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An Interior Regularity Property for the Solution to a Linear Elliptic System with Singular Coefficients in the Lower-Order Term

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Abstract: This paper deals with the interior higher differentiability of the solution u to the Dirichlet problem related to system $-\text{div}(A(x)Du) + B(x,u) = f$ on a bounded Lipschitz domain Ω in $\mathbb{R}^n$. The matrix $A(x)$ lies in the John and Nirenberg space BMO. The lower-order term $B(x,u)$ is controlled with respect to the spatial variable by a function $b(x)$ belonging to the Marcinkiewicz space $L^{n,\infty}$. The novelty here is the presence of a singular coefficient in the lower-order term.

Keywords: linear elliptic system; BMO space; Hodge decomposition

MSC: 30H35; 35B65; 47J20

1. Introduction

We investigate the higher integrability of the gradient of the weak solution of the elliptic system

$$-\text{div}(A(x)Du) + B(x,u) = f \tag{1}$$

in a bounded domain Ω of $\mathbb{R}^n$, $n > 2$ where $A(x) = (A_{ij}(x))$ is a symmetric, positive-definite matrix with measurable coefficients and f a vector field in $L^2(\Omega, \mathbb{R}^n)$.

Let $B : (x,u) \in \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory vector field such that

$$|B(x,u) - B(x,u')| \leqslant b(x)|u - u'| \tag{2}$$

$$\langle B(x,u) - B(x,u'), u - u' \rangle \geqslant 0 \tag{3}$$

$$B(x,0) = 0 \tag{4}$$

where $b(x)$ is a nonnegative real function belonging to the Marcinkiewicz space $L^{n,\infty}(\Omega)$.

The matrix is elliptic, i.e.,

$$\langle A(x)Y, Y \rangle \geqslant \|Y\|^2 \tag{5}$$

for every matrix $Y \in \mathbb{R}^{n\times n}$.

We emphasize that no extra differentiability can be obtained for solutions even if the data $f(x)$ and $b(x)$ are smooth, without any differentiability assumption on the operator $A(x)$.

Here, we assume that the entries of the matrix A lie in the John–Nirenberg space BMO and there exists a nonnegative function $K(x)$ belonging to $L^{n,\infty}(\Omega)$ such that

$$|A(x + he_i) - A(x)| \leqslant K(x)|h|, \quad i = 1, ..., n \tag{6}$$

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for a.e. $x \in \Omega$ and $h \in \mathbb{R}$ such that $x + he_i \in \Omega$.

Condition (6) states that the derivative $D_{x_i}A$ has the same integrability as $K(x)$.

The functions

$$A(x) = \left(\frac{e^{-|x|}}{\Lambda} - \Lambda \log |x| \right) \mathrm{Id},$$
$$K(x) = \frac{e^{-|x|}}{\Lambda} + \Lambda \frac{1}{|x|}, \tag{7}$$
$$b(x) = \frac{\lambda}{|x|}$$

where Λ and λ are positive constants, satisfying, for example, assumptions (2)–(6).

Early contributions on the second-order regularity of solutions of equations with discontinuity in the coefficients are due to Miranda [1,2] and consider coefficients in the Sobolev class $W^{1,n}$. Chiarenza, Frasca, Longo [3] and Chiarenza [4] established regularity results for equations in a nondivergence form with VMO coefficients.

In 2018, Giannetti and Moscariello [5] studied the Dirichlet problem for the second-order elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f$$

under the assumption that f is in L^2 and that the coefficients a_{ij} are measurable and bounded functions with the first derivatives in the Marcinkiewicz class $L^{n,\infty}$ and have a sufficiently small distance to L^∞. They proved the solvability of the problem in $W^{2,2} \cap W_0^{1,2^\star}$, where $2^\star = \frac{2n}{n-2}$. A higher integrability result for the gradient of the solution is proved when $f \in L^p$, $p > 2$.

We underline that the role of the weak L^n structure is crucial to ensure the higher integrability of Du as shown in example 1.2 of [5]. Reference [6] assumed the smallness of the BMO norm.

Reference [7] studied the Dirichlet problem for a uniformly elliptic equation of type (1) introducing a hypothesis that relates the coefficient of the lower-order term $b(x)$ with the right-hand side f. A regularizing effect on the solution of the Dirichlet problem in the case of uniformly elliptic equations follows (for more details, see [8,9]).

Reference [10] established a higher differentiability result also for an equation of the type in (1). A recent significant development is due to Stroffolini [11] who studied the Dirichlet problem for a linear system with coefficients in BMO and obtained regularity results for the minimizers of functionals of the Calculus of Variations applied to linear and nonlinear systems with a principal part as in (1).

In [12], Moscariello and Pascale established a higher differentiability result for this class of systems. They extended their result in [13] for some nonlinear elliptic systems with growth coefficients in BMO.

In the present paper, the novelty consists of studying local regularity properties of solutions to linear systems with the presence of a lower-order term as in (1).

Definition 1. *A vector field u in the Sobolev space $W_0^{1,2}(\Omega, \mathbb{R}^n)$ is a weak solution of the Dirichlet problem*

$$\begin{cases} -div(A(x)Du) + B(x,u) = f & in\, \Omega \\ u = 0 & on\, \partial\Omega \end{cases} \tag{8}$$

if it is verified that

$$\int_\Omega \langle A(x)Du(x), D\varphi(x) \rangle \, dx + \int_\Omega B(x,u)\varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx, \tag{9}$$

$\forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$.

It is useful to set

$$K(x) = K_1(x) + K_2(x) \tag{10}$$

with $K_1 \in L^\infty$ and $K_2 \in L^{n,\infty}$. Our main result is the following:

Theorem 1. *Let Ω be a Lipschitz domain. Let us assume (2)–(6). Then, the problem (8) admits a unique solution $u \in W_0^{1,2}(\Omega, \mathbb{R}^n)$. Moreover, there exists $\varepsilon > 0$ depending on n such that if*

$$\|K_2\|_{L^{n,\infty}} < \varepsilon, \tag{11}$$

then $u \in W_{loc}^{2,2}(\Omega, \mathbb{R}^n)$ and

$$\int_{B_R} |D^2 u|^2 \, dx \leqslant C \left[\int_{B_{2R}} \left(\frac{1}{R^2} + 1 \right) \left(|Du|^2 + |u|^2 + |f|^2 \right) dx \right]$$

for every ball $B_{2R} \subset\subset \Omega$ and for C depending on n, $\|K_2\|_{L^{n,\infty}}$, the BMO-norm of A and $\|b\|_{L^{n,\infty}}$.

Let us discuss condition (11). First of all, we underline that condition (11) does not imply that the norm of $K(x)$ is small in $L^{n,\infty}$. Indeed, by easy calculations (for more details, see [12]), if $K(x)$ is the function in the example (7), condition (11) reduces to $\Lambda < \varepsilon \omega_n^{-\frac{1}{n}}$.

On the other hand,

$$\|K\|_{L^{n,\infty}} \geqslant \frac{C}{\Lambda}$$

where $C = C(n)$, and so the norm of K will be large for the small ε.

As is known, L^∞ is not dense in $L^{n,\infty}$. Notice that condition (11) is equivalent to saying that there exists a function $K_1 \in L^\infty(\Omega)$ such that

$$\|K - K_1\|_{L^{n,\infty}} < \varepsilon.$$

The condition (11) measures how far $K(x)$ is from the bounded functions and it is satisfied whenever $K(x)$ lies in the Lorentz space $L^{n,q}$, $1 < q < \infty$. Finally, we point out that our Theorem applies also for $A(x) \in W^{1,n}$ since by embedding $W^{1,n} \subset BMO$ (see Theorem 2).

A similar condition occurs in the study of nonlinear elliptic and parabolic equations with conventional terms [14,15]. See also [16]. We point out that Theorem 1 is new even in the case that $A(x)$ is a bounded matrix.

2. Notations and Preliminary Results

2.1. BMO Spaces

Definition 2 ([17,18]). *Let Ω be a cube or the entire space $\mathbb{R}^n$. The $BMO(\Omega)$ space consists of all functions g that are integrable on every cube $Q \subset \Omega$ with sides parallel to those of Ω and satisfy*

$$\|g\|_\star = \sup_Q \left\{ \frac{1}{|Q|} \int_Q |g - g_Q| \, dx \right\} < \infty,$$

where $g_Q = \dfrac{1}{|Q|} \displaystyle\int_Q g(y) dy$ and $|Q|$ denotes the Lebesgue measure of Q.

The functional $\| \cdot \|_\star$ does not define a norm since it vanishes on constant functions. BMO becomes a Banach space if we identify functions which differ a.e. by a constant.

Therefore, the function $\|\cdot\|_\star$ is properly a norm on the quotient space of BMO functions modulo the space of constant functions on the domain considered.

Bounded functions are in BMO. On the other hand, BMO contains unbounded functions. An example of BMO functions is given by the following:

$$g(x) = \log|x|, \quad x \in B(0,1).$$

We also recall that L^∞ is not dense in BMO.

Another example of the BMO function is contained in [19].

Lemma 1. *Let $M\mu$ denote the Hardy–Littlewood maximal function of a Radon measure μ in $\mathbb{R}^n$, and suppose $M\mu(x)$ is finite and positive at some point x, then $\log(M\mu)$ belongs to $BMO(\mathbb{R}^n)$.*

Theorem 2 ([17])**.** *For any cube $Q \subset \mathbb{R}^n$, the following inclusion holds with continuous embedding:*

$$W^{1,n}(Q) \hookrightarrow BMO(Q). \tag{12}$$

2.2. Lorentz Spaces

Let Ω be a bounded domain in $\mathbb{R}^n$. Given $1 < p, q < \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions g defined on Ω for which the quantity

$$\|g\|^q_{L^{p,q}} \;=\; p\int_0^\infty |\Omega_t(g)|^{\frac{q}{p}} t^{q-1}\, dt$$

is finite, where $\Omega_t(g) = \{x \in \Omega : |g(x)| > t\}$ and $|\Omega_t|$ is the Lebesgue measure of Ω_t. Remark that $\|\cdot\|_{L^{p,q}}$ is equivalent to a norm. Endowed with this norm, $L^{p,q}$ becomes a Banach space.

For $p = q$, the Lorentz space $L^{p,p}$ coincides with the Lebesgue space L^p. For $q = \infty$, the class $L^{p,\infty}$ consists of all measurable functions g defined on Ω such that

$$\|g\|^p_{L^{p,\infty}} = \sup_{t>0} t^p|\Omega_t(g)| < \infty.$$

$L^{p,\infty}$ coincides with the Marcinkiewicz class, weak L^p. The following inclusions for Lorentz spaces hold

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega),$$

whenever $1 \leqslant q < p < r \leqslant \infty$.

Let us recall the following *Hölder inequality in Lorentz spaces.*

Theorem 3 ([20])**.** *If $0 < p_1, p_2, p < \infty$ and $0 < q_1, q_2, q \leqslant \infty$ obey $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*

$$\|fg\|_{L^{p,q}} \leqslant \|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} \tag{13}$$

whenever the right-hand side norms are finite.

The *Sobolev embedding theorem in Lorentz spaces will be fundamental.*

Theorem 4 ([21])**.** *Let us assume that $1 < p < n$, $1 \leqslant q \leqslant p$, and then any function $u \in W^{1,1}_0(\Omega)$ such that $|\nabla u| \in L^{p,q}(\Omega)$ actually belongs to $L^{p^\star,q}(\Omega)$ and*

$$\|u\|_{L^{p^\star,q}} \leqslant C_{n,p}\|\nabla u\|_{L^{p,q}}, \tag{14}$$

where $p^\star = \frac{np}{n-p}$ and $C_{n.p} = \omega_n^{-\frac{1}{n}}\frac{p}{n-p}$. Here, ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

By using Theorem 4 and the Hölder inequality, we immediately obtain

$$\int_D |k(x)\varphi\nabla\psi|\,dx \leqslant C_{n,2}\,\|k\|_{L^{n,\infty}(D)}\|\nabla\varphi\|_{L^2(D)}\|\nabla\psi\|_{L^2(D)} \tag{15}$$

for any $\varphi \in W_0^{1,2}(D)$ and ψ such that $\nabla\psi \in L^2(D)$.

We define the distance of the given $f \in L^{p,\infty}$ to L^∞ as

$$\mathrm{dist}_{L^{p,\infty}}(f, L^\infty) \;=\; \inf_{g\in L^\infty}\|f-g\|_{L^{p,\infty}}.$$

We underline that assuming $\|K_2\|_{L^{n,\infty}} < \varepsilon$ is equivalent to $\inf\limits_{K_1\in L^\infty}\|K-K_1\|_{L^{n,\infty}} < \varepsilon$.

2.3. Difference Quotient

To prove a higher differentiability result for solutions to (1), we will introduce the finite difference operator in order to apply the difference quotient method.

Definition 3 ([22])**.** *For every vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$, the finite difference operator and the difference quotient are, respectively, defined by*

$$\tau_{i,h}F(x) = F(x+he_i) - F(x),$$

$$\Delta_h F(x) = \frac{\tau_{i,h}F(x)}{h}, \quad h \in \mathbb{R}\setminus\{0\}$$

where $h \in \mathbb{R}$, e_i is the unit vector in the x_i direction and $i \in \{1,\dots,n\}$.

The function $\Delta_h f$ is defined in the set

$$\Delta_h\Omega = \{x \in \Omega : x + he_i \in \Omega\}.$$

We list some elementary properties of the finite difference operator.

Proposition 1 ([22])**.** *Let f and g be two functions such that $F, G \in W^{1,p}(\Omega;\mathbb{R}^N)$, with $p \geq 1$, and let us consider*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then, we have the following:

$(d1)$ *$\tau_h F \in W^{1,p}(\Omega)$ and*

$$D_i(\tau_h F) = \tau_h(D_i F).$$

$(d2)$ *If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F\,\tau_h G\,dx = -\int_\Omega G\,\tau_{-h}F\,dx.$$

$(d3)$ *The following equality holds:*

$$\tau_h(FG)(x) = F(x+he_s)\tau_h G(x) + G(x)\tau_h F(x).$$

The next Lemma is a kind of integral version of the Lagrange Theorem. The following results will be useful in the sequel.

Lemma 2 ([22]). *If* $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $i \in \{1, \dots, n\}$ *and* $F, D_i F \in L^p(B_R)$, *then*

$$\int_{B_\rho} |\tau_h F(x)|^p \, dx \le |h|^p \int_{B_R} |D_i F(x)|^p \, dx.$$

Moreover,

$$\int_{B_\rho} |F(x + he_i)|^p \, dx \le \int_{B_R} |F(x)|^p \, dx.$$

Let us recall the Sobolev embedding property.

Lemma 3 (see Lemma 8.2 in [22]). *Let* $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ *with* $1 < p < +\infty$. *Suppose that there exist* $\rho \in (0, R)$ *and* $C > 0$ *such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_h F(x)|^p \, dx \le C^p |h|^p,$$

for every h *with* $|h| < \frac{R-\rho}{2}$. *Then,* $F \in W^{1,p}(B_\rho; \mathbb{R}^N) \cap L^{\frac{np}{n-p}}(B_\rho; \mathbb{R}^N)$. *Moreover,*

$$||DF||_{L^p(B_\rho)} \le C$$

and

$$||F||_{L^{\frac{np}{n-p}}(B_\rho)} \le c\Big(C + ||F||_{L^p(B_R)}\Big),$$

with $c \equiv c(n, N, p)$.

We recall an iteration Lemma. It finds a remarkable application in the so-called hole-filling method.

Lemma 4 (see Lemma 6.1 [22]). *Let* $h : [\rho, R_0] \to \mathbb{R}$ *be a nonnegative bounded function and* $0 < \vartheta < 1$, $A, B \ge 0$ *and* $\beta > 0$. *Suppose that*

$$h(r) \le \vartheta h(d) + \frac{A}{(d-r)^\beta} + B,$$

for all $\rho \le r < d \le R_0$. *Then,*

$$h(\rho) \le \frac{cA}{(R_0 - \rho)^\beta} + B,$$

where $c = c(\vartheta, \beta) > 0$.

3. Proof of Theorem 1

In this section, we prove our main result: Theorem 1. In the first part of the proof, we show the uniqueness of the solution. In the second part, we establish an a priori estimate for the second derivatives of the solutions. In the third part, we construct the suitable approximating problems proving that the a priori estimate is preserved when we pass to the limit.

Step 1. The uniqueness

Let u and v be two solutions of problem (8). We use $w = u - v$ as the test function for the solution u and the solution v, respectively:

$$\int_\Omega \langle A(x)Du, Du - Dv\rangle + B(x,u)(u-v)\, dx = \int_\Omega f(u-v)\, dx \tag{16}$$

and

$$\int_\Omega \langle A(x)Dv, Du - Dv\rangle + B(x,v)(u-v)\,dx \;=\; \int_\Omega f(u-v)\,dx \tag{17}$$

By subtracting (17) from (16), we obtain

$$\int_\Omega \langle A(x)(Du - Dv), Du - Dv\rangle + [B(x,u) - B(x,v)]\,(u-v)\,dx \;=\; 0$$

Then, from (5) and (3), we obtain

$$\|Du - Dv\|^2_{L^2} \;\leqslant\; \int_\Omega \langle A(x)(Du - Dv), Du - Dv\rangle dx \;\leqslant\; 0$$

and so $u = v$ a.e.

Step 2. The a priori estimate

Theorem 5. *Let Ω be a Lipschitz domain. If u is the solution of (8) in $W^{2,2}_{loc}(\Omega,\mathbb{R}^n)$, then there exists ε, depending only on n, such that if $\|K_2\|_{L^{n,\infty}} < \varepsilon$, the following estimate holds*

$$\int_{B_R} |D^2u|^2\,dx \;\leqslant\; C\left[\int_{B_{2R}}\left(\frac{1}{R^2}+1\right)\left(|Du|^2 + |u|^2 + |f|^2\right)dx\right] \tag{18}$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant C depending on n, $\|K_2\|_{L^{n,\infty}}$, the BMO-norm of A and $\|b\|_{L^{n,\infty}}$.

Proof. Let us fix a ball $B_R \subset \Omega$ and arbitrary radii $R < s < t < 2R$, with R small enough. Let $u \in W^{1,2}_0(\Omega,\mathbb{R}^n)$ be a solution of (8) and then in particular for any $\varphi \in C^\infty_0(B_t,\mathbb{R}^n)$

$$\int_{B_t} \langle A(x)Du(x), D\varphi(x)\rangle\,dx + \int_{B_t} B(x,t)\varphi(x)\,dx = \int_{B_t} f(x)\varphi(x)\,dx$$

i.e., u solves the equation

$$-\mathrm{div}(A(x)Du) + B(x,u) = f \qquad \text{in } B_t. \tag{19}$$

Since $u \in W^{2,2}(B_t,\mathbb{R}^n)$, $A(x)Du \in L^2(B_t,\mathbb{R}^{n\times n})$. Now, by Hodge decomposition, we decompose uniquely

$$ADu = D\Psi + H \tag{20}$$

with H a divergence-free vector field and $\Psi \in W^{1,2}_0(B_t,\mathbb{R}^{n\times n})$ (see [23] (p. 148), relation (2.8) and line 17 of p. 149).

By (19) and (20),

$$\mathrm{div}D\Psi = \Delta\Psi = B(x,u) \,-\, f \quad \text{ in } B_t$$

and, by the classical theory,

$$\|D\Psi\|_{L^2(B_t)} \;\leqslant\; \|B(\cdot,u)\|_{L^2(B_t)} \,+\, \|f\|_{L^2(B_t)}. \tag{21}$$

Let T be the projection operator of a vector field onto a gradient field. Notice that $H = ADu - D\Psi$ where $D\Psi = T(ADu)$, i.e.,

$$H = ATDu - T(ADu).$$

Then, using Theorem 3.1 and Lemma 3.3 of [11], we conclude that H is in $L^2(B_t,\mathbb{R}^n)$ and

$$\|H\|_{L^2(B_t)} \;\leqslant c\,\|A\|_\star\|Du\|_{L^2(B_t)} \tag{22}$$

where $c = c(n)$. Finally, from (19) and relations (20), (21) and (22), we obtain

$$\|ADu\|_{L^2(B_t)} \leqslant c(n)\Big(\|A\|_\star \|Du\|_{L^2(B_t)} + \|f\|_{L^2(B_t)} + \|B(\cdot,u)\|_{L^2(B_t)}\Big). \tag{23}$$

Let $\eta \in C_0^\infty(B_t)$, $0 \leqslant \eta \leqslant 1$ be a cut-off function such that $\eta \equiv 1$ on B_s and $|D\eta| \leq \frac{c}{t-s}$. Since u is a weak solution of (1), we are able to use $\varphi = \tau_{-h}(\eta^2\tau_h u)$ as a test function in (9) obtaining

$$\begin{aligned}\int_{B_t} \langle A(x)Du, D(\tau_{-h}(\eta^2\tau_h u))\rangle dx \quad &+ \quad \int_{B_t} \langle B(x,u), \tau_{-h}(\eta^2\tau_h u)\rangle dx \\ &= \quad \int_{B_t} \langle f(x), \tau_{-h}(\eta^2\tau_h u)\rangle dx\end{aligned}$$

and using the properties of difference quotients

$$\begin{aligned}\int_{B_t} \langle \tau_h(A(x)Du), D(\eta^2\tau_h u)\rangle dx \quad &= \quad -\int_{B_t} \langle B(x,u), \tau_{-h}(\eta^2\tau_h u)\rangle dx \\ &+ \quad \int_{B_t} \langle f(x), \tau_{-h}(\eta^2\tau_h u)\rangle dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{B_t} \eta^2\langle \tau_h(A(x)Du), D\tau_h u\rangle dx + 2\int_{B_t} \eta\langle \tau_h(A(x)Du), \nabla\xi \otimes \tau_h u\rangle \\ = \quad &-\int_{B_t} \langle B(x,u), \tau_{-h}(\eta^2\tau_h u)\rangle dx + \int_{B_t} \langle f(x), \tau_{-h}(\eta^2\tau_h u)\rangle dx.\end{aligned} \tag{24}$$

We remark that

$$\tau_h(A(x)Du) = A(x+he_i)D\tau_h u + (\tau_h A(x))Du.$$

Then, from (24), we obtain

$$\begin{aligned}\int_{B_t} \eta^2\langle A(x+he_i)D\tau_h u, D\tau_h u\rangle\, dx = \quad &- \quad \int_{B_t} \eta^2\langle(\tau_h A(x))Du, D\tau_h u\rangle dx \\ &- \quad 2\int_{B_t} \eta\langle A(x+he_i)D\tau_h u, \nabla\eta \otimes \tau_h u\rangle dx \\ &- \quad 2\int_{B_t} \eta\langle \tau_h A(x))Du(x), \nabla\eta \otimes \tau_h u\rangle dx \\ &- \quad \int_{B_t} [B(x,u)]\tau_{-h}(\eta^2\tau_h u) dx \\ &+ \quad \int_{B_t} [f(x)]\tau_{-h}(\eta^2\tau_h u) dx \\ &= \quad I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{25}$$

ESTIMATE OF I_1

To estimate I_1, we first use property $(d1)$ and assumption (6). In the same spirit of (15), by the Hölder inequality in Lorentz spaces (Theorem 3) and Young's inequality with a constant $\nu \in (0,1)$,

$$\begin{aligned}|I_1| \quad &\leqslant \quad \int_{B_t} \eta^2|h|K(x)|Du||D\tau_h u|\, dx \\ &\leqslant \quad \frac{\nu}{2}\int_{B_t} \eta^2|D\tau_h u|^2\, dx + \frac{1}{2\nu}\int_{B_t} |h|^2|K_2|^2|\eta Du|^2\, dx\end{aligned}$$

$$
\begin{aligned}
&+ \int_{B_t} \eta^2 \|K_1\|_{L^\infty} |h||Du||D\tau_h u|\, dx \\
&\leqslant \frac{\nu}{2} \int_{B_t} \eta^2 |D\tau_h u|^2\, dx + \frac{|h|^2}{2\nu} \|K_2\|^2_{L^{n,\infty}(B_t)} \|\eta Du\|^2_{L^{2^\star,2}(B_t)} \\
&+ \frac{\nu}{2} \int_{B_t} \eta^2 |D\tau_h u|^2\, dx + \frac{|h|^2}{2\nu} \int_{B_t} \eta^2 \|K_1\|^2_{L^\infty} |Du|^2\, dx.
\end{aligned}
$$

By Theorem 4, we obtain the following:

$$
\begin{aligned}
|I_1| &\leqslant \nu \int_{B_t} \eta^2 |D\tau_h u|^2\, dx \\
&+ \frac{|h|^2}{2\nu} C^2_{2,n} \|K_2\|^2_{L^{n,\infty}} \|D(\eta Du)\|^2_{L^2(B_t)} \\
&+ \frac{|h|^2}{2\nu} \int_{B_t} \eta^2 \|K_1\|^2_{L^\infty} |Du|^2\, dx.
\end{aligned} \tag{26}
$$

ESTIMATE OF I_2

$$
\begin{aligned}
|I_2| &\leqslant 2\int_{B_t - B_s} |\eta||A(x+he_i)\tau_h u||D\tau_h u||D\eta|\, dx \\
&\leqslant \int_{B_t - B_s} \eta^2 |D\tau_h u|^2 dx + \int_{B_t - B_s} |D\eta|^2 |A(x+he_i)\tau_h u|^2\, dx.
\end{aligned}
$$

Since $|D\eta| \leq \frac{1}{t-s}$, we obtain

$$
|I_2| \leqslant \int_{B_t - B_s} \eta^2 |D\tau_h(u)|^2\, dx + \frac{1}{(t-s)^2} \int_{B_t - B_s} |A(x+he_i)\tau_h u|^2 dx. \tag{27}
$$

ESTIMATE OF I_3

By Young's inequality, assumption (6), Lemma 3 and Hölder inequality (13)

$$
\begin{aligned}
|I_3| &\leqslant 2\int_{B_t} |\eta||A(x+he_i) - A(x)||Du||D\eta||\tau_h u|\, dx \\
&\leqslant \int_{B_t} |\eta|^2 |A(x+he_i) - A(x)|^2 |Du|^2\, dx + \int_{B_t - B_s} |D\eta|^2 |\tau_h u|^2\, dx \\
&\leqslant 2\int_{B_t} |h|^2 |\eta|^2 |K_2|^2 |Du|^2 dx + 2\int_{B_t} |h|^2 |\eta|^2 \|K_1\|^2_{L^\infty} |Du|^2\, dx \\
&+ \int_{B_t - B_s} |D\eta|^2 |\tau_h u|^2\, dx \\
&\leqslant 2|h|^2 \|K_2\|^2_{L^{n,\infty}} \|\eta Du\|^2_{L^{2^\star,2}} + 2|h|^2 \|K_1\|^2_{L^\infty} \int_{B_t} |\eta|^2 |Du|^2 dx \\
&+ \frac{1}{(t-s)^2} \int_{B_t - B_s} |\tau_h u|^2 dx.
\end{aligned}
$$

Applying Theorem 4, we obtain the following:

$$
\begin{aligned}
|I_3| &\leqslant 4C^2_{n,2} |h|^2 \|K_2\|^2_{L^{n,\infty}} \|\eta D^2 u\|^2_{L^2} \\
&+ 2|h|^2 \|K_1\|^2_{L^\infty} \int_{B_t} \eta^2 |Du|^2 dx \\
&+ \frac{1}{(t-s)^2} \int_{B_t - B_s} |\tau_h u|^2 dx
\end{aligned}
$$

$$+ \quad 4|h|^2 C_{n,2}^2 \frac{1}{(t-s)^2} \|K_2\|_{L^{n,\infty}}^2 \int_{B_t - B_s} |Du|^2 dx. \tag{28}$$

ESTIMATE OF I_4

By using Young's inequality, assumption (2) and Hölder inequality (13)

$$\begin{aligned} |I_4| &\leqslant |h|^2 \int_{B_t} |B(x,u)||\Delta_{-h}(\eta^2 \Delta_h u)| dx \\ &\leqslant 2|h|^2 \int_{B_t} b|\eta(x-he_i)u||\Delta_{-h}\eta \Delta_h u + \eta(x-he_i)\Delta_{-h}\Delta_h u| dx \\ &\leqslant \frac{|h|^2}{\nu} \int_{B_t} b^2 |\eta(x-he_i)u|^2 \, dx + 2\nu |h|^2 \int_{B_t} |\Delta_{-h}\eta|^2 |\Delta_h u|^2 dx \\ &+ 2\nu |h|^2 \int_{B_t} \eta^2(x-he_i)|\Delta_{-h}\Delta_h u|^2 dx \\ &\leqslant 2\frac{|h|^2}{\nu} \|b\|_{L^{n,\infty}}^2 \|\eta(x-he_i)u\|_{L^{2^*,2}}^2 + 2\nu|h|^2 \int_{B_t} |\Delta_{-h}\eta|^2 |\Delta_h u|^2 dx \\ &+ 2\nu|h|^2 \int_{B_t} \eta^2(x-he_i)|\Delta_{-h}\Delta_h u|^2 dx. \end{aligned} \tag{29}$$

By using Theorem 4 in the first integral in the right-hand side, we obtain

$$\begin{aligned} |I_4| &\leqslant 2\frac{|h|^2}{\nu} C_{n,2}^2 \|b\|_{L^{n,\infty}}^2 \|D(\eta u)\|_{L^2}^2 + 2\nu|h|^2 \frac{1}{(t-s)^2} \int_{B_t} |\Delta_h u|^2 dx \\ &+ 2\nu|h|^2 \int_{B_t} \eta^2(x-he_i)|\Delta_{-h}\Delta_h u|^2 dx \end{aligned}$$

and so

$$\begin{aligned} |I_4| &\leqslant 4\frac{|h|^2}{\nu} C_{n,2}^2 \|b\|_{L^{n,\infty}} \|\eta Du\|_{L^2}^2 + 4\frac{|h|^2}{\nu} C_{n,2}^2 \|b\|_{L^{n,\infty}} \|u D\eta\|_{L^2}^2 \\ &+ 2\nu|h|^2 \frac{1}{(t-s)^2} \int_{B_t} |\Delta_h u|^2 dx \\ &+ 2\nu|h|^2 \int_{B_t} \eta^2(x-he_i)|\Delta_{-h}\Delta_h u|^2 dx. \end{aligned}$$

ESTIMATE OF I_5

$$\begin{aligned} |I_5| &\leqslant |h|^2 \int_{B_t} |f||\Delta_{-h}(\eta^2 \Delta_h u)| dx \\ &\leqslant \frac{\nu |h|^2}{2} \int_{B_t} |\Delta_{-h}(\eta^2 \Delta_h u)|^2 \, dx + \frac{|h|^2}{\nu} \int_{B_t} |f|^2 \, dx. \end{aligned} \tag{30}$$

By arguing as above, we obtain

$$\begin{aligned} |I_5| &\leqslant \frac{\nu}{2}|h|^2 \int_{B_t} \eta^2 |\Delta_{-h}\Delta_h u|^2 dx + \nu|h|^2 \frac{1}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx \\ &+ \frac{|h|^2}{\nu} \int_{B_t} |f|^2 dx. \end{aligned}$$

By combining (26)–(30), dividing by $|h|^2$, as $h \to 0$, we obtain the following:

$$\begin{aligned} \int_{B_t} \eta^2 |D^2 u|^2 dx &\leqslant \frac{7}{2}\nu \int_{B_t} \eta^2 |D^2 u|^2 \, dx \\ &+ \left(\frac{1}{\nu} + 4\right) C_{n,2}^2 \|K_2\|_{L^{n,\infty}}^2 \int_{B_t} |\eta D^2 u|^2 dx \end{aligned}$$

$$
\begin{aligned}
&+ \int_{B_t - B_s} \eta^2 |D^2 u|^2 dx \\
&+ \frac{1}{(t-s)^2} \int_{B_t - B_s} |A(x) Du|^2 dx \\
&+ \left(\frac{1}{\nu} + 4\right) \|K_2\|^2_{L^{n,\infty}} C^2_{n,2} \frac{1}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx \\
&+ \frac{1}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx \\
&+ \frac{4}{\nu} C^2_{n,2} \|b\|_{L^{n,\infty}} \frac{1}{(t-s)^2} \int_{B_t - B_s} |u|^2\, dx \\
&+ \frac{2\nu}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx \\
&+ \left[\left(2 + \frac{1}{2\nu}\right) \|K_1\|^2_{L^\infty} + \frac{4}{\nu} C^2_{n,2} \|b\|_{L^{n,\infty}}\right] \int_{B_t} \eta^2 |Du|^2 dx. \\
&+ \frac{1}{\nu} \int_{B_t} |f|^2 dx. \qquad (31)
\end{aligned}
$$

We can estimate $\|ADu\|_{L^2(B_t)}$ by using (23). Indeed, from inequality (13) and Theorem 4,

$$
\begin{aligned}
\|bu\|^2_{L^2(B_t)} &\leqslant \|b\|^2_{L^{n,\infty}(B_t)} \|u\|^2_{L^{2^\star,2}(B_t)} \\
&\leqslant c \|b\|^2_{L^{n,\infty}(B_t)} \left(\|u\|^2_{L^2(B_t)} + \|Du\|^2_{L^2(B_t)}\right).
\end{aligned}
$$

Then, by assumptions (2) and (4),

$$
\begin{aligned}
\|ADu\|^2_{L^2(B_t)} &\leqslant c\left[\|A\|^2_\star \|Du\|^2_{L^2(B_t)} + \|b\|^2_{L^{n,\infty}(B_t)} \left(\|u\|^2_{L^2(B_t)} + \|Du\|^2_{L^2(B_t)}\right)\right] \\
&+ c\|f\|^2_{L^2(B_t)} \qquad (32)
\end{aligned}
$$

where $c = c(n)$.

Here, we first choose the number ν, such that $1 - \frac{7}{2}\nu > 0$, and then $\|K_2\|_{L^{n,\infty}}$ is not large enough so that we are able to reabsorb the integrals on the right-hand side, including those related to $\|\eta D^2 u\|_{L^2(B_t)}$, into the integrals on the left-hand side. Obviously, from the definition of the distance, this is equivalent to requiring that $\|K_2\|_{L^{n,\infty}} < \varepsilon$ for a suitable $\varepsilon > 0$. More precisely, if ν is such that $\left(1 - \frac{7\nu}{2}\right) > 0$, since $\eta = 1$ on B_s, on account of (32), for

$$
\|K_2\|^2_{L^{n,\infty}} < \frac{(1 - \frac{7}{2}\nu)}{C^2_{n,2}\left(\frac{1}{\nu} + 4\right)}
$$

we obtain from (31)

$$
\begin{aligned}
C \int_{B_s} |D^2 u|^2 dx &\leqslant \int_{B_t - B_s} |D^2 u|^2 dx \\
&+ \frac{c}{(t-s)^2} \left(\|A\|^2_\star + \|b\|^2_{L^{n,\infty}} + \|K_2\|^2_{L^{n,\infty}}\right) \\
&\cdot \int_{B_t} \left(|Du|^2 + |u|^2 + |f|^2\right) dx \\
&+ c\left(\|K_1\|^2_{L^\infty} + \|b\|_{L^{n,\infty}} + 1\right)
\end{aligned}
$$

$$\cdot \int_{B_t}\left(|Du|^2+|u|^2+|f|^2\right)dx$$

where $C = (1-\frac{7}{2}\nu) - \left(\frac{1}{\nu}+4\right)C^2_{n,2}\|K_2\|^2_{L^{n,\infty}}$ and $c = c(n,\nu)$.

Now, we fill the hole:

$$\begin{aligned}\int_{B_s}|D^2u|^2dx &\leqslant \frac{1}{C+1}\int_{B_t}|D^2u|^2dx \\ &+ \frac{\mathcal{A}}{(t-s)^2}\int_{B_t}\left(|Du|^2+|u|^2+|f|^2\right)dx \\ &+ \mathcal{B}\int_{B_t}\left(|Du|^2+|u|^2+|f|^2\right)dx\end{aligned}$$

where

$$\mathcal{A} = \frac{c}{C+1}\left(\|A\|^2_\star + \|b\|^2_{L^{n,\infty}} + \|K_2\|^2_{L^{n,\infty}}\right)$$

and

$$\mathcal{B} = \frac{c}{C+1}\left(\|K_1\|^2_{L^\infty} + \|b\|_{L^{n,\infty}} + 1\right).$$

By applying Lemma 4, we obtain the estimate (18).

Step 3. The approximation

We first extend the matrices $A(x)$ and $b(x)$ to $\mathbb{R}^n$. We put zero outside of Ω. Then, we take $\rho \in C_0^\infty(\mathbb{R}^n)$ such that supp$\rho \subset \overline{B_1(0)}$, $\rho \geqslant 0$, $\rho \neq 0$ and ∞ and we consider the convolution $A_N = A \star \rho_N$, with $\rho_N = \frac{N^n\rho(Nx)}{\int\rho}$.

$$A_N(x) = \int_{\mathbb{R}^n} A(y)\rho_N(x-y)\,dy, \quad x \in \overline{\Omega}.$$

We notice the following:

1. $A_N \in C^\infty(\overline{\Omega},\mathbb{R}^{n\times n}) \cap L^\infty(\Omega,\mathbb{R}^{n\times n})$;
2. $\langle A_N Y, Y\rangle \geqslant \|Y\|^2$;
3. $|A_N(x+he_i) - A_N(x)| \leqslant K_N(x)|h|$, where $K_N(x) = (K\star\rho_N)(x)$;
4. $K_N(x) \in L^{n,\infty}$;
5. $\|A_N\|_\star \leqslant \|A\|_\star$;
6. A_N converges to A in L^2.

For all positive integers N, let $u_N \in W_0^{1,2}(\Omega,\mathbb{R}^n)$ be the solution of the Dirichlet problems:

$$\begin{cases} -\text{div}(A_N Du_N) + B(x,u_N) = f & \text{in } \Omega \\ u_N = 0 & \text{on } \partial\Omega \end{cases}$$

that converge weakly in $W_0^{1,2}$ and strongly in L^2 to u (see [11]). A result contained in [24] guarantees that $u_N \in W^{2,2}_{loc}(\Omega,\mathbb{R}^n)$. Let us consider as the test function $\varphi = u_N$

$$\int_\Omega \langle A_N(x)Du_N, Du_N\rangle dx + \int_\Omega B(x,u_N)u_N dx = \int_\Omega f u_N dx$$

since by (3) $\langle B(x,u_N),u_N\rangle \geqslant 0$, we obtain

$$\int_\Omega |Du_N|^2dx \leqslant \int_\Omega \langle A_N(x)Du_N, Du_N\rangle dx \leqslant \int_\Omega f u_N dx$$

$\{u_N\}$ is bounded in $W_0^{1,2}(\Omega)$ and there exists a function $u \in W_0^{1,2}(\Omega)$ and a subsequence, still denoted $\{u_N\}$ such that u_N converge weakly to a function $u \in W_0^{1,2}(\Omega)$ and $u_N \to u$ in $L^2(\Omega)$.

$$\begin{aligned} \left|\int_\Omega \langle A_N(x)Du_N - A(x)Du, D\varphi\rangle dx\right| &= \left|\int_\Omega (f - B(x,u_N))\varphi dx - \int_\Omega (f - B(x,u))\varphi dx\right| \\ &\leqslant \int_\Omega |B(x,u) - B(x,u_N)|\,|\varphi| dx \\ &\leqslant \|\varphi\|_\infty \int_\Omega |b||u_N - u|\, dx. \end{aligned} \tag{33}$$

Since $b \in L^{n,\infty} \subset L^2$ and $u_N \to u$ in L^2,

$$\int_\Omega |b||u_N - u| dx \leqslant \|b\|_{L^2}\|u - u_N\|_{L^2} \to 0.$$

The last relation in (33) also implies

$$\int_\Omega B(x,u_N)\,\varphi\, dx \to \int_\Omega B(x,u)\,\varphi\, dx$$

and then u solves problem (8).

Let $\varepsilon > 0$ be the number fixed in (18), and let us assume

$$\|K_2\|_{L^{n,\infty}} < \varepsilon.$$

We notice that from Lemma $A.4$ in [25], we have

$$\|K_N - K_1\|_{L^{n,\infty}} \leqslant \|K_2\|_{L^{n,\infty}} < \varepsilon.$$

More precisely, if $K_1 \in L^\infty(\Omega)$ is a function such that $\|K_2\|_{L^{n,\infty}} < \varepsilon$, we obtain

$$\|K_N - K_1\|_{L^{n,\infty}} \leqslant \|K - K_1\|_{L^{n,\infty}} + \|(K_1)_N - K_1\|_{L^{n,\infty}}.$$

Since $K_1 \in L^p(\Omega)$ for every $p \geqslant n$, thanks to Theorem 3 the second term on the right-hand side of the previous inequality goes to 0 as $N \to +\infty$. Then, we can assume

$$\|K_N - K_1\|_{L^{n,\infty}} < \varepsilon$$

for N sufficiently large.

Now, arguing as in Theorem 5,

$$\begin{aligned} \int_{B_s} |D^2u_N|^2 dx &\leqslant \frac{1}{C+1}\int_{B_t} |D^2u_N|^2 dx \\ &+ \frac{\mathcal{A}}{(t-s)^2}\int_{B_t}\left(|Du_N|^2 + |u_N|^2 + |f|^2\right) dx \\ &+ \mathcal{B}\int_{B_t}\left(|Du_N|^2 + |u_N|^2 + |f|^2\right) dx \end{aligned}$$

where

$$\mathcal{A} = \frac{c}{C+1}\left(\|A\|_\star^2 + \|b\|_{L^{n,\infty}}^2 + \|K_2\|_{L^{n,\infty}}^2\right)$$

and

$$\mathcal{B} = \frac{c}{C+1}\left(\|K_1\|_{L^\infty}^2 + \|b\|_{L^{n,\infty}} + 1\right).$$

Applying Lemma 4, we obtain

$$\int_{B_R} |D^2 u_N|^2 \, dx \leqslant C\left[\left(\frac{1}{R^2}+1\right)\left(\int_{B_{2R}}\left(|Du_N|^2+|u_N|^2+|f|^2\right)dx\right)\right].$$

We deduce that $|D^2 u_N|$ is a bounded sequence in $L^2(B_R)$. Then, by compactness, up to a sequence, $|D^2 u_N|$ converges weakly to $|D^2 u|$ in $L^2(B_R, \mathbb{R}^n)$ and, by the semicontinuity of the norm with respect to weak convergence, we obtain

$$\begin{aligned} \int_{B_R} |D^2 u|^2 dx &\leqslant \liminf_N \int_{B_R} |D^2 u_N|^2 dx \\ &\leqslant c\left[\left(\frac{1}{R^2}+1\right)\left(\int_{B_{2R}}\left(|Du|^2+|u|^2+|f|^2\right)dx\right)\right] \end{aligned}$$

where $c = c(n, \|K_2\|_{L^{n,\infty}}, \|A\|_\star, \|b\|_{L^{n,\infty}})$. □

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References

1. Miranda, C. Alcune limitazioni integrali per le soluzioni delle equazioni lineari ellittiche di secondo ordine. *Ann. Mat. Pura Appl.* **1960**, *49*, 375–384. [CrossRef]
2. Miranda, C. Sulle equazioni ellittiche del secondo ordine di tipo non variazionale con coefficienti non discontinui. *Ann. Mat. Pura Appl.* **1963**, *63*, 353–386. [CrossRef]
3. Chiarenza, F.; Frasca, M.; Longo, P. $W^{2,,p}$ solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. *Trans. Am. Math. Soc.* **1993**, *336*, 841–853.
4. Chiarenza, F. L^p regularity for systems of PDEs with coefficients in VMO. *Nonlinear Anal. Funct. Spaces Appl.* **1994**, *5*, 1–32.
5. Giannetti, F.; Moscariello, G. $W^{2,2}-$solvability of the Dirichlet problem for a class of elliptic equations with discontinuous coefficients. *Rend. Lincei Mat. Appl.* **2018**, *29*, 557–577.
6. Greco, L.; Moscariello, G.; Radice, T. Nondivergence Elliptic Equations with unbounded coefficients. *Discret. Contin. Dyn. Syst.* **2009**, *11*, 131–143. [CrossRef]
7. Capone, C.; Radice, T. A regularity result for a class of elliptic equations with lower order terms. *J. Elliptic Parabol. Equ.* **2020**, *6*, 751–771. [CrossRef]
8. Arcoya, D.; Boccardo, L. Regularizing effect of the interplay between coefficients in some elliptic equations. *J. Funct. Anal.* **2015**, *268*, 1153–1166. [CrossRef]
9. Arcoya, D.; Boccardo, L. Regularizing effect of L^q interplay between coefficients in some elliptic equations. *J. Math. Pures Appl.* **2018**, 111, 106–125. [CrossRef]
10. Radice, T. A regularity result for non uniformly elliptic equations with lower order terms. *Stud. Math.* **2024**, *276*, 1–17. [CrossRef]
11. Stroffolini, B. Elliptic systems of PDE with BMO coefficients. *Potential Anal.* **2001**, *15*, 285–299. [CrossRef]
12. Moscariello, G.; Pascale, G. Second order regularity for a linear elliptic system having BMO coefficients. *Milan J. Math.* **2021**, *89*, 413–432. [CrossRef]
13. Moscariello, G.; Pascale, G. Higher differentiability and integrability for some nonlinear elliptic systems with growth coefficients in BMO. *Calc. Var.* **2024**, *63*, 80. [CrossRef]
14. Farroni, F.; Greco, L.; Moscariello, G.; Zecca, G. Noncoercive quasilinear elliptic operators with singular lower order terms. *Calc. Var. Partial. Differ. Equ.* **2021**, *60*, 83. [CrossRef]
15. Farroni, F.; Greco, L.; Moscariello, G.; Zecca, G. Noncoercive parabolic obstacle problems. *Adv. Nonlinear Anal.* **2023**, *12*, 20220322. [CrossRef]

16. Radice, T.; Zecca, G. Existence and uniqueness for nonlinear elliptic equations with unbounded coefficients. *Ric. Mat.* **2014**, *63*, 355–367. [CrossRef]
17. Brezis, H.; Nirenberg, L. Degree theory and *BMO*, part I: Compact manifolds without boundaries. *Sel. Math. New Ser.* **1995**, *1*, 197–263. [CrossRef]
18. John, F.; Nirenberg, L. On functions of bounded mean oscillation. *Comm. Pure Appl. Math* **1961**, *14*, 415–426. [CrossRef]
19. Coifman R.; Rochberg R. Another characterization of BMO. *Proc. AMS* **1980**, *79*, 249–254. [CrossRef]
20. O'Neil, R. Fractional Integration in Orlicz spaces I. *Am. Math. Soc.* **1965**, *115*, 300–328. [CrossRef]
21. Alvino, A. Sulla disuguaglianza di Sobolev in spazi di Sobolev. *Boll. Unione Mat. Ital.* **1977**, *14*, 148–156.
22. Giusti, E. *Direct Method in the Calculus of Variations*; World Scientific: Singapore, 2003
23. Iwaniec, T.; Sbordone C. Weak minima of variational integrals. *J. Reine Angew. Math.* **1994**, *454*, 143–162.
24. Morrey, C.B., Jr. Second order elliptic systems of differential equations. *A*nn. Math. Stud. **1954** *33*, 101–159.
25. Benilanm, P.; Brezis, H.; Crandall, M. A semilinear equation in $L^1(\mathbb{R}^N)$. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **1975**, *4*, 523–555.