Journal of Algebra 615 (2023) 77–102

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Addition-deletion results for the minimal degree of a Jacobian syzygy of a union of two curves

Alexandru Dimca [a,b,1], Giovanna Ilardi [c,*], Gabriel Sticlaru [d]

[a] *Université Côte d'Azur, CNRS, LJAD, France*
[b] *Simion Stoilow Institute of Mathematics, P.O. Box 1-764, RO-014700 Bucharest, Romania*
[c] *Dipartimento Matematica Ed Applicazioni "R. Caccioppoli", Università Degli Studi Di Napoli "Federico II", Via Cintia - Complesso Universitario Di Monte S. Angelo, 80126 - Napoli, Italy*
[d] *Faculty of Mathematics and Informatics, Ovidius University Bd. Mamaia 124, 900527 Constanta, Romania*

ARTICLE INFO

Article history:
Received 28 July 2022
Available online 27 October 2022
Communicated by Steven Dale Cutkosky

MSC:
primary 14H50
secondary 14B05, 13D02, 32S22

Keywords:
Plane curve
Derivations
Jacobian syzygy
Free curve
Nearly free curve
Jacobian module
Tjurina number

ABSTRACT

Let $C : f = 0$ be a reduced curve in the complex projective plane. The minimal degree $mdr(f)$ of a Jacobian syzygy for f, which is the same as the minimal degree of a derivation killing f, is an important invariant of the curve C, for instance it can be used to determine whether C is free or nearly free. In this note we study the relations of this invariant $mdr(f)$ with a decomposition of C as a union of two curves C_1 and C_2, without common irreducible components. When all the singularities that occur are quasihomogeneous, a result by Schenck, Terao and Yoshinaga yields finer information on this invariant in this setting. Using this, we give some geometrical criteria, *the first ones of this type in the existing literature as far as we know*, for a line to be a jumping line for the rank

* Corresponding author.
E-mail addresses: dimca@unice.fr (A. Dimca), giovanna.ilardi@unina.it (G. Ilardi), gabriel.sticlaru@gmail.com (G. Sticlaru).

[1] This work has been partially supported by the Romanian Ministry of Research and Innovation, CNCS - UEFISCDI, grant PN-III-P4-ID-PCE-2020-0029, within PNCDI III.

https://doi.org/10.1016/j.jalgebra.2022.10.019
0021-8693/

Quasihomogeneous singularity

2 vector bundle of logarithmic vector fields along a reduced curve C.

1. Introduction

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables x, y, z with complex coefficients. We denote by $\partial_x, \partial_y, \partial_z$ the partial derivations with respect to x, y, z respectively. Let $Der(S) = \{\partial = a\partial_x + b\partial_y + c\partial_z \ : \ a, b, c \in S\}$ be the free S-module of $\mathbb{C}$-derivations of the polynomial ring S.

Definition 1.1. Let $g \in S$ be a polynomial. The S-module $D(g)$ of derivations of S *preserving the principal ideal* $(g) \subset S$ is by definition

$$D(g) = \{\partial \in Der(S) \ : \ \partial g \in (g)\}.$$

Moreover, the S-module $D_0(g)$ of derivations of S *killing the polynomial* g is by definition

$$D_0(g) = \{\partial \in Der(R) \ : \ \partial g = 0\}.$$

When g is a homogeneous polynomial, then both modules $D(g)$ and $D_0(g)$ are graded S-modules and one has

$$D(g) = D_0(g) \oplus S(-1) \cdot E,$$

where $E = x\partial_x + y\partial_y + z\partial_z$ denotes the *Euler derivation.* The curve $C_g : g = 0$ in $\mathbb{P}^2$ is said to be *free* if $D(g)$ or, equivalently, $D_0(g)$ is a free graded S-module.

Note that $D_0(g)$ can be identified with the S-module of all Jacobian relations for g, namely to

$$AR(g) = \{(a, b, c) \in S^3 \ : \ ag_x + bg_y + cg_z = 0\},$$

where $g_u = \partial_u g$, for $u = x, y, z$. An important numerical invariant associated to a reduced curve $C : f = 0$ in the projective plane $\mathbb{P}^2$ is the *minimal degree of a derivation killing* f or, equivalently, the *minimal degree of a Jacobian relation (syzygy) for* f. This is defined by

$$mdr(f) = \min\{s \in \mathbb{N} : \ D_0(f)_s \neq 0\} = \min\{s \in \mathbb{N} : \ AR(f)_s \neq 0\}.$$

It can be used for instance to characterize the free or the nearly free curves, see (2.4) and (2.5) below. In this note we study the relations of this invariant with a decomposition of C as a union of two curves $C_1 : f_1 = 0$ and $C_2 : f_2 = 0$, without common irreducible components. In particular, we would like to relate $r = mdr(f) = mdr(f_1 f_2)$ to $r_j = mdr(f_j)$ for $j = 1, 2$. The case when C_1 is a line arrangement and C_2 is a line was studied in detail in [2].

In section 2 we recall some basic notations and facts, for instance the definition of the Jacobian module $N(f)$ and of free, nearly free and plus one generated curves which play a key role in this paper.

Then we consider in section 3 the case when C_2 is a line and C_1 is any reduced curve, not having C_2 as a component. We study in Theorem 3.3 the behavior of our invariant $mdr(f)$ when C_2 is a member of a pencil of lines in $\mathbb{P}^2$, under the assumption that we know not only r_1, but also a non trivial derivation in $D_0(f_1)_{r_1}$. Several examples are given in section 4.

We study in section 5 the general case of two curves C_1 and C_2, and get bounds for $r = mdr(f)$ in terms of the degrees $d_j = \deg(f_j)$ and of the invariants r_j for $j = 1, 2$, see Theorem 5.1. As an example, we discuss in Proposition 5.5 all the possibilities when both C_1 and C_2 are smooth conics.

Finally, in section 6, *we assume that all the singularities of C_1 and C are quasihomogeneous and that C_2 is a smooth curve (most of the time C_2 is also supposed to be rational).* Under this assumption, we may use a key result by Schenck, Terao and Yoshinaga, see [25], to get finer information on r. Our Theorem 6.2 gives a description of the cohomology exact sequence associated to the short sheaf exact sequence obtained in [25], paying special attention to the description of the morphisms between the corresponding cohomology groups.

This approach was already used in [25] to relate the freeness of C_1 to the freeness of C. Here we show that even when the curve C_1 is not free, one can obtain valuable information on r using this approach. This idea works best when the Jacobian module $N(f_1)$ is small, and this explains why we consider mostly free and nearly free curves C_1. Sometimes the determination of r is rather easy, using just the knowledge of the numerical invariant r_1, as in most examples in section 6. In Example 6.16 we present a situation where one needs to use the morphisms in the exact sequence given by Theorem 6.2, namely the multiplication by f_2^2 between the two Jacobian modules $N(f_1)$ and $N(f)$. As a by-product, under the assumption for this final section, we get lower bounds on the number of points in the intersection $C_1 \cap C_2$ in terms of r_1 when the curve C_1 is free or nearly free and C_2 is either a line or a smooth conic, see Corollary 6.6 and Corollary 6.9.

The study of the jumping lines of the rank 2 vector bundle $T\langle C\rangle$ of logarithmic vector fields along a reduced curve C is a classical subject in Algebraic Geometry, see for instance [15,16,24]. At the end of the paper we give some geometrical criteria, *the first ones of this type in the existing literature as far as we know*, for a line L in $\mathbb{P}^2$ to be a jumping line for the vector bundle $T\langle C\rangle$, see Theorem 6.19 and Example 6.20 where this is applied to Thom-Sebastiani curves.

We would like to thank Laurent Busé, Piotr Pokora, Ştefan Tohăneanu and Masahiko Yoshinaga for useful discussions related to this paper.

2. Prerequisites

In this section we recall some basic facts, see for instance [5,12]. For any degree e reduced homogeneous polynomial $g \in S_e$, let $N(g) = \widehat{J}_g/J_g$ be the Jacobian module of g, with J_g the Jacobian ideal of g in S, spanned by the partial derivatives g_x, g_y, g_z of g, and $\widehat{J}_g$ the saturation of the ideal J_g with respect to the maximal ideal $\mathbf{m} = (x, y, z)$ in S. We set $n(g)_j = \dim N(g)_j$, $T_g = 3(e-2)$ and recall that we have

$$n(g)_0 \leq n(g)_1 \leq \ldots \leq n(g)_{\lfloor \frac{T_g}{2} \rfloor -1} \leq n(g)_{\lfloor \frac{T_g}{2} \rfloor} \geq n(g)_{\lfloor \frac{T_g}{2} \rfloor +1} \geq \ldots \geq n(g)_{T_g}. \tag{2.1}$$

For a reduced curve $C_g : g = 0$, we consider the following invariants

$$\sigma(C_g) = \min \{j : n(g)_j \neq 0\} = \operatorname{indeg}(N(f)) \text{ and } \nu(C_g) = \max \{n(g)_j\}_j \,.$$

The self duality of the graded S-module $N(g)$ implies $n(g)_j = n(g)_{T_g - j}$, for any integer j, see [26]. In particular $n(g)_k > 0$ exactly when $\sigma(C_g) \leq k \leq T_g - \sigma(C_g)$.

The form of the minimal graded free resolution for the Milnor algebra $M(g) = S/J_g$ is

$$0 \to \oplus_{i=1}^{m-2} S(-e_i) \to \oplus_{i=1}^{m} S(1-e-d'_i) \to S^3(1-e) \to S, \tag{2.2}$$

with $e_1 \leq e_2 \leq \ldots \leq e_{m-2}$ and $1 \leq d'_1 \leq d'_2 \leq \cdots \leq d'_m$. In this case the curve C_g is said to be an *m-syzygy curve* with exponents $(d'_1, \ldots, d'_m)$. The first degree $r_g = d'_1$ is denoted by $mdr(g)$ and is the *minimal degree of a Jacobian relation (syzygy) for g*. It follows from [20, Lemma 1.1] that one has

$$e_j = e + d'_{j+2} - 1 + \epsilon_j,$$

for $j = 1, \ldots, m-2$ and some integers $\epsilon_j \geq 1$. The minimal resolution of $N(g)$ obtained from (2.2), by [20, Proposition 1.3], is

$$0 \to \oplus_{i=1}^{m-2} S(-e_i) \to \oplus_{i=1}^{m} S(-\ell_i) \to \oplus_{i=1}^{m} S(d'_i - 2(e-1)) \to \oplus_{i=1}^{m-2} S(e_i - 3(e-1)),$$

where $\ell_i = e + d'_i - 1$. It follows that

$$\sigma(C_g) = 3(e-1) - e_{m-2} = 2(e-1) - d'_m - \epsilon_{m-2}. \tag{2.3}$$

The following are important special cases, see [1,11,12]. Here $\tau(C_g)$ is the total Tjurina number of the curve C_g, which is the same as the degree of the Jacobian ideal J_g.

(1) C_g is a free curve if and only if $m = 2$ and $d'_1 + d'_2 = e - 1$. In this case $\nu(C_g) = 0$ and $N(g) = 0$. The degrees (d'_1, d'_2) are the *exponents* of the free curve C_g. Moreover, a reduced curve C_g is free if and only if

$$\tau(C_g) = (e-1)^2 - r_g(e - r_g - 1), \tag{2.4}$$

see [7,17].

(2) C_g is a nearly free curve if and only if $m = 3$ and $d'_1 + d'_2 = e$, $d'_3 = d'_2$. In this case $\nu(C_g) = 1$ and $\sigma(C_g) = e + d'_1 - 3$. The degrees (d'_1, d'_2) are the *exponents* of the nearly free curve C_g. Moreover, C_g is nearly free if and only if

$$\tau(C_g) = (e-1)^2 - r_g(e - r_g - 1) - 1, \tag{2.5}$$

see [7].

(3) C_g is a plus one generated curve if and only if $m = 3$ and $d'_1 + d'_2 = e$, $d'_3 > d'_2$, see [1] for the case C_g a line arrangement and [12] for the general case. In this case $\nu(C_g) = d'_3 - d'_2 + 1$ and $\sigma(C_g) = 2e - d'_3 - 3$.

3. Adding a line to a reduced curve

Consider a reduced plane curve $C_1 : f_1 = 0$ of degree d_1 in $\mathbb{P}^2$ such that $mdr(f_1) = r_1$. Let L be a line in $\mathbb{P}^2$, which is not an irreducible component of C_1 and consider the curve $C = C_1 \cup L : f = 0$. Then C has degree $d = d_1 + 1$, and we denote $r = mdr(f)$. In this section we analyze the relation between r and r_1, starting with the following result.

Proposition 3.1. *With the above notation, one has $r_1 \leq r \leq r_1 + 1$.*

Proof. Choose a coordinate system on $\mathbb{P}^2$ such that the line L is given by $z = 0$, and hence $f = zf_1$. Let

$$af_x + bf_y + cf_z = 0 \tag{3.1}$$

be a Jacobian syzygy of minimal degree r for f, and

$$a_1 f_{1x} + b_1 f_{1y} + c_1 f_{1z} = 0 \tag{3.2}$$

a Jacobian syzygy of minimal degree r_1 for f_1. Note that one has

$$f_x = zf_{1x},\ f_y = zf_{1y} \text{ and } f_z = zf_{1z} + f_1 = \frac{1}{d_1} xf_{1x} + \frac{1}{d_1} yf_{1y} + \frac{d}{d_1} zf_{1z}. \tag{3.3}$$

Using (3.1) we get

$$azf_{1x} + bzf_{1y} + c(zf_{1z} + f_1) = 0,$$

and hence the polynomial c is divisible by z, so we can write $c = zc'$. Indeed, note that f_1 is not divisible by z by our assumptions. With this notation, and using (3.3), we get after division by z the following equation.

$$(a + \frac{1}{d_1}c'x)f_{1x} + (b + \frac{1}{d_1}c'y)f_{1y} + \frac{d}{d_1}c'zf_{1z} = 0. \tag{3.4}$$

This implies $r_1 \leq r$. Similarly, using (3.2) and (3.3) we get

$$(d_1a_1z - c_1x)f_x + (d_1b_1z - c_1y)f_y + c_1d_1zf_z = 0. \tag{3.5}$$

Note that this is a non trivial syzygy, namely one cannot have

$$d_1a_1z - c_1x = d_1b_1z - c_1y = c_1d_1z = 0.$$

This implies $r \leq r_1 + 1$. □

Remark 3.2. With the above notation, if z divides c_1, the coefficient of f_{1z} in (3.2), then all the coefficients in (3.5) are divisible by z, and hence after simplification by z we get $r = r_1$ in this case. When $\dim D_0(f_1)_{r_1} > 1$, there is a choice of the syzygy (3.2) within a linear system, and some choices may be better than others, i.e. for the good ones z divides c_1, see Example 4.4 below for such a situation.

To say more about the value of r, it is convenient to look not only at a single line L, but at all the lines in a pencil. The pencil we consider is formed by all the lines in $\mathbb{P}^2$ passing through a point $p \in \mathbb{P}^2$, which may or may not be on the curve C_1. We choose a coordinate system on $\mathbb{P}^2$ such that $p = (1 : 0 : 0)$, hence a line in the pencil has the equation $L_u : sy + tz = 0$ for some $u = (s : t) \in \mathbb{P}^1$. Assume that (3.1) and (3.2) are minimal degree Jacobian syzygies for $f = (sy + tz)f_1$ and respectively for f_1, with respect to this coordinate system. Note that the coefficients a_1, b_1, c_1 are known and independent of u, since they depend only on C_1 and the choice of the coordinate system. Let

$$r = d'_1(f) \leq d'_2(f) \leq \cdots \leq d'_m(f)$$

be the degrees of a minimal set of generators for $AR(f)$ coming from the resolution (2.2) of the Milnor algebra $M(f)$, which depend in general on u, see Example 4.2 below. Elementary computations similar to those done above yield the following syzygy

$$A_uf_x + B_uf_y + C_uf_z = 0, \tag{3.6}$$

where $A_u = d(sy + tz)a_1 - x(sb_1 + tc_1)$, $B_u = d(sy + tz)b_1 - y(sb_1 + tc_1)$ and finally $C_u = d(sy + tz)c_1 - z(sb_1 + tc_1)$. Using this syzygy, we can prove the following result.

Theorem 3.3. *With the above notation, if $sy + tz$ is a factor of $sb_1 + tc_1$, then $r = r_1$. If $sy + tz$ is not a factor of $sb_1 + tc_1$, then either*

(1) $r = r_1 + 1$, *or*

(2) *$r = r_1$ and $d_2'(f) \le r + 1$.*

Moreover, the case (2) *is impossible when $2r_1 < d_1 - 1$, or when $2r_1 = d_1 - 1$ and C is not free.*

Proof. The first claim is obvious. Indeed, when $sy + tz$ is a factor of $sb_1 + tc_1$, the coefficients A_u, B_u and C_u can be divided by $sy+tz$, and the syzygy (3.6) yields a syzygy of degree r_1. Since $r \ge r_1$ by Proposition 3.1, we get $r = r_1$. Assume now that $sy + tz$ is not a factor of $sb_1 + tc_1$. Then we claim that the syzygy (3.6) is primitive, i.e. it is not a multiple of a syzygy of strictly lower degree. In other words, we have to show that A_u, B_u and C_u have no common factor in this case. Note that $yA_u - xB_u = d(sy+tz)(ya_1 - xb_1)$, $zA_u - xC_u = d(sy + tz)(za_1 - xc_1)$ and $zB_u - yC_u = d(sy + tz)(zb_1 - yc_1'$. Let D be a common irreducible factor of A_u, B_u and C_u, supposed to be a homogeneous polynomial of degree > 0. It is clear that D cannot be $sy + tz$, since $sy + tz$ is not a factor of $sb_1 + tc_1$. Hence D has to divide the polynomials $m_{12} = ya_1 - xb_1$, $m_{13} = za_1 - xc_1$ and $m_{23} = zb_1 - yc_1$. Recall now the construction of the Bourbaki ideal $B(C_1, \rho_1')$ associated to the curve C_1 and to the minimal degree syzygy ρ_1' given by (3.2), as described in [13, Section 5]. It follows that the Bourbaki ideal $B(C_1, \rho_1')$ is contained in the principal ideal generated by D. This is a contradiction, since the Bourbaki ideal $B(C_1, \rho_1')$ defines a subscheme which is either empty (when C_1 is a free curve), or zero-dimensional, see [13, Theorem 5.1].

Therefore the syzygy (3.6) is indeed primitive. It follows that either $r = r_1 + 1$, or $r = r_1$ and $d_2(f) \le r + 1$. Note that in this latter case we have

$$d_1 = d - 1 \le d_1'(f) + d_2'(f) \le r_1 + r_1 + 1 = 2r_1 + 1.$$

Indeed, recall that $d - 1 = d_1'(f) + d_2'(f)$ exactly when C is free, and $d - 1 < d_1'(f) + d_2'(f)$ otherwise, see for instance [27]. □

Proposition 3.4. *With the notation from Theorem 3.3, we have the following equivalent properties.*

(1) *$sy + tz$ is a factor of $sb_1 + tc_1$ for infinitely many $u = (s : t) \in \mathbb{P}^1$;*
(2) *$sy + tz$ is a factor of $sb_1 + tc_1$ for all $u = (s : t) \in \mathbb{P}^1$;*
(3) *the reduced curve $C_1 : f_1 = 0$ is the union of the curve $h = 0$ with a pencil of lines $g = 0$ passing through the point $p = (1 : 0 : 0)$.*

Proof. The fact that (2) implies (1) is clear. First we show that (1) implies (3). Note that $sy+tz$ is a factor of sb_1+tc_1 for infinitely many $u = (s : t) \in \mathbb{P}^1$ if and only if there is a polynomial h of degree $r_1 - 1$ such that $b_1 = yh$ and $c_1 = zh$. Replacing these values in (3.2) we conclude that f_{1x} is divisible by h, say $f_{1x} = hg$, with $\deg g = d_1 - r_1 \ge 1$. If we divide the syzygy (3.2) by h, we get

$$a_1 g + y f_{1y} + z f_{1z} = 0 \tag{3.7}$$

or, equivalently,

$$a_1 g + d_1 f_1 - x f_{1x} = 0. \tag{3.8}$$

It follows that g is a common factor of f_1 and f_{1x}. To conclude the proof of the implication (1) $\implies$ (3) we use the following result, communicated to us by Laurent Busé.

Lemma 3.5. *With the above notation, assume that $g = G.C.D.(f_1, f_{1x})$ has degree ≥ 1. Then g is a homogeneous polynomial in y and z only, and*

$$f_1(x, y, z) = g(y, z)h(x, y, z),$$

for some homogeneous polynomial $h \in S$. In geometric terms, the reduced curve C_1 : $f_1 = 0$ is the union of the curve $h = 0$ with a pencil of lines $g = 0$ passing through the point $p = (1 : 0 : 0)$.

Proof. Let A be an irreducible common factor of f_1 and f_{1x}, such that $f_1 = AU$ for $U \in S$. This implies $f_{1x} = A_x U + AU_x$, and hence, if $A_x \neq 0$, then A has to divide U. Indeed, A cannot divide A_x since $\deg A_x < \deg A$. But this contradicts the fact that $C_1 : f_1 = 0$ is a reduced curve. Hence $A_x = 0$, in other words A is a homogeneous polynomial in y and z only. Since g is a product of such polynomials, the claim is proved. □

Finally we show that (3) implies (2). Assume that $g = G.C.D.(f_1, f_{1x})$ has degree ≥ 1, then one can define a_1 using the above equation (3.8). Then, if we multiply the equation (3.7) by $h = f_{1x} g^{-1}$, we get a primitive syzygy of the form (3.2), where $b_1 = yh$ and $c_1 = zh$. □

4. Examples

Example 4.1. Assume C_1 is an irreducible nodal curve and $L_u : sy + tz = 0$ is a line such that $C = C_1 \cup L_u$ is nodal. Then it is known that $r_1 = d_1 - 1$ and $r = d - 2 = d_1 - 1$, see [10,18]. Note that one has in this case $d_2(f) = r + 1$, see [10, Theorem 4.1]. Hence the case (2) of Theorem 3.3 might occur.

Example 4.2. Consider the rational cuspidal curve $C_1 : f_1 = xy^{d_1 - 1} + z^{d_1} = 0$, $d_1 \geq 3$, which is nearly free, and $L_u : sy + tz = 0$ a line passing through the singular point $p = (1 : 0 : 0)$. Then the syzygy (3.2) becomes

$$(d_1 - 1)x f_{1x} - y f_{1y} = 0.$$

Hence $sb_1 + tc_1 = -sy$ is divisible by $sy + tz$ only for $(s : t) = (1 : 0)$ and for $(s : t) = (0 : 1)$, and we get in these cases $r = r_1 = 1$ using Theorem 3.3 as we see now. The curve $C' : f = xy^{d_1} + yz^{d_1} = 0$ corresponding to $(s : t) = (1 : 0)$ is free, the two generating syzygies being

$$(d_1)^2 x f_x - d_1 y f_y + z f_z = 0$$

and

$$d_1 z^{d_1-1} f_x - y^{d_1-1} f_z = 0$$

satisfying $d'_1(f) + d'_2(f) = 1 + (d_1 - 1) = d - 1$. The curve $C'' : f = xy^{d_1-1}z + z^{d_1+1} = 0$ corresponding to $(s : t) = (0 : 1)$, is nearly free with exponents $d'_1(f) = 1$, $d'_2(f) = d'_3(f) = d-1$. Indeed, note that the curve C'' has two singularities, namely $p = (1 : 0 : 0)$ and $q = (0 : 1 : 0)$. The singularity at q is a simple node A_1, and the singularity at p is given in local coordinates $y' = y/x$ and $z' = z/x$ by $(y')^{d_1-1}z' + (z')^{d_1+1} = 0$. This is a quasi homogeneous singularity, with weights $wt(z') = d^{-1}$ and $wt(y') = d_1[(d_1 - 1)d]^{-1}$. It follows that

$$\tau(C'', p) = \mu(C'', p) = d^2 - 3d + 1$$

and hence the total Tjurina number of C'' is given by

$$\tau(C'') = \tau(C'', q) + \tau(C'', p) = d^2 - 3d + 2.$$

The fact that C'' is nearly free follows now from (2.5).

For $d_1 \geq 4$ and for $L_u : y + z = 0$, we have $r = r' + 1 = 2$ by Theorem 3.3, since $2r_1 < d_1 - 1$ in this case. The corresponding curves C_u are again nearly free, but this time with exponents $d'_1(f) = 2$, $d'_2(f) = d'_3(f) = d - 2$. To see this, one notes that a curve C_u in this family has two singularities, a node and a semi quasi homogeneous singularity $(C_u, p) : g(y', z') = g_0(y', z') + g_+(y', z') = 0$, where g_0 is quasi homogeneous and $g_+(y', z')$ is the sum of two monomials of strictly higher degree. Working in the Milnor algebra $M(g_0)$, we see that the Tjurina algebra of g is isomorphic to the quotient $M(g_0)/(y^{d_1})$. This implies that

$$\mu(C_u, p) = (d_1)^2 - d_1 - 1 \text{ and } \tau(C_u, p) = (d_1 - 1)^2 + 1.$$

It follows that $\tau(C_u) = (d_1 - 1)^2 + 2 = (d - 2)^2 + 2 = (d - 1)^2 - 2(d - 3) - 1$, showing that C_u is nearly free by (2.5).

Example 4.3. Let $C_1 : f_1 = (y^2 - 2xy + z^2)(y^2 + 4xy + z^2) = 0$, be the union of two smooth conics tangent at one point $p = (1 : 0 : 0)$ and meeting transversely at $q_\pm = (0 : 1 : \pm i)$. Then using Singular we see that $r_1 = 2$ and a minimal degree derivation is given by

$$\partial' = xz\partial_x - yz\partial_y + y^2\partial_z.$$

Then the equation $sy + tz$ of a line L passing through the tangency point p divides

$$sb_1 + tc_1 = y(ty - sz)$$

if and only if either $(s : t) = (1 : 0)$ or $(s : t) = (1 : \pm i)$.

The case $(s : t) = (1 : 0)$ corresponds to a common tangent $y = 0$ to the two conics at p. Using Singular, we see that the corresponding curve

$$C : f = yf_1 = y(y^2 - 2xy + z^2)(y^2 + 4xy + z^2) = 0$$

is free with exponents $(2, 2)$, in particular $r = 2 = r_1$ as predicted by Theorem 3.3.

The case $(s : t) = (1 : \pm i)$ corresponds to a line joining the tangency point p to one of the two nodes $q_\pm$ of C_1. Using Singular, we see that the corresponding curve

$$C : f = (y \pm iz)f_1 = (y \pm iz)(y^2 - 2xy + z^2)(y^2 + 4xy + z^2) = 0$$

is nearly free with exponents $(2, 3)$, in particular, again $r = 2 = r_1$ as predicted by Theorem 3.3.

Finally, to see what happens when $sy + tz$ does not divide $sb_1 + tc_1 = y(ty - sz)$, namely when the line through p is general, we consider the special case $(s : t) = (1 : 1)$. Using Singular, we see that the corresponding curve

$$C : f = (y + z)f_1 = (y + z)(y^2 - 2xy + z^2)(y^2 + 4xy + z^2) = 0$$

is a maximal Tjurina curve of type $(d, r) = (5, 3)$, see [14] for the definition and the properties of such curves, and in particular C has exponents $(3, 3, 3, 3)$. Hence $r = 3 = r_1 + 1$. Note that we can show that for any line $L : sy + tz = 0$ with $t \neq 0$, the singularity of C at p is of type D_6. Indeed, it follows easily that this singularity is semi weighted homogeneous of type $(2, 1; 5)$, where $wt(y) = 2$ and $wt(z) = 1$. The claim follows using [6, Corollary (7.39)]. In particular, when $sy + tz$ does not divide $sb_1 + tc_1 = y(ty - sz)$, we always have $\tau(C) = 10$, since there are 4 nodes A_1 on C in addition to the D_6 singularity.

Example 4.4. Let $C_1 : f_1 = (y^2 - xz)^2 + y^2z^2 + z^4 = 0$ be the curve considered in [12, Example 4.1]. This curve is plus one generated with exponents $(d'_1, d'_2, d'_3) = (2, 2, 3)$, in particular $\dim D_0(f_1)_2 = 2$. If we choose the right element in $D_0(f_1)_2$, namely

$$\partial' = (2xy + 3yz)\partial_x + (xz + 2z^2)\partial_y - yz\partial_z,$$

then z divides the coefficient of ∂_z, and it follows that $r = r_1 = 2$ by Theorem 3.3.

5. The general case: the union of two curves

Let $C_1 : f_1 = 0$ and $C_2 : f_2 = 0$ be two reduced curves in $\mathbb{P}^2$, without common irreducible components. We denote $d_j = \deg f_j$ and $r_j = mdr(f_j)$ for $j = 1, 2$. Consider now the union of the two curves $C : f = f_1 f_2 = 0$, and let $d = d_1 + d_2 = \deg f$ and $r = mdr(f)$.

Theorem 5.1. *With the above notation, one has the following.*

(1) *If $\delta_1 \in D_0(f_1)$, then*

$$\delta = f_2\delta_1 - \frac{\delta_1(f_2)}{d}E \in D_0(f),$$

where $E = x\partial_x + y\partial_y + z\partial_z$ denotes the Euler derivation. In particular

$$r \leq \min\{r_1 + d_2, r_2 + d_1\}.$$

(2) *$D_0(f) \subset D(f_1) \cap D(f_2)$. More precisely, for $\delta \neq 0$, one has $\delta \in D_0(f)$ if and only if δ can be written in a unique way in the form*

$$\delta = \frac{h}{d_1}E + \delta_1 = -\frac{h}{d_2}E + \delta_2,$$

where $h \in S$ and $\delta_j \in D_0(f_j)$ are non-zero derivations. In particular

$$r \geq \max\{r_1, r_2\}.$$

Proof. To prove (1), first we check that $\delta(f) = 0$. Then we note that $\delta \neq 0$ if $\delta_1 \neq 0$. Indeed, if $\delta_1(f_2) = 0$, then clearly $\delta = f_2\delta_1 \neq 0$. When $\delta_1(f_2) \neq 0$, note that

$$\delta(f_1) = \frac{d_1 f_1 \delta_1(f_2)}{d} \neq 0.$$

The last claim follows by noting that if δ_1 is a homogeneous derivation then also δ is a homogeneous derivation. Moreover, the roles played by f_1 and f_2 are symmetric.

To prove (2), start with $\delta \in D_0(f)$ and hence

$$\delta(f) = f_2\delta(f_1) + f_1\delta(f_2) = 0.$$

If $\delta(f_1) = 0$, then $\delta(f_2) = 0$ and hence $\delta \in D_0(f_1) \cap D_0(f_2)$. If $\delta(f_1) \neq 0$, then f_2 divides the product $f_1\delta(f_2)$. Since f_1 and f_2 have no common factor by our assumptions, it follows that f_2 divides $\delta(f_2)$, hence $\delta \in D(f_2)$. This is possible only if $\delta \in D(f_1)$ as well. It follows that we can write $\delta \in D_0(f)$ in the form

$$\delta = h_j E + \delta_j$$

where $h_j \in S$ and $\delta_j \in D_0(f_j)$. Clearly $\delta_j \neq 0$, since otherwise $\delta(f) \neq 0$. Then $\delta(f_1) = d_1 h_1 f_1$ and $\delta(f_2) = d_2 h_2 f_2$. It follows that

$$0 = \delta(f) = \delta(f_1) f_2 + f_1 \delta(f_2) = f_1 f_2 (d_1 h_1 + d_2 h_2).$$

Then one implication in the claim follows by taking $h = d_1 h_1 = -d_2 h_2$. The other implication is obvious. □

Remark 5.2. The inequality $r \geq \max\{r_1, r_2\}$ was already noticed in [4, Proposition 3.2. (ii)], where the S-module $D_0(g) = AR(g)$ is denoted by $\mathrm{Syz}(J_g)$ and $mdr(g)$ is denoted by $\mathrm{indeg}(\mathrm{Syz}(J_g))$. Note also that in [4] one works over the polynomial ring in n-variables with coefficients in an arbitrary infinite field. The corresponding result for a product $f = f_1 f_2 \cdot \ldots \cdot f_m$ of $m \geq 2$ forms in n-variables is considered in [4, Proposition 3.5]. Interesting information on the invariant $\mathrm{indeg}(\mathrm{Syz}(J_f))$ when $C : f = 0$ is the union of several smooth plane curves meeting transversally is given in [28, Proposition 3.6].

Corollary 5.3. *With the above notations, $r = mdr(f)$ is the minimal integer s such that either $D_0(f_1)_s \cap D_0(f_2)_s \neq 0$, or $D_0(f_1)_s + D_0(f_2)_s$ contains a non-zero multiple of the Euler derivation E.*

Proof. The first case corresponds to $h = 0$ in Theorem 5.1 (2), while the second case corresponds to $h \neq 0$. □

Example 5.4. Let $C_1 : f_1 = x^2 + y^2 - z^2 = 0$ and $C_2 : f_1 = x^2 + y^2 - 4z^2 = 0$ be two smooth conics with 2 tacnodes as in Proposition 5.5 (3). Hence $d_1 = d_2 = 2$, $r_1 = r_2 = 1$. Note that $y\partial_x - x\partial_y \in D_0(f_1)_1 \cap D_0(f_2)_1$. Therefore, according to Corollary 5.3 we have $r = 1$, see also Proposition 5.5, (3).

Consider next the case $C_1 : f_1 = xyz = 0$ and $C_2 : f_2 = xy + yz + xz = 0$. Then C_2 is a smooth conic circumscribed in the triangle C_1. Using Singular, we see that $r = 2$ and $D_0(f)_2$ is spanned by

$$\delta = 2x(y - z)\partial_x - y(3y + 2z)\partial_y + z(2y + 3z)\partial_z$$

and

$$\delta' = x(3x + 4y - 2z)\partial_x - y(2x + 6y + 2z)\partial_y + z(-2x + 4y + 3z)\partial_z.$$

Then $\delta(f_1) = xyz(y + 3z) = d_2 h_2 f_1$, which implies $h = -d_2 h_2 = -(y + 3z)$. Similarly $\delta'(f_1) = xyz(-x + 2y - z) = d_2 h_2 f_1$, which implies that in this case $h = -d_2 h_2 = x - 2y + z$. It follows that in this case $D_0(f_1)_2 \cap D_0(f_2)_2 = 0$. Therefore, both situations may occur in Corollary 5.3. The fact that $r = 2$ in this case is discussed from another view-point, without the use of Singular, in Example 6.12. The curve C has three D_4 singularities and hence $\tau(C) = 12$. Using the characterization of free curves in (2.4), it follows that C is a free curve.

Let C_1 and C_2 be smooth conics, hence $d_1 = d_2 = 2$ and $r_1 = r_2 = 1$. For $C = C_1 \cup C_2$, Theorem 5.1 gives us $1 \leq r \leq 3$. We have the following precise result.

Proposition 5.5. *The two conics C_1 and C_2 can be in one of the following four situations.*

(1) $|C_1 \cap C_2| = 4$, *and then all the intersection points are nodes for C. In this case $r = 2$.*
(2) $|C_1 \cap C_2| = 3$, *and then one intersection point is a tacnode and the other two intersection points are nodes for C. In this case $r = 2$.*
(3) $|C_1 \cap C_2| = 2$. *Then the two intersection points are either two tacnodes for C, and in this case $r = 1$ and the curve C is nearly free with exponents $(1, 3)$, or a node A_1 and a singularity A_5 for C, and in this case $r = 2$ and the curve C is nearly free with exponents $(2, 2)$.*
(4) $|C_1 \cap C_2| = 1$, *and then the intersection point is a singularity A_7 for C, $r = 1$ and C is a free curve.*

Computations with Singular suggest that in case (1) the curve $C = C_1 \cup C_2$ is a 4-syzygy curve with exponents $(2, 3, 3, 3)$, and in case (2) the curve $C = C_1 \cup C_2$ is a plus one generated curve with exponents $(2, 2, 3)$.

Proof. The claim (1) follows from [10, Theorem 4.1]. For the claim (2) we use the inequalities involving r and the Tjurina number $\tau(C)$ due to du Plessis and Wall, see [17]. In case (2) we have $\tau(C) = 5$. We know that

$$5 = \tau(C) \geq (d-1)(d-1-r) = 3(3-r).$$

This implies $r \geq 2$. For $r = 3$ we also have

$$5 = \tau(C) \leq (d-1)(d-1-r) + r^2 - \binom{2r-d+2}{2} = 3,$$

a contradiction. So the only possibility is $r = 2$. In case (3), when the contact between C_1 and C_2 consists of two tacnodes, using the results in [22], we see that a pair of conics in this situation is projectively equivalent to a pair of conics of the form

$$C_1 : f_1 = x^2 - y^2 - z^2 = 0 \text{ and } C_2 : f_1 = x^2 - y^2 - kz^2 = 0, \tag{5.1}$$

with $k \in \mathbb{C}^*$, $k \neq 0$. We have $\tau(C) = 6$, and the same approach as in Example 5.4 gives $r = 1$ in this case. Since we have $\tau(C) = 6$ in this case, the equality

$$\tau(C) = (d-1)(d-1-r) + r^2 - 1,$$

holds, and it follows from (2.5) that C is nearly free. Assume now that the smooth conics C_1 and C_2 have a contact of type A_5 at $(0:0:1)$. Choosing the coordinates, we may assume that $C_1 : f_1 = yz - x^2 = 0$. Then it is easy to see that the other conic has an equation of the form $C_2 = f_2 = yz - x^2 + a \cdot xy + b \cdot y^2 = 0$ for some $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Since $\tau(C) = 6$, we get as above $r \leq 2$. To show that $r > 1$ one can use Theorem 5.1 (2), since we have a simple description of $D(f_1)_1$. We see that $\delta_1(f_2) \notin \mathbb{C} \cdot f_2$, for any $\delta_1 \in D_0(f_1)_1$. It follows that $r = 2$ and we get the nearly freeness of C as above.

In the case (4), it follows from [3, Proposition 1.3], that the equations of the two conics can be chosen as follows

$$C_1 : f_1 = x(x+y) + yz - a \cdot y^2 = 0 \text{ and } C_2 : f_2 = x(x+y) + yz + a \cdot y^2 = 0$$

for some $a \in \mathbb{C}^*$. It follows that $(f_1)_x = (f_2)_x$, $(f_1)_z = (f_2)_z$, $\delta = (f_1)_z\partial_x - (f_1)_x\partial_z \in D_0(f_1)_1 \cap D_0(f_1)_1 \neq 0$, and hence $r = 1$. Moreover, the freeness of C follows from [7,17]. $\square$

Example 5.6. Let $C_1 : f_1 = (x^2 + y^2 - z^2)(x^2 + y^2 - 4z^2) = 0$ be the union of two smooth conics with 2 tacnodes as in Proposition 5.5 (3). Hence $d_1 = 4$, $r_1 = 1$. Let $C_2 : f_2 = (x - z)(3y^2 - (x + 2z)^2) = 0$ be the union of 3 lines, forming a triangle which is inscribed in the conic C_2 and circumscribed to the conic C_1. Then $d_2 = 3$, $r_2 = 1$. For $C = C_1 \cup C_2$, Theorem 5.1 gives us $1 \leq r \leq 3$. Using Singular we see that $r = 3$, see also Example 6.13 below for a different approach. In fact, C is a free curve with exponents $(3, 3)$ as follows from (2.4), see also [8].

6. The case of quasihomogeneous singularities

Consider the sheafification

$$E_C := \widetilde{AR(f)} = \widetilde{D_0(f)}$$

of the graded S-module $AR(f) = D_0(f)$, which is a rank two vector bundle on $\mathbb{P}^2$, see [26] for details. Moreover, recall that

$$E_C = T\langle C\rangle(-1), \tag{6.1}$$

where $T\langle C\rangle$ is the sheaf of logarithmic vector fields along C as considered for instance in [19,21,9]. One has, for any integer k,

$$H^0(\mathbb{P}^2, E_C(k)) = D_0(f)_k \text{ and } H^1(\mathbb{P}^2, E_C(k)) = N(f)_{k+d-1}, \tag{6.2}$$

where $d = \deg(f)$, for which we refer to [26, Proposition 2.1]. Return now to the setting of the previous section, where $C = C_1 \cup C_2$ and $f = f_1 f_2$, and recall the following result, see [25, Theorem 1.6 and Remark 1.8].

Theorem 6.1. *With the above notation, assume that C_2 is an irreducible curve, and that all singularities of C_1, C_2 and C are quasihomogeneous. If $C_1 \cap C_2$ is contained in the smooth part of C_2, then there is an exact sequence of sheaves on $\mathbb{P}^2$ given by*

$$0 \to E_{C_1}(1 - d_2) \xrightarrow{f_2} E_C(1) \to i_{2*}\mathcal{F} \to 0$$

where $i_2 : C_2 \to \mathbb{P}^2$ is the inclusion and $\mathcal{F}$ a torsion free sheaf on C_2. Moreover, when C_2 is smooth, then one has $\mathcal{F} = \mathcal{O}_{C_2}(-K_{C_2} - R)$, where K_{C_2} is the canonical divisor on C_2 and R is the reduced scheme of $C_1 \cap C_2$.

For simplicity, in this note we consider only the case C_2 smooth. If we set

$$\mathcal{O}_{C_2}(1) = i_2^*\mathcal{O}_{\mathbb{P}^2}(1),$$

then one can write $\mathcal{O}_{C_2}(1) = \mathcal{O}_{C_2}(D)$, where the divisor D corresponds to the intersection of a line in $\mathbb{P}^2$ with the curve C_2, and hence $\deg D = d_2$. With this notation, by tensoring the above exact sequence with $\mathcal{O}_{\mathbb{P}^2}(k-1)$, for any integer k, we get the exact sequence

$$0 \to E_{C_1}(k - d_2) \xrightarrow{f_2} E_C(k) \to i_{2*}\mathcal{O}_{C_2}(-K_{C_2} - R + (k-1)D) \to 0. \tag{6.3}$$

By taking the corresponding long cohomology sequence and using (6.2), we get the following result.

Theorem 6.2. *With the above notation, assume that C_2 is a smooth curve, and that all singularities of C_1 and C are quasihomogeneous. Then there is an exact sequence for any integer k given by*

$$0 \to D_0(f_1)_{k-d_2} \xrightarrow{\phi_k} D_0(f)_k \to H^0(C_2, \mathcal{O}_{C_2}(-K_{C_2} - R + (k-1)D)) \to$$
$$\to N(f_1)_{k-d_2+d_1-1} \xrightarrow{\psi_k} N(f)_{k+d-1} \to H^1(C_2, \mathcal{O}_{C_2}(-K_{C_2} - R + (k-1)D)),$$

where the morphism $\phi_k : D_0(f_1)_{k-d_2} \to D_0(f)_k$ is given by

$$\phi_k(\delta_1) = f_2\delta_1 - \frac{\delta_1(f_2)}{d}E$$

for $\delta_1 \in D_0(f_1)$ and ψ_k is induced by the multiplication by f_2^2. In particular, if

$$(k+2)d_2 < d_2^2 + |R|,$$

then the morphism ϕ_k is an isomorphism and ψ_k is a monomorphism.

Proof. The exact sequence above is part of the long cohomology exact sequence associated to the exact sequence of sheaves (6.3). It remains to explain the claims about the morphisms ϕ_k and ψ_k. Using the identification $D_0(g) = D(g)/SE$, valid for any homogeneous polynomial $g \in S$, it is shown in [25] that the morphism $E_{C_1}(1-d_2) \to E_C(1)$ is induced by the multiplication by f_2. In terms of the modules $D_0(g)$, this is precisely the mapping $D_0(f_1)(-d_2) \to D_0(f)$ given by

$$\phi : \delta_1 \mapsto \delta = f_2\delta_1 - \frac{\delta_1(f_2)}{d}E,$$

as constructed in Theorem 5.1. To explain the formula for ψ_k, consider the diagram of graded S-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_0(f_1)(-d_2) & \xrightarrow{\iota} & S^3(-d_2) & \xrightarrow{\nabla f_1} & J_{f_1}(d_1-d_2-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f_2^2} & & \\
0 & \longrightarrow & D_0(f) & \xrightarrow{\iota} & S^3 & \xrightarrow{\nabla f} & J_f(d-1) & \longrightarrow & 0
\end{array}$$

Here ι are the obvious inclusions, ϕ is the morphism defined above and its extension to a map $Der(S)(-d_2) = S^3(-d_2) \to Der(S) = S^3$ given by the same formula, $\nabla f : S^3 \to J_f$ is the map $(a,b,c) \mapsto af_x + bf_y + cf_z$ and similarly for ∇f_1, while

$$f_2^2 : J_{f_1}(d_1 - d_2 - 1) \to J_f(d-1)$$

is the multiplication by f_2^2. A simple computation shows that this diagram is commutative. Since $N(f_1) = \widehat{J}_{f_1}/J_{f_1}$ and $N(f) = \widehat{J}_f/J_f$, it follows that the morphism $N(f_1)_{k-d_2+d_1-1} \xrightarrow{\psi_k} N(f)_{k+d-1}$ induced by the long cohomology exact sequence, and hence coming from ϕ, is nothing else but multiplication by f_2^2. The final inequality says that

$$\deg(-K_{C_2} - R + (k-1)D) < 0,$$

and so the claim follows from the exact sequence. To see this, recall that

$$\deg K_{C_2} = 2g_{C_2} - 2 = d_2^2 - 3d_2,$$

where g_{C_2} is the genus of the smooth curve C_2. □

Remark 6.3. The formula for ψ_k given in Theorem 6.2 implies the following fact: for any $h \in \widehat{J}_{f_1}$, one has $f_2^2 h \in \widehat{J}_f$. When C_1 is a smooth curve, then $\widehat{J}_{f_1} = S$ and this situation occurs already in [4, Proposition 3.2. (i)], where S is the polynomial ring in n-variables with coefficients in an arbitrary infinite field.

Corollary 6.4. *With the above notation, assume that C_2 is a smooth curve, and that all singularities of C_1 and C are quasihomogeneous. Let R be the reduced scheme of $C_1 \cap C_2$. If*

$$|R| > (r_1 + 1)d_2,$$

then $r = r_1 + d_2$. This applies in particular when C_2 is a generic curve and $r_1 \neq d_1 - 1$.

Proof. The hypothesis $|R| > (r_1 + 1)d_2$ implies that the inequality

$$(k + 2)d_2 < d_2^2 + |R|$$

holds for all $k \leq r_1 + d_2 - 1$. Using Theorem 6.2 and the definition of r_1, it follows that $D_0(f)_k = 0$ for all $k \leq r_1 + d_2 - 1$. The exact sequence in Theorem 6.2 also implies that $D_0(f)_{r_1+d_2} \neq 0$, which proves our claim. When C_2 is a generic curve, then $C_1 \cap C_2$ consists of $d_1 d_2$ nodes for C and the claim is clear. □

Example 6.5. Let C_1 be a reduced curve satisfying $r_1 \neq d_1 - 1$ and such that all the singularities of C_1 are quasihomogeneous. Then for any point $p \notin C_1$ and any line C_2 through p such that C_2 meets transversally C_1 at smooth points, one has $r = r_1 + 1$. The claim follows from Corollary 6.4, since $d_2 = 1$ and $C_1 \cap C_2$ consists of d_1 nodes for C in this case. This result should be compared to Theorem 3.3. Moreover, Example 4.1 shows that the restriction $r_1 \neq d_1 - 1$ is necessary. If we take $p \in C_1$, then the condition that (C, p) is quasihomogeneous *limits drastically the choices* for the line C_2 passing through p. Consider the rational cuspidal curve $C_1 : f_1 = xy^{d_1-1} + z^{d_1} = 0$, with $d_1 > 2$. We have seen in Example 4.2 that, if we take C_2 to be the line through the singular point $p = (1 : 0 : 0)$ given by $y = 0$ or $z = 0$, then (C, p) is quasihomogeneous, and $r = r_1$ in these two cases. In fact, in these cases $|R| = 1$ and Corollary 6.4 does not apply. When C_2 is given by $sy + tz = 0$ with $st \neq 0$, then the singularity (C, p) is not quasihomogeneous, as we have seen in Example 4.2 for the case $s = t = 1$.

Assume from now on that $|R| \leq (r_1 + 1)d_2$, or equivalently $(k + 2)d_2 \geq d_2^2 + |R|$ and set

$$k_0 = d_2 - 2 + \left\lceil \frac{|R|}{d_2} \right\rceil.$$

To simplify the discussion, we also assume that C_2 is a smooth rational curve, hence $d_2 \in \{1, 2\}$. It follows that k_0 is the smallest integer k such that $H^0(C_2, \mathcal{O}_{C_2}(-K_{C_2} - R + (k - 1)D)) \neq 0$. If we assume in addition that C_1 is a free curve, then $N(f_1) = 0$ and Theorem 6.2 implies the following.

Corollary 6.6. *With the above notation and assumptions, if in addition C_1 is a free curve and C_2 is rational, then $|R| \leq (r_1+1)d_2$ implies $k_0 \geq r_1$ and*

$$r_1 \leq r = k_0 \leq r_1 + d_2 - 1.$$

In particular, $|R| > (r_1+1)d_2 - d_2^2$, that is we have the following cases.

(1) *Let $C_1 : f_1 = 0$ be a free curve and L be a line such that C_1 and $C_1 \cup L$ have only quasihomogeneous singularities. Then*

$$|C_1 \cap L| > r_1 = mdr(f_1).$$

(2) *Let $C_1 : f_1 = 0$ be a free curve and Q be a smooth conic such that C_1 and $C_1 \cup Q$ have only quasihomogeneous singularities. Then*

$$|C_1 \cap Q| > 2r_1 - 2, \text{ where } r_1 = mdr(f_1).$$

Proof. Note that $r \geq r_1$ implies $k_0 \geq r_1$, which yields in particular the last claim. □

When C_2 is a line, then $r = r_1$ and $|R| = r_1 + 1$ in these conditions, a known result when C_1 is a line arrangement, see for instance [2, Theorem 3.6 (2)].

Example 6.7. Consider $C_1 : xyz(x-y)(y-z)(x-z) = 0$, which is free with $d_1 = 6$ and $r_1 = 2$. Let C_2 be a general conic passing through 2 triple points and 2 double points of C_1, for instance $C_2 : x^2 + z^2 - xy - yz = 0$. Then $d_2 = 2$, $r_2 = 1$ and $|R| = 6$. Corollary 6.6 implies $r = k_0 = 3$. It follows that the curve C is free with exponents $(3,4)$ by (2.4). Indeed, this curve C has 3 nodes, 4 ordinary triple points and 2 ordinary quadruple points, hence $\tau(C) = 37$.

The application to the exact sequence (6.3) to study free curves goes back to [25]. Now we show that this sequence gives valuable information even when C_1 is not a free curve. We start with the case C_2 is a line, hence we have to decide by Proposition 3.1 or by Theorem 3.3 whether $r = r_1$ or $r = r_1 + 1$.

Corollary 6.8. *With the above notation, assume that C_2 is a line and that all singularities of C_1 and C are quasihomogeneous. Let R be the reduced scheme of $C_1 \cap C_2$. If*

$$|R| \leq r_1 + 1,$$

then there is the following exact sequence

$$0 \to D_0(f)_{r_1} \to H^0(C_2, \mathcal{O}_{C_2}(r_1 + 1 - |R|)) \to N(f_1)_{r_1+d_1-2} \to N(f)_{r_1+d_2+d-2} \to 0.$$

Proof. The proof is as above, using Theorem 6.2 for $k = r_1$ and the fact that $H^1(C_2, \mathcal{O}_{C_2}(\ell)) = 0$ if $\ell \geq 0$. □

Corollary 6.9.

(1) *Let $C_1 : f_1 = 0$ be a nearly free curve and L be a line such that C_1 and $C_1 \cup L$ have only quasihomogeneous singularities. Then*

$$|C_1 \cap L| \geq r_1 = mdr(f_1).$$

(2) *Let $C_1 : f_1 = 0$ be a nearly free curve and Q be a smooth conic such that C_1 and $C_1 \cup Q$ have only quasihomogeneous singularities. Then*

$$|C_1 \cap Q| \geq 2r_1 - 1, \text{ where } r_1 = mdr(f_1).$$

Proof. The proof is as above, using Theorem 6.2 for $k = r_1 - 1$ and the fact that $\dim H^(C_2, \mathcal{O}_{C_2}(\ell)) \geq 2 > \nu(C_1) = 1$ if $\ell \geq 1$. For $d_2 = 2$, we use the stronger fact that $\sigma(C_1) = d_1 + r_1 - 3$. □

Example 6.10. Let C_1 be a nearly free curve having only quasihomogeneous singularities. Let C_2 be a line such that $|R| = r_1$, the minimal possible value, and $C = C_1 \cup C_2$ has again only quasihomogeneous singularities. Then in the exact sequence of Corollary 6.8 we have

$$\dim H^0(C_2, \mathcal{O}_{C_2}(r_1 + 1 - |R|)) \geq 2 = \dim H^0(C_2, \mathcal{O}_{C_2}(1)) > \dim N(f_1)_{r_1+d_1-2} = 1.$$

The last equality follows from the equality $\sigma(C_1) = d_1 + r_1 - 3$, see [11, Corollary 2.17]. This implies $r = r_1$ in this situation.

A first explicit example of such a situation is provided by the curves C_1 discussed in Example 4.2 with the line C_2 given by $y = 0$ or $z = 0$, when $r_1 = 1$.

A second example is provided by the quartic with 3 cusps

$$C_1 : x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z) = 0,$$

which is nearly free with $r_1 = 2$, see [11, Example 2.13] and $C_2 : z = 0$, a line joining 2 cusps. The curve C has in this case one cusp A_2 and two D_5 singularities, hence has only quasihomogeneous singularities. Since $|R| = 2 = r_1$, the above discussion applies and it follows that $r = r_1 = 2$. Using [7,17], it follows that the obtained quintic curve

$$C : x^2y^2z + y^2z^3 + x^2z^3 - 2xyz^2(x + y + z) = 0,$$

is free with exponents $(2, 2)$.

As a third example, consider C_1 to be the union of two smooth conics tangent to each other in two points, as in Proposition 5.5. Let C_2 be the line joining these two points. Then C has two D_6 singularities, $2 = |R| > r_1 = 1$. Hence *Corollary 6.8 cannot be used to conclude.* Note that using the equation (5.1), we see that $y\partial_x + x\partial_y \in D(f_1)_1 \cap D(f_2)_1 \neq 0$, and hence $r = 1$. Using [7] it follows that this curve C is nearly free with exponents $(1, 4)$.

Here is the version of Corollary 6.8 when C_2 is a smooth conic. Here we know already that $r_1 \leq r \leq r_1 + 2$ by Theorem 5.1.

Corollary 6.11. *With the above notation, assume that C_2 is a smooth conic and that all singularities of C_1 and C are quasihomogeneous. Let R be the reduced scheme of $C_1 \cap C_2$. If*

$$|R| \leq 2(r_1 + 1),$$

then there is the following exact sequences.

$$0 \to D_0(f)_{r_1} \to H^0(C_2, \mathcal{O}_{C_2}(2r_1 - |R|)) \to N(f_1)_{r_1+d_1-3} \to N(f)_{r_1+d_2+d-3}$$

and

$$0 \to D_0(f)_{r_1+1} \to H^0(C_2, \mathcal{O}_{C_2}(2r_1 + 2 - |R|)) \to N(f_1)_{r_1+d_1-2} \to N(f)_{r_1+d_2+d-2} \to 0.$$

Proof. Use Theorem 6.2 for $k = r_1$ and for $k = r_1 + 1$. □

Example 6.12. Consider next the case $C_1 : f_1 = xyz = 0$ and $C_2 : f_2 = xy + yz + xz = 0$. Then C_2 is a smooth conic circumscribed in the triangle C_1, as in the second part of Example 5.4. In this case $r_1 = 1$ and $|R| = 3$, hence we can apply Corollary 6.11. The first exact sequence implies that $D_0(f)_1 = 0$, and the second exact sequence implies that

$$\dim D_0(f)_2 = 2 = \dim H^0(C_2, \mathcal{O}_{C_2}(1))$$

since C_1 is a free curve, and hence $N(f_1) = 0$.

Example 6.13. Let $C_1 : f_1 = (x - z)(3y^2 - (x + 2z)^2)(x^2 + y^2 - 4z^2) = 0$ be a smooth conic Q circumscribed in a triangle Δ as in Example 6.12. Let $C_2 : f_2 = (x^2 + y^2 - z^2) = 0$ be a conic inscribed in the triangle Δ and tangent to the conic Q in two points. Then $d_2 = 2$, $r_2 = 1$. In this case $r_1 = 2$ and $|R| = 5$, hence we can apply Corollary 6.11. The first exact sequence implies that $D_0(f)_2 = 0$, and the second exact sequence implies that

$$\dim D_0(f)_3 = 3 = \dim H^0(C_2, \mathcal{O}_{C_2}(1))$$

since C_1 is a free curve, and hence $N(f_1) = 0$.

Example 6.14. Let C_1 be a nearly free curve having only quasihomogeneous singularities. Let C_2 be a smooth conic such that either $|R| \leq 2r_1 - 1$ or $|R| = 2r_1 + 1$ and $C = C_1 \cup C_2$ has again only quasihomogeneous singularities. Then, when $|R| = 2r_1 - 1$, we get exactly as in Example 6.10 that $r = r_1$. Assume now that $|R| = 2r_1 + 1$. In the first exact sequence of Corollary 6.11 we have

$$H^0(C_2, \mathcal{O}_{C_2}(2r_1 - |R|)) = H^0(C_2, \mathcal{O}_{C_2}(-1)) = 0,$$

and hence $D_0(f)_{r_1} = 0$. In the second exact sequence of Corollary 6.11 we have

$$2 = \dim H^0(C_2, \mathcal{O}_{C_2}(2r_1 + 2 - |R|)) = H^0(C_2, \mathcal{O}_{C_2}(1)) > \dim N(f_1)_{r_1+d_1-3} = 1.$$

The last equality follows from the equality $\sigma(C_1) = d_1 + r_1 - 3$, see [11, Corollary 2.17]. This implies $r = r_1 + 1$ in this situation.

To have an explicit example, we consider again the quartic C_1 with 3 cusps from Example 6.10, and take now C_2 to be a smooth generic conic passing through the 3 cusps, then the resulting curve C will have 3 D_5 singularities and 2 nodes A_1. It follows that $|R| = 5 = 2r_1 + 1$. It follows that in this case $r = r_1 + 1 = 3$. As an explicit example, one can take

$$C : (xy + yz + xz)(x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z)) = 0,$$

which is a plus one generated curve with exponents $(3, 3, 4)$.

Example 6.15. Let C_1 be a nearly free curve having only quasihomogeneous singularities. Let C_2 be a smooth conic such that $|R| = 2r_1 + 2$. In the first exact sequence of Corollary 6.11 we have

$$\dim H^0(C_2, \mathcal{O}_{C_2}(2r_1 - |R|)) = \dim H^0(C_2, \mathcal{O}_{C_2}(-2)) = 0,$$

and in the second exact sequence of Corollary 6.11 we have

$$\dim H^0(C_2, \mathcal{O}_{C_2}(2r_1 + 2 - |R|)) = \dim H^0(C_2, \mathcal{O}_{C_2}) = 1.$$

If $N(f_1)_{r_1+d_1-2} = 0$, then we have $r = r_1 + 1$. We consider the following explicit situation. Let $C_1 : f_1 = x^2y^2 + z^4 - xz^3 - 2xyz^3 = 0$, which has an A_4-singularity at $p = (0 : 1 : 0)$ and an A_2-singularity at $q = (1 : 0 : 0)$. Then C_1 is a nearly free curve with $d_1 = 4$ and $r_1 = 2$, see [11, Example 2.13]. Let $C_2 : xy + yz + xz = 0$ be a smooth conic, passing transversely through p and q and meeting C_1 transversally at another 4 points. It follows that $|R| = 2 + 4 = 6 = 2r_1 + 2$ and C has 4 nodes A_1, one D_5 singularity and one D_7-singularity in all. Note that

$$\sigma(C_1) = d_1 + r_1 - 3 = T_{f_1}/2 = 3.$$

This implies that $N(f_1)_4 = 0$, exactly what we need to conclude that $r = r_1 + 1 = 3$.

Example 6.16. We end with an example in order to conclude via Corollary 6.11 we need to analyze the morphism

$$N(f_1)_{r_1+d_1-3} \xrightarrow{f_2^2} N(f)_{r_1+d-1}.$$

Let $C_1 : f_1 = (x^2 - 2xz + y^2)(x^2 + 2xz + y^2) = 0$, hence C_1 is a pair of smooth conics tangent at one point, as in Proposition 5.5 (2). This curve C_1 is a plus one generated curve with exponents $(2, 2, 3)$. Hence $d_1 = 4$ and $r_1 = 2$. Let $C_2 : x^2 + y^2 - 4z^2 = 0$, a circle tangent to each circle in C_1 at one point and passing through the 2 nodes of C_1. Hence C has 3 singularities A_3 and 2 singularities D_4. It follows that $|R| = 4 = 2r_1$. In the first exact sequence of Corollary 6.11 we have

$$\dim H^0(C_2, \mathcal{O}_{C_2}(2r_1 - |R|)) = \dim H^0(C_2, \mathcal{O}_{C_2}) = 1,$$

and $\dim N(f_1)_{r_1+d_1-3} = 2$. More precisely, a basis of $N(f_1)_{r_1+d_1-3}$ is given by the monomials xyz and xz^2. A computation using Singular shows that the kernel of the morphism

$$N(f_1)_{r_1+d_1-3} \xrightarrow{f_2^2} N(f)_{r_1+d-1}$$

is 1-dimensional, generated by $xyz - xz^2$. As a result $D_0(f)_2 = 0$. In the second exact sequence of Corollary 6.11 we have

$$\dim H^0(C_2, \mathcal{O}_{C_2}(2r_1 + 2 - |R|)) = \dim H^0(C_2, \mathcal{O}_{C_2}(2)) = 3$$

and $\dim N(f_1)_{r_1+d_1-2} = 1$. Therefore we have $r = r_1 + 1 = 3$. A computation with Singular shows that C is a plus one generated curve with exponents $(3, 3, 4)$.

6.17. An application to the jumping lines of the rank 2 vector bundle E_{C_1}

For a reduced plane curve C and a line L in $\mathbb{P}^2$, the pair of integers $(d_1^L(C), d_2^L(C))$ such that $d_1^L(C) \leq d_2^L(C)$ and $E_C|_L \simeq \mathcal{O}_L(-d_1^L(C)) \oplus \mathcal{O}_L(-d_2^L(C))$ is called the (ordered) splitting type of E_C along L, see for instance [23]. For a generic line L_0, the corresponding splitting type $(d_1^{L_0}(C), d_2^{L_0}(C))$ is known to be constant, see [23, Definition 2.2.3 and Lemma 3.2.2]. A line L in $\mathbb{P}^2$ is called a jumping line for E_C or, equivalently, for $T\langle C\rangle$, if

$$d_1^{L_0}(C) - d_1^L(C) > 0.$$

The following result relates the splitting type of E_C along a line $L : \alpha_L = 0$, to the Lefschetz properties of the Jacobian module $N(f)$ with respect to the multiplication by α_L, see [13, Proposition 4.1].

Proposition 6.18. *For any reduced curve $C : f = 0$ and any line $L : \alpha_L = 0$ in $\mathbb{P}^2$, we have $d_1^L(C) = \min\{mdr(f), k(f, L)\}$, where*

$$k(f, L) = \min\{k \in \mathbb{N} \ : \ N(f)_{k+d-2} \stackrel{\cdot\alpha_L}{\to} N(f)_{k+d-1} \text{ is not injective } \}.$$

Using this result, we give now an easy geometric way to check that a line is a jumping line, under some conditions.

Theorem 6.19. *Let $C_1 : f_1 = 0$ be a reduced curve and $C_2 : f_2 = 0$ be a line in $\mathbb{P}^2$. Assume that all the singularities of C_1 and of $C = C_1 \cup C_2$ are quasihomogeneous, and let R be the reduced scheme of $C_1 \cap C_2$. If $|R| < r_1 + 1$, then for any k satisfying $|R| - 1 \le k < r_1$, the morphism*

$$\psi_k' : N(f_1)_{k+d_1-2} \xrightarrow{f_2} N(f_1)_{k+d_1-1}$$

is not injective and one has

$$d_1^{C_2}(C_1) = k(f_1, C_2) \le |R| - 1.$$

Moreover, if one of the following two conditions holds

(1) *either $2r_1 < d_1$, or*
(2)

$$2r_1 \ge d_1 \text{ and } |R| - 1 < \left\lfloor \frac{d_1 - 1}{2} \right\rfloor,$$

then C_2 is a jumping line for the rank two vector bundle E_{C_1}.

Proof. Note that the condition $k \ge |R| - 1$ implies that

$$H^0(C_2, \mathcal{O}_{C_2}(-K_{C_2} - R + (k-1)D)) = H^0(C_2, \mathcal{O}_{C_2}(k + 1 - |R|)) \ne 0$$

in the exact sequence from Theorem 6.2. On the other hand, the condition $k < r_1$ implies that $D_0(f)_k = 0$. Hence, using Theorem 6.2, we see that the morphism

$$\psi_k : N(f_1)_{k+d_1-2} \xrightarrow{f_2^2} N(f)_{k+d_1},$$

is not injective. To prove our claim, it is enough to show that

$$\ker \psi_k \subset \ker \psi_k'.$$

Let $h \in \ker \psi_k$ be (the representative of) some element in this kernel. This means that $f_2^2 h \in J_f$, in other words there is a derivation $\delta \in Der(S)$ such that

$$f_2^2 h = \delta(f) = f_2\delta(f_1) + f_1\delta(f_2).$$

Since f_1 and f_2 have no common factor, this means that $\delta(f_2)$ is divisible by f_2, say $\delta(f_2) = f_2 g$ for some $g \in S$. Dividing the above relation by f_2 we get

$$f_2 h = \delta(f_1) + f_1 g,$$

which implies $f_2 h \in J_{f_1}$. Hence $h \in \ker \psi'_k$ as we claimed. Now we prove the final claims in Theorem 6.19. Since we know that ψ'_k is not injective for $|R| - 1 \leq k < r_1$, it follows by Proposition 6.18 that

$$d_1^{C_2}(C_1) = k(f_1, C_2) \leq |R| - 1.$$

Assume now $2r_1 < d_1$. First note that the curve C_1 cannot be a free curve in view of Corollary 6.6 (1) saying that in this case $|R| > r_1$. If C_1 is a nearly free curve, then the exponents $r = d'_1 \leq d'_2$ verify $d'_1 + d'_2 = d_1$, and hence the condition $2r_1 < d_1$ implies $d'_1 < d'_2$. Using for instance [13, Example 4.8] we see that in this case $d_1^{L_0}(C_1) = r_1$. The same equality holds for all the other non free reduced plane curves C_1 satisfying $2r_1 < d_1$, see [13, Corollary 4.5]. This fact implies that C_2 is a jumping line for E_{C_1} in the case (1). The claim in case (2) follows from [13, Corollary 4.6 and Example 4.8]. Indeed, these results imply that we have

$$d_1^{L_0}(C_1) = \left\lfloor \frac{d_1 - 1}{2} \right\rfloor$$

when $2r_1 \geq d_1$. □

Example 6.20. Let $C_1 : f_1 = 0$ be a Thom-Sebastiani plane curve, i.e. a curve such that $f_1(x, y, z) = g(x, y) + z^{d_1}$, where g is a homogeneous polynomial of degree d_1 in $S' = \mathbb{C}[x, y]$. Assume that

$$g = \ell_1^{k_1} \cdots \ell_m^{k_m},$$

where the linear forms $\ell_j \in S'$ are distinct and $m \geq 2$. It follows that C_1 is a 3-syzygy curve with exponents $r_1 = d'_1 = m - 1$ and $d'_2 = d'_3 = d_1 - 1$ for $m \geq 3$ and C_1 is nearly free with exponents $r_1 = d'_1 = 1$ and $d'_2 = d_1 - 1$ for $m = 2$, see [12, Example 4.5].

Assume first that $2(m-1) < d_1$ and take the line C_2 to be given by one of the factors of g, say $C_2 : f_2 = \ell_1 = 0$. Then it is easy to check that all the singularities of C_1 and of $C = C_1 \cup C_2$ are quasihomogeneous. To do this, one can assume that $\ell_1 = x$ and hence R is the point $(0 : 1 : 0)$. It follows that $|R| = 1 < r_1 + 1$, and Theorem 6.19 (1) implies that the line C_2 is a jumping line for E_{C_1} and moreover

$$d_1^{C_2}(C_1) = k(f_1, C_2) = 0.$$

Note that the inequality $d_1^{C_2}(C_1) = k(f_1, C_2) \geq 0$ holds in general, see for instance [13, Proposition 2.5].

If we assume now that $2(m-1) \geq d_1 \geq 3$, then we get the same result using Theorem 6.19 (2). In this way we have found out m jumping lines $\ell_j = 0$ for $j = 1, \ldots, m$ for the vector bundle E_{C_1}.

Recall that when C_1 is a free curve, then all the lines L in $\mathbb{P}^2$ are not jumping lines for the vector bundle E_{C_1}. With this in mind, the following result, which is a reformulation of Theorem 6.19, can be regarded as a generalization of Corollary 6.6 (1).

Corollary 6.21. *Let $C_1 : f_1 = 0$ be a reduced curve and L be a line in $\mathbb{P}^2$, which is not a jumping line for the vector bundle E_{C_1}. Let $d_1 = \deg(f_1)$ and $r_1 = mdr(f_1)$. If all the singularities of C_1 and of $C_1 \cup L$ are quasihomogeneous, then*

$$|C_1 \cap L| > d_1^L(C_1) = \begin{cases} r_1 & \text{if } \ 2r_1 < d_1, \\ \left\lfloor \frac{d_1-1}{2} \right\rfloor & \text{if } \ 2r_1 \geq d_1. \end{cases}$$

Data availability

Data will be made available on request.

References

[1] T. Abe, Plus-one generated and next to free arrangements of hyperplanes, Int. Math. Res. Not. 2021 (12) (2021) 9233–9261, https://doi.org/10.1093/imrn/rnz099.

[2] T. Abe, A. Dimca, G. Sticlaru, Addition–deletion results for the minimal degree of logarithmic derivations of hyperplane arrangements and maximal Tjurina line arrangements, J. Algebraic Comb. 54 (2021) 739–766, https://doi.org/10.1007/s10801-020-00986-9.

[3] J.W. Bruce, P.J. Giblin, A stratification of the space of plane quartic curves, Proc. Lond. Math. Soc. 42 (1981) 270–298.

[4] R. Burity, A. Simis, S. Tohaneanu, On the Jacobian ideal of central arrangements, arXiv:2101.02735.

[5] A. Cerminara, A. Dimca, G. Ilardi, On the Hilbert vector of the Jacobian module of a plane curve, Port. Math. 76 (2020) 311–325.

[6] A. Dimca, Topics on Real and Complex Singularities, Vieweg Advanced Lecture in Mathematics, Friedr. Vieweg und Sohn, Braunschweig, 1987.

[7] A. Dimca, Freeness versus maximal global Tjurina number for plane curves, Math. Proc. Camb. Philos. Soc. 163 (2017) 161–172.

[8] A. Dimca, P. Pokora, On conic-line arrangements with nodes, tacnodes, and ordinary triple points, J. Algebraic Comb. 56 (2022) 403–424, https://doi.org/10.1007/s10801-022-01116-3.

[9] A. Dimca, E. Sernesi, Syzygies and logarithmic vector fields along plane curves, J. Éc. Polytech. Math. 1 (2014) 247–267.

[10] A. Dimca, G. Sticlaru, Koszul complexes and pole order filtrations, Proc. Edinb. Math. Soc. 58 (2015) 333–354.

[11] A. Dimca, G. Sticlaru, Free and nearly free curves vs. rational cuspidal plane curves, Publ. Res. Inst. Math. Sci. 54 (2018) 163–179.

[12] A. Dimca, G. Sticlaru, Plane curves with three syzygies, minimal Tjurina curves, and nearly cuspidal curves, Geom. Dedic. 207 (2020) 29–49.

[13] A. Dimca, G. Sticlaru, On the jumping lines of bundles of logarithmic vector fields along plane curves, Publ. Mat. 64 (2020) 513–542.

[14] A. Dimca, G. Sticlaru, Jacobian syzygies, Fitting ideals, and plane curves with maximal global Tjurina numbers, Collect. Math. 73 (2022) 391–409, https://doi.org/10.1007/s13348-021-00325-6.
[15] I. Dolgachev, M. Kapranov, Schur quadrics, cubic surfaces and rank 2 vector bundles over the projective plane, Astérisque 218 (1993) 111–144.
[16] I. Dolgachev, M. Kapranov, Arrangements of hyperplanes and vector bundles on P^n, Duke Math. J. 71 (1993) 633–664.
[17] A.A. du Plessis, C.T.C. Wall, Application of the theory of the discriminant to highly singular plane curves, Math. Proc. Camb. Philos. Soc. 126 (1999) 259–266.
[18] Ph. Ellia, Quasi complete intersections and global Tjurina number of plane curves, J. Pure Appl. Algebra 224 (2020) 423–431.
[19] D. Faenzi, J. Vallès, Logarithmic bundles and line arrangements, an approach via the standard construction, J. Lond. Math. Soc. 90 (2014) 675–694.
[20] S.H. Hassanzadeh, A. Simis, Plane Cremona maps: saturation and regularity of the base ideal, J. Algebra 371 (2012) 620–652.
[21] S. Marchesi, J. Vallès, Nearly free curves and arrangements: a vector bundle point of view, Math. Proc. Camb. Philos. Soc. 170 (2019) 51–74.
[22] G. Megyesi, Configurations of conics with many tacnodes, Tohoku Math. J. (2) 52 (4) (2000) 555–577.
[23] C. Okonek, M. Schneider, H. Spindler, Vector Bundles on Complex Projective Spaces, Progress in Math., vol. 3, Birkhauser, 1980.
[24] H. Schenck, Elementary modifications and line configurations in P^2, Comment. Math. Helv. 78 (3) (2003) 447–462.
[25] H. Schenck, H. Terao, M. Yoshinaga, Logarithmic vector fields for curve configurations in $\mathbb{P}^2$ with quasihomogeneous singularities, Math. Res. Lett. 25 (2018) 1977–1992.
[26] E. Sernesi, The local cohomology of the Jacobian ring, Doc. Math. 19 (2014) 541–565.
[27] A. Simis, S.O. Tohăneanu, Homology of homogeneous divisors, Isr. J. Math. 200 (2014) 449–487.
[28] S.O. Tohăneanu, On freeness of divisors on $\mathbb{P}^2$, Commun. Algebra 41 (2013) 2916–2932.