A NEW PROOF OF COMPACTNESS IN $G(S)BD$

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Abstract. We prove a compactness result in GBD which also provides a new proof of the compactness theorem in $GSBD$, due to Chambolle and Crismale [5, Theorem 1.1]. Our proof is based on a Fréchet-Kolmogorov compactness criterion and does not rely on Korn or Poincaré-Korn inequalities.

1. Introduction

In this paper we prove a compactness result in GBD, which in particular provides an alternative proof of the compactness theorem in $GSBD$ obtained by Chambolle and Crismale in [5, Theorem 1.1]. Referring to Section 2 for the notation used below, the theorem reads as follows.

Theorem 1.1. *Let $U \subseteq \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$ and let $u_k \in GBD(U)$ be such that*

$$\sup_{k\in\mathbb{N}} \hat{\mu}_{u_k}(U) < +\infty\,. \tag{1.1}$$

Then, there exists a subsequence, still denoted by u_k, such that the set

$$A := \{x \in U : \ |u_k(x)| \to +\infty \text{ as } k \to \infty\}$$

has finite perimeter, $u_k \to u$ a.e. in $U \setminus A$ for some function $u \in GBD(U)$ with $u = 0$ in A. Furthermore,

$$\mathcal{H}^{n-1}(\partial^* A) \leq \lim_{\sigma\to\infty} \liminf_{k\to\infty} \mathcal{H}^{n-1}(J^\sigma_{u_k})\,, \tag{1.2}$$

where $J^\sigma_{u_k} := \{x \in J_{u_k} \ : \ |[u_k(x)]| \geq \sigma\}$.

We notice that the main difference with [5] is that we do not request equi-integrability of the approximate symmetric gradient $e(u_k)$ and boundedness of the measure of the jump sets J_{u_k}, but only boundedness of $\hat{\mu}_{u_k}(U)$, which is the natural assumption for sequences in $GBD(U)$. Hence, when passing to the limit, the absolutely continuous and the singular parts of $\hat{\mu}_{u_k}$ could interact. For this reason, it is not possible to get weak L^1-convergence of the approximate symmetric gradients or lower-semicontinuity of the measure of the jump.

Nevertheless, we are able to recover the lower-semicontinuity (1.2) for the set A where $|u_k| \to +\infty$. In particular, formula (1.2) highlights that the emergence of the singular set A results from an uncontrolled jump discontinuity along the sequence u_k. Hence, an equi-boundedness of the measure of the super-level sets $J^\sigma_{u_k}$, i.e.,

for every $\varepsilon > 0$ there exists $\sigma_\varepsilon \in \mathbb{N}$ such that $\mathcal{H}^{n-1}(J^\sigma_{u_k}) < \varepsilon$ for $\sigma \geq \sigma_\varepsilon$ and $k \in \mathbb{N}$,

guarantees $\partial^* A = \emptyset$.

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The $GSBD$-result [5, Theorem 1.1] is recovered by replacing (1.1) with

$$\sup_{k\in\mathbb{N}} \int_U \phi(|e(u_k)|)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k}) < +\infty\,, \tag{1.3}$$

for a positive function ϕ with superlinear growth at infinity. The novelty of our proof, presented in Section 3, concerns the compactness part of Theorem 1.1. It is based on the Fréchet-Kolmogorov criterion and makes no use of Korn or Korn-Poincaré type of inequalities [3] (see also [2, 7, 8]), which are instead the key tools of [5]. The remaining lower-semicontinuity results of [5, Theorem 1.1] can be obtained by standard arguments.

2. Preliminaries and notation

We briefly recall here the notation used throughout the paper. For $d, k \in \mathbb{N}$, we denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the Lebesgue and the k-dimensional Hausdorff measure in $\mathbb{R}^d$, respectively. Given $F \subseteq \mathbb{R}^d$, we indicate with $\dim_{\mathcal{H}}(F)$ the Hausdorff dimension of F. For every compact subsets F_1 and F_2 of $\mathbb{R}^d$, $\mathrm{dist}_{\mathcal{H}}(F_1, F_2)$ stands for the Hausdorff distance between F_1 and F_2. We denote by $\mathbb{1}_E$ the characteristic function of a set $E \subseteq \mathbb{R}^d$. For every measurable set $\Omega \subseteq \mathbb{R}^d$ and every measurable function $u\colon \Omega \to \mathbb{R}^d$, we further set J_u the set of approximate discontinuity points of u and

$$J_u^\sigma := \{x \in J_u : |[u](x)| \geq \sigma\} \qquad \sigma > 0\,,$$

where $[u](x) := u^+(x) - u^-(x)$, $u^\pm(x)$ being the unilateral approximate limit of u at x.

For $m, \ell \in \mathbb{N}$ we denote by $\mathbb{M}^{m\times\ell}$ the space of $m \times \ell$ matrices with real coefficients, and set $\mathbb{M}^m := \mathbb{M}^{m\times m}$. The symbol $\mathbb{M}^m_{sym}$ (resp. $\mathbb{M}^m_{skw}$) indicates the subspace of $\mathbb{M}^m$ of squared symmetric (resp. skew-symmetric) matrices of order m. We further denote by $SO(m)$ the set of rotation matrices.

Let us now fix $n \in \mathbb{N} \setminus \{0\}$. For every $\xi \in \mathbb{S}^{n-1}$, π_ξ stands for the projection over the subspace $\xi^\perp$ orthogonal to ξ. For every measurable set $V \subseteq \mathbb{R}^n$, every $\xi \in \mathbb{S}^{n-1}$, and every $y \in \mathbb{R}^n$, we set

$$\Pi^\xi := \{z \in \mathbb{R}^n : z \cdot \xi = 0\}\,, \qquad V^\xi_y := \{t \in \mathbb{R} : y + t\xi \in V\}\,.$$

For $V \subseteq \mathbb{R}^n$ measurable, $\xi \in \mathbb{S}^{n-1}$, and $y \in \mathbb{R}^n$ we define

$$\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi \qquad \text{for every } t \in V^\xi_y\,.$$

For every open bounded subset U of $\mathbb{R}^n$, the space $GBD(U)$ of generalized functions of bounded deformation [6] is defined as the set of measurable functions $u\colon U \to \mathbb{R}^n$ which admit a positive Radon measure $\lambda \in \mathcal{M}^+_b(U)$ such that for every $\xi \in \mathbb{S}^{n-1}$ one of the two equivalent conditions is satisfied [6, Theorem 3.5]:

- for every $\theta \in C^1(\mathbb{R}; [-\frac{1}{2}; \frac{1}{2}])$ such that $0 \leq \theta' \leq 1$, the partial derivative $D_\xi(\theta(u \cdot \xi))$ is a Radon measure in U and $|D_\xi(\theta(u \cdot \xi))|(B) \leq \lambda(B)$ for every Borel subset B of U;
- for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$ the function $\hat{u}^\xi_y$ belongs to $BV_{loc}(U^\xi_y)$ and

$$\int_{\Pi^\xi} \big|(D\hat{u}^\xi_y)\big|\big(B^\xi_y \setminus J^1_{\hat{u}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\hat{u}^\xi_y}\big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \leq \lambda(B) \tag{2.1}$$

  for every Borel subset B of U.

A function u belongs to $GSBD(U)$ if $\hat{u}^\xi_y \in SBV_{loc}(U^\xi_y)$ and (2.1) holds. Every function $u \in GBD(U)$ admits an approximate symmetric gradient $e(u) \in L^1(U; \mathbb{M}^n_{sym})$. The jump set J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable with approximate unit normal vector ν_u. We will also use measures $\hat{\mu}^\xi, \hat{\mu}_u \in \mathcal{M}^+_b(U)$ defined in [6, Definitions 4.10

and 4.16] for $u \in GBD(U)$ and $\xi \in \mathbb{S}^{n-1}$. We further refer to [6] for an exhaustive discussion on the fine properties of functions in $GBD(U)$.

3. Proof of Theorem 1.1

This section is devoted to the presentation of an alternative proof of Theorem 1.1, based on the Fréchet-Kolmogorov compactness criterion. We start by giving two definitions.

Definition 3.1. Let $\Xi = \{\xi_1, \dots, \xi_n\}$ denote an orthonormal basis of $\mathbb{R}^n$. We define

$$S_{\Xi,0} := \bigcup_{\xi \in \Xi} \{x \in \mathbb{R}^n : |x| = 1, \ x \in \Pi^\xi\}.$$

Given $\delta > 0$ we define the δ-neighborhood of $S_{\Xi,0}$ as

$$S_{\Xi,\delta} := \{x \in \mathbb{R}^n : |x| = 1, \ \ \mathrm{dist}(x, S_{\Xi,0}) < \delta\}.$$

Definition 3.2. In order to simplify the notation, given a family $\mathcal{K}$ and a positive natural number m, we denote by $\mathcal{K}_m$ the set consisting of all subsets of $\mathcal{K}$ containing exactly m-elements of $\mathcal{K}$, i.e.

$$\mathcal{K}_m := \{\mathcal{Z} \in \mathrm{P}(\mathcal{K}) : \#\mathcal{Z} = m\}.$$

In order to prove Theorem 1.1, we need the following two lemmas, which allow us to construct a suitable orthonormal basis of $\mathbb{R}^n$ that will be used to test the Fréchet-Kolmogorov compactness criterium.

Lemma 3.3. *Let $M \in \mathbb{N}$ be such that $M \geq n$ and consider a family $\mathcal{K} := \{\Xi_1, \dots, \Xi_M\}$ of orthonormal bases of $\mathbb{R}^n$ such that for every $\mathcal{Z} \in \mathcal{K}_n$*

$$\bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} = \emptyset\,. \tag{3.1}$$

Then, there exists a further orthonormal basis $\Sigma = \{\xi_1, \dots, \xi_n\}$ such that for every $\mathcal{Z} \in \mathcal{K}_{n-1}$

$$S_{\Sigma,0} \cap \bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} = \emptyset\,. \tag{3.2}$$

Proof. First of all notice that whenever $\mathcal{Z} \in \mathcal{K}_n$ is such that

$$\bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} = \emptyset$$

then we have

$$\mathcal{H}^0\Big(\bigcap_{\Xi \in \mathcal{X}} S_{\Xi,0}\Big) < +\infty \qquad \text{for every } \mathcal{X} \in \mathcal{Z}_{n-1}. \tag{3.3}$$

Indeed, let us suppose by contradiction that (3.3) does not hold for some $\mathcal{X} \in \mathcal{Z}_{n-1}$. Since for $\Xi \in \mathcal{X}$ we have that each $S_{\Xi,0}$ is a finite union of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ intersected with $\mathbb{S}^{n-1}$, the equality $\mathcal{H}^0\Big(\bigcap_{\Xi \in \mathcal{X}} S_{\Xi,0}\Big) = +\infty$ implies that

$$\dim_{\mathcal{H}}\Big(\bigcap_{\Xi \in \mathcal{X}} S_{\Xi,0}\Big) \geq 1\,.$$

As a consequence we get

$$\dim_{\mathcal{H}}\Big(\bigcap_{\Xi \in \mathcal{X}} \bigcup_{\xi \in \Xi} \{\xi^\perp\}\Big) \geq 2\,.$$

Hence, if we denote by $\overline{\Xi}$ the basis contained in $\mathcal{Z} \setminus \mathcal{X}$, then by using Grassmann's formula

$$\dim(V) + \dim(W) - \dim(V \cap W) = \dim(V + W) \leq n\,,$$

which is valid for each couple V, W of vector subspaces of $\mathbb{R}^n$, we deduce

$$\dim_{\mathcal{H}} \Big(\bigcup_{\xi \in \overline{\Xi}} \{\xi^{\perp}\} \cap \bigcap_{\Xi \in \mathcal{X}} \bigcup_{\xi \in \Xi} \{\xi^{\perp}\} \Big) \geq 1\,,$$

hence

$$\bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} \neq \emptyset$$

which is a contradiction to the assumption (3.1).

Fix $\{e_1, \dots, e_n\}$ an orthonormal basis of $\mathbb{R}^n$ and let $SO(n)$ be the group of special orthogonal matrices. It can be endowed with the structure of an $\big(\frac{n^2-n}{2}\big)$-dimensional submanifold of $\mathbb{R}^{n^2}$. We can identify an element $O \in SO(n)$ with an $(n \times n)$-matrix whose columns are the vectors of an orthonormal basis Ξ written with respect to $\{e_1, \dots, e_n\}$ and viceversa.

In order to show the existence of Σ satisfying (3.2) we prove the following stronger condition: given $\mathcal{Z} \in \mathcal{K}_{n-1}$, for $\mathcal{H}^{(n^2-n)/2}$-a.e. choice of Σ we have that

$$S_{\Sigma,0} \cap \bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} = \emptyset\,. \tag{3.4}$$

This easily implies the existence of an orthonormal basis Σ satisfying (3.2), as the choice of $\mathcal{Z} \in \mathcal{K}_{n-1}$ is finite. To show (3.4), for every $i \in \{1, \dots, n\}$ let us define the smooth map $\Lambda_i \colon SO(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\} \to \mathbb{S}^{n-1}$ as

$$\Lambda_i(\Sigma, y) := \sum_{j<i} y_j \xi_j + \sum_{j>i} y_{j-1} \xi_j\,,$$

where ξ_j denotes the j-th column vector of the matrix representing Σ. In order to show (3.4), we claim that it is enough to prove that for every $x \in \mathbb{S}^{n-1}$ we have

$$\mathcal{H}^{(n^2-n)/2}\big(\pi_{SO(n)}(\{\Lambda_i^{-1}(x)\})\big) = 0 \qquad \text{for } i \in \{1, \dots, n\}, \tag{3.5}$$

where $\pi_{SO(n)} \colon SO(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\} \to SO(n)$ is the canonical projection map. Indeed, if Σ does not belong to $\pi_{SO(n)}(\{\Lambda_i^{-1}(x)\})$ for every $x \in \bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0}$ and for every $i \in \{1, \dots, n\}$, then by using the definition of the map Λ_i we deduce immediately that Σ satisfies $S_{\Sigma,0} \cap \bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} = \emptyset$. Therefore, if (3.5) holds, then the set (remember that $\bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0}$ is a discrete set)

$$\bigcup_{i=1}^{n} \bigcup_{x \in \bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0}} \pi_{SO(n)}(\{\Lambda_i^{-1}(x)\})$$

is of $\mathcal{H}^{(n^2-n)/2}$-measure zero and (3.4) holds true. Thus, $\mathcal{H}^{(n^2-n)/2}$-a.e. Σ satisfies (3.2).

To prove (3.5) it is enough to show that the differential of Λ_i has full rank at every point $z \in SO(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\}$. Indeed, this implies that $\Lambda_i^{-1}(x)$ is an $\big(\frac{n^2-n-2}{2}\big)$-dimensional submanifold for every $x \in \mathbb{S}^{n-1}$, which ensures the validity of (3.5) since

$$\#\big(\{\pi^{-1}_{SO(n)}(\Xi)\} \cap \{\Lambda_i^{-1}(x)\}\big) = 1, \quad x \in \mathbb{S}^{n-1},$$
$$\frac{n^2-n-2}{2} < \frac{n^2-n}{2} = \dim_{\mathcal{H}}(SO(n)) \quad (n \geq 2)\,.$$

Notice that Λ_i is the restriction to $SO(n)\times\{y\in\mathbb{R}^{n-1}:\,|y|=1\}$ of the map $\tilde\Lambda_i\colon\mathbb{M}^n\times\mathbb{R}^{n-1}\to\mathbb{R}^n$ defined as

$$\tilde\Lambda_i(\Theta,y):=\sum_{j<i}y_j\theta_j+\sum_{j>i}y_{j-1}\theta_j\,,$$

where θ_j is the j-th column vector of the matrix $\Theta\in\mathbb{M}^n$. To show that the differential of Λ_i has full rank everywhere, it is enough to check that for every $z\in SO(n)\times\{y\in\mathbb{R}^{n-1}:\,|y|=1\}$ the differential of $\tilde\Lambda_i$ restricted to $\mathrm{Tan}(SO(n)\times\{y\in\mathbb{R}^{n-1}:\,|y|=1\},z)$ has rank equal to $n-1$. By using the relation

$$\tilde\Lambda_i(M\Theta,y)=M\tilde\Lambda_i(\Theta,y)\,,$$

valid for every $M\in\mathbb{M}^n$, we can reduce ourselves to the case $z=(\mathrm{I},\overline{y})$, where I denotes the identity matrix and $\overline{y}\in\mathbb{R}^{n-1}$ is such that $|\overline{y}|=1$. It is well known that

$$\mathrm{Tan}(SO(n)\times\{\zeta\in\mathbb{R}^{n-1}:\,|\zeta|=1\},z)\cong\mathbb{M}^n_{skw}\times\mathrm{Tan}(\{\zeta\in\mathbb{R}^{n-1}:\,|\zeta|=1\},\overline{y})\,,$$

where $\mathbb{M}^n_{skw}$ denotes the space of skew symmetric matrices. Using that $\mathbb{R}^{n^2+n-1}\cong\mathbb{M}^n\times\mathbb{R}^{n-1}$, we identify a point $Z\in\mathbb{R}^{n^2+n-1}$ as $Z=((x^i_j)^n_{i,j=1},y_1,\dots,y_{n-1})$. A direct computation shows that the differential of Λ_i at the point $(\mathrm{I},\overline{y})$ acting on the vector Z is given by

$$d\tilde\Lambda_i(\mathrm{I},\overline{y})[Z]=\sum_{l=1}^n\sum_{j<i}(x^j_l\overline{y}_j+\delta_{jl}y_j)e_l+\sum_{j>i}(x^j_l\overline{y}_{j-1}+\delta_{jl}y_{j-1})e_l\,.$$

It is better to introduce the matrix $P_i\in\mathbb{M}^{n\times(n-1)}$ defined as

$$(P_i)^m_k:=\begin{cases}\delta_{km} & \text{if } 1\le m<i\,,\\ \delta_{k-1m} & \text{if } i\le m\le n-1\,.\end{cases}$$

Roughly speaking, given $X\in\mathbb{M}^{l\times n}$, the product XP_i is the matrix in $\mathbb{M}^{l\times(n-1)}$ obtained by removing from X the i-th column, while given $Y\in\mathbb{M}^{(n-1)\times l}$, the product P_iY is the matrix in $\mathbb{M}^{n\times l}$ obtained by adding a new row made of zero entries at the i-th position. With this definition the linear map $d\Lambda_i(\mathrm{I},\overline{y})(\cdot)$ can be rewritten more compactly as

$$d\Lambda_i(\mathrm{I},\overline{y})[(X,y)]=XP_i\overline{y}+P_iy,\quad X\in\mathbb{M}^n_{skw},\quad y\in\mathrm{Tan}(\{\zeta\in\mathbb{R}^{n-1}:\,|\zeta|=1\},\overline{y})\,.\tag{3.6}$$

Given $O\in SO(n-1)$ such that $O\tilde e_1=\overline{y}$ ($\{\tilde e_1,\dots,\tilde e_{n-1}\}$ denotes the reference orthonormal basis of $\mathbb{R}^{n-1}$), we can rewrite the system as

$$d\Lambda_i(\mathrm{I},\overline{y})[(X,y)]=XP_iO\tilde e_1+P_iy,\;\;X\in\mathbb{M}^n_{skw},\;\;y\in\mathrm{Tan}(\{\zeta\in\mathbb{R}^{n-1}:|\zeta|=1\},\overline{y}).\tag{3.7}$$

Hence, by the well known relation

$$\dim(V)-\dim(\mathrm{Im}[\alpha])=\dim(\ker[\alpha])\,,\tag{3.8}$$

valid for every linear map $\alpha\colon V\to W$ and every finite dimensional vector spaces V and W, if we want to prove that $d\Lambda_i(\mathrm{I},\overline{y})$ has full rank, i.e.

$$\dim(\mathrm{Im}[(\cdot)P_iO\tilde e_1+P_i(\cdot)])=n-1\,,\tag{3.9}$$

since

$$n-1\ge\dim(\mathrm{Im}[(\cdot)P_iO\tilde e_1+P_i(\cdot)])\ge\dim(\mathrm{Im}[(\cdot)P_iO\tilde e_1])$$

(where the first inequality comes from $\mathrm{Im}[d\Lambda_i(\mathrm{I},\overline{y})]\subset\mathrm{Tan}(\mathbb{S}^{n-1},\Lambda_i(\mathrm{I},\overline{y}))$), it is enough to show that

$$\dim(\mathrm{Im}[(\cdot)P_iO\tilde e_1])=n-1\,.\tag{3.10}$$

Again by relation (3.8) we can reduce ourselves to find the dimension of the kernel of the map $\mathbb{M}^n_{skw} \ni X \mapsto XP_iO\tilde{e}_1$. But this dimension can be easily computed to be

$$\dim(\ker[(\cdot)P_iO\tilde{e}_1]) = \sum_{k=1}^{n-2} k = \frac{(n-2)(n-1)}{2}\,,$$

which immediately implies (3.10). □

Remark 3.4. By a standard argument from linear algebra it is possible to construct n orthonormal bases of $\mathbb{R}^n$, say $\mathcal{K} = \{\Xi_1, \dots, \Xi_n\}$ satisfying

$$\bigcap_{\Xi \in \mathcal{K}} S_{\Xi,0} = \emptyset\,.$$

Moreover, given $U \subset SO(n)$ open, then Ξ_i can be chosen in such a way that

$$\Xi_i \in U, \quad i \in \{1, \dots, n\}\,.$$

Therefore, Lemma 3.3, and in particular condition (3.4), tells us that for every $M \in \mathbb{N}$ ($M \geq n$) we can always find a family of orthonormal bases of $\mathbb{R}^n$, say $\mathcal{K} = \{\Xi_1, \dots, \Xi_M\}$, satisfying (3.1) and

$$\Xi_i \in U, \quad i \in \{1, \dots, M\}\,.$$

Lemma 3.5. *Let $A \subset \mathbb{R}^n$ be a measurable set with $\mathcal{L}^n(A) < \infty$, let $(B_k)_{k=1}^\infty$ be measurable subsets of A, and let $(v_k)_{k=1}^\infty$ be measurable functions $v_k \colon B_k \to \mathbb{S}^{n-1}$. Then, given a sequence $\epsilon_h \searrow 0$, there exist a sequence $\delta_h \searrow 0$ with $\delta_h > 0$, a map $\phi \colon \mathbb{N} \to \mathbb{N}$, and an orthonormal basis Ξ of $\mathbb{R}^n$ such that, up to passing through a subsequence on k, $\mathcal{L}^n(v_k^{-1}(S_{\Xi,\delta_h})) \leq \epsilon_h$ for every $k \geq \phi(h)$.*

Proof. We claim that for every natural number $N \geq n$, for every $j \in \{0, 1, \dots, n-1\}$, for every $\varepsilon > 0$, and for every open set $U \subset SO(n)$ there exist $\delta > 0$ and a family of orthonormal bases $\mathcal{K} := \{\Xi_1, \dots, \Xi_N\} \subseteq U$, such that, up to subsequences on k, we have

$$\mathcal{L}^n\Big(v_k^{-1}\Big(\{x \in \bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,\delta} : \mathcal{Z} \in \mathcal{K}_{n-j}\}\Big)\Big) \leq \varepsilon, \quad k = 1, 2, \dots, \tag{3.11}$$

$$\Xi \in U, \quad \Xi \in \mathcal{K}\,. \tag{3.12}$$

Clearly the pair $(\delta, \mathcal{K})$ depends on (N, j, ε), but we do not emphasize this fact. We proceed by induction on j. The case $j = 0$: given $N \in \mathbb{N}$, $\varepsilon > 0$, and any open set $U \subset SO(n)$, we can make use of Lemma 3.3 and Remark 3.4 to find N orthonormal bases $\mathcal{K} = \{\Xi_1, \dots, \Xi_N\} \subseteq U$ such that

$$\bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,0} = \emptyset \qquad \text{for } \mathcal{Z} \in \mathcal{K}_n\,.$$

Being $S_{\Xi,0}$ closed sets, there exists $\delta > 0$ such that

$$\bigcap_{\Xi \in \mathcal{Z}} S_{\Xi,\delta} = \emptyset \qquad \text{for } \mathcal{Z} \in \mathcal{K}_n.$$

Hence, (3.11) is satisfied with $j = 0$ and (3.12) holds true.

We want to prove the same for $0 < j \leq n-1$. For this purpose we fix a natural number $M \geq n$, a parameter $\varepsilon > 0$, and an open set $U \subset SO(n)$. By using the induction hypothesis, we may suppose that (3.11) and (3.12) hold true for $j-1$. This means that given $N \geq n$, $\tilde{\varepsilon} > 0$ (to be chosen later), we find $\delta > 0$ and orthonormal bases $\mathcal{K} = \{\Xi_1, \dots, \Xi_N\}$ such that (3.11) and (3.12) hold true for $j-1$. Choose $\mathcal{Z} \in \mathcal{K}_M$ and consider the following set

$$S^{n-j}_{\mathcal{Z},\delta} := \bigcup_{q \in \mathcal{Z}_{n-j}} \bigcap_{\Xi \in q} S_{\Xi,\delta} \,. \tag{3.13}$$

which is the union of all the possible $(n-j)$-intersections of sets of the form $S_{\Xi,\delta}$ for $\Xi \in \mathcal{Z}$.

We recall the following identity valid for any finite family of subsets of A, say $(B)_{l=1}^L$, which reads as

$$\mathcal{L}^n\Big(\bigcup_{l=1}^L B_l\Big) = \sum_{l=1}^L \mathcal{L}^n(B_l) - \sum_{l_1<l_2}^L \mathcal{L}^n(B_{l_1} \cap B_{l_2}) + \ldots + (-1)^{L-1}\mathcal{L}^n\Big(\bigcap_{l=1}^L B_l\Big) \,. \tag{3.14}$$

Now we partition $\mathcal{K}$ into N/M disjoint subsets (without loss of generality we may choose N to be an integer multiple of M) each of which belongs to $\mathcal{K}_M$. We call this partition $\mathcal{P}$. By construction, any l-intersection of sets of the form $S^{n-j}_{\mathcal{Z},\delta}$ with $\mathcal{Z} \in \mathcal{P}$ can be written as the union of $\binom{M}{n-j}^l$ sets each of which, thanks to the fact that (we use that $\mathcal{P}$ is a partition)

$$Z_1, Z_2 \in \mathcal{P} \Rightarrow Z_1 \cap Z_2 = \emptyset \,,$$

is the intersection of at least $n-(j-1)$ different sets of the form $S_{\Xi,\delta}$ with $\Xi \in \mathcal{K}$. Taking this last fact into account, if we replace the sets B_j with $v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})$ and $L = N/M$ in identity (3.14), we obtain

$$\mathcal{L}^n\Big(\bigcup_{\mathcal{Z}\in\mathcal{P}} v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})\Big) \geq \sum_{\mathcal{Z}\in\mathcal{P}} \mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})) - \sum_{l=2}^{N/M} \binom{M}{n-j}^l \tilde{\varepsilon}, \quad k = 1, 2, \ldots, \tag{3.15}$$

where we have used the inductive hypothesis (3.11) for $j-1$ to estimate the remaining terms in the right hand-side of (3.14).

Now suppose that for every $\mathcal{Z} \in \mathcal{K}_M$ it holds true for some k

$$\mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})) > \varepsilon \,, \tag{3.16}$$

then inequality (3.15) implies

$$\mathcal{L}^n\Big(\bigcup_{\mathcal{Z}\in\mathcal{P}} v_k^{-1}(S_{\mathcal{Z},\delta})\Big) > \frac{N}{M}\varepsilon - \sum_{l=2}^{N/M} \binom{M}{n-j}^l \tilde{\varepsilon} \,. \tag{3.17}$$

Therefore, if we choose N sufficiently large in such a way that

$$\frac{N}{M}\varepsilon \geq 2\mathcal{L}^n(A) \,,$$

and $\tilde{\varepsilon} > 0$ such that

$$\sum_{l=2}^{N/M} \binom{M}{n-j}^l \tilde{\varepsilon} < \mathcal{L}^n(A) \,,$$

then (3.17) implies that for every k there exists $\mathcal{Z}^k \in \mathcal{P}$ for which (3.16) does not hold, i.e.,

$$\mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z}^k,\delta})) \leq \varepsilon, \quad k = 1, 2, , \ldots ,$$

where we have used that B_k, the domain of v_k, is contained in A. Being $\mathcal{P}$ a finite family, we may suppose that, up to subsequences on k, we find a common $\mathcal{Z} \in \mathcal{P}$ for which

$$\mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})) \leq \varepsilon, \quad k = 1, 2, , \ldots . \tag{3.18}$$

Taking into account the definition of $S^{n-j}_{\mathcal{Z},\delta}$ (3.13), formula (3.18) gives our claim for j. Finally, by induction, this implies the validity of our claim for every $j \in \{0,\dots,n\}$.

Now we prove the lemma. For $j = n-1$ the claim says in particular that we find an orthonormal basis Ξ_0 and $\delta_0 > 0$ such that, up to pass to a subsequence on k, we have

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_0,\delta_0})) \le \epsilon_0, \quad k = 1,2,\dots .$$

Notice that by using a continuity argument, we find a neighborhood U_0 of Ξ_0 in $SO(n)$ such that

$$S_{\Xi,\delta_0/2} \Subset S_{\Xi_0,\delta}, \quad \Xi \in U_0 .$$

By applying again the claim we find an orthonormal basis $\Xi_1 \in U_0$ and $\tilde\delta_1 > 0$ such that, up to pass to a further subsequence on k, we have

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_1,\tilde\delta_1})) \le \epsilon_1, \quad k = 1,2,\dots .$$

Hence if we set $\delta_1 := \min\{\tilde\delta_1, \delta_0/2\}$ we obtain as well

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_1,\delta_1})) \le \epsilon_1, \quad k = 1,2,\dots,$$
$$S_{\Xi_1,\delta_1} \Subset S_{\Xi_0,\delta_0} .$$

Proceeding again by induction, we find for every $h = 1,2,\dots$ an orthonormal basis Ξ_h, $\delta_h > 0$, and a subsequence $(k^h_\ell)_\ell$, such that

$$\mathcal{L}^n(v_{k^h_\ell}^{-1}(S_{\Xi_h,\delta_h})) \le \epsilon_h, \quad \ell = 1,2,\dots,$$
$$S_{\Xi_h,\delta_h} \Subset S_{\Xi_{h-1},\delta_{h-1}} ,$$
$$(k^h_\ell)_\ell \subset (k^{h-1}_\ell)_\ell .$$

If we denote with abuse of notation the diagonal sequence $(k^h_h)_h$ simply as k, then we can find a map $\phi\colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_h,\delta_h})) \le \epsilon_h, \quad k \ge \phi(h) \tag{3.19}$$
$$S_{\Xi_h,\delta_h} \Subset S_{\Xi_{h-1},\delta_{h-1}} . \tag{3.20}$$

Being the family $(S_{\Xi_h,0})_h$ made of compact subsets of $\mathbb{S}^{n-1}$, then it is relatively compact with respect to the Hausdorff distance. This means that, up to a subsequence on h, we find an orthonormal basis Ξ such that

$$\lim_{h\to\infty} \mathrm{dist}_{\mathcal{H}}(S_{\Xi_h,0}, S_{\Xi,0}) = 0 .$$

By using (3.20) and the fact that S_{Ξ_h,δ_h} are relatively open subsets of $\mathbb{S}^{n-1}$, this last convergence tells us that for every h the compact inclusion $S_{\Xi,0} \Subset S_{\Xi_h,\delta_h}$ holds true. But this implies that up to defining suitable $\delta'_h > 0$ with $\delta'_h \le \delta_h$, we can write

$$S_{\Xi,\delta'_h} \Subset S_{\Xi_h,\delta_h}, \quad h \in \mathbb{N}.$$

Finally, with abuse of notation we set $\delta_h := \delta'_h$ for every h. Then (3.19) implies

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi,\delta_h})) \le \epsilon_h, \quad k \ge \phi(h), \quad h \in \mathbb{N}.$$

This gives the desired result. □

Remark 3.6. Given $U \subset \mathbb{R}^n$, $u \in GBD(U)$, and $\sigma \ge 1$, we have that

$$\mathcal{H}^{n-1}(J^\sigma_u) \le 4n\hat\mu_u(U) . \tag{3.21}$$

Indeed, given $\epsilon > 0$, one can consider a partition of $\mathbb{S}^{n-1}$ into a finite family of measurable sets $\{S_1,\dots,S_M\}$ such that for every $m = 1,\dots,M$ there exists an orthonormal basis $\Xi_m = \{\xi^m_1,\dots,\xi^m_n\}$ with $\xi \cdot \xi^m_i \ge 1/4$ for every $\xi \in S_m$ and for every

$i, j \in \{1, \dots, n\}$ and $m \in \{1, \dots, M\}$. Consider then the partition of J_u^σ given by $\{B_1, \dots, B_M\}$ where $B_m := \{x \in J_u^\sigma : [u(x)]/|[u(x)]| \in S_m\}$. We then have

$$\begin{aligned}\mathcal{H}^{n-1}(J_u^\sigma) &\le \sum_{m=1}^M \sum_{\xi \in \Xi_m} \int_{B_m} |\nu_u \cdot \xi| \, \mathrm{d}\mathcal{H}^{n-1} = \sum_{m=1}^M \sum_{\xi \in \Xi_m} \int_{\Pi_\xi} \mathcal{H}^0((B_m)^\xi_y) \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \sum_{m=1}^M \sum_{\xi \in \Xi_m} \int_{\Pi_\xi} \mathcal{H}^0(J^1_{4\hat{u}^\xi_y} \cap (B_m)^\xi_y) \, \mathrm{d}\mathcal{H}^{n-1}(y) = \sum_{m=1}^M \sum_{\xi \in \Xi_m} \hat{\mu}^\xi_{4u}(B_m) \\ &\le n \sum_{m=1}^M \hat{\mu}_{4u}(B_m) \le n\hat{\mu}_{4u}(U) \le 4n\mu_u(U),\end{aligned}$$

where we have used that $|[4\hat{u}^\xi_y](t)| \ge 1$ for every $t \in J_{4\hat{u}^\xi_y} \cap (B_m)^\xi_y$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ with $\xi \in \Xi_m$.

Remark 3.7. Let $U \subset \mathbb{R}^n$ and $u \in GBD(U)$. Given $\xi \in \mathbb{S}^{n-1}$ and $\sigma > 1$ if we introduce the map $\hat{\mu}^\xi_\sigma : \mathcal{B}(U) \to \overline{\mathbb{R}}$ as

$$\hat{\mu}^\xi_\sigma(B) := \int_{\Pi^\xi} |D\hat{u}^\xi_y|(B^\xi_y \setminus J^\sigma_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^\sigma_{\hat{u}^\xi_y}) \, \mathrm{d}\mathcal{H}^{n-1}(y), \quad B \in \mathcal{B}(U), \tag{3.22}$$

then we have $\hat{\mu}^\xi_\sigma \in \mathcal{M}^+_b(U)$. More precisely, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$\begin{aligned}&|D\hat{u}^\xi_y|(B \setminus J^\sigma_{\hat{u}^\xi_y}) + \mathcal{H}^0(B \cap J^\sigma_{\hat{u}^\xi_y}) \\ &\le |D\hat{u}^\xi_y|(B \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B \cap J^1_{\hat{u}^\xi_y}) + (\sigma - 1)\mathcal{H}^0(B \cap (J^1_{\hat{u}^\xi_y} \setminus J^\sigma_{\hat{u}^\xi_y})), \quad B \in \mathcal{B}(U^\xi_y),\end{aligned}$$

(notice that for $\mathcal{H}^{n-1}$-a.e. y the right hand side is a finite measure thanks to Remark 3.6). By using the inclusion $J^1_{\hat{v}^\xi_y} \subset (J^1_v)^\xi_y$, valid for every $v \in GBD(U)$ for every $\xi \in \mathbb{S}^{n-1}$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, we deduce

$$\hat{\mu}^\xi_\sigma(B) \le \hat{\mu}^\xi(B) + (\sigma - 1) \int_{B \cap J^1_u} |\nu_u \cdot \xi| \, \mathrm{d}\mathcal{H}^{n-1}, \quad B \in \mathcal{B}(U). \tag{3.23}$$

Finally, Remark 3.6 and the definition of $\hat{\mu}^\xi$ (see [6, Definition 4.10]) imply that the right-hand side of (3.23) is a finite measure, and so is $\hat{\mu}^\xi_\sigma$.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\tau(t) := \arctan(t)$. We claim that for every $i \in \{1, \dots, n\}$ the family $(\tau(u_k \cdot e_i))_k$ is relatively compact in $L^1(U)$, where $\{e_i\}_{i=1}^n$ denotes a suitable orthonormal basis of $\mathbb{R}^n$. Now given $\epsilon_h \searrow 0$, by using Lemma 3.5, there exists $\delta_h \searrow 0$ such that if we define $B_k := \{|u_k| \neq 0\}$ and $v_k : B_k \to \mathbb{S}^{n-1}$ as $v_k := u_k/|u_k|$, then

$$\mathcal{L}^n\big(v_k^{-1}(S_{\Xi,\delta_h})\big) \le \epsilon_h \qquad \text{for every } k \ge \phi(h)$$

for a suitable orthonormal basis Ξ and a suitable map $\phi : \mathbb{N} \to \mathbb{N}$.

In order to simplify the notation, let us denote $\Xi = \{e_1, \dots, e_n\}$. Fix $i \in \{1, \dots, n\}$ and set $\xi^t_j := \frac{\sqrt{t}}{\sqrt{t+t^2}} e_i + \frac{t}{\sqrt{t+t^2}} e_j \in \mathbb{S}^{n-1}$ for every $j \neq i$ and $t > 0$. Notice that

$$|\xi^t_j - e_i| \le \sqrt{2t} \qquad \text{and} \qquad \left| \frac{\xi^t_j - e_i}{|\xi^t_j - e_i|} - e_j \right| \le \sqrt{2t}\,. \tag{3.24}$$

We define $U_t := \{x \in U : \operatorname{dist}(\partial U, x) > t\}$. Since we want to apply Fréchet-Kolmogorov Theorem, we have to estimate for $x \in U_t$

$$\begin{aligned}|\tau(u_k(x+te_j)\cdot e_i) &- \tau(u_k(x)\cdot e_i)| \\ &\le |\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x+te_j)\cdot \xi^t_j)| \\ &\quad + |\tau(u_k(x+te_j)\cdot \xi^t_j) - \tau(u_k(x-\sqrt{t}e_i)\cdot \xi^t_j)| \\ &\quad + |\tau(u_k(x-\sqrt{t}e_i)\cdot \xi^t_j) - \tau(u_k(x-\sqrt{t}e_i)\cdot e_i)| \\ &\quad + |\tau(u_k(x-\sqrt{t}e_i)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,.\end{aligned}$$

Now notice that by definition of S_{Ξ,δ_h} (see Definition 3.1), there exists a positive constant $c = c(\delta_h)$ such that for every $x \in U \setminus v_k^{-1}(S_{\Xi,\delta_h/2})$ and every $i, j \in \{1,\dots,n\}$

$$|u_k(x)\cdot e_i| \ge c(\delta_h)\,|u_k(x)\cdot e_j| \qquad \text{for every } k \text{ and } h\,. \tag{3.25}$$

Moreover, by taking into account (3.24), we deduce the existence of a dimensional parameter $\bar t > 0$ such that

$$|z\cdot\xi^t_j|^2 \ge 2^{-1}|z\cdot e_i|^2 \qquad t\le\bar t,\ z\in\mathbb{R}^n,\ i,j\in\{1,\dots,n\} \tag{3.26}$$

$$\left|z\cdot\frac{\xi^t_j - e_i}{|\xi^t_j - e_i|}\right| \le 2|z\cdot e_j| \qquad t\le\bar t,\ z\in\mathbb{R}^n,\ i,j\in\{1,\dots,n\}. \tag{3.27}$$

For every $t \le \bar t$, if $x \in U_t$ and $x \notin v_k^{-1}(S_{\Xi,\delta_h/2}) - te_j$, by using (3.24) and (3.25)-(3.27), we can write

$$\begin{aligned}|\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x+te_j)\cdot\xi^t_j)| &= \left|\int_{u_k(x+te_j)\cdot e_i}^{u_k(x+te_j)\cdot\xi^t_j}\frac{\mathrm{d}s}{1+s^2}\right| \\ \le \max\left\{\frac{\sqrt{2t}}{1+|u_k(x+te_j)\cdot e_i|^2},\frac{\sqrt{2t}}{1+|u_k(x+te_j)\cdot\xi^t_j|^2}\right\}&\left|u_k(x+te_j)\cdot\frac{\xi^t_j-e_i}{|\xi^t_j-e_i|}\right| \\ \le \max\left\{\frac{\sqrt{2t}}{1+|u_k(x+te_j)\cdot e_i|^2},\frac{\sqrt{2t}}{1+2^{-1}|u_k(x+te_j)\cdot e_i|^2}\right\}&\left|u_k(x+te_j)\cdot\frac{\xi^t_j-e_i}{|\xi^t_j-e_i|}\right| \\ \le \frac{2\sqrt{2t}}{1+2^{-1}|u_k(x+te_j)\cdot e_i|^2}|u_k(x+te_j)\cdot e_j| \le \frac{2\sqrt{t}}{c(\delta_h)}&\end{aligned} \tag{3.28}$$

and analogously if $x \in U_t$ and $x \notin v_k^{-1}(S_{\Xi,\delta_h/2}) + \sqrt{t}e_i$

$$|\tau(u_k(x-\sqrt{t}e_i)\cdot\xi^t_j) - \tau(u_k(x-\sqrt{t}e_i)\cdot e_i)| \le \frac{2\sqrt{t}}{c(\delta_h)}\,. \tag{3.29}$$

Hence, from (3.28) and (3.29) we infer that for every $t \le \bar t$

$$\int_{U_t}|\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x+te_j)\cdot\xi^t_j)|\,\mathrm{d}x \le |U|\frac{2\sqrt{t}}{c(\delta_h)} + \pi\epsilon_h\,,$$

and

$$\int_{U_t}|\tau(u_k(x-\sqrt{t}e_i)\cdot e_i) - \tau(u_k(x-\sqrt{t}e_i)\cdot\xi^t_j)|\,\mathrm{d}x \le |U|\frac{2\sqrt{t}}{c(\delta_h)} + \pi\epsilon_h\,.$$

Moreover, setting $s_t := \sqrt{t+t^2}$ we can write

$$\begin{aligned}\int_{U_t}&|\tau(u_k(x+te_j)\cdot\xi^t_j) - \tau(u_k(x-\sqrt{t}e_i)\cdot\xi^t_j)|\,\mathrm{d}x \\ &= \int_{U_t}|\tau(u_k(x-\sqrt{t}e_i+s_t\xi^t_j)\cdot\xi^t_j) - \tau(u_k(x-\sqrt{t}e_i)\cdot\xi^t_j)|\,\mathrm{d}x\end{aligned} \tag{3.30}$$

$$= \int_{U_t+\sqrt{t}e_i} |\tau(u_k(x+s_t\xi_j^t)\cdot\xi_j^t) - \tau(u_k(x)\cdot\xi_j^t)|\,\mathrm{d}x$$
$$\le \int_{\Pi_{\xi_j^t}} \Big(\int_{(U_t+\sqrt{t}e_i)_y^{\xi_j^t}} |D\tau(\hat{u}_y^{\xi_j^t})|((s,s+s_t))\,\mathrm{d}s\Big)\mathrm{d}\mathcal{H}^{n-1}(y)\,.$$

By a mollification argument, we have that

$$\int_{\Pi_{\xi_j^t}} \Big(\int_{(U_t+\sqrt{t}e_i)_y^{\xi_j^t}} |D\tau(\hat{u}_y^{\xi_j^t})|((s,s+s_t))\,\mathrm{d}s\Big)\mathrm{d}\mathcal{H}^{n-1}(y)$$
$$= \int_{\Pi_{\xi_j^t}} \Big(\int_0^{s_t} |D\tau(\hat{u}_y^{\xi_j^t})|((U_t+\sqrt{t}e_i)_y^{\xi_j^t}+\lambda)\,\mathrm{d}\lambda\Big)\mathrm{d}\mathcal{H}^{n-1}(y)\,,$$

so that we obtain from (3.30) that

$$\int_{U_t} |\tau(u_k(x+te_j)\cdot\xi_j^t) - \tau(u_k(x-\sqrt{t}e_i)\cdot\xi_j^t)|\,\mathrm{d}x$$
$$\le \int_{\Pi_{\xi_j^t}} \Big(\int_0^{s_t} |D\tau(\hat{u}_y^{\xi_j^t})|((U_t+\sqrt{t}e_i)_y^{\xi_j^t}+\lambda)\,\mathrm{d}\lambda\Big)\mathrm{d}\mathcal{H}^{n-1}(y)$$
$$\le \int_0^{s_t} \Big(\int_{\Pi_{\xi_j^t}} |D\tau(\hat{u}_y^{\xi_j^t})|(U_y^{\xi_j^t})\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)\mathrm{d}\lambda \le \pi s_t\hat{\mu}_{u_k}(U)\,.$$

Analogously,

$$\int_{U_t} |\tau(u_k(x-\sqrt{t}e_i)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,\mathrm{d}x \le \pi\sqrt{t}\hat{\mu}_{u_k}(U)\,.$$

Summarizing, we have shown that if t_h is such that $t_h \in (0,\bar{t}]$ and

$$|U|\frac{2\sqrt{t_h}}{c(\delta_h)} \le \epsilon_h \qquad \text{and} \qquad \pi s_{t_h}\hat{\mu}_{u_k}(U) \le \epsilon_h\,,$$

then for every $t \le t_h$ we have for every $e_j \in \Xi$

$$\int_{U_t} |\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,\mathrm{d}x \le 10\epsilon_h \qquad \text{for every } k \ge \phi(h)\,.$$

As a consequence, there exists a positive constant $L = L(n)$ such that

$$\int_{U_t} |\tau(u_k(x+t\xi)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,\mathrm{d}x \le L(n)\epsilon_h \qquad \xi\in\mathbb{S}^{n-1},\quad k\ge\phi(h),\quad t\le t_h.$$

Since the index i chosen at the beginning was arbitrary, this means also that if we consider the diffeomorphism $\psi\colon \mathbb{R}^n \to (-\pi/2,\pi/2)^n$ defined by $\psi(x) := (\tau(x_1),\dots,\tau(x_n))$, then

$$\int_{U_t} |\psi(u_k(x+t\xi)) - \psi(u_k(x))|\,\mathrm{d}x \le L'(n)\epsilon_h, \qquad \xi\in\mathbb{S}^{n-1},\quad k\ge\phi(h),\quad t\le t_h.$$

By Fréchet-Kolmogorov Theorem, this last inequality implies that the sequence $\psi(u_k)$ is relatively compact in $L^1(U;\mathbb{R}^n)$. Hence, we can pass to another subsequence, still denoted by $\psi(u_k)$, such that $\psi(u_k)\to v$ as $k\to\infty$ strongly in $L^1(U;\mathbb{R}^n)$. By eventually passing through another subsequence, we may suppose $\psi(u_k(x))\to v(x)$ a.e. in U as $k\to\infty$. As a consequence, there exists a measurable $u\colon U\to\overline{\mathbb{R}}$ such that $u_k(x)\to u(x)$ as $k\to\infty$ a.e. in $U\setminus\{x\in U : v(x)\in\partial(-\frac{\pi}{2},\frac{\pi}{2})^n\}$. Moreover, $|u_k(x)|\to+\infty$ if and only if for at least one index i, $u_k(x)\cdot e_i\to\pm\infty$ (clearly $\tau(u\cdot e_i)=v_i$) or equivalently if and only if $x \in \{x \in U : v(x) \in \partial(-\frac{\pi}{2}, \frac{\pi}{2})^n\}$. Thus, we obtain that $u_k \to u$ a.e. in $U \setminus A$ as $k \to \infty$.

To show that $A := \{x \in U : |u_k(x)| \to +\infty\}$ has finite perimeter the argument follows that in [4]. We give a sketch of the proof.

It is easy to check that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ it holds true

$$x \in A \quad \text{if and only if} \quad \lim_{k\to\infty} \tau(u_k(x)\cdot\xi) = \pm\frac{\pi}{2}\,, \quad \text{for a.e. } x \in U\,. \tag{3.31}$$

Now fix $\sigma \geq 1$. First of all using also (3.31) we can follow a standard measure theoretic argument which shows that we can extract a subsequence, still denoted as $(u_k)_k$, such that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ it holds true

$$\tau((\hat{u}_k)^\xi_y) \to v^\xi_y := \begin{cases} \tau(\hat{u}^\xi_y) & \text{on } U^\xi_y \setminus A^\xi_y \\ \pm\frac{\pi}{2} & \text{on } A^\xi_y, \end{cases} \quad \text{in } L^1(U^\xi_y)\,. \tag{3.32}$$

Fix $\epsilon > 0$. By Fatou Lemma and Remarks 3.6 and 3.7 we estimate

$$\begin{aligned}
&\int_{\Pi_\xi} \liminf_{k\to\infty} [\epsilon\,|D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^\sigma_{(\hat{u}_k)^\xi_y})]\,\mathrm{d}\mathcal{H}^{n-1}(y) \qquad (3.33)\\
&\leq \int_{\Pi_\xi} \liminf_{k\to\infty} [\epsilon\,|D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap (J^\sigma_{u_k})^\xi_y)]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\leq \limsup_{k\to\infty} \Big(\epsilon\,\hat{\mu}^\xi_{u_k}(U) + \epsilon(\sigma-1)\int_{U\cap J^1_{u_k}} |\nu_{u_k}\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}\Big) + \liminf_{k\to\infty}\int_{U\cap J^\sigma_{u_k}} |\nu_{u_k}\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}\\
&\leq \epsilon \sup_{k\in\mathbb{N}} (1+4n(\sigma-1))\hat{\mu}_{u_k}(U) + \liminf_{k\to\infty}\int_{U\cap J^\sigma_{u_k}} |\nu_{u_k}\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1} < +\infty\,.
\end{aligned}$$

For $\mathcal{H}^{n-1}$-a.e. y we can thus consider a subsequence depending on y but still denoted by $(u_k)_k$ such that

$$\sup_{k\in\mathbb{N}} \epsilon\,|D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^\sigma_{(\hat{u}_k)^\xi_y}) < +\infty\,. \tag{3.34}$$

Now we study the behavior of a sequence of one dimensional functions satisfying (3.34). Let $(a,b) \subset \mathbb{R}$ be a non-empty open interval and suppose that $(f_k)_k$ is a sequence in $BV_{\mathrm{loc}}((a,b))$ satisfying

$$\sup_{k\in\mathbb{N}} |Df_k|((a,b) \setminus J^\sigma_{f_k}) + \mathcal{H}^0(J^\sigma_{f_k}) < \infty. \tag{3.35}$$

We write $f_k = f^1_k + f^2_k$ for $f^1_k, f^2_k \colon (a,b) \to \mathbb{R}$ defined as

$$f^1_k(t) := Df_k((a,t) \setminus J^\sigma_{f_k}) \quad \text{and} \quad f^2_k(t) := f_k(a) + Df_k((a,t) \cap J^\sigma_{f_k}).$$

We study the convergence of f^1_k and f^2_k separately.

Inequality (3.35) tells us that up to extract a further not relabelled subsequence

$$f^1_k \to f^1 \text{ pointwise a.e. for some } f^1 \in BV((a,b)) \text{ as } k \to \infty. \tag{3.36}$$

As for $(f^2_k)_k$, by inequality (3.35) we may suppose that, up to extract a further not relabelled subsequence, there exists a finite set $J \subset [a,b]$ such that

$$\mathcal{H}^0(J) \leq \sup_{k\in\mathbb{N}} \mathcal{H}^0(J^\sigma_{f_k})\,, \tag{3.37}$$

$$J^\sigma_{f_k} \to J \qquad \text{in Hausdorff distance as } k \to \infty. \tag{3.38}$$

Then, (3.37)–(3.38) together with the fact that by construction f_k^2 is a piecewise constant function allows us to deduce that any pointwise limit function f^2 for $(f_k^2)_k$ must be of the form

$$f^2(t) = \sum_{l=1}^{M} \alpha_l \mathbb{1}_{(a_l,a_{l+1})}(t) \qquad \text{for } t \in (a,b),$$

for a suitable $M \leq \mathcal{H}^0(J \cap (a,b)) + 1$, for suitable $\alpha_l \in \mathbb{R} \cup \{\pm\infty\}$ with $\alpha_l \neq \alpha_{l+1}$, and for suitable $a_l \in J$ with $a_l < a_{l+1}$ and $a_1 = a$, $a_{\mathcal{H}^0(J\cap(a,b))+2} = b$. Up to extract a further not relabelled subsequence we may suppose $f_k^2 \to f^2$ pointwise a.e.. Now if $\alpha_l \in \{\pm\infty\}$ and $l \neq 1$ and $l \neq \mathcal{H}^0(J \cap (a,b)) + 1$, we set

$$T_{l,k} := \{t \in J^\sigma_{f_k^2} : |t - a_l| \leq 1/2 \min_{t_1,t_2\in J} |t_1 - t_2|\},$$
$$T_{l+1,k} := \{t \in J^\sigma_{f_k^2} : |t - a_{l+1}| \leq 1/2 \min_{t_1,t_2\in J} |t_1 - t_2|\},$$

while if $l = 1$ we set

$$T_{l,k} := \{t \in J^\sigma_{f_k^2} : |t - a_{l+1}| \leq 1/2 \min_{t_1,t_2\in J} |t_1 - t_2|\},$$

and if $l = M$ we set

$$T_{l,k} := \{t \in J^\sigma_{f_k^2} : |t - a_l| \leq 1/2 \min_{t_1,t_2\in J} |t_1 - t_2|\}.$$

By (3.38) we have $T_{l,k} \neq \emptyset$ for every but sufficiently large k and thanks to the definition of $T_{l,k}$ any sequence $(t_{l,k})_k$ with $t_{l,k} \in T_{l,k}$ is such that $t_{l,k} \to \alpha_l$ as $k \to \infty$. We claim that for every $l \in \{1, \dots, M\}$ there exists one of such sequences $(t_{l,k})_k$ such that

$$\lim_{k\to\infty} |[f_k^2(t_{l,k})]| = +\infty. \tag{3.39}$$

Suppose by contradiction that there exists l and a subsequence k_j such that

$$\sup_{j\in\mathbb{N}} \max_{t\in T_{l,k_j}} |[f_{k_j}^2(t)]| < +\infty.$$

Then, we are in the following situation: we choose one of the endpoints a_l or a_{l+1}, for example a_l, (in the case $l = 1$ we choose a_{l+1} and in the case $l = M$ we choose a_l) and the sequence $v_j := f_{k_j}^2 \llcorner (a_l - \frac{1}{2}\min_{t_1,t_2\in J} |t_1 - t_2|, a_l + \frac{1}{2}\min_{t_1,t_2\in J} |t_1 - t_2|)$ satisfies

$$\begin{aligned}
&v_j \text{ is piecewise constant,}\\
&J_{v_j} = T_{l,k_j} \quad \text{and} \quad J_{v_j} \to a_l \text{ in Hausdorff distance as } j \to \infty,\\
&\sup_{j\in\mathbb{N}} \mathcal{H}^0(T_{l,k_j}) < +\infty, \qquad \sup_{j\in\mathbb{N}} \max_{t\in J_{v_j}} |[v_j](t)| < +\infty.
\end{aligned}$$

It is easy to see that the previous conditions are in contradiction with the fact that $f^2 \llcorner (a_l - 1/2\min_{t_1,t_2\in J} |t_1 - t_2|, a_l + 1/2\min_{t_1,t_2\in J} |t_1 - t_2|)$, i.e. the pointwise limit of v_j, is such that f^2 has a non finite jump point at a_l. This proves our claim. Our claim implies in particular that, being $(f_k^1)_k$ equibounded, then the sequence $t_{l,k}$ satisfying (3.39) is actually contained for every but sufficiently large k in $J^\sigma_{f_k}$ (roughly speaking the jumps of f_k^1 cannot compensate a non-bounded sequence of jumps of f_k^2). Clearly, being the interval $\{t : |t - a_l| < \frac{1}{2}\min_{t_1,t_2\in J} |t_1 - t_2|\}$ pairwise disjoints for $l \in \{2, \dots M\}$ (we are avoiding the end points a and b), then we have actually proved the following lower semi-continuity property

$$\mathcal{H}^0(\partial^*\{f = \pm\infty\}) = \mathcal{H}^0(\{t \in (a,b) \cap J_f : |[f(t)]| = \infty\}) \leq \liminf_{k\to\infty} \mathcal{H}^0(J^\sigma_{f_k}), \tag{3.40}$$

where $f := f_1 + f_2$. Notice that the set J_f is well defined since f is the sum of a (bounded) BV function and a piecewise constant function which might assume values $\pm\infty$, but jumps only at finitely many points.

Having this in mind we can come back to our original problem. Fix $\xi \in \mathbb{S}^{n-1}$ satisfying (3.32). Given $y \in \Pi^\xi$ for which (3.32) and (3.34) hold true we can pass through a not relabelled subsequence (depending on y) for which the following liminf

$$\liminf_{k\to\infty} [\epsilon\, |D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^\sigma_{(\hat{u}_k)^\xi_y})]$$

is actually a limit. Passing through a further not relabelled subsequence, we may also suppose that (3.40) holds true in each connected component of U^ξ_y, i.e.

$$\mathcal{H}^0(\partial^*\{v^\xi_y = \pm\pi/2\}) \le \liminf_{k\to\infty} \mathcal{H}^0(J^\sigma_{(\hat{u}_k)^\xi_y}) .$$

Notice that $|v^\xi_y| < \pi/2$ a.e. on $U^\xi_y \setminus A^\xi_y$, hence $\{v^\xi_y = \pm\pi/2\} = A^\xi_y$ a.e. and so $\partial^*\{v^\xi_y = \pm\pi/2\} = \partial^* A^\xi_y$. In particular

$$\mathcal{H}^0(\partial^* A^\xi_y) \le \liminf_{k\to\infty} \mathcal{H}^0(J^\sigma_{(\hat{u}_k)^\xi_y}) \tag{3.41}$$

Therefore, by passing through suitable subsequences, each depending on y, when computing the liminf inside the left-hand side integral of (3.33) and by using (3.41) we infer

$$\int_{\Pi^\xi} \mathcal{H}^0(\partial^* A^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \tag{3.42}$$
$$\le \epsilon \sup_{k\in\mathbb{N}} (1 + 4n(\sigma-1))\hat{\mu}_{u_k}(U) + \liminf_{k\to\infty} \int_{U\cap J^\sigma_{u_k}} |\nu_{u_k}\cdot\xi|\, \mathrm{d}\mathcal{H}^{n-1} .$$

The arbitrariness of ξ implies that (3.42) holds for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$. Hence, we deduce that A has finite perimeter in U. In addition, by taking the integral on $\mathbb{S}^{n-1}$ on both sides of (3.42) we infer

$$\alpha_n \mathcal{H}^{n-1}(\partial^* A) \le \epsilon n\omega_n (1 + 4n(\sigma-1)) \sup_{k\in\mathbb{N}} \hat{\mu}_{u_k}(U) + \alpha_n \liminf_{k\to\infty} \mathcal{H}^{n-1}(J^\sigma_{u_k}) ,$$

where $\alpha_n := \int_{\mathbb{S}^{n-1}} |\nu\cdot\xi|$. Moreover, the arbitrariness of $\epsilon > 0$ tells us

$$\mathcal{H}^{n-1}(\partial^* A) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(J^\sigma_{u_k}) .$$

Finally, by the arbitrariness of $\sigma \ge 1$ and by the fact that $J^{\sigma_1} \subset J^{\sigma_2}$ for $\sigma_1 \ge \sigma_2$ we conclude (1.2).

In order to show that u can be extended to the whole of U as a function in $GBD(U)$, we define the sequence of $GBD(U)$ functions by

$$\tilde{u}_k(x) := \begin{cases} u_k(x) & \text{if } x \in U \setminus A , \\ 0 & \text{if } x \in A . \end{cases}$$

Clearly, if we define v as

$$v(x) := \begin{cases} u(x) & \text{if } x \in U \setminus A , \\ 0 & \text{if } x \in A . \end{cases} \tag{3.43}$$

then we have $\tilde{u}_k \to v$ a.e. in U and

$$\sup_{k\in\mathbb{N}} \hat{\mu}_{\tilde{u}_k}(U) \le \sup_{k\in\mathbb{N}} \hat{\mu}_{u_k}(U) + \mathcal{H}^{n-1}(\partial^* A) < +\infty .$$

Therefore, by using the technique developed in [1, 6] we can conclude $v \in GBD(U)$. □

Remark 3.8. Under the additional assumption (1.3) with $u_k \in GSBD(U)$, we can obtain the further information $e(u_k)\mathbb{1}_{U\setminus A} \rightharpoonup e(u)$ in $L^1(U;\mathbb{M}^n_{sym})$ thanks to $e(\tilde{u}_k) \rightharpoonup e(u)$ in $L^1(U;\mathbb{M}^n_{sym})$ together with the fact $e(u_k)\mathbb{1}_{U\setminus A} = e(\tilde{u}_k)$ for every $k \in \mathbb{N}$. Moreover, (3.40) can be modified in the following way:

$$\mathcal{H}^0(J_f \cup \partial^*\{f = \pm\infty\}) \le \liminf_{k\to\infty} \mathcal{H}^0(J_{f_k})\,,$$

from which it is possible to deduce that

$$\mathcal{H}^{n-1}(J_u \cup \partial^* A) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k})\,.$$

Condition (1.3) would also imply that in (3.33) we actually control

$$\int_{\Pi_\xi} \liminf_{k\to\infty} \Big[\int_{U^\xi_y} \epsilon\, \phi(|(\dot{u}_k)^\xi_y(t)|)\, \mathrm{d}t + \mathcal{H}^0(U^\xi_y \cap J_{(\hat{u}_k)^\xi_y})\Big]\, \mathrm{d}\mathcal{H}^{n-1}(y) < +\infty\,,$$

where $(\dot{u}_k)^\xi_y$ denotes the absolutely continuous part of $D(\hat{u}_k)^\xi_y$. This in turns allows us to use the well known compactness result for SBV functions in one variable to deduce that the pointwise limit function f^1 in (3.36) belongs to $SBV((a,b))$. For this reason, the techniques of [1, 6] can be adapted to deduce $v \in GSBD(U)$ (see (3.43) for the definition of v). The convergence of $e(u_k)$ to $e(u)$ in $L^2(\Omega\setminus A;\mathbb{M}^n_{sym})$ follows instead by the arguments of [5, pp. 10–11].

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