k)}] + 2\varepsilon_{\alpha\beta 3}(\nu_{\alpha 3\beta} + J_{\alpha 3\beta})\}\mathrm{d}a. \end{aligned} \tag{74}$$

The constant $c_k$ are determined by the system (73). First, we have to find the solutions of the plane problems $A^{(k)}$ $(k = 1, 2, 3, 4)$. Then, we can determine the constants $D_{mn}$ from (41) and the solution of the problem $(\mathcal{P}_2)$. Next, we determine the constants $b_k$ from the system (70), (72). Now, we can solve the problem $(\mathcal{P}_1)$ and to find from (73) the constants $c_k$ $(k = 1, 2, 3, 4)$. The above results can be used to study the case when the temperature field is a polynomial in the axial coordinate.

## 6 Application

In this section, we use the solution (23) to investigate the thermoelastic deformation of a chiral circular cylinder subjected to a uniform temperature. We consider a homogeneous cylinder that occupies the domain $B = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < a^2, 0 < x_3 < h\}$, $(a > 0)$. We assume that the mechanical loads are absent. In this case, we have

$$T = T^*, \quad F_j = 0, \quad M_j = 0, \tag{75}$$

where $T^*$ is a prescribed constant. We seek the solution of the problem $(\mathcal{T})$ in the form

$$w_\alpha = E_1 x_\alpha, \quad w_3 = 0, \quad \psi = E_2, \tag{76}$$

where $E_1$ and $E_2$ are arbitrary constants. From (28), (27) and (76), we obtain

$$\begin{aligned} &\eta_{\alpha\beta} = E_1 \delta_{\alpha\beta}, \eta_{\alpha 3} = 0, \xi_{ijk} = 0, \\ &t_{\alpha\beta} = [2(\lambda + \mu)E_1 + dE_2 - bT^*]\delta_{\alpha\beta}, t_{\alpha 3} = 0, m_{\alpha\beta j} = 0, \\ &\pi_\alpha = 0, \gamma = 2dE_1 + \xi E_2 - \beta T^*, m_{3\alpha\beta} = f E_1 \varepsilon_{\beta\alpha 3}, m_{3\alpha 3} = 0, \\ &m_{33j} = 0, P_\alpha^* = [2(\lambda + \mu)E_1 + dE_2 - bT^*]n_\alpha, P_3^* = 0, R_i^* = 0. \end{aligned} \tag{77}$$

The equilibrium equations (34) and the boundary conditions (38) are satisfied if the constants $E_1$ and $E_2$ are given by

$$E_1 = (\beta d - b\xi)T^*/(2D),\ E_2 = [db - \beta(\lambda + \mu)]T^*/D, \tag{78}$$

where

$$D = d^2 - \xi(\lambda + \mu). \tag{79}$$

The positive definiteness of the elastic potential implies that $D \neq 0$, Thus, the solution of the problem $(\mathcal{T})$ is given by (76). It follows from (40) and (77) that

$$F_3^* = \pi a^2(2\lambda E_1 + dE_2 - bT^*),\ M_j^* = 0. \tag{80}$$

In a similar way, we can study the problems $A^{(3)}$ and $A^{(4)}$. The solution of the problem $A^{(3)}$ is

$$u_\alpha^{(3)} = S_1 x_\alpha,\ u_3^{(3)} = 0,\ \varphi^{(3)} = S_2, \tag{81}$$

where

$$S_1 = (\lambda\xi - d^2)/(2D), \quad S_2 = \mathrm{d}\mu/D,$$

and $D$ is given by (79). The problem $A^{(4)}$ has the following solution

$$u_\alpha^{(4)} = K_1 x_\alpha,\ u_3^{(4)} = 0,\ \varphi^{(3)} = K_2, \tag{82}$$

where

$$K_1 = -f\xi/D, \quad K_2 = 2bf/D.$$

It follows from (81), (82) and (41) that

$$\begin{aligned} &D_{\alpha 3} = 0,\ D_{\alpha 4} = 0,\ D_{33} = \pi\mu a^2[3d^2 - (3\lambda + 2\mu)\xi]/D,\\ &D_{44} = \mu\frac{\pi a^4}{2} + 4(2a_4 - \alpha_5 - K_1 f)\pi a^2,\\ &D_{34} = D_{43} = 2\pi a^2 f[3d^2 - (3\lambda + 2\mu)\xi]/D. \end{aligned} \tag{83}$$

In view of symmetry of $D_{mn}$, we obtain $D_{3\alpha} = 0$ and $D_{4\alpha} = 0$. Thus, with the help of (82) and (83) we find that the system (40) reduces to

$$D_{\alpha\beta}a_\beta = 0,\ D_{33}a_3 + D_{34}a_4 = -F_3^*,\ D_{43}a_3 + D_{44}a_4 = 0. \tag{84}$$

It follows from (42) that this system uniquely determines the constants $a_k$. We find that

$$a_1 = a_2 = 0,\ a_3 = -D_{44}F_3^* q,\ a_4 = D_{34}F_3^* q, \tag{85}$$

where

$$q^{-1} = D_{33}D_{44} - D_{34}^2.$$

Let us note that we have solved the problem without using the solutions of the problems $A^{(1)}$ and $A^{(2)}$.

In view of (23), (76), (81), (82) and (85), we see that the solution of the problem is given by

$$\begin{aligned} &u_\alpha = \varepsilon_{3\beta\alpha}a_4 x_\beta x_3 + a_3 S_1 x_\alpha + a_4 K_1 x_\alpha + w_\alpha,\\ &u_3 = a_3 x_3,\ \varphi = a_3 S_2 + a_4 K_2 + \psi. \end{aligned} \tag{86}$$

From (86), we conclude that the thermal field produces torsion, extension and a variation of microdilatation. In the case of an achiral material, we have $f = 0$ and (83) implies that $D_{34} = 0$. Then, from (85) we obtain $a_4 = 0$. In this case, the thermal field does not produce torsion.

## 7 Conclusions

The results presented in this paper can be summarized as follows:

a. In the context of the strain gradient theory of porous thermoelastic solids, we study the deformation of isotropic chiral cylinders subjected to a temperature field that is linear in the axial coordinate.
b. We introduce the problem of generalized plane deformation of a chiral cylinder, and we express the equilibrium equations in terms of displacements and microdilatation function.
c. We establish the solution of the problem when the temperature distribution is independent of the axial coordinate. It is shown that the thermal field produces extension, bending and torsion.
d. We study the thermoelastic deformation of cylinders subjected to a thermal field which is linear in the axial coordinate. We present a method to reduce the three-dimensional problem to the study of plane problems.
e. We use the results to study the deformation of a circular cylinder subjected to a uniform temperature variation. In contrast to the case of achiral materials, the thermal field in chiral cylinders produces torsional effects.

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