Anti Boolean Conditions
for sets of subgroups of a group

preprint version, january 2025

Ulderico Dardano *and* Fausto De Mari
ulderico.dardano@unina.it , fausto.demari@unina.it
Università degli Studi di Napoli "Federico II"

Abstract

We show that, in the universe of generalised soluble groups and for the most common generalisations χ of normality, the condition that in the set of *non*-χ subgroups of a group there are no copies of the boolean algebra of subsets of a countable set guarantiees that the structure of the group is under control. This incorporates many previuos results from the wider literature on the topic. Moreover, as an application, we point out how some specific antichains control the cardinality of the whole set of subgroups.

Mathematics Subject Classification (2020): 20F22, 20F19, 20E07

Keywords: order embedding, antichain, scattered poset, generalised normal subgroup, minimax group.

1 Introduction and statement of results

The study of groups G with "many" subgroups having a certain property χ is a standard topic in the theory of groups (see [6] for a survey dating 2009 and then [4]). Often one sets restrictions on the set $\bar{\chi}(G)$ of all $\bar{\chi}$:=*non*-χ subgroups of G in order to evaluate how far G is from having all χ-subgroups. In the literature, such restrictions appears to be mainly of two types:

• *chain conditions*, i.e., restrictions on the order type of linearly ordered subsets of $\bar{\chi}(G)$, as introduced by Baer and Zaicev (see [4, 5] for details);

• *restictions on the cardinality* of $\bar{\chi}(G)$ (as for example in [2]).

With the aim of viewing these two approches in the same framework, in this note we introduce a new condition: the *Anti Boolean Condition*, ABC for short. We say that a partially ordered set, poset for short, $\mathcal{P}$ has ABC if there is no order-embedding $f : P(\mathbb{N}) \to \mathcal{P}$, where $P(\mathbb{N})$ denotes the standard (boolean) lattice of subsets of $\mathbb{N}$ (or any other countable infinite set with a harmless abuse of terminology). Recall that an *order embedding* is a map $f : S \to T$ between ordered sets such that $x \leq y$ if and only if $f(x) \leq f(y)$, for all $x, y \in S$.

Clearly, all intervals in an ABC poset have ABC. Moreover, by standard Set Theory results we have that the following posets all embed into $P(\mathbb{N})$:

- any countable (including non linear) poset,
- the real line $\mathbb{R}$,
- any antichain with cardinality at most $|\mathbb{R}| = |P(\mathbb{N})| = 2^{\aleph_0}$.

Here, as usual, by an *antichain* in a poset, we mean a subset of pairwise incomparable elements.

Returning to groups, we say that a group G has the *Anti Boolean Condition on $\bar{\chi}$-subgrops*, ABC-$\bar{\chi}$ for short, if the poset $\bar{\chi}(G)$ has ABC. When $\chi = T$, an always satisfied property, we just write ABC for ABC-T. It is not difficult to see that *a soluble-by-finite group has ABC on all subgroups, if and only if it is minimax* (see below), that is it has a finite series whose factors are either cyclic or quasicyclic or finite. The class of such groups has been widely studied, is resonably well understood and, in the universe of generalised radical group, it may be characterised as in Theorem 2.(0) below. Recall that a *generalised radical* group is a group having an ascending (normal) series with locally (nilpotent or finite) factors. As usual in the literature, and along the same lines as in [4, 5], we will consider some of the most popular generalization of normality in the role of property χ, and the chain restrictions (1-5) explicitly described in the following:

Framework Statement *For a generalised radical group G and a subgroup property χ, the following conditions are equivalent:*

($\grave{1}$) Min-∞-$\bar{\chi}$: the weak minimal condition on non-χ subgroups, i.e., the set of non-χ subgroups of G, with respect to the order relation of being a subgroup with infinite index, does not contain any subset which is order isomorphic to the poset of negative integers $-\mathbb{N}$;

($\acute{1}$) Max-∞-$\bar{\chi}$: the weak maximal condition on non-χ subgroups, i.e., the set of non-χ subgroups of G, with respect to the order relation of being a subgroup with infinite index, does not contain any subset which is order isomorphic to the poset of positive integers $\mathbb{N}$;

(2) DCC-∞-$\bar{\chi}$: the weak double chain condition on non-χ subgroups, i.e., the set of non-χ subgroups of G, with respect to the order relation of being a subgroup with infinite index, does not contain any subset which is order isomorphic to the poset of integers $\mathbb{Z}$;

(3) QCC-$\bar{\chi}$: the rational chain condition on non-χ subgroup, i.e., the poset of non-χ subgroups of G, with respect to the usual ordering, does not contain any subset which is order isomorphic to the poset of rational numbers $\mathbb{Q}$;

(4) RCC-$\bar{\chi}$: the real chain condition on non-χ subgroups, i.e,. the poset of non-χ subgroups of G, with respect to the usual ordering, does not contain any subset which is order isomorphic to the poset of real numbers $\mathbb{R}$;

(5) ABC-$\bar{\chi}$: the Anti Boolean Condition on non-χ subgroups, i.e., the poset of non-χ subgroups of G, with respect to the usual ordering, does not contain any subset which is order isomorphic to the boolean algebra $\mathcal{P}(\mathbb{N})$;

It is clear that $(\grave{1})\vee(\acute{1}) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) = ABC\text{-}\bar{\chi} \Leftarrow ABC\text{-}\bar{\chi}_1 \Leftarrow ABC$, whenever $\chi \Leftarrow \chi_1$, for any group G. Properties $(\grave{1})$, $(\acute{1})$ and (2) have been shown to be equivalent for χ=T and G locally (soluble-by-finite) in [15].

A poset as in (3), i.e., which does not contain copies of $\mathbb{Q}$ (ore equivalently a non-empty dense totally ordered countable subset), is usually called *scattered.* Note that an order condition equivalent to (3) has been introduced in [13] in connection with the so-called Krull dimension of a module, and then further studied in [7] and in the very recent preprint [14], by using the more complex concept of *deviation*, an ordinal number associated to a well behaved poset. However, since *a poset has deviation if and only if it is scattered* (see [3], 6.1.3), definition (3) is sufficient for our purposes.

Due to the existence of Tarski groups, this kind of investigations are usually restricted to some universe of generalised soluble groups. In [4, 5] the authors introduced QCC and RCC and showed that, excluding the novelty of item (5), the Framework Statement holds for generalised radical groups. Here, by Theorem 2, we show that we can safely add condition (5), that is ABC.

Given conditions ($\grave{1}$), ($\acute{1}$) and (2), one is tempted to introduce the conditions QCC-∞ , RCC-∞ and ABC-∞, where the suffix -∞ means again that one is no longer considering the usual setwise order relation $\leq$ among subgroups, but the stronger relation, say $\lll$, of being either equal or a subgroup with infinite index, instead. Then one might rewrite the whole Framework Statement by considering the order relation $\lll$ and discarding about the suffix -∞. However, this would be pointless, as the next Proposition shows.

Proposition 1 *For any group and any subgroup property χ, the conditions QCC-∞-χ, RCC-∞-χ and ABC-∞-χ are equivalent to QCC-χ, RCC-χ and ABC-χ, resp.*

For details and definitions which are not recalled here we refer to [11] and [4, 5], where as χ we considered the standard properties as in the next Definition. They are all generalization of normality, (n). On the other hand, apart from the fact that they generalise normality and that "$(pr) \Rightarrow (m)$", there are no further implication statements among them (in the general case of infinite groups).

Definitions. *A subgroup H of a group G is called:*

- *(n)* normal, *if H coincides with its conjugates in G, that is $H^g = H$ for all $g \in G$;*
- *(an)* almost normal, *if H has finitely many conjugates in G, that is $|G : N_G(H)| < \aleph_0$;*
- *(nn)* nearly normal, *if H has finite index in a normal subgroup N of G;*
- *(m)* modular, *if $\langle A, H \cap B\rangle = \langle A, H\rangle \cap B$ holds for $A, B \leq G$ whenever $A \leq B$ and $\langle H, A \cap B\rangle = \langle A, H\rangle \cap B$ holds for $A, B \leq G$ whenever $H \leq B$;*
- *(pr)* permutable, *if $\langle H, K\rangle = HK = KH$ for each subgroup $K \leq G$;*
- *(pn)* pronormal, *if for each $g \in G$ there is $k \in \langle H, H^g\rangle$ such that $H^k = H^g$;*
- *(sn)* subnormal, *if there is a finite series between H and G.*

Here we show that in the result in [4, 5] we can safely add condition (5), as it follows from our comprehensive statement, Theorem 1, whose proof is given in sect. 2.

Theorem 2 *Let G be a generalised radical group. Then:*

(0) G *has ABC if and only if it is a soluble-by-finite minimax group;*

(0′) G *has ABC-sn if and only if it is minimax, provided G is soluble-by-finite;*

(1) G *has ABC-$\bar{\chi}$ if and only if G satisfies Min-∞-$\bar{\chi}$, for χ equal to any of all above properties n, nn, an, m, pm, pn;*

(2) G *has ABC-$\bar{\chi}$ if and only if G has ABC or all subgroups have χ, where χ is any of the properties n, m, pr, pn;*

(2′) G *has ABC-$\bar{sn}$ if and only if G has ABC or all subgroups have sn, provided G is soluble and periodic.*

Unfortunately, (2′) does not hold for non-periodic groups. For a counterexample, see the comments before Lemma 5.1 in [4].

As a next step we propose an other application of this "boolean" approach: the study of the cardinality of the antichains of subgroups. Recall that, on the one hand, in [1] groups whose antichains of subgroups have cardinality at most n (for some fixed $n \in \mathbb{N}$) have been studied. On the other hand, in [2] Theorem 2.7 *locally (soluble-by-finite) groups with countably many subgroups (CMS) have been characterised*, as in the statement (2) of Corollary 4 below.

Even if the case of a Prüfer p-group $\mathcal{C}_{p^\infty}$ shows that in an infinite group we may have that all antichains of subgroups consist of at most a single element, we will see that, however, the cardinality of some "strong" antichains of subgroups is *a further tool for distinguishing whether the group has CMS*, as in the next corollary. By a *strong antichain* we mean a set of pairwise *strongly incommensurable* subgroups of a group G, where two subgroups H and K are strongly incommensurable if $H \cap K$ has infinite index in both of them, that is $H \gg H \cap K \ll K$. Note that our use of the expression *strong antichain* is similar but not equal to the one used in group theory. Let us first remark a simple and fundamental fact, which will be proved in this note.

Remark 3 *A group has a* strong *antichain (of normal subgroups) with cardinality $2^{\aleph_0}$, provided it is either the direct product $G = \mathrm{Dr}_{i \in I}\, G_i$ of an infinite family of non-trivial subgroups or the direct square $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ of a Prüfer p-group.*

Corollary 4 *For a generalised radical group G, the following are equivalent:*

(1) G *has no subgroup strong antichain with cardinality $2^{\aleph_0}$;*

(2) G *is soluble-by-finite minimax with no (subnormal) section of type $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$;*

(3) G *has only countably many subgroups.*

Further, if G is soluble-by-finite, these are equivalent also to:

(1′) G *has no strong antichain with cardinality $2^{\aleph_0}$ consisting of subnormal subgroups.*

2 Proofs

Proof of Proposition 1. It is trivial that QCC-χ, RCC-χ and ABC-χ imply QCC-∞-χ, RCC-∞-χ and ABC-∞-χ, respectively, since the relation $\lll$ is stronger then the relation $\leq$.

About the converse, to settle the cases QCC and RCC, assume that a (nontrivial) dense poset $\mathcal{P}$ (as $\mathbb{Q}$ and $\mathbb{R}$) does not embed in $\chi(G)$ with respect to $\lll$, but -by contradiction- there is a usual order embedding w.r.t. $\leq$, say $e : \mathcal{P} \to \chi(G)$. Thus, for any $i < j$ in $\mathcal{P}$, by density, there exists an ascending countable chain $i < i_1 < \cdots < i_n < \cdots < j$ of elements of $\mathcal{P}$. Therefore there is a corresponding chain $H_i < H_{i_1} < \cdots < H_{i_n} < \cdots < H_j$. Thus, $H_i \ll H_j$ for all $i < j$ and e is an embeddig also w.r.t. $\lll$, which is the desired contradiction.

In the case of ABC, suppose $\chi(G)$ has ABC-∞ while $e : P(\mathbb{N}) \to \chi(G)$ is a usual order embedding (w.r.t. $\leq$). Take any bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and consider the induced order-preserving bijection $\bar{f} : P(\mathbb{N} \times \mathbb{N}) \to P(\mathbb{N})$. Then let $g : X \in P(\mathbb{N}) \to X \times \mathbb{N} \in P(\mathbb{N} \times \mathbb{N})$. We have that

$$e' := e\bar{f}g : P(\mathbb{N}) \to \chi(G)$$

is an order embedding as a composition of order embeddings. We will reach a contradiction by showing that if $X \subset Y$, then the index of $e'(X)$ in $e'(Y)$ is infinite. To this aim, take $y \in Y \setminus X$ and set $X_n = (X \times \mathbb{N}) \cup (\{y\} \times \{0, 1, ..., n\})$ for $n \in \mathbb{N}$. Then $g(X) \subset X_n \subset X_{n+1} \subset g(Y)$ where inequalities are strict. By appling the order embedding $e\bar{f}$ we have $e'(X) \subset e\bar{f}(X_n) \subset e\bar{f}(X_{n+1}) \subset e'(Y)$ for each n, so that $e'(X) \ll e'(Y)$. ■

Before proving our main Theorem, we consider a proposition which is indipendent from solvability conditions.

Proposition 5 *Any group with a finite series whose factors satysfy either Min or Max has QCC, hence ABC.*

Proof. Since either Min or Max imply QCC and, in turn, QCC-groups are ABC, it is enough to show that if N and G/N satisfy QCC, then G satisfies QCC as well. Let by contradiction $i \in \mathbb{Q} \to G_i \in \mathcal{L}(G)$ be an order embedding. Then the increasing map $i \mapsto G_i \cap N$ is not injective. Hence there are $i < j$ such that $G_i \cap N = G_j \cap N$. Similarly, $G_x \cap N = G_j \cap N$ for each $x \in [i, j]$. Note that $G_x/G_x \cap N \simeq G_xN/N$ for every $x \in]i, j[$, therefore the map $x \in]i, j[\to G_xN/N \in \mathcal{L}(G/N)$ is an order embedding. Since $]i, j[$ and $\mathbb{Q}$ have the same order type, we have reached the contradiction. ■

Proof of Theorem 2.(0). The sufficiency follow from Proposition 5. To prove necessity, first recall that any abelian group which is not minimax has an homomorphic image which is the direct product of infinitely many non-trivial subgroups (see for instance Lemma 3.2 of [7]). Then an abelian ABC-group is minimax. By a celebrated result by Shunkov [12], it follows that any locally finite ABC-group is a Chernikov group (and hence it is soluble-by-finite). Therefore any generalised radical ABC-group is radical-by-finite by Proposition 2.3 of [4] and so it is a soluble-by-finite minimax group by a theorem by Baer (see [10] Part 2, Theorem 10.35). ■

Proof of Theorem 2.(0′). It is now enough to note that the abelian normal factors of G have full ABC and apply the above result. ■

To prove the remaing parts of Theorem 2, we can proceed in the same way as in [4] by ignoring Proposition 2.1 and replacing RCC by ABC in all statements but Lemma 2.5 and 4.1 (and related statements and proofs) which can be replaced by the next two statements respectively.

Lemma 6 *Let G be a group with a section H/K which is the direct product of an infinite collection $(H_\lambda/K)_{\lambda\in\Lambda}$ of non-trivial subgroups, and let L be a subgroup of G such that $L\cap H \leq K$ and $\langle H_\lambda, L\rangle = H_\lambda L$ for each λ. If the interval $[H/K]$ has ABC on non-χ subgroups of G, then there exists a normal subgroup H^* of H containing K such that $LH^* = H^*L$ is a χ-subgroup of G.*

Proof. Clearly we may assume $\Lambda = \mathbb{N}$ and H_i/K is infinite for each $i \in \mathbb{N}$. As above, we have that the following map is an order embedding.

$$I \in P(\mathbb{N}) \mapsto H_I/K := \langle H_i/K \mid i \in I\rangle \in \mathcal{L}(H/K)$$

Therefore there is some $I \in P(\mathbb{N})$ such that if $H^* := H_I$, then the subgroup LH^* is a χ-subgroup of G. ■

Lemma 7 *Let G be a group with a section H/K which is the direct product of an infinite collection $(H_\lambda/K)_{\lambda\in\Lambda}$ of non-trivial subgroups, and let x be an element of G such that $\langle x\rangle\cap H \leq K$. If G has ABC-$\bar{m}$ (resp. ABC-$\bar{pr}$), then there exists a subgroup L of H such that both L and $\langle x, L\rangle$ are modular (resp. permutable) subgroups of G.*

Proof. Note that the map $f : I \in P(\mathbb{N}) \mapsto \langle L_I, x\rangle \in \mathcal{L}(G)$, where $L_I := \langle H_i \mid i \in I\rangle$, is increasing, clearly. Moreover, since $\langle x\rangle \cap H \leq K \leq L_I \leq H$, each subgroup L_I is modular by Lemma 2.6 in [4]. From $L_I \leq \langle L_I, x\rangle \cap H = \langle L_I, \langle x\rangle \cap H\rangle \leq L_I$, it follows that the map f is also injective, i.e. an order embedding. Thus there is some modular (resp. permutable) subgroup $L = L_I$ such that $f(I) = \langle x, L\rangle$ is modular (resp. permutable) as well. ■

Proof of Theorem 2.(1). The result follows now as in [4] for what concerns properties $\chi = n, an, nn$ and as in [5] for properties m, pr. ■

Proof of Theorem 2.(2). For what concerns the properties n, m, pr, pn, apply Theorem 2 together with Theorems 3.11, 4.4, 4.6 in [4] and Theorem 2.23 in [5], respectively.

For what concerns property pn, note that all statements of [5] remain valid if one substutite the sentences "the is no $\mathbb{R}$-chain of non-pronormal sungroups" or "RCC on non-pronormal subgroups" by the condition ABC-$\bar{pn}$. ■

To complete the proof of Theorem 2.(2′) we state a Lemma.

Lemma 8 *Let $G = AB$ a periodic Baer group, where A is an abelian normal subgroup of G and B is abelian and divisible. If G has ABC-$\bar{sn}$, then G is abelian.*

Proof. Let G be a counterexample. Since any subnormal abelian divisible subgroup of a periodic Baer group G, is contained in the centre of G (see for instance [8], Lemma 5.1), we have that B is not subnormal in G. Hence G is not nilpotent.

On the other hand B is a direct product of a family of Prüfer subgroups and one of them, say P, does not centralize A. Hence also AP is couterexample and so, without loss of generality, it can be assumed that $B = P$ is a Prüfer group. Since $A \cap B \leq C_B(A) \leq B_G$ and G/B_G is still a counterexample, we may assume also that $A \cap B = C_B(A) = B_G = \{1\}$.

Since A is normal and abelian, each subgroup of A is subnornal in G. Hence it has ABC, whence it is a Chernikov group by Theorem 1.(0). Thus the whole $G = AB$ is a Chernikov group. Hence G is nilpotent, as it is a Baer group with the minimal condition (apply again the argument of the first paragraph of this proof keeping in mind that $B \leq Z(G)$ and $G/B = AB/B$ is abelian). This gives the desired contradiction. ■

Proof of Theorem 2.(2′). To prove the statement about the property sn, note that one can proceed in the same way as in sect. 5 of [4] but one has to replace Lemma 5.4 in [4] by Lemma 8 above. ■

The proof of Theorem 2 is now completed.

Proof of Remark 3. Let $G = Dr_{i \in I}\, G_i$. By assemblying the sugroups G_i, we can assume that $I = \mathbb{N}$ or even $I = \mathbb{Q}$, as the latter is countable. Then, for all $x \in \mathbb{R}$, the subgroups $\hat{G}_x = \mathrm{Dr}_{x \leq i < x+1}\, G_i$ form a strong countable antichain for G, since if $x < x'$ we have that $\hat{G}_x \cap \hat{G}_{x'} = \langle G_i \mid x' \leq i < x+1 \rangle$ has infinite index $2^{\aleph_0}$ in both $\hat{G}_x$ and $\hat{G}_{x'}$.

On the other hand, if $G = A \times B \simeq \mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ is the direct product of two Prüfer groups, then any two distinct complements A^* of B, that is any two distinct subgroups A^* such that $A \times B = A^* \times B$, are pairwise strongly incommensurable, as they are both Prüfer groups (and are distinct). Moreover, the set that they form is well-known to be equipotent to $Hom(B, A) \simeq End(\mathcal{C}_{p^\infty})$, which has cardinality $2^{\aleph_0}$, being equipotent to the ring of p-adic integers (see for example [11] sect. 11.1). ■

Proof of Corollary 4. If (1) holds, then by Proposition 1 the group G has ABC and by Theorem 2.(0) we know that G is a soluble-by-finite minimax group. Thus (2) holds by Remark 3. In turn, if (2) holds, then (3) holds by the above quoted result (Theorem 2.7 in [2]). Finally, it is trivial that (3) implies (1).

Further, if G_0 is any soluble subgroup of G of finite index, then G_0 is minimax by Theorem 2.(0′). Hence G_0 has (2) and, by the above, only countably many subgroups. Then the same holds for the whole of G (see [2]. Lemma 2.1), as desired. ■

References

[1] R. Brand , The Dilworth number of subgroup lattices, Arch. Math., Vol. 50, (1988) 502-510

[2] G. Cutolo and H. Smith, Groups with countably many subgroups, J. Algebra, 424 (2016), 399-417, DOI: 10.1016/j.jalgebra.2015.10.007

[3] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, John Eiley and Sons, Chichester (1988).

[4] U. Dardano and F. De Mari, A real chain condition for groups, J. Algebra **642** (2024), 451–469. doi.org/10.1016/j.jalgebra.2023.12.019

[5] U. Dardano and F. De Mari, Groups with the real chain condition on non-pronormal subgroups, Int. J. Group Theory **14** No.1 (2025), 9-18, doi.org/10.22108/IJGT.2023.138453.1858

[6] M.R. Dixon and I.Y. Subbotin, Groups with finiteness conditions on some subgroup system: a contemporary stage, *Algebra Discrete Math.* **4** (2009), 29-54.

[7] F. de Giovanni, L.A. Kurdachenko and A. Russo, Groups with pronormal deviation, *J. Algebra* **613**(2023), 32-45.

[8] L.A. Kurdachenko and H. Smith, Groups with the weak minimal condition for non-subnormal subgroups, *Annali Mat.* **173** (1997), 299-312.

[9] L.A. Kurdachenko and V. Walter, Baer groups satisfying weak chain conditions, *Ricerche Mat.* **53** (2004), 1-5 (2005).

[10] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin-New York (1972).

[11] D.J.S. Robinson, *A course in the theory of groups*, Springer, Berlin-New York (1996).

[12] V.P. Shunkov, Locally finite groups with a minimality condition for abelian subgroups. *Algebra and Logic* **9** (1970), 129-141.

[13] A.V. Tushev, On deviation in groups, *Illinois J. Math.* **47** (2003),539-550.

[14] A.V. Tushev, On Krull dimension of modules over rings of minimax abelian groups, doi.org/10.48550/arXiv.2412.19344

[15] D.I. Zaicev, On the theory of minimax groups, *Ukrainian Math. J.* **23** (1971), 536-542.