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Journal of Differential Equations 414 (2025) 746–772

Journal of Differential Equations

www.elsevier.com/locate/jde

Normalized solutions for a nonlinear Dirac equation

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Received 15 November 2023; revised 26 August 2024; accepted 16 September 2024

Abstract

We prove the existence of a normalized, stationary solution $\psi : \mathbb{R}^3 \to \mathbb{C}^4$ with frequency $\omega > 0$ of the nonlinear Dirac equation. The result covers the case in which the nonlinearity is the gradient of a function of the form

$$F(\Psi) = a|(\Psi, \gamma^0\Psi)|^{\frac{\alpha}{2}} + b|(\Psi, \gamma^1\gamma^2\gamma^3\Psi)|^{\frac{\alpha}{2}}$$

with $\alpha \in (2, \frac{8}{3}]$, $b \geq 0$ and $a > 0$ sufficiently small. Here γ^i, $i = 0, \ldots, 3$ are the 4×4 Dirac's matrices.

We find the solution as a critical point of a suitable functional restricted to the unit sphere in L^2, and ω turns out to be the corresponding Lagrange multiplier.

MSC: 35Q40; 35P30; 47J10; 49J35

Keywords: Nonlinear Dirac equation; Critical point theory; Min-Max methods; Normalized solutions

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1 Margherita Nolasco was partially supported by the Grant MIUR-PRIN-20227HX33Z – "Pattern formation in nonlinear phenomena".

https://doi.org/10.1016/j.jde.2024.09.029

1. Introduction

The nonlinear Dirac equation is a simplified model describing self-interacting fermions (electrons). The equation we consider is the following:

$$(-i\gamma^\mu \partial_\mu + m)\Psi = \gamma\beta\nabla F(\Psi) \qquad \text{in } \mathbb{R}\times\mathbb{R}^3 \tag{1.1}$$

where $\Psi\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^4$, $m>0$ is the mass of the electron, γ^μ the 4×4 Dirac matrices

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^k = \beta\alpha^k \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1,\dots,3,$$

σ^k the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\gamma\in\mathbb{R}$ is a constant, $\gamma>0$. The nonlinear self-interaction of the electron is described by the function $\gamma\nabla F(\Psi)$. One can find a discussion of the motivation and history of the models leading to the nonlinear Dirac equation in [22] where in particular the model nonlinearity

$$F(\Psi) = |(\Psi, \beta\Psi)|^{\frac{\alpha}{2}} + b|(\Psi, \gamma^1\gamma^2\gamma^3\Psi)|^{\frac{\alpha}{2}}$$

is discussed with $\alpha = 4$ (in case $b=0$ this is the so called Soler model).

We prove existence of localized stationary solutions of fixed L^2 norm for a class of nonlinearities which includes those of the form above provided $\alpha\in(2,\frac{8}{3}]$ (see below the precise assumptions).

A normalized stationary solution is a solution of the form

$$\Psi(t,x) = \mathrm{e}^{-i\omega t}\,\psi(x), \qquad \int_{\mathbb{R}^3} |\Psi|^2 = 1.$$

Assuming $F(\mathrm{e}^{i\theta}\phi) = F(\phi)$ for all $\theta\in\mathbb{R}$, $\phi\in\mathbb{C}^4$ we have that ψ solves

$$\begin{cases} (-i\boldsymbol{\alpha}\cdot\nabla + m\beta - \omega)\,\psi = \gamma\nabla F(\psi) \\ \int |\psi|^2 = 1 \end{cases}. \tag{1.2}$$

We will find solutions for problem (1.2) as critical points of a suitable functional under the constraint of fixed L^2 norm. The frequency ω will arise as the Lagrange multiplier associated to the constraint.

The use of variational methods in the study of the existence of stationary solutions of nonlinear equations and systems involving the Dirac operator goes back to the pioneering work of Esteban and Séré in [13], and several later developments, see for example [1,9]. Also related problems like the Maxwell-Dirac and Klein-Gordon-Dirac problems have been tackled using variational methods: see the review paper [12] where many of these results are presented. In all these works the frequency ω is assigned, and the typical result is that for any $\omega \in (0, m)$ there is a solution of the equation.

The search of normalized solutions, in particular such that $\int |\psi|^2 = 1$, is quite natural since $|\Psi(t, x)|^2$ can be interpreted as the probability density of the position of the electron at point x and time t. The problem is also an interesting mathematical one, and a lot of work has been devoted to finding normalized solutions for nonlinear Schrödinger equations, see [23,6,2,18,19] and more recently [16,17,15,20,11]. The main approach to finding normalized solutions for this class of problems is by minimization (usually a nontrivial one) or other min-max procedures of a suitable functional on the unit sphere of L^2.

Very little is known about the existence via variational methods of normalized solutions for equations involving the Dirac operator. In this case the functional associated is strongly indefinite and a direct minimization on the sphere in L^2 is not possible.

A first result on this topic is the one by Buffoni and Jeanjean [5], dealing with semilinear elliptic equations giving rise to a strongly indefinite functional. Later Esteban and Séré in [14] proved the existence of infinitely many solutions for the Dirac-Fock equation, which describes the self-interaction of N electrons around a fixed nucleus. A solution for this problem is a set of N orthonormal functions which solve an equation with the Dirac operator and a nonlocal nonlinear term. The orthonormal solutions are found with variational methods using a penalization method. A similar penalization method has been used in the paper [4] to find normalized solutions for a strongly indefinite problem not involving the Dirac operator.

Recent results on the existence of normalized solutions for strongly indefinite problems involving the Dirac operator are contained in [7], dealing with the problem of an electron interacting with a nuclei and its own electric field, [21] where the existence of a normalized solutions for the Dirac-Maxwell system is proved, [8] dealing with a Klein-Gordon-Dirac system and [10], where Ding, Yu and Zhao consider a nonlinearity of the form $F(\phi) = K(x)|\phi|^\alpha$, $\alpha \in (2, \frac{8}{3})$ with $K(x) \to 0$ as $|x| \to +\infty$.

Let us now state our assumptions: we will assume that $F \in C^2(\mathbb{C}^4, \mathbb{R})$ is such that the following holds for some $\alpha \in (2, \frac{8}{3}]$ and $R > 0$

$$\nabla F(0) = D^2 F(0) = 0, \tag{H1}$$

$$|D^2 F(\phi)| \leq |\phi|^{\alpha-2} \quad \text{for all } |\phi| \geq R, \tag{H2}$$

$$(\nabla F(\phi), \phi) \geq \alpha F(\phi) \geq 0 \quad \text{for all } \phi \in \mathbb{C}^4, \tag{H3}$$

and also that, for some $\rho > 0$, $\bar{\gamma} > 0$ and $\nu \in (\frac{\alpha}{2}, \frac{3}{2})$

$$F(\phi) \geq \bar{\gamma} |(\phi, \beta\phi)|^{\frac{\alpha}{2}} \qquad \text{for all } \phi \in \mathbb{C}^4,\ |\phi| < \rho, \tag{H4}$$

$$|\nabla F(\phi)| \leq |\phi|^\nu \quad \text{for all } \phi \in \mathbb{C}^4,\ |\phi| < \rho. \tag{H5}$$

We also assume that $\xi > 3$ is such that for all $\zeta > 0$ there is a C_ζ such that for all $\phi \in \mathbb{C}^4$

$$|\nabla F(\phi)| \leq \left(\zeta + C_\zeta F(\phi)^{1/\xi}\right) |\phi|. \tag{H6}$$

Remark 1.3. Let us remark that the assumptions (H1), (H2), (H3), (H6) are the same as in the paper [13]. The restriction on the exponent α is quite natural when looking for normalized solutions. The assumption (H4) is similar to assumption (H9) of the same paper [13].

Let us also remark that (H5) holds if

$$|D^2F(\phi)| \le |\phi|^{\alpha-2} \quad \text{for all } \phi \in \mathbb{C}^4.$$

Remark 1.4. Follows from (H1) and (H2) that for all $\epsilon > 0$ there are μ_ϵ and δ_ϵ such that, for all $\phi, \varphi \in \mathbb{C}^4$

$$|D^2F(\phi)| \le \epsilon + \mu_\epsilon |\phi|^{\alpha-2} \tag{1.5}$$

$$|\nabla F(\phi)| \le \epsilon|\phi| + \mu_\epsilon |\phi|^{\alpha-1} \tag{1.6}$$

$$0 \le F(\phi) \le \frac{\epsilon}{2}|\phi|^2 + \frac{\mu_\epsilon}{\alpha}|\phi|^\alpha \tag{1.7}$$

$$|\nabla F(\phi+\varphi) - \nabla F(\phi)| \le \left(\epsilon + \mu_\epsilon(|\phi|^{\alpha-2} + |\varphi|^{\alpha-2})\right)|\varphi| \tag{1.8}$$

We will write $\mu = \mu_1$ and use the fact that, for all $\phi, \varphi \in \mathbb{C}^4$

$$|D^2F(\phi)| \le 1 + \mu|\phi|^{\alpha-2} \tag{1.9}$$

$$|\nabla F(\phi)| \le |\phi| + \mu|\phi|^{\alpha-1} \tag{1.10}$$

$$0 \le F(\phi) \le \frac{1}{2}|\phi|^2 + \frac{\mu}{\alpha}|\phi|^\alpha \tag{1.11}$$

$$|\nabla F(\phi+\varphi) - \nabla F(\phi)| \le \left(1 + \mu(|\phi|^{\alpha-2} + |\varphi|^{\alpha-2})\right)|\varphi|. \tag{1.12}$$

Our final assumptions give the size of the coefficient γ in front of the nonlinear term ∇F. We let that $\gamma_0 > 0$ be such that

$$\gamma_0\left(S_2^2 + \mu S_3^{3(\alpha-2)} S_2^{3\alpha-8}\right) < \frac{1}{16}, \tag{1.13}$$

$$\gamma_0\left(S_2^2 + \mu S^2_{\frac{4}{4-\alpha}}\right) < \frac{1}{8} \tag{1.14}$$

where $m > 0$ is the mass of our electron, μ is the constant in (1.9)-(1.12), S_q is the constant relative to the Sobolev embedding of $H^{1/2}$ into L^q (with the norm (2.1), equivalent to the usual one) and α is the exponent in (H2).

Our result is

Theorem 1.15. *Let $m > 0$, assume F satisfies (H1)–(H6), and let $\gamma \in (0, \gamma_0]$.*
Then there is $\omega \in (0, m)$ and $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ solutions of problem (1.2).

As far as we know this is the first result on the existence of *normalized* solutions for Dirac equation with a nonlinear interaction of Soler type. Esteban and Séré in [13] deal with essentially the same equation, and find for all assigned $\omega \in (0, m)$ a solution which is not normalized. With respect to the result of [10] we have a different nonlinear term, and in particular our result covers the case in which $F(\phi)$ is a pure power $\frac{1}{\alpha(\alpha-1)}|\phi|^\alpha$ or $\frac{1}{\alpha(\alpha-1)}|(\phi, \beta\phi)|^{\alpha/2}$, with exponent $\alpha \in (2, \frac{8}{3}]$.

We will find such a solution as a critical point of the functional

$$I(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} (H\psi, \psi) - \gamma \int_{\mathbb{R}^3} F(\psi)$$

restricted on the manifold $|\psi|_2^2 = 1$. Here

$$H = -i\boldsymbol{\alpha} \cdot \nabla + m\beta.$$

The solutions are found as critical points of the strongly indefinite functional I restricted to the unit sphere in L^2, and, following the method introduced in [7,21,8], the solution will be found via the following min-max procedure: we fix an L^2 normalized function w in the positive energy subspace of H and we maximize I over the ball of radius 1 in the subspace spanned by w and the negative energy subspace of H. Such a maximum being unique and a smooth function $\psi(w)$, we then proceed to find a minimizer of $w \mapsto I(\psi(w))$.

Let us also point out that $F(\phi)$ is not coercive but only satisfies assumption (H4), making it harder to deduce the estimates necessary to prove the result.

2. Notation and background results

We denote with $|u|_p^p = \int_{\mathbb{R}^3} |u(x)|^p$ the norm in $L^p(\mathbb{R}^3, \mathbb{C}^4)$ and with $(u \mid v) = \int_{\mathbb{R}^3} (u(x), v(x))$ the scalar product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$, where $(\xi, \eta) = \sum_{i=1}^4 \bar{\xi}_i \eta_i$ and $|\xi|^2 = (\xi, \xi)$ are the scalar product and the norm in $\mathbb{C}^4$. With $\Re z$ we denote the real part of $z \in \mathbb{C}$.

We will work in the Hilbert space $X = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ with scalar product

$$\langle \psi_1 \mid \psi_2 \rangle = \int_{\mathbb{R}^3} \sqrt{|\xi|^2 + m^2}\, (\hat{\psi}_1(\xi), \hat{\psi}_2(\xi))\, d\xi$$

and corresponding norm

$$\|\psi\|^2 = \int_{\mathbb{R}^3} \sqrt{|\xi|^2 + m^2}\, |\hat{\psi}(\xi)|^2\, d\xi, \tag{2.1}$$

where $\hat{\psi}(\xi) = \mathcal{F}\psi(\xi)$ is the Fourier transform of $\psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. The norm in (2.1) is equivalent to the usual one (given by (2.1) with $m = 1$).

Let us also recall some properties (see [24]) of the Dirac operator

$$H = -i\boldsymbol{\alpha} \cdot \nabla + m\beta.$$

H is a first order, self-adjoint operator on $H^1(\mathbb{R}^3, \mathbb{C}^4)$ with purely absolutely continuous spectrum given by

$$\sigma(H) = (-\infty, -m] \cup [m, +\infty).$$

One can define orthogonal projectors $\Lambda_\pm$ on the positive and negative part of the spectrum of H. These projections are such that

$$H\Lambda_\pm = \Lambda_\pm H = \pm\sqrt{-\Delta + m^2}\Lambda_\pm = \pm\Lambda_\pm\sqrt{-\Delta + m^2}$$

and

$$\langle \Lambda_+\psi \mid \Lambda_-\eta\rangle = (\Lambda_+\psi \mid \Lambda_-\eta) = 0$$

so that

$$\begin{aligned}\int (\psi(x), H\psi(x))\,dx &= \int (\Lambda_+\psi(x), \Lambda_+ H\psi(x))\,dx + \int (\Lambda_-\psi(x), \Lambda_- H\psi(x))\,dx \\ &= \left|(-\Delta + m^2)^{1/4}\Lambda_+\psi\right|_2^2 - \left|(-\Delta + m^2)^{1/4}\Lambda_-\psi\right|_2^2 \\ &= \|\Lambda_+\psi\|^2 - \|\Lambda_-\psi\|^2.\end{aligned}$$

We also let $X_\pm = \Lambda_\pm X$ and, for $\lambda \in (0,1]$, $\Sigma^\lambda = \left\{ \psi \in X \;\middle|\; |\psi|_2^2 = \lambda \right\}$ and $\Sigma^\lambda_\pm = \left\{ \psi \in X_\pm \;\middle|\; |\psi|_2^2 = \lambda \right\}$, $\Sigma_\pm = \Sigma^1_\pm$.

With S_p we will denote the best constant for the Sobolev embedding of $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ (with norm given by (2.1)) in $L^p(\mathbb{R}^3; \mathbb{C}^4)$ for $2 \le p \le 3$:

$$|\psi|_p \le S_p\|\psi\|.$$

Let us remark that, with our choice of the norm in X, we have that $m\,|\psi|_2^2 \le \|\psi\|^2$ and $S_2 \le \frac{1}{\sqrt{m}}$ (actually $S_2 = \frac{1}{\sqrt{m}}$).

3. Maximization

We are interested in solutions of problem (1.2), which are functions having L^2 norm equal one. In order to find them, we will study this problems under the constraint of L^2 norm equal to λ, with $\lambda \in (0,1]$, as in [18].

Let $I\colon H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \to \mathbb{R}$

$$I(\psi) = \frac{1}{2}\|\Lambda_+\psi\|^2 - \frac{1}{2}\|\Lambda_-\psi\|^2 - \gamma \int_{\mathbb{R}^3} F(\psi).$$

We denote, for $\lambda \in (0,1]$

$$B_\lambda = \left\{ \eta \in X_- \;\middle|\; |\eta|_2^2 < \lambda \right\}.$$

For $\eta \in B_\lambda$ and $w \in \Sigma_+$ we let

$$a(\eta) = \sqrt{\lambda - |\eta|_2^2} \qquad \text{and} \qquad \psi = a(\eta)w + \eta \in \Sigma^\lambda.$$

We will look, given $w \in \Sigma_+$, for a maximizer of the functional J_w^λ defined on B_λ

$$J_w^\lambda(\eta) = I(a(\eta)w + \eta) = \frac{1}{2}\|a(\eta)w\|^2 - \frac{1}{2}\|\eta\|^2 - \gamma \int_{\mathbb{R}^3} F(\psi).$$

Since $da(\eta)[\xi] = -a(\eta)^{-1}\Re(\eta \,|\, \xi)$, the derivative of J_w^λ is given, for all $\xi \in X_-$, by

$$\begin{aligned} dJ_w^\lambda(\eta)[\xi] &= dI(a(\eta)w + \eta)[da(\eta)[\xi]w + \xi] = dI(\psi)[h_\xi] \\ &= \Re\langle a(\eta)w \,|\, da(\eta)[\xi]w\rangle - \Re\langle \eta \,|\, \xi\rangle - \gamma \int_{\mathbb{R}^3} (\nabla F(\psi), h_\xi) \\ &= -\Re(\eta \,|\, \xi)\|w\|^2 - \Re\langle \eta \,|\, \xi\rangle - \gamma \int_{\mathbb{R}^3} (\nabla F(\psi), h_\xi) \end{aligned} \tag{3.1}$$

(here $h_\xi = da(\eta)[\xi]w + \xi$) and, in particular

$$dJ_w^\lambda(\eta)[\eta] = -|\eta|_2^2\,\|w\|^2 - \|\eta\|^2 - \gamma \int_{\mathbb{R}^3} (\nabla F(\psi), h_\eta).$$

Lemma 3.2. *For all $w \in \Sigma_+$ and $\eta \in B_\lambda$ we have*

$$\|\eta\|^2 \le a(\eta)^2\|w\|^2 - 2J_w^\lambda(\eta), \tag{3.3}$$

and for all $\eta \in B_\lambda$ such that $J_w^\lambda(\eta) \ge 0$ we have that

$$\frac{1}{a(\eta)}\left|\int (\nabla F(\psi), w)\right| \le C_{\lambda,\alpha}\|w\|^2, \tag{3.4}$$

where

$$C_{\lambda,\alpha} = 4\left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S_3^{3(\alpha-2)} S_2^{3\alpha-8}\right).$$

Moreover, if $J_w^\lambda(\eta) \ge 0$, $|\eta|_2^2 \ge \frac{\lambda}{2}$ we have that

$$dJ_w^\lambda(\eta)[\eta] < -\|\eta\|^2 < 0. \tag{3.5}$$

Proof. We have that for $\eta \in B_\lambda$ and $\psi = a(\eta)w + \eta$

$$\frac{1}{2}\|\eta\|^2 \le \frac{1}{2}\|\eta\|^2 + \gamma \int F(\psi) = \frac{1}{2}\|a(\eta)w\|^2 - J_w^\lambda(\eta)$$

and (3.3) follows.

Let us now assume that $J_w^\lambda(\eta) \ge 0$, so that $\|\eta\|^2 \le a(\eta)^2\|w\|^2$.

Since

$$a(\eta) - da(\eta)[\eta] = a(\eta) + a(\eta)^{-1}\,|\eta|_2^2 = \lambda a(\eta)^{-1}$$

we have that

$$dJ_w^\lambda(\eta)[\eta] = -\,|\eta|_2^2\,\|w\|^2 - \|\eta\|^2 - \gamma\int(\nabla F(\psi),\psi) + \gamma\frac{\lambda}{a(\eta)^2}\int(\nabla F(\psi), a(\eta)w).$$

Let's estimate the last term. Using (1.10) we have

$$\left|\int(\nabla F(\psi), a(\eta)w)\right| \le \int\left(|\psi| + \mu|\psi|^{\alpha-1}\right)|a(\eta)w| \le |\psi|_2\,|a(\eta)w|_2 + \mu\,|\psi|_\alpha^{\alpha-1}\,|a(\eta)w|_\alpha\,.$$

Since $|\psi|_2^2 = \lambda$ and $|a(\eta)w|_2^2 \le \lambda$ we deduce that

$$|\psi|_\alpha^\alpha = \int|\psi|^\alpha \le \int|\psi|^{3\alpha-6}|\psi|^{6-2\alpha} \le |\psi|_3^{3\alpha-6}\,|\psi|_2^{6-2\alpha} = \lambda^{3-\alpha}\,|\psi|_3^{3\alpha-6}$$

and

$$|a(\eta)w|_\alpha^\alpha \le \lambda^{3-\alpha}\,|a(\eta)w|_3^{3\alpha-6}$$

so that

$$\begin{aligned}\left|\int(\nabla F(\psi), a(\eta)w)\right| &\le |\psi|_2\,|a(\eta)w|_2 + \mu\lambda^{(3-\alpha)}\,|\psi|_3^{\frac{3(\alpha-2)(\alpha-1)}{\alpha}}\,|a(\eta)w|_3^{\frac{3(\alpha-2)}{\alpha}}\\ &\le S_2^2\|\psi\|\|a(\eta)w\| + \mu S_3^{3(\alpha-2)}\lambda^{(3-\alpha)}\|\psi\|^{\frac{3(\alpha-2)(\alpha-1)}{\alpha}}\|a(\eta)w\|^{\frac{3(\alpha-2)}{\alpha}}\\ &\le S_2^2\|\psi\|^2 + \mu S_3^{3(\alpha-2)}\lambda^{(3-\alpha)}\|\psi\|^{3(\alpha-2)}.\end{aligned}$$

Since $\|\psi\|^2 \ge S_2^2\,|\psi|_2^2 = S_2^2\lambda$ and $3(\alpha-2)\in(0,2]$ for all $\alpha\in(2,\frac{8}{3}]$ we have that

$$\|\psi\|^{3(\alpha-2)} \le (S_2\sqrt{\lambda})^{3\alpha-8}\|\psi\|^2 \le 4(S_2\sqrt{\lambda})^{3\alpha-8}a(\eta)^2\|w\|^2$$

and

$$\frac{\lambda}{a(\eta)^2}\left|\int(\nabla F(\psi), a(\eta)w)\right| \le 4\lambda\left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}}S_3^{3(\alpha-2)}S_2^{3\alpha-8}\right)\|w\|^2 = \lambda C_{\alpha,\lambda}\|w\|^2.$$

We finally deduce that

$$dJ_w^\lambda(\eta)[\eta] \le \left(-\,|\eta|_2^2 + \gamma\lambda C_{\alpha,\lambda}\right)\|w\|^2 - \|\eta\|^2 - \gamma\int(\nabla F(\psi),\psi) < -\|\eta\|^2$$

if $|\eta|_2^2 > \frac{\lambda}{2}$ and since by (1.13) γ is such that $\gamma C_{\alpha,\lambda} < \frac{1}{4}$ for all $\lambda\in(0,1]$. $\square$

Remark 3.6. It follows from the Lemma 3.2 that if η_n is a Palais-Smale sequence for J_w^λ such that $J_w^\lambda(\eta_n) \geq 0$, then $|\eta_n|_2^2 < \frac{\lambda}{2}$ for all $n \in \mathbb{N}$ large enough.

Lemma 3.7. *Let $\eta_n \in B_\lambda$ be a Palais-Smale sequence for J_w^λ, that is*

$$J_w^\lambda(\eta_n) \to c \geq 0, \qquad dJ_w^\lambda(\eta_n) \to 0.$$

Then η_n converges, up to a subsequence, to a critical point η of J_w^λ.

Proof. Follows form the Lemma 3.2 and Remark 3.6 that $|\eta_n|_2^2 < \frac{\lambda}{2}$ and that $\|\eta_n\| \leq a(\eta_n)\|w\| \leq \sqrt{\lambda}\|w\|$ is bounded, hence $\eta_n \rightharpoonup \eta$ (up to a subsequence).

From

$$\begin{aligned} o(1) = dJ_w^\lambda(\eta_n)[\eta_n - \eta] &= -\Re(\eta_n \mid \eta_n - \eta)\|w\|^2 - \Re\langle \eta_n \mid \eta_n - \eta\rangle \\ &\quad - \gamma \int (\nabla F(\psi_n), da(\eta_n)[\eta_n - \eta]w + \eta_n - \eta) \\ &= -|\eta_n - \eta|_2^2\,\|w\|^2 - \Re(\eta \mid \eta_n - \eta)\|w\|^2 - \|\eta_n - \eta\|^2 - \Re\langle \eta \mid \eta_n - \eta\rangle \\ &\quad - \gamma \int (\nabla F(\psi_n), \eta_n - \eta) - \gamma da(\eta_n)[\eta_n - \eta] \int (\nabla F(\psi_n), w) \end{aligned}$$

we deduce, using also the fact that $|\eta_n|_2^2 < \frac{\lambda}{2}$, that

$$\begin{aligned} &|\eta_n - \eta|_2^2\,\|w\|^2 + \|\eta_n - \eta\|^2 \\ &\quad = -\gamma \int (\nabla F(\psi_n), \eta_n - \eta) + \gamma a(\eta_n)^{-1}\Re(\eta_n \mid \eta_n - \eta) \int (\nabla F(\psi_n), w) + o(1) \\ &\qquad = -\gamma \int (\nabla F(\psi_n), \eta_n - \eta) + \gamma a(\eta_n)^{-1}\,|\eta_n - \eta|_2^2 \int (\nabla F(\psi_n), w) + o(1). \end{aligned}$$

From the estimate (3.4) follows

$$(1 - \gamma C_{\alpha,\lambda})\,|\eta_n - \eta|_2^2\,\|w\|^2 + \|\eta_n - \eta\|^2 \leq -\gamma \int (\nabla F(\psi_n), \eta_n - \eta) + o(1).$$

We will prove that $\liminf_{n\to+\infty} \int (\nabla F(\psi_n), \eta_n - \eta) \geq 0$ applying concentration compactness to the sequence of functions $\eta_n - \eta$.

In case we have vanishing, that is

$$\forall R > 0 \qquad \lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |\eta_n - \eta|^2 = 0,$$

we have that $|\eta_n - \eta|_q \to 0$ as $n \to +\infty$ for all $q \in (2, 3)$ (see [18] or [25, Lemma 1.21]).

For all $\epsilon > 0$ we have that

$$\left|\int (\nabla F(\psi_n), \eta_n - \eta)\right| \le \int \left(\epsilon|\psi_n| + \mu_\epsilon|\psi_n|^{\alpha-1}\right)|\eta_n - \eta|$$
$$\le \epsilon\,|\eta_n - \eta|_2 + \mu_\epsilon\,|\psi_n|_\alpha^{\alpha-1}\,|\eta_n - \eta|_\alpha \le \sqrt{2\lambda}\epsilon + o(1)$$

so that $\lim_{n\to+\infty}\int(\nabla F(\psi_n), \eta_n - \eta) = 0$ if vanishing occurs.

If we have dichotomy then for all $\epsilon > 0$ there is a sequences $R_n \to +\infty$ such that

$$\int\limits_{\mathbb{R}^3\setminus B_{R_n}(0)} |\eta_n - \eta|_2^2 \ge \sigma - \epsilon,$$

where (up to a subsequence)

$$\sigma = \lim_{n\to+\infty}\int\limits_{\mathbb{R}^3} |\eta_n - \eta|^2.$$

We have that

$$\begin{aligned}
&\int (\nabla F(\psi_n), \eta_n - \eta) \\
&= \int\limits_{B_{R_n}(0)} (\nabla F(\psi_n), \eta_n - \eta) + \int\limits_{\mathbb{R}^3\setminus B_{R_n}(0)} (\nabla F(\eta_n), \eta_n) \\
&\quad + \int\limits_{\mathbb{R}^3\setminus B_{R_n}(0)} (\nabla F(\psi_n) - \nabla F(\eta_n), \eta_n) - \int\limits_{\mathbb{R}^3\setminus B_{R_n}(0)} (\nabla F(\psi_n), \eta) \\
&\ge -c(\sqrt{\epsilon} + \epsilon^{\frac{2(3-\alpha)}{\alpha}}) \\
&\quad + \int\limits_{\mathbb{R}^3\setminus B_{R_n}(0)} (\nabla F(\psi_n) - \nabla F(\eta_n), \eta_n) - \int\limits_{\mathbb{R}^3\setminus B_{R_n}(0)} (\nabla F(\psi_n), \eta)
\end{aligned}$$

where we have used the fact that $(\nabla F(\eta_n), \eta_n) \ge 0$ and

$$\begin{aligned}
&\left|\int\limits_{B_{R_n}(0)} (\nabla F(\psi_n), \eta_n - \eta)\right| \\
&\le |\psi_n|_2 \left(\int\limits_{B_{R_n}(0)} |\eta_n - \eta|^2\right)^{\frac{1}{2}} + \mu\,|\psi_n|_\alpha^{\alpha-1}\left(\int\limits_{B_{R_n}(0)} |\eta_n - \eta|^\alpha\right)^{\frac{1}{\alpha}}
\end{aligned}$$

$$\leq \left(\int_{B_{R_n}(0)} |\eta_n - \eta|^2 \right)^{\frac{1}{2}} + \mu \, |\psi_n|_\alpha^{\alpha-1} \, |\eta_n - \eta|_3^{\frac{3(\alpha-2)}{\alpha}} \left(\int_{B_{R_n}(0)} |\eta_n - \eta|^2 \right)^{\frac{2(3-\alpha)}{\alpha}}$$

$$\leq c(\sqrt{\epsilon} + \epsilon^{\frac{2(3-\alpha)}{\alpha}})$$

for some constant $c > 0$.

We now observe that

$$\int_{\mathbb{R}^3 \setminus B_{R_n}(0)} (\nabla F(\psi_n) - \nabla F(\eta_n), \eta_n) \leq \int_{\mathbb{R}^3 \setminus B_{R_n}(0)} |\nabla F(\psi_n) - \nabla F(\eta_n)||\eta_n|$$

$$\leq \int_{\mathbb{R}^3 \setminus B_{R_n}(0)} a(\eta_n)|w||\eta_n|$$

$$+ \mu \int_{\mathbb{R}^3 \setminus B_{R_n}(0)} \left(|\eta_n|^{\alpha-2} + a(\eta_n)^{\alpha-2}|w|^{\alpha-2} \right) a(\eta_n)|w||\eta_n|$$

$$\leq \sqrt{\lambda} \, |\eta_n|_2 \left(\int_{\mathbb{R}^3 \setminus B_{R_n(0)}} |w|^2 \right)^{\frac{1}{2}} + \mu \sqrt{\lambda} \, |\eta_n|_\alpha^{\alpha-1} \left(\int_{\mathbb{R}^3 \setminus B_{R_n}(0)} |w|^\alpha \right)^{1/\alpha}$$

$$+ \mu \lambda^{\frac{\alpha-1}{2}} \, |\eta_n|_\alpha \left(\int_{\mathbb{R}^3 \setminus B_{R_n}(0)} |w|^\alpha \right)^{(\alpha-1)/\alpha} = o(1)$$

where we have used (1.12).

A similar argument shows that

$$\int_{\mathbb{R}^3 \setminus B_{R_n}(0)} (\nabla F(\psi_n), \eta) = o(1),$$

and also when dichotomy occurs we find that

$$(1 - \gamma C_{\alpha,\lambda}) \, |\eta_n - \eta|_2^2 \, \|w\|^2 + \|\eta_n - \eta\|^2 \leq c\epsilon + o(1).$$

So in every case we have that and $\eta_n \to \eta$, with η critical point of J_w^λ. $\square$

We now show that the functional J_w^λ is concave in the set where it is positive in the L^2-ball of radius $\sqrt{\frac{\lambda}{2}}$.

Lemma 3.8. *Let $\eta \in B_\lambda$ be such that $|\eta|_2^2 < \frac{\lambda}{2}$ and $J_w^\lambda(\eta) \geq 0$. Then*

$$d^2 J_w^\lambda(\eta)[\xi,\xi] \leq -\frac{3}{4}\|\xi\|^2 \qquad \text{for all } \xi \in X_-.$$

Proof. In order to compute the second derivative we denote $\psi = a(\eta)w + \eta$ and $h_\xi = da(\eta)[\xi]w + \xi$ and observe that

$$d^2 a(\eta)[\xi,\zeta] = -a(\eta)^{-1}\left(\Re(\xi \,|\, \zeta) + \frac{\Re(\eta \,|\, \xi)\Re(\eta,\zeta)}{\lambda - |\eta|_2^2}\right).$$

We have

$$\begin{aligned} d^2 J_w^\lambda(\eta)[\xi,\xi] &= -|\xi|_2^2\,\|w\|^2 - \|\xi\|^2 - \gamma\int (D^2F(\psi)h_\xi, h_\xi) \\ &\quad - \gamma\int (\nabla F(\psi), d^2a(\eta)[\xi,\xi]w) \\ &= -|\xi|_2^2\,\|w\|^2 - \|\xi\|^2 - \gamma\int (D^2F(\psi)h_\xi, h_\xi) \\ &\quad + \frac{\gamma}{a(\eta)}\left(|\xi|_2^2 + \frac{(\eta\,|\,\xi)^2}{\lambda - |\eta|_2^2}\right)\int (\nabla F(\psi), w) \end{aligned}$$

Then, using (3.4) and (1.9):

$$\begin{aligned} d^2 J_w^\lambda(\eta)[\xi,\xi] &\leq -|\xi|_2^2\,\|w\|^2 - \|\xi\|^2 + \gamma\int |h_\xi|^2 + \gamma\mu\int |\psi|^{\alpha-2}|h_\xi|^2 \\ &\quad + \frac{\lambda|\xi|_2^2 - |\xi|_2^2|\eta|_2^2 + (\eta\,|\,\xi)^2}{\lambda - |\eta|_2^2}\gamma C_{\alpha,\lambda}\|w\|^2 \\ &\leq -(1-\gamma S_2^2)\,|\xi|_2^2\,\|w\|^2 - (1-\gamma S_2^2)\|\xi\|^2 \\ &\quad + \gamma\mu|\psi|_2^{\alpha-2}|h_\xi|^2_{\frac{4}{4-\alpha}} + 2\,|\xi|_2^2\,\gamma C_{\alpha,\lambda}\|w\|^2 \\ &\leq -\left(1-\gamma(2C_{\alpha,\lambda} + S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}})\right)|\xi|_2^2\,\|w\|^2 \\ &\quad - \left(1-\gamma(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}})\right)\|\xi\|^2 \\ &\leq -\frac{1}{4}|\xi|_2^2\,\|w\|^2 - \frac{3}{4}\|\xi\|^2 \leq -\frac{3}{4}\|\xi\|^2, \end{aligned}$$

since $2\gamma C_{\alpha,\lambda} < \frac{1}{2}$ by (1.13) and by (1.14). $\square$

Lemma 3.9. *Let $w \in \Sigma_+$, $\eta \in B_\lambda$ and $\psi = a(\eta)w + \eta \in \Sigma^\lambda$. Then we have that*

$$\int F(\psi) \geq \int F(\sqrt{\lambda}w) + \int (\nabla F(aw), \eta) - \left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}}\right) |\eta|_2^2 \, \|w\|^2$$
$$- \left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^\alpha_{\frac{4}{4-\alpha}}\right) \|\eta\|^2 - \mu\lambda^{\frac{\alpha-2}{2}} \int |w|^{\alpha-2}|\eta|^2 \quad (3.10)$$

Proof. The result follows, using (1.12) from

$$\begin{aligned}
&\int F(\psi) - \int F(a(\eta)w)) - \int (\nabla F(a(\eta)w), \eta) = \\
&= \int (\nabla F(a(\eta)w + \theta\eta) - \nabla F(a(\eta)w), \eta) \\
&\geq - \int (1 + \mu|a(\eta)w|^{\alpha-2} + \mu|\eta|^{\alpha-2})|\eta|^2 \\
&\geq - \int |\eta|^2 - \mu\lambda^{\frac{\alpha-2}{2}} \int |w|^{\alpha-2}|\eta|^2 - \mu \int |\eta|^\alpha \\
&\geq - |\eta|_2^2 - \mu\lambda^{\frac{\alpha-2}{2}} \int |w|^{\alpha-2}|\eta|^2 - \mu \, |\eta|_2^{\alpha-2} \, |\eta|^2_{\frac{4}{4-\alpha}} \\
&\geq -(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}})\|\eta\|^2 - \mu\lambda^{\frac{\alpha-2}{2}} \int |w|^{\alpha-2}|\eta|^2
\end{aligned}$$

for some $\theta(x) \in [0, 1]$ and

$$\begin{aligned}
&\int (F(a(\eta)w) - F(\sqrt{\lambda}w)) = \int (\nabla F(a(\eta)w + \theta(a(\eta) - \sqrt{\lambda})w), (a(\eta) - \sqrt{\lambda})w) \\
&\geq - \int |a(\eta) + \theta(a(\eta) - \sqrt{\lambda})||w||a(\eta) - \sqrt{\lambda}||w| \\
&\quad - \mu \int |a(\eta) + \theta(a(\eta) - \sqrt{\lambda})|^{\alpha-1}|w|^{\alpha-1}|a(\eta) - \sqrt{\lambda}||w| \\
&\geq -\sqrt{\lambda}|\sqrt{\lambda} - a(\eta)| \int |w|^2 - \mu\lambda^{\frac{\alpha-1}{2}} |\sqrt{\lambda} - a(\eta)| \int |w|^\alpha \\
&\geq -(\sqrt{\lambda} - a(\eta)) \left(\sqrt{\lambda}\, |w|_2^2 + \mu\lambda^{\frac{\alpha-1}{2}} \int |w|^\alpha\right) \\
&\geq -\frac{\lambda - a(\eta)^2}{\sqrt{\lambda} + a(\eta)} \left(\sqrt{\lambda}\, |w|_2^2 + \mu\lambda^{\frac{\alpha-1}{2}} \, |w|_2^{(\alpha-2)} \, |w|^2_{4/(4-\alpha)}\right) \\
&\geq -\left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}}\right) |\eta|_2^2 \, \|w\|^2. \quad \square
\end{aligned}$$

We let, for all $w \in \Sigma_+$

$$\mathcal{E}_\lambda(w) = \sup_{\eta \in B_\lambda} J_w^\lambda(\eta).$$

Lemma 3.11. *For all $w \in \Sigma_+$ we have*

$$0 < \frac{\lambda}{2S_2^2}\left(1-\gamma\left(S_2^2+2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}}\right)\right)$$
$$\leq \frac{\lambda}{2}\left(1-\gamma\left(S_2^2+2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}}\right)\right)\|w\|^2 \leq \mathcal{E}_\lambda(w) \leq \frac{\lambda}{2}\|w\|^2.$$

Proof. We have, using (1.11), that

$$\begin{aligned}
\mathcal{E}_\lambda(w) \geq J_w^\lambda(0) &= \frac{1}{2}\|\sqrt{\lambda}w\|^2 - \gamma\int F(\sqrt{\lambda}w) \\
&\geq \frac{\lambda}{2}\|w\|^2 - \gamma\frac{\lambda}{2}|w|_2^2 - \gamma\frac{\mu}{\alpha}\lambda^{\frac{\alpha}{2}}|w|_2^{\alpha-2}|w|^2_{\frac{4}{4-\alpha}} \\
&\geq \frac{\lambda}{2}\|w\|^2 - \gamma\lambda\left(\frac{S_2^2}{2}+\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2 \\
&\geq \frac{\lambda}{2S_2^2}\left(1-\gamma\left(S_2^2+2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}}\right)\right)
\end{aligned}$$

which is positive by (1.14) and

$$J_w^\lambda(\eta) = \frac{1}{2}\|a(\eta)w\|^2 - \frac{1}{2}\|\eta\|^2 - \gamma\int F(\psi) \leq \frac{\lambda}{2}\|w\|^2. \quad \square$$

The following proposition is analogous to [7, Proposition 4.5], [21, Proposition 3.6] and [8, Proposition 2.9]. Since in the present situation the functional J_w^λ is, by Lemma 3.8, concave in the region where $J_w^\lambda(\eta) \geq 0$ and $|\eta|_2^2 < \frac{\lambda}{2}$ the proof is actually simpler.

Proposition 3.12. *For every $w \in \Sigma_+$ and $\lambda > 0$ there is a unique $\eta(w) \in B_\lambda$ such that*

$$J_w^\lambda(\eta(w)) = \max_{\eta\in B_\lambda} J_w^\lambda(\eta) = \mathcal{E}_\lambda(w).$$

$\eta(w)$ is a critical point of J_w^λ on B_λ such that $|\eta(w)|_2^2 < \frac{\lambda}{2}$ and

$$\|\eta(w)\|^2 \leq \lambda\gamma\left(S_2^2+2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}}S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2. \tag{3.13}$$

Moreover the map

$$w \in X_+ \setminus \{0\} \mapsto \gamma(w) = \eta(|w|_2^{-1}w) \in B_\lambda$$

is smooth.

Proof. We can find, by Lemma 3.11 and using Ekeland's variational principle a maximizing Palais-Smale sequence η_n at positive level.

Then, by Lemma 3.7, $\eta_n \to \eta$ (up to a subsequence), with

$$dJ_w^\lambda(\eta) = 0, \qquad J_w^\lambda(\eta) = \mathcal{E}_\lambda(w) > 0.$$

Using Lemma 3.11 we deduce that

$$\frac{\lambda}{2}\left(1 - \gamma\left(S_2^2 + 2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}}\right)\right)\|w\|^2 \le \mathcal{E}_\lambda(w) = J_w^\lambda(\eta) \le \frac{\lambda}{2}\|w\|^2 - \frac{1}{2}\|\eta\|^2$$

from which (3.13) follows.

As a consequence we have that for all $w \in \Sigma_+$ the convex set

$$\mathcal{B} = \left\{ \eta \in B_\lambda \;\middle|\; |\eta|_2^2 < \frac{1}{2} \text{ and } \|\eta\|^2 \le 2\lambda\gamma\left(S_2^2 + 2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2 \right\}$$

contains all the maxima of J_w^λ in B_λ. We now show that $J_w^\lambda(\eta) \ge 0$ for all $\eta \in \mathcal{B}$.

We have that

$$\begin{aligned}
\int F(a(\eta)w + \eta) &\le \frac{1}{2}\,|a(\eta)w + \eta|_2^2 + \frac{\mu}{\alpha}\,|a(\eta)w + \eta|_\alpha^\alpha \\
&\le \frac{\lambda}{2} + \frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}}\,|a(\eta)w + \eta|^2_{\frac{4}{4-\alpha}} \\
&\le \frac{\lambda}{2}S_2^2\|w\|^2 + \frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}}\,\|a(\eta)w + \eta\|^2 \\
&\le \left(\frac{\lambda}{2}S_2^2 + \frac{\mu}{\alpha}\lambda^{\frac{\alpha}{2}} S^2_{\frac{4}{4-\alpha}} + \frac{1}{2}\frac{\mu}{\alpha}\lambda^{\frac{\alpha}{2}} S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2 \\
&\le \lambda\left(\frac{1}{2}S_2^2 + \mu S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2
\end{aligned}$$

We then have

$$\begin{aligned}
J_w^\lambda(\eta) &= \frac{1}{2}a(\eta)^2\|w\|^2 - \frac{1}{2}\|\eta\|^2 - \gamma\int F(a(\eta)w + \eta) \\
&\ge \frac{\lambda}{4}\|w\|^2 - \lambda\gamma\left(S_2^2 + 2\frac{\mu}{\alpha}\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2 - \gamma\lambda\left(\frac{1}{2}S_2^2 + \mu S^2_{\frac{4}{4-\alpha}}\right)\|w\|^2 \\
&> \frac{\lambda}{4}\|w\|^2 - \frac{\lambda}{8}\|w\|^2 - \frac{\lambda}{8}\|w\|^2 = 0
\end{aligned}$$

The strict concavity of J_w^λ in the convex region $\mathcal{B}$ implies that such a maximum is unique.

The properties of the map $w \mapsto \gamma(w) = \eta(|w|_2^{-1} w)$ follows exactly as in [8, Proposition 2.9]. $\square$

Exactly as in [8] we can consider the smooth functional $\mathcal{E}_\lambda \colon X_+ \setminus \{0\} \to \mathbb{R}$ defined as

$$\mathcal{E}_\lambda(w) = J^\lambda_{P(w)}(\gamma(w)) = \sup_{\eta \in B_\lambda} J^\lambda_{P(w)}(\eta),$$

where, as in the proof of 3.12, $P(w) = \frac{w}{|w|_2}$ and $\gamma(w) = \eta(Pw)$ and deduce that for all $w \in \Sigma_+$ and $v \in X_+$.

$$d\mathcal{E}_\lambda(w)[v] = a(\gamma(w))dI(\psi_w)[v] - a(\gamma(w))^2\omega(\psi_w)(w \mid v) \tag{3.14}$$

where

$$\omega(\psi_w) = a(\gamma(w))^{-1}dI(\psi_w)[w]. \tag{3.15}$$

We also deduce that, for all $h = v + \xi \in X$, $v \in X_+$, $\xi \in X_-$ and with ψ, a and ω as above we have

$$\begin{aligned} dI(\psi)[h] - \omega(\psi \mid h) &= dI(\psi)[v] + dI(\psi)[\xi] - \omega(a(\eta)w + \eta \mid v + \xi) \\ &= dI(\psi)[v] + \left(-da(\eta)[\xi] - (w \mid v) - \frac{(\eta \mid \xi)}{a(\eta)}\right) dI(\psi)[w] \\ &= dI(\psi)[v - (w \mid v)w] = \frac{1}{a(\eta)} d\mathcal{E}_\lambda(w)[v] \end{aligned}$$

which shows that w is a critical point of $\mathcal{E}_\lambda$ if and only if $\psi = a(\eta(w))w + \eta(w)$ is a critical point for I under the constraint $|\psi|_2^2 = \lambda$.

The following is essentially Proposition 2.13 of [8]:

Proposition 3.16. *Let $w_0 \in \Sigma_+$ be a critical point of $\mathcal{E}_\lambda$ restricted on the manifold Σ_+. Then w_0 is a critical point for $\mathcal{E}_\lambda$ on X_+ and the function*

$$\psi_0 = a(\eta(w_0))w_0 + \eta(w_0) \in \Sigma^\lambda$$

is a critical point for I on the manifold Σ and satisfies

$$dI(\psi_0)[h] = \omega(\psi_0 \mid h) \qquad \text{for all } h \in X \tag{3.17}$$

where $\omega = \omega(\psi_0) \in \mathbb{R}$,

$$(1 - \gamma C_{\alpha,\lambda})\|w_0\|^2 \le \omega(\psi_0) \le 2I(\psi_{w_0}) = 2\mathcal{E}_\lambda(w_0).$$

Moreover, if $\psi_0 \in \Sigma^\lambda$ satisfies (3.17) for some $\omega > 0$, then $w = |\Lambda_+\psi_0|_2^{-1}\,\Lambda_+\psi_0$ is a critical point for $\mathcal{E}_\lambda(w)$.

Proof. We only have to prove the estimate on the Lagrange multiplier. The other points follow as in proof of Proposition 2.13 of [8]. Using (3.4) we immediately deduce

$$\omega(\psi_0) = a(\eta(w_0))^{-1} dI(\psi_0)[w_0] \geq (1 - \gamma C_{\alpha,\lambda})\|w_0\|^2.$$

We also have, using (H3), that

$$\omega(\psi_0) = dI(\psi_0)[\psi_0] \leq 2I(\psi_0). \quad \square$$

4. The constrained minimization

We introduce now the following minimization problem:

$$e(\lambda) = \inf_{w\in\Sigma_+} \mathcal{E}_\lambda(w) = \inf_{w\in\Sigma_+} \left\{ \frac{1}{2} a(\eta(w))^2 \|w\|^2 - \frac{1}{2}\|\eta(w)\|^2 - \gamma \int F(\psi_w) \right\}$$

where $\eta(w)$ is given in Proposition 3.12, $a(\eta(w)) = \sqrt{\lambda - |\eta(w)|_2^2}$ and $\psi_w = a(\eta(w))w + \eta(w)$.

The next lemma contains an estimate which will be essential in proving (via the concentration compactness lemma [18,19]) convergence of minimizing sequences for $\mathcal{E}_\lambda$.

Lemma 4.1. *For all* $\lambda \in (0, 1]$ *we have that* $0 < e(\lambda) < \frac{\lambda m}{2}$.

Proof. From Lemma 3.11 we have that

$$e(\lambda) \geq \frac{\lambda}{2S_2^2} \left(1 - \gamma \left(S_2^2 + 2\frac{\mu}{\alpha} \lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}} \right)\right) > 0.$$

Using Lemma 3.9 we deduce that

$$\begin{aligned}
\mathcal{E}_\lambda(w) = I(\psi_w) &= \frac{1}{2}a(\eta(w))^2\|w\|^2 - \frac{1}{2}\|\eta(w)\|^2 - \gamma \int F(\psi_w) \\
&\leq \frac{\lambda}{2}\|w\|^2 - \frac{1}{2}|\eta(w)|_2^2\,\|w\|^2 - \frac{1}{2}\|\eta(w)\|^2 - \gamma \int F(\sqrt{\lambda}w) \\
&\quad - \gamma \int (\nabla F(aw), \eta(w)) + \gamma \left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}} \right) |\eta(w)|_2^2\,\|w\|^2 \\
&\quad + \gamma \left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}} \right) \|\eta(w)\|^2 + \gamma\mu\lambda^{\frac{\alpha-2}{2}} \int |w|^{\alpha-2}|\eta(w)|^2
\end{aligned}$$

Since $\gamma \left(S_2^2 + \mu\lambda^{\frac{\alpha-2}{2}} S^2_{\frac{4}{4-\alpha}} \right) < \frac{1}{4}$ by (1.14) we have, for all $w \in \Sigma_+$:

$$\begin{aligned}
\mathcal{E}_\lambda(w) \leq \frac{\lambda}{2}\|w\|^2 - \frac{1}{4}|\eta(w)|_2^2\,\|w\|^2 - \frac{1}{4}\|\eta(w)\|^2& \\
- \gamma \int F(\sqrt{\lambda}w) - \gamma \int (\nabla F(aw), \eta(w)) + \gamma\mu\lambda^{\frac{\alpha-2}{2}} \int |w|^{\alpha-2}|\eta(w)|^2&.
\end{aligned}$$

Fix $w_1 \in H^1(\mathbb{R}^3, \mathbb{C}) \cap C^1(\mathbb{R}^3, \mathbb{C})$ such that $|w_1|_2 = 1$ and let $w = \binom{w_1}{0} \in \mathbb{C}^4$ and $w_\epsilon(x) = \epsilon^{3/2} w(\epsilon x)$. Then $|w_\epsilon|_2 = 1$ for all $\epsilon > 0$ and $|w_\epsilon(x)|_\infty = \epsilon^{3/2}\,|w(x)|_\infty \to 0$ as $\epsilon \to 0$. We also have that $|(w(x), \beta w(x))| = |w_1(x)|^2$.

The same computation of Lemma 2.16 in [8] show that

$$\begin{aligned}
&\|w_\epsilon\|^2 - m\,|w_\epsilon|_2^2 \le \frac{\epsilon^2}{2m}\int |q|^2 |\hat{w}_1(q)|^2 \\
&\|w_\epsilon\|^2 \le m + C\epsilon^2 \\
&\|w_\epsilon - \Lambda_+ w_\epsilon\|^2 \le \frac{\epsilon^2}{4m}\int |p|^2 |\hat{w}_1(p)|^2 \\
&\big|1 - |\Lambda_+ w_\epsilon|_2\big|^2 \le \frac{\epsilon^2}{4m}\int |p|^2 |\hat{w}_1(p)|^2.
\end{aligned}$$

We deduce from this that for $\epsilon > 0$ small enough $|\Lambda_+ w_\epsilon|_2 > \frac{1}{2}$.

Let

$$\varphi_\epsilon(x) = |\Lambda_+ w_\epsilon|_2^{-1}\,\Lambda_+ w_\epsilon(x) \in \Sigma_+.$$

We have that

$$\begin{aligned}
&\|\varphi_\epsilon\| \le |\Lambda_+ w_\epsilon|_2^{-1}\,\|w_\epsilon\| \le \sqrt{m} + C\epsilon \\
&\|w_\epsilon - \varphi_\epsilon\| \le \frac{\epsilon}{2\sqrt{m}}(2 + \|\varphi_\epsilon\|)\left(\int |p|^2 |\hat{w}_1(p)|^2\right)^{1/2} = \frac{\epsilon}{\sqrt{m}}(2 + \|\varphi_\epsilon\|)\,|\nabla w_1|_2 \\
&|\varphi_\epsilon - w_\epsilon|_2 \le \frac{\epsilon}{\sqrt{m}}\left(\int |p|^2 |\hat{w}_1(p)|^2\right)^{1/2} = \frac{\epsilon}{\sqrt{m}}\,|\nabla w_1|_2
\end{aligned}$$

and hence, for ϵ small enough

$$\|\varphi_\epsilon\| \le \sqrt{m} + 1, \qquad \|\eta(\varphi_\epsilon)\| \le a(\eta(\varphi_\epsilon))\|\varphi_\epsilon\| \le \sqrt{\lambda}(\sqrt{m} + 1).$$

We now estimate the functional for small $\epsilon > 0$:

$$\begin{aligned}
\mathcal{E}_\lambda(\varphi_\epsilon) \le{}& \frac{\lambda m}{2} + \frac{\lambda}{2}(\|\varphi_\epsilon\|^2 - m\,|\varphi_\epsilon|_2^2) - \frac{1}{4}\,|\eta(\varphi_\epsilon)|_2^2\,\|\varphi_\epsilon\|^2 - \frac{1}{4}\|\eta(\varphi_\epsilon)\|^2 \\
&- \gamma\int F(\sqrt{\lambda}\varphi_\epsilon) - \gamma\int (\nabla F(a\varphi_\epsilon), \eta(\varphi_\epsilon)) + \gamma\mu(\sqrt{\lambda})^{\alpha-2}\int |\varphi_\epsilon|^{\alpha-2}|\eta(\varphi_\epsilon)|^2
\end{aligned} \tag{4.2}$$

The first term can be estimated as follows:

$$\|\varphi_\epsilon\|^2 - m\,|\varphi_\epsilon|_2^2 \le \frac{\epsilon^2}{2m}(3 + \|\varphi_\epsilon\|)^2\,|\nabla w_1|_2^2 \le \frac{\epsilon^2}{2m}(4 + \sqrt{m})^2\,|\nabla w_1|_2^2 \le c_1\epsilon^2.$$

Using (H4) and (H5) and since $|w_\epsilon|_\infty \to 0$ we have that

$$\begin{aligned}
\int F(\sqrt{\lambda}\varphi_\epsilon) &= \int F(\sqrt{\lambda}w_\epsilon) + \int (\nabla F(\sqrt{\lambda}w_\epsilon), \varphi_\epsilon - w_\epsilon) \\
&\quad + \int (\nabla F(\sqrt{\lambda(}w_\epsilon + \theta(\varphi_\epsilon - w_\epsilon))) - \nabla F(\sqrt{\lambda}w_\epsilon), \varphi_\epsilon - w_\epsilon) \\
&\geq \bar{\gamma}\lambda^{\frac{\alpha}{2}} \int |(w_\epsilon, \beta w_\epsilon)|^{\frac{\alpha}{2}} - \lambda^{\frac{\nu}{2}} \, |w_\epsilon|_{2\nu}^{\nu} \, |\varphi_\epsilon - w_\epsilon|_2 \\
&\quad - \int |\varphi_\epsilon - w_\epsilon|^2 - \mu \int |w_\epsilon|^{\alpha-2} |\varphi_\epsilon - w_\epsilon|^2 \\
&\quad - \mu \int (|\varphi_\epsilon| + |w_\epsilon|)^{\alpha-2} |\varphi_\epsilon - w_\epsilon|^2 \\
&\geq \bar{\gamma}\lambda^{\frac{\alpha}{2}} \int |(w_\epsilon, \beta w_\epsilon)|^{\frac{\alpha}{2}} - \lambda^{\frac{\nu}{2}} \, |w_\epsilon|_{2\nu}^{\nu} \, |\varphi_\epsilon - w_\epsilon|_2 \\
&\quad - \int |\varphi_\epsilon - w_\epsilon|^2 - 2\mu \int |w_\epsilon|^{\alpha-2} |\varphi_\epsilon - w_\epsilon|^2 \\
&\quad - 2\mu \int |\varphi_\epsilon|^{\alpha-2} |\varphi_\epsilon - w_\epsilon|^2 \\
&\geq \epsilon^{\frac{3(\alpha-2)}{2}} \bar{\gamma}\lambda^{\frac{\alpha}{2}} \int |w_1|^\alpha - \lambda^{\frac{\nu}{2}} \epsilon^{\frac{3(\nu-1)}{2}} \, |w_1|_{2\nu}^{\nu} \, |\varphi_\epsilon - w_\epsilon|_2 \\
&\quad - |\varphi_\epsilon - w_\epsilon|_2^2 - 2\mu \left(|w_\epsilon|_\alpha^{\alpha-2} + |\varphi_\epsilon|_\alpha^{\alpha-2} \right) |\varphi_\epsilon - w_\epsilon|_\alpha^2 \\
&\geq \epsilon^{\frac{3(\alpha-2)}{2}} \bar{\gamma}\lambda^{\frac{\alpha}{2}} \int |w_1|^\alpha - c_2 \epsilon^{\frac{3\nu+1}{2}} - c_3 \epsilon^2.
\end{aligned}$$

For the next term we observe that

$$\int (\nabla F(a\varphi_\epsilon), \eta(\varphi_\epsilon)) \leq \int |\nabla F(a\varphi_\epsilon) - \nabla F(aw_\epsilon)||\eta(\varphi_\epsilon)| + \int |\nabla F(aw_\epsilon)||\eta(\varphi_\epsilon)|.$$

Let us analyze the first term in this expression: we have

$$\begin{aligned}
&\int |\nabla F(a\varphi_\epsilon) - \nabla F(aw_\epsilon)||\eta(\varphi_\epsilon)| \\
&\leq \int \left(|a(\varphi_\epsilon - w_\epsilon)| + |aw_\epsilon| + \mu |a(\varphi_\epsilon - w_\epsilon)|^{\alpha-2} + \mu |aw_\epsilon|^{\alpha-2} \right) a |\varphi_\epsilon - w_\epsilon| |\eta(\varphi_\epsilon)| \\
&\leq a^2 \int \left(|(\varphi_\epsilon - w_\epsilon)|^2 + |w_\epsilon||\varphi_\epsilon - w_\epsilon| \right) |\eta(\varphi_\epsilon)| \\
&\quad + \mu a^{\alpha-1} \int \left(|\varphi_\epsilon - w_\epsilon|^{\alpha-1} + |w_\epsilon|^{\alpha-2} |\varphi_\epsilon - w_\epsilon| \right) |\eta(\varphi_\epsilon)| \\
&\leq a^2 \, |\varphi_\epsilon - w_\epsilon|_3^2 \, |\eta(\varphi_\epsilon)|_3 + a^2 \, |w_\epsilon|_3 \, |\varphi_\epsilon - w_\epsilon|_3 \, |\eta(\varphi_\epsilon)|_3 \\
&\quad + \mu a^{\alpha-1} \, |\varphi_\epsilon - w_\epsilon|_\alpha^{\alpha-1} \, |\eta(\varphi_\epsilon)|_\alpha + \mu a^{\alpha-1} \epsilon^{\frac{3(\alpha-2)}{2}} \, |w_1|_\infty^{\alpha-2} \, |\varphi_\epsilon - w_\epsilon|_2 \, |\eta(\varphi_\epsilon)|_2
\end{aligned}$$

$$
\begin{aligned}
&\leq a^2\, |\eta(\varphi_\epsilon)|_3 \left(S_3^2 \|\varphi_\epsilon - w_\epsilon\|^2 + |w_\epsilon|_3\, S_3 \|\varphi_\epsilon - w_\epsilon\| \right) \\
&\quad + \mu a^{\alpha-1} S_\alpha^\alpha \|\eta(\varphi_\epsilon)\| \|\varphi_\epsilon - w_\epsilon\|^{\alpha-1} \\
&\quad + \mu a^{\alpha-1} \epsilon^{\frac{3(\alpha-2)}{2}}\, |w_1|_\infty^{\alpha-2}\, |\varphi_\epsilon - w_\epsilon|_2\, |\eta(\varphi_\epsilon)|_2 \\
&\leq \lambda^{\frac{3}{2}} S_3^2 (\sqrt{m}+1) \left(\epsilon^2 S_3 \frac{(3+\sqrt{m})^2}{m}\, |\nabla w_1|_2^2 + \epsilon^{\frac{3}{2}}\, |w_1|_3\, \frac{3+\sqrt{m}}{\sqrt{m}}\, |\nabla w_1|_2 \right) \\
&\quad + \epsilon^{\alpha-1} \mu \lambda^{\frac{2\alpha-1}{2}} S_\alpha^\alpha (\sqrt{m}+1) \left(\frac{3+\sqrt{m}}{\sqrt{m}} \right)^{\alpha-1} |\nabla w|_2^{\alpha-1} \\
&\quad + \epsilon^{\frac{3\alpha-4}{2}} \mu \lambda^{\frac{2\alpha-1}{2}} S_2^2\, |w_1|_\infty^{\alpha-2}\, \frac{(\sqrt{m}+1)}{\sqrt{m}}\, |\nabla w_1|_2 \\
&\leq c_4 \epsilon^2 + c_5 \epsilon^{\frac{3}{2}} + c_6 \epsilon^{\alpha-1} + c_7 \epsilon^{\frac{3\alpha-4}{2}}
\end{aligned}
$$

while for the second term we have

$$
\begin{aligned}
\int |\nabla F(a w_\epsilon)| |\eta(\varphi_\epsilon)| \leq a^\nu \int |w_\epsilon|^\nu |\eta(\varphi_\epsilon)| &\leq \frac{1}{2} a^{2\nu}\, |w_\epsilon|_{2\nu}^{2\nu} + \frac{1}{2}\, |\eta(\varphi_\epsilon)|_2^2 \\
&\leq \epsilon^{3(\nu-1)} \frac{\lambda^\nu}{2}\, |w_1|_{2\nu}^{2\nu} + \frac{S_2^2}{2} \|\eta(\varphi_\epsilon)\|^2 \leq c_8 \epsilon^{3(\nu-1)} + \frac{S_2^2}{2} \|\eta(\varphi_\epsilon)\|^2
\end{aligned}
$$

The last term in (4.2) can be estimated as follows:

$$
\int |\varphi_\epsilon|^{\alpha-2} |\eta(\varphi_\epsilon)|^2 \leq \epsilon^{\frac{3(\alpha-2)}{2}}\, |\varphi_1|_\infty^{\alpha-2}\, |\eta(\varphi_\epsilon)|_2^2 \leq \epsilon^{\frac{3(\alpha-2)}{2}}\, |\varphi_1|_\infty^{\alpha-2}\, S_2^2 \|\eta(\varphi_\epsilon)\|^2
$$

We therefore deduce that

$$
\begin{aligned}
\mathcal{E}_\lambda(\varphi_\epsilon) \leq \frac{\lambda m}{2} + c_1 \epsilon^2 - \frac{1}{4}\, |\eta(\varphi_\epsilon)|_2^2\, \|\varphi_\epsilon\|^2 - \frac{1}{4} \|\eta(\varphi_\epsilon)\|^2 - \epsilon^{\frac{3(\alpha-2)}{2}} \gamma \bar{\gamma} \lambda^{\frac{\alpha}{2}} \int |w_1|^\alpha \\
+ c_2 \gamma \epsilon^{\frac{3\nu+1}{2}} + c_3 \gamma \epsilon^2 + c_4 \gamma \epsilon^2 + c_5 \gamma \epsilon^{\frac{3}{2}} + c_6 \gamma \epsilon^{\alpha-1} + c_7 \gamma \epsilon^{\frac{3\alpha-4}{2}} \\
+ c_8 \gamma \epsilon^{3(\nu-1)} + \gamma \frac{S_2^2}{2} \|\eta(\varphi_\epsilon)\|^2 + \epsilon^{\frac{3(\alpha-2)}{2}} \gamma\, |\varphi_1|_\infty^{\alpha-2}\, S_2^2 \|\eta(\varphi_\epsilon)\|^2
\end{aligned}
$$

Since $\gamma S_2^2 < \frac{1}{4}$ by (1.14) and

$$
\frac{3(\alpha-2)}{2} < \min\left\{ 2, \frac{3\nu+1}{2}, \frac{3}{2}, \alpha-1, \frac{3\alpha-4}{2}, 3(\nu-1) \right\} \qquad \text{for all } \alpha \in (2, \frac{8}{3}]
$$

we deduce that

$$
\mathcal{E}_\lambda(\varphi_\epsilon) < \frac{\lambda m}{2}
$$

for ϵ small enough and for all $\lambda \in (0, 1]$ and hence $e(\lambda) < \frac{\lambda m}{2}$. □

Proposition 4.3. *For all $\lambda \in (0,1)$ and $\theta > 1$ such that $\theta\lambda < 1$ we have that*

$$e(\theta\lambda) < \theta e(\lambda). \tag{4.4}$$

Proof. Let $\theta > 1$ be such that $\lambda\theta \in (0,1)$. Take $w \in \Sigma_+$ and let $\eta_{\theta\lambda}(w) \in B_{\theta\lambda}$ be the function whose existence follows from Proposition 3.12. Let $\psi_{\theta\lambda}(w) = a_{\theta\lambda}(\eta_{\theta\lambda}(w))w + \eta_{\theta\lambda}(w) \in \Sigma_{\theta\lambda}$. We have that

$$\mathcal{E}_{\theta\lambda}(w) = \frac{1}{2}\|a_{\theta\lambda}(\eta_{\theta\lambda}(w))w\|^2 - \frac{1}{2}\|\eta_{\theta\lambda}(w)\|^2 - \gamma \int F(\psi_{\theta\lambda}(w)).$$

We recall that from assumption (H3) follows that

$$F(\sigma x) \geq \sigma^\alpha F(x) \geq \sigma^2 F(x) + \left(\sigma^\alpha - \sigma^2\right) F(x) \qquad \text{for all } x \in \mathbb{C}^4, \sigma \geq 1.$$

We have that for all $\eta \in B_{\theta\lambda}$

$$a_{\lambda\theta}(\eta) = \sqrt{\lambda\theta - |\eta|_2^2} = \sqrt{\theta}\sqrt{\lambda - \left|\frac{\eta}{\sqrt{\theta}}\right|_2^2} = \sqrt{\theta} a_\lambda(\frac{\eta}{\sqrt{\theta}})$$

and hence we have that

$$\begin{aligned}
e(\theta\lambda) \leq \mathcal{E}_{\theta\lambda}(w) &= \theta\left[\frac{1}{2}\|a_\lambda(\frac{\eta_{\theta\lambda}(w)}{\sqrt{\theta}})w\|^2 - \frac{1}{2}\|\frac{\eta_{\theta\lambda}(w)}{\sqrt{\theta}}\|^2 - \frac{\gamma}{\theta}\int F\left(\sqrt{\theta}\frac{\psi_{\theta\lambda}(w)}{\sqrt{\theta}}\right)\right] \\
&\leq \theta\left[\frac{1}{2}\|a_\lambda(\frac{\eta_{\theta\lambda}(w)}{\sqrt{\theta}})w\|^2 - \frac{1}{2}\|\frac{\eta_{\theta\lambda}(w)}{\sqrt{\theta}}\|^2 - \gamma\int F\left(\frac{\psi_{\theta\lambda}(w)}{\sqrt{\theta}}\right)\right] \\
&\quad - \gamma\theta\left(\theta^{(\alpha-2)/2} - 1\right)\int F\left(\frac{\psi_{\theta\lambda}(w)}{\sqrt{\theta}}\right) \\
&\leq \theta\mathcal{E}_\lambda(w) - \gamma\theta\left(\theta^{(\alpha-2)/2} - 1\right)\int F\left(\frac{\psi_{\theta\lambda}(w)}{\sqrt{\theta}}\right).
\end{aligned}$$

We claim that for all $w \in \Sigma_+$ such that $\mathcal{E}_\lambda(w) < \frac{1}{2}(\frac{\lambda m}{2} + e(\lambda))$

$$\int F\left(\frac{\psi_{\theta\lambda}(w)}{\sqrt{\theta}}\right) \geq \delta > 0.$$

Assuming the claim, let $w_n \in \Sigma_+$ be a sequence which minimize $\mathcal{E}_\lambda(w)$, $\mathcal{E}_\lambda(w_n) \to e(\lambda) < \frac{1}{2}(\frac{\lambda m}{2} + e(\lambda))$. We have that for all $\theta > 1$

$$e(\theta\lambda) \leq \mathcal{E}_{\theta\lambda}(w_n) \leq \theta\mathcal{E}_\lambda(w_n) - \theta\left(\theta^{(\alpha-2)/2} - 1\right)\delta = \theta e(\lambda) - \theta\left(\theta^{(\alpha-2)/2} - 1\right)\delta + o(1) < \theta e(\lambda)$$

and the proposition follows.

To prove the claim, we assume it does not hold and that there is a sequence $w_n \in \Sigma_+$ such that $\mathcal{E}_\lambda(w_n) < \frac{1}{2}(\frac{\lambda m}{2} + e(\lambda))$ and

$$\int F\left(\frac{\psi_{\theta\lambda}(w_n)}{\sqrt{\theta}}\right) \to 0.$$

Then also $\int F(\psi_{\theta\lambda}(w_n)) \to 0$ and we deduce from (H6) that for all $\zeta > 0$

$$\begin{aligned}\left|\int (\nabla F(\psi_{\theta\lambda}(w_n)), w_n)\right| &\le \int (\zeta + C_\zeta F(\psi_{\theta\lambda}(w_n))^{\frac{1}{\xi}})|\psi_{\theta\lambda}(w_n)||w_n| \\ &\le \zeta\, |\psi_{\theta\lambda}(w_n)|_2\, |w_n|_2 + C_\zeta\, |\psi_{\theta\lambda}(w_n)|_{\frac{2\xi}{\xi-1}}\, |w_n|_{\frac{2\xi}{\xi-1}} \int F(\psi_{\theta\lambda}(w_n)) \le C\zeta + o(1)\end{aligned}$$

which implies that

$$\lim_{n\to+\infty} \left|\int (\nabla F(\psi_{\theta\lambda}(w_n)), w_n)\right| = 0$$

and similarly

$$\lim_{n\to+\infty} \left|\int (\nabla F(\psi_{\theta\lambda}(w_n)), \eta_{\theta\lambda}(w_n))\right| = 0$$

Since

$$0 = -\,|\eta_{\theta\lambda}(w_n)|_2^2\, \|w_n\|^2 - \|\eta_{\theta\lambda}(w_n)\|^2 - \gamma \int (\nabla F(\psi_{\theta\lambda}(w_n)), -\frac{|\eta_{\theta\lambda}(w_n)|_2^2}{a_{\theta\lambda}(\eta_{\theta\lambda}(w_n))} w_n + \eta_{\theta\lambda}(w_n))$$

we deduce from the boundedness of $\|w_n\|$ and $\|\eta_{\theta\lambda}(w_n)\|$ that $\|\eta_{\theta\lambda}(w_n)\| \to 0$ and hence

$$\begin{aligned}\frac{\theta}{2}(\frac{\lambda m}{2} + e(\lambda)) &> \theta\mathcal{E}_\lambda(w_n) \ge \mathcal{E}_{\theta\lambda}(w_n) \\ &= \frac{\theta\lambda}{2}\|w_n\|^2 - \frac{1}{2}\,|\eta_{\theta\lambda}(w_n)|_2^2\, \|w_n\|^2 - \frac{1}{2}\|\eta_{\theta\lambda}(w_n)\|^2 - \gamma \int F(\psi_{\theta\lambda}(w_n)) \\ &\ge \frac{\theta\lambda m}{2} + o(1),\end{aligned}$$

a contradiction which proves the claim. $\square$

Lemma 4.5. *Let $\omega \in (0, m)$, $C > 0$ and $\psi \in X$ be such that*

$$\begin{cases} dI(\psi)[h] = \omega(\psi \mid h) & \text{for all } h \in X \\ \|\psi\| \le C \end{cases} \tag{4.6}$$

Then there exists two positive constants, c_1 and c_2, which depend only on ω and C, such that

$$\|\psi\| \ge c_1 > 0, \qquad I(\psi) \ge c_2 > 0.$$

Proof. Taking $h = \Lambda_+\psi - \Lambda_-\psi$ in (4.6) and $\epsilon > 0$ we have

$$\begin{aligned}
\|\Lambda_+\psi\|^2 + \|\Lambda_-\psi\|^2 + \omega|\Lambda_-\psi|_2^2 &= \omega|\Lambda_+\psi|_2^2 + \int (\nabla F(\psi), \Lambda_+\psi - \Lambda_-\psi) \\
&\le \frac{\omega}{m}\|\Lambda_+\psi\|^2 + \int (\epsilon|\psi| + \mu_\epsilon|\psi|^{\alpha-1})(|\Lambda_+\psi| + |\Lambda_-\psi|) \\
&\le \frac{\omega}{m}\|\Lambda_+\psi\|^2 + 2\epsilon\,|\psi|_2^2 + \mu_\epsilon\,|\psi|_\alpha^{\alpha-1}\,(|\Lambda_+\psi|_\alpha + |\Lambda_-\psi|_\alpha) \\
&\le \frac{\omega}{m}\|\Lambda_+\psi\|^2 + 2\epsilon C_2^2\|\psi\|^2 + \mu_\epsilon C_\alpha^\alpha\|\psi\|^\alpha.
\end{aligned}$$

If $\epsilon > 0$ is such that $2\epsilon C_2^2 < \frac{1-\frac{\omega}{m}}{2}$ we deduce that

$$\frac{1}{2}\left(1 - \frac{\omega}{m}\right)\|\psi\|^2 \le \mu_\epsilon C_\alpha^\alpha\|\psi\|^\alpha$$

and

$$\|\psi\|^{\alpha-2} \ge \frac{1}{2\mu_\epsilon C_\alpha^\alpha}\left(1 - \frac{\omega}{m}\right).$$

Let us now estimate the critical level:

$$\begin{aligned}
I(\psi) &= I(\psi) - \frac{1}{2}dI(\psi)[\psi] + \frac{1}{2}\omega|\psi|_2^2 \\
&= \int \left(\frac{1}{2}(\nabla F(\psi), \psi) - F(\psi)\right) + \frac{1}{2}\omega|\psi|_2^2 \\
&\ge \left(\frac{\alpha}{2} - 1\right)\int F(\psi) + \frac{1}{2}\omega|\psi|_2^2.
\end{aligned}$$

We claim that there is $\delta > 0$ such that $\int F(\psi) \ge \delta > 0$ for all ψ for which (4.6) holds. If not, we find a sequence ψ_n of solutions such that $\int F(\psi_n) \to 0$. From

$$0 = dI(\psi_n)[\Lambda_\pm\psi_n] - \omega|\Lambda_\pm\psi_n|_2^2$$

we deduce that

$$\begin{aligned}
\|\Lambda_+\psi_n\|^2 - \omega|\Lambda_+\psi_n|_2^2 &= \int (\nabla F(\psi_n), \Lambda_-\psi_n) \\
\|\Lambda_-\psi_n\|^2 + \omega|\Lambda_-\psi_n|_2^2 &= -\int (\nabla F(\psi_n), \Lambda_-\psi_n).
\end{aligned}$$

As in the proof of 4.3, using (H6) we can deduce from $\int F(\psi_n) \to 0$ that

$$\int (\nabla F(\psi_n), \Lambda_\pm\psi_n) \to 0$$

and hence also

$$\|\psi_n\| \to 0,$$

contradiction which proves the claim. $\square$

Proof of theorem. By Ekeland's variational principle, there exists a sequence $w_n \in \Sigma_+$ such that

$$\mathcal{E}_1(w_n) \to e(1), \qquad \sup_{v\in\Sigma_+} |d\mathcal{E}_1(w_n)[v]| \to 0.$$

Since $\mathcal{E}_1(w_n) \to e(1)$ we deduce from Lemma 3.11 that the sequence w_n is bounded. Follows from Proposition 3.2 that also $\eta_n = \eta(w_n)$ and $\psi_n = a(\eta_n)w_n + \eta_n$ are bounded in X. Letting $\omega_n = a(\eta_n)^{-1} dI(\psi_n)[w_n] \geq (1 - C_{\alpha,1})$ we have that

$$dI(\psi_n)[h] - \omega_n(\psi_n \,|\, h) = \frac{1}{a(\eta_n)} d\mathcal{E}_\lambda(w_n)[v] \to 0 \qquad \text{for all } h = v + \xi \in X.$$

We can assume that (up to a subsequence) $\psi_n \rightharpoonup \psi$ in X and that $\omega_n \to \omega$, with $\omega \in (1 - C_{\alpha,1}, 2\mathcal{E}_1(w_n)) \subset (1 - C_{\alpha,1}, m)$ for n sufficiently large. Then we have that for all $h \in X$

$$dI(\psi_n)[h] - \omega_n(\psi_n \,|\, h) = \langle \psi_n \,|\, \Lambda_+ h\rangle - \langle \psi_n \,|\, \Lambda_- h\rangle - \gamma \int (\nabla F(\psi_n), h) - \omega_n(\psi_n \,|\, h) \to 0$$

since we have that

$$\int (\nabla F(\psi_n) - \nabla F(\psi), h) \to 0$$

we deduce that

$$dI(\psi)[h] - \omega(\psi \,|\, h) = 0 \qquad \text{for all } h \in X.$$

The weak convergence does not, however, preserve the L^2 norm, so we only know that $|\psi|_2 \leq |\psi_n|_2 = 1$ (it could even be that $\psi = 0$).

To conclude we will now apply the concentration-compactness principle, see [18,19]. First of all let us show that no vanishing occurs. By contradiction, assume that

$$\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |\psi_n|^2 = 0.$$

Then we know, see [18] or [25, Lemma 1.21], that $\psi_n \to 0$ in $L^p(\mathbb{R}^3)$ for $2 < p < 3$. Then for any $\epsilon > 0$

$$\left|\int (\nabla F(\psi_n), w_n)\right| \leq \epsilon \int |\psi_n||w_n| + \mu_\epsilon \int |\psi_n|^{\alpha-1}|w_n| \leq \epsilon + \mu_\epsilon\, |\psi_n|_\alpha^{\alpha-1}\, |w_n|_\alpha$$

and

$$\lim_{n\to+\infty}\left|\int(\nabla F(\psi_n), w_n)\right| = 0.$$

We deduce from

$$\epsilon_n = dI(\psi_n)[w_n] - \omega_n a(\eta_n)\,|w_n|_2^2 = a(\eta_n)(\|w_n\|^2 - \omega_n\,|w_n|_2^2) - \int(\nabla F(\psi_n), w_n)$$

that for all $\lambda \in (0, 1]$

$$\epsilon_n = a(\eta_n)(\|w_n\|^2 - \omega_n\,|w_n|_2^2) \geq a(\eta_n)(m - \omega_n)\,|w_n|_2^2 \geq \sqrt{\frac{1}{2}}(m - \omega_n) > 0$$

a contradiction since, by Proposition 3.16 and Lemma 4.1 we have that $\omega_n \leq 2\mathcal{E}_1(w_n)$ and $e(1) < \frac{m}{2}$. Hence vanishing does not occur.

Since $\|\psi\| \leq \|\psi_n\|$ and $I(\psi) \geq c_2$ (by Lemma 4.5) we deduce from the concentration compactness principle that there exists $p \geq 1$ functions $\phi_1, \ldots, \phi_p \in X$, critical points for I under the constraint $|\psi|_2^2 = \mu_i \in (0, 1]$ (satisfying (3.17) with $\omega = \lim_n \omega_n > 0$) and p sequences of points $x_{i,n} \in \mathbb{R}^3$, $i = 1, \ldots, p$ such that $|x_{i,n} - x_{j,n}| \to +\infty$ for all $i \neq j$ as $n \to +\infty$ and

$$\|\psi_n - \sum_{i=i}^{p}\phi_i(\cdot - x_{i,n})\| \to 0 \quad \text{as } n \to +\infty.$$

From this follows also that $|\psi_n|_2^2 = 1 = \sum_{i=1}^{p}\mu_i$.

We then observe that

$$\begin{aligned}
\|\Lambda_+\psi_n\|^2 - \|\Lambda_-\psi_n\|^2 &= \langle\psi_n \,|\, \Lambda_+\psi_n - \Lambda_-\psi_n\rangle \\
&= \langle\psi_n - \sum_{i=i}^{p}\phi_i(\cdot - x_{i,n}) \,|\, \Lambda_+\psi_n - \Lambda_-\psi_n\rangle \\
&\quad + \sum_{i=i}^{p}\langle\phi_i(\cdot - x_{i,n}) \,|\, \Lambda_+\psi_n - \Lambda_-\psi_n\rangle \\
&= \sum_{i=i}^{p}\big(\langle\Lambda_+\phi_i(\cdot - x_{i,n}) \,|\, \psi_n\rangle - \langle\Lambda_-\phi_i(\cdot - x_{i,n}) \,|\, \psi_n\rangle\big) + o(1) \\
&= \sum_{i=i}^{p}\Big(\|\Lambda_+\phi_i\|^2 - \|\Lambda_-\phi_i\|^2\Big) + o(1)
\end{aligned}$$

and also

$$\int F(\psi_n) = \sum_{i=1}^{p}\int F(\phi_i(x)) + o(1).$$

(this can be deduced from the Brezis-Lieb lemma, see [3, Theorem 2]). Finally we have that

$$e(1) = I(\psi_n) + o(1) = \sum_{i=1}^{p} I(\phi_i) + o(1) \tag{4.7}$$

and

$$0 = dI(\phi_i)[h] - \omega(\phi_i \,|\, h) \qquad \text{for all } h \in X,$$

with $\omega > 0$. Follows from Proposition 3.16 that $w_i = |\Lambda_+\phi_i|_2^{-1}\,\Lambda_+\phi_i \in \Sigma_+$ is a critical point for $\mathcal{E}_{\mu_i}$ and $\mathcal{E}_{\mu_i}(w_i) = I(\phi_i)$.

Since

$$\mathcal{E}_{\mu_i}(w_i) \geq e(\mu_i)$$

we deduce from (4.4) that if $p > 1$ (i.e. if dichotomy occurs)

$$\begin{aligned} e(1) = \sum_{i=1}^{p} I(\phi_i) + o(1) = \sum_{i=1}^{p} \mathcal{E}_{\mu_i}(w_i) + o(1) \geq \sum_{i=1}^{p} e(\mu_i) + o(1) \\ > \sum_{i=1}^{p} \mu_i e(\frac{1}{\mu_i}\mu_i) + o(1) = e(1)\sum_{i=1}^{p} \mu_i + o(1), \end{aligned}$$

a contradiction.

Since there is no vanishing or dichotomy, our sequence ψ_n converges – up to a translation – strongly in X to a critical point $\psi \in X$ of (3.17) such that $|\psi|_2 = 1$ and the theorem follows. □

Data availability

No data was used for the research described in the article.

Acknowledgment

The authors sincerely thank the referee for several useful comments and in particular for pointing out a mistake in a first version of the manuscript.

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