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A family of surfaces with $p_g = q = 2$, $K^2 = 7$ and Albanese map of degree 3

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We study a family of surfaces of general type with $p_g = q = 2$ and $K^2 = 7$, originally constructed by Cancian and Frapporti by using the Computer Algebra System MAGMA. We provide an alternative, computer-free construction of these surfaces, that allows us to describe their Albanese map and their moduli space.

1 Introduction

In recent years, the work of several authors on the classification of irregular algebraic surfaces (that is, surfaces S with $q(S) > 0$) produced a considerable amount of results, see for example the survey papers [2], [19] for a detailed bibliography on the subject.

In particular, surfaces of general type with $\chi(\mathcal{O}_S) = 1$, that is, $p_g(S) = q(S)$ were investigated. For these surfaces, [11, Théorème 6.1] implies $p_g \leq 4$. Surfaces with $p_g = q = 4$ and $p_g = q = 3$ are nowadays completely classified, see [3], [10], [13], [26]. On the other hand, for the the case $p_g = q = 2$, which presents a very rich and subtle geometry, we have so far only a partial understanding of the situation; we refer the reader to [23], [24], [25] for an account on this topic and recent results.

As the title suggests, in this paper we consider a family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$. The existence of such surfaces was originally established in [7] with the help of the Computer Algebra System MAGMA [6]; the present work provides an alternative, computer-free construction of them, that allows us to describe their Albanese map and their moduli space.

Our results can be summarized as follows, see Theorem 3.7.

Main Theorem. *There exists a 3-dimensional family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ such that, for all elements $S \in \mathcal{M}$, the canonical class K_S is ample and the Albanese map $\alpha : S \to A$ is a generically finite triple cover of a principally polarized abelian surface (A, Θ), simply branched over a curve D_A numerically equivalent to 4Θ having an ordinary sextuple point and no other singularities. The family $\mathcal{M}$ provides a generically smooth, irreducible, open and normal subset of the Gieseker moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,7}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$.*

In particular, this means that $\mathcal{M}$ provides a dense open set of a generically smooth, irreducible component of $\mathcal{M}^{\mathrm{can}}_{2,2,7}$. Furthermore, denoting by $\mathcal{M}_2$ the coarse moduli space of curves of genus 2, there exists a quasi-finite, surjective morphism $\varsigma : \mathcal{M} \to \mathcal{M}_2$ of degree 40 (see Proposition 3.9).

Let us explain now how the paper is organized. In Section 2 we explain our construction in detail and we compute the invariants of the resulting surfaces (Proposition 2.5); moreover we study their Albanese map, giving a precise description of its image and of its branch curve (Proposition 2.8). It is worth pointing out that the general surface S contains no irrational pencils (Proposition 2.9).

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Section 3 is devoted to the study of the first-order deformations of the surfaces in $\mathcal{M}$ and to the description of the corresponding subset in $\mathcal{M}^{\text{can}}_{2,2,7}$. A key point in our analysis is showing that for all elements in $S \in \mathcal{M}$ we have $h^1(S, T_S) = 3$, see Proposition 3.6.

Since the degree of the Albanese map is in this case a topological invariant (Proposition 3.1), it follows that these surfaces lie in a different connected component of the moduli space than the only other known example with the same invariants, namely the surface with $p_g = q = 2$ and $K^2 = 7$ constructed in [28], whose Albanese map is a generically finite double cover of an abelian surface with polarization of type $(1, 2)$, see Remark 3.8. Hence the family $\mathcal{M}$ provides a substantially new piece in the fine classification of minimal surfaces of general type with $p_g = q = 2$.

Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers. By *surface* we mean a projective, non-singular surface S, and for such a surface K_S denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler–Poincaré characteristic*.

If C is a smooth curve, we identify $\text{Pic}^0(C)$ with the Jacobian variety $J(C)$ by means of the canonical isomorphism provided by the Abel–Jacobi map, see [5, Theorem 11.1.3]. Furthermore, we write $\text{Sym}^n(C)$ for the n-th symmetric product of C.

Given a finite group G acting on a vector space V, we denote by V^G the G-invariant subspace.

2 The construction

Let V_2 and V_3 be the two hypersurfaces of $\mathbb{P}^3$ defined by

$$V_2 := \{x_2 x_3 + r(x_0, x_1) = 0\}, \quad V_3 := \{x_2^3 + x_3^3 + s(x_0, x_1) = 0\}, \tag{2.1}$$

where $r, s \in \mathbb{C}[x_0, x_1]$ are general homogeneous forms of degree 2 and 3, respectively. Then $C_4 := V_2 \cap V_3$ is a smooth, canonical curve of genus 4. Denoting by ξ a primitive third root of unity, we see that C_4 admits a free action of the cyclic group $\langle \xi \rangle \cong \mathbb{Z}/3\mathbb{Z}$, defined by

$$\xi \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : \xi x_2 : \xi^2 x_3] \tag{2.2}$$

and the quotient $C_2 := C_4/\langle \xi \rangle$ is a smooth curve of genus 2.

Proposition 2.1 *All étale Galois triple covers of a smooth curve of genus 2 can be obtained in this way.*

P r o o f . Let C_2 be any smooth curve of genus 2 and choose any étale $\mathbb{Z}/3\mathbb{Z}$-cover $c : C_4 \to C_2$. Thus C_4 is a smooth curve of genus 4 and we can choose a fixed-point free automorphism $\varphi : C_4 \to C_4$ generating the Galois group of the cover.

The curve C_4 cannot be hyperelliptic, otherwise its ten Weierstrass points would be an invariant set by any automorphism, which is impossible because any orbit of c consists of three distinct points. Hence the canonical divisor K_{C_4} is very ample and defines an embedding of C_4 in $\mathbb{P}^3 = \mathbb{P}H^0(C_4, K_{C_4})$, whose image (that we still denote by C_4) is the complete intersection of a (uniquely determined) quadric hypersurface V_2 and a cubic hypersurface V_3. It remains to show that we can choose V_2 and V_3 as in (2.1).

Pushing down the canonical line bundle of C_4 to C_2 gives a decomposition of $H^0(C_4, K_{C_4})$ into $\mathbb{Z}/3\mathbb{Z}$-eigenspaces, namely

$$H^0(C_4, K_{C_4}) = H^0(C_2, K_{C_2}) \oplus H^0(C_2, K_{C_2} + \eta) \oplus H^0(C_2, K_{C_2} + 2\eta) \tag{2.3}$$

where η is a non-trivial, 3-torsion divisor on C_2. The first summand in (2.3) has dimension 2, whereas the others have dimension 1; so we can choose a basis x_0, x_1, x_2, x_3 of $H^0(C_4, K_{C_4})$ such that x_0, x_1 generate $H^0(C_2, K_{C_2})$ whereas x_2 and x_3 generate $H^0(C_2, K_{C_2} + \eta)$ and $H^0(C_2, K_{C_2} + 2\eta)$, respectively. This means that, using homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ in $\mathbb{P}^3$, the action of $\mathbb{Z}/3\mathbb{Z} = \langle \xi \rangle$ can be written as in (2.2).

We start by looking at the invariant quadrics in the homogeneous ideal of C_4. There are four invariant monomials of degree 2, namely

$$x_0^2, \; x_0 x_1, \; x_1^2, \; x_2 x_3, \tag{2.4}$$

hence the invariant subspace $\big(\mathrm{Sym}^2 H^0(C_4, K_{C_4})\big)^{\langle\xi\rangle}$ of $\mathrm{Sym}^2 H^0(C_4, K_{C_4})$ has dimension 4. On the other hand, the subspace of invariant quadrics in the homogeneous ideal of C_4 is the kernel of the surjective map

$$\big(\mathrm{Sym}^2 H^0(C_4, K_{C_4})\big)^{\langle\xi\rangle} \longrightarrow H^0(C_4, 2K_{C_4})^{\langle\xi\rangle} \cong H^0(C_2, 2K_{C_2}) \cong \mathbb{C}^3,$$

hence it has dimension 1. In other words, the unique quadric V_2 containing C_4 is invariant, hence the polynomial defining V_2 is a linear combination of the monomials in (2.4). The coefficient of x_2x_3 cannot vanish, or V_2 would be reducible, so V_2 is as in (2.1).

Let us look now at the invariant cubics in the homogeneous ideal of C_4. There are eight invariant monomials of degree 3, namely

$$x_0^3,\ x_0^2x_1,\ x_0x_1^2,\ x_1^3,\ x_0x_2x_3,\ x_1x_2x_3,\ x_2^3,\ x_3^3,$$

hence the invariant subspace $\big(\mathrm{Sym}^3 H^0(C_4, K_{C_4})\big)^{\langle\xi\rangle}$ of $\mathrm{Sym}^3 H^0(C_4, K_{C_4})$ has dimension 8. On the other hand, the subspace of invariant cubics in the homogeneous ideal of C_4 is the kernel of the surjective map

$$\big(\mathrm{Sym}^3 H^0(C_4, K_{C_4})\big)^{\langle\xi\rangle} \longrightarrow H^0(C_4, 3K_{C_4})^{\langle\xi\rangle} \cong H^0(C_2, 3K_{C_2}) \cong \mathbb{C}^5,$$

hence it has dimension 3. In particular, this implies that the general invariant cubic hypersurface V_3 containing C_4 is not a multiple of the quadric V_2. Adding suitable scalar multiples of x_0V_2 and x_1V_2 in order to get rid of the monomials $x_0x_2x_3$ and $x_1x_2x_3$, and changing coordinates by multiplying x_2 and x_3 by suitable constants we obtain an equation of V_3 as in (2.1) and we are done. □

Let us consider now the product $C_4 \times C_4 \subset \mathbb{P}^3 \times \mathbb{P}^3$, and write $\boldsymbol{x} = [x_0 : x_1 : x_2 : x_3]$ for the homogeneous coordinates in the first factor and $\boldsymbol{y} = [y_0 : y_1 : y_2 : y_3]$ for those in the second factor. Then the action of $\langle\xi\rangle$ on C_4 induces an action of $H := \langle \xi_x, \xi_y, \sigma\rangle$ on $C_4 \times C_4$, where

$$\xi_x(\boldsymbol{x}, \boldsymbol{y}) := (\xi \cdot \boldsymbol{x}, \boldsymbol{y}), \quad \xi_y(\boldsymbol{x}, \boldsymbol{y}) := (\boldsymbol{x}, \xi \cdot \boldsymbol{y}), \quad \sigma(\boldsymbol{x}, \boldsymbol{y}) := (\boldsymbol{y}, \boldsymbol{x}).$$

Clearly ξ_x and ξ_y commute, whereas $\sigma\xi_x = \xi_y\sigma$ and $\sigma\xi_y = \xi_x\sigma$, so H is a semi-direct product of the form

$$H = \langle \xi_x, \xi_y\rangle \rtimes \langle\sigma\rangle \cong (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}.$$

In particular, $|H| = 18$ and every element $h \in H$ can be written in a unique way as $h = \sigma^k \xi_x^i \xi_y^j$, where $k \in \{0, 1\}$ and $i, j \in \{0, 1, 2\}$.

Lemma 2.2 *The non-trivial elements of H having fixed points on $C_4 \times C_4$ are precisely the three elements of order* 2

$$h_i := \sigma \xi_x^i \xi_y^{3-i}, \quad i = 0, 1, 2.$$

More precisely, the element h_i fixes pointwise the smooth curve

$$\Gamma_i := \big\{ \big(\boldsymbol{x}, \xi^i \cdot \boldsymbol{x}\big) \mid \boldsymbol{x} \in C_4 \big\},$$

that is, the graph of the automorphism of C_4 defined by $\boldsymbol{x} \mapsto \xi^i \cdot \boldsymbol{x}$. The three curves Γ_0, Γ_1 and Γ_2 are isomorphic to C_4, pairwise disjoint and their self-intersection equals -6.

Proof. Let $h = \sigma^k \xi_x^i \xi_y^j$ be an element of H. If $k = 0$ then $h(\boldsymbol{x}, \boldsymbol{y}) = \big(\xi^i \cdot \boldsymbol{x}, \xi^j \cdot \boldsymbol{y}\big)$ so, since the action of ξ on C_4 is free, h has fixed points if and only if it is trivial. Thus we can assume $k = 1$, in which case we have

$$\sigma \xi_x^i \xi_y^j (\boldsymbol{x}, \boldsymbol{y}) = \big(\xi^j \cdot \boldsymbol{y}, \xi^i \cdot \boldsymbol{x}\big).$$

Hence $(\boldsymbol{x}, \boldsymbol{y})$ is a fixed point for h if and only if $i + j \equiv 0 \pmod 3$ and $\boldsymbol{y} = \xi^i \cdot \boldsymbol{x}$, that is $(\boldsymbol{x}, \boldsymbol{y}) \in \Gamma_i$.

A straightforward computation using the relations $\sigma^2 = 1$ and $\xi_x\sigma = \sigma\xi_y$ shows that the order of h_i is 2.

The curve Γ_0 is the diagonal of $C_4 \times C_4$, hence it is isomorphic to C_4 and satisfies $(\Gamma_0)^2 = 2 - 2g(C_4) = -6$. The same is true for the curves Γ_1 and Γ_2, because they are the translate of Γ_0 by the action of ξ_y and ξ_x, respectively. Finally, Γ_i and Γ_j are disjoint for $i \neq j$, because ξ acts freely on C_4. □

Lemma 2.2 implies that the quotient map $C_4 \times C_4 \to (C_4 \times C_4)/H$ is ramified exactly over the three curves Γ_i, with ramification index 2 on each of them. We factor such a map through the quotient by the normal abelian subgroup $\langle \xi_x, \xi_y \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2$. This subgroup acts separately on the two factors, whereas σ exchanges them, so we get

$$(C_4 \times C_4)/\langle \xi_x, \xi_y \rangle \cong C_2 \times C_2, \quad (C_4 \times C_4)/H \cong \mathrm{Sym}^2(C_2).$$

Thus the surface $B = (C_4 \times C_4)/H$ contains a unique rational curve, namely the (-1)-curve E corresponding to the unique g^1_2 of C_2. Denoting by $\pi : B \to A$ the blow-down of E, we see that A is an abelian surface isomorphic to the Jacobian variety $J(C_2)$.

Remark 2.3 Because of Proposition 2.1, all Jacobians of smooth curves of genus 2 can be obtained in this way.

Let us denote now by ξ_{xy} the element $\xi_x\xi_y$ and set $G := \langle \xi_{xy}, \sigma \rangle$; then G is a non-normal, abelian subgroup of H, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Setting

$$T := (C_4 \times C_4)/\langle \xi_{xy} \rangle, \quad S := (C_4 \times C_4)/G,$$

and writing $t : C_4 \times C_4 \to T$ and $f : C_4 \times C_4 \to S$ for the corresponding projection maps, we have the following commutative diagram:

$$\begin{array}{ccccc}
C_4 \times C_4 & & & \longrightarrow & \\
 & \overset{t}{\searrow} & & & \searrow \\
 & & T = (C_4 \times C_4)/\langle \xi_{xy} \rangle & \xrightarrow{\ \gamma\ } & (C_4 \times C_4)/\langle \xi_x, \xi_y \rangle \cong C_2 \times C_2 \\
 & \underset{f}{\searrow} & \downarrow u & & \downarrow v \\
 & & S = (C_4 \times C_4)/G & \xrightarrow{\ \beta\ } & (C_4 \times C_4)/H = B \cong \mathrm{Sym}^2(C_2) \\
 & & & \overset{\alpha}{\searrow} & \downarrow \pi \\
 & & & & A.
\end{array} \tag{2.5}$$

The morphism $u : T \to S$ is a double cover, induced by the involution σ exchanging the two coordinates in $C_4 \times C_4$.

We first compute the invariants of T.

Lemma 2.4 *The surface S is a minimal surface of general type with*

$$p_g(T) = 6, \quad q(T) = 4, \quad K^2_T = 24.$$

P r o o f . By standard calculations we have

$$p_g(C_4 \times C_4) = 16, \quad q(C_4 \times C_4) = 8, \quad K^2_{C_4 \times C_4} = 72.$$

The group $\langle \xi_{xy} \rangle \cong \mathbb{Z}/3\mathbb{Z}$ acts diagonally and freely on $C_4 \times C_4$, hence T is a so-called *quasi-bundle*, see for instance [27, Section 3]. Therefore we obtain

$$K^2_T = \frac{1}{3} K^2_{C_4 \times C_4} = 24, \quad \chi(\mathcal{O}_T) = \frac{1}{3}\chi(\mathcal{O}_{C_4 \times C_4}) = 3, \quad q(T) = g(C_2) + g(C_2) = 4,$$

so $p_g(T) = 6$. Note that by Noether's formula this implies $c_2(T) = 12$. Finally, T is a minimal surface of general type because it a finite, étale quotient of the minimal surface of general type $C_4 \times C_4$. □

The three curves $\Gamma_i \subset C_4 \times C_4$ are ξ_{xy}-invariant, hence their images $\Sigma_i := t(\Gamma_i) \subset T$ are three curves isomorphic to C_2 and such that $(\Sigma_i)^2 = \frac{1}{3}(\Gamma_i)^2 = -2$. Moreover, the curve Γ_0 is also σ-invariant, whereas Γ_1 and Γ_2 are switched by the action of σ. Then $D_S := u(\Sigma_0)$ and $R := u(\Sigma_1) = u(\Sigma_2)$ are two disjoint curves in S, both isomorphic to C_2, such that $(D_S)^2 = -4$ and $R^2 = -2$. Note that D_S is the branch locus of the double cover $u : T \to S$.

We can now compute the invariants of S.

Proposition 2.5 *The surface S is a minimal surface of general type with*

$$p_g(S) = 2, \quad q(S) = 2, \quad K^2_S = 7.$$

The morphism $\beta : S \to B$ is a non-Galois triple cover, simply ramified over R and simply branched over the diagonal $D_B \subset B$. Finally, S contains no rational curves (in particular, K_S is ample) and contains a smooth elliptic curve, namely $Z := \beta^* E$ (which satisfies $Z^2 = -3$).

P r o o f . We start by proving the last claim. The two smooth curves D_B and E intersect transversally at the six points corresponding to the six Weierstrass points of C_2. The preimage of $Z := \beta^* E$ on $C_4 \times C_4$ is the disjoint union of three smooth curves isomorphic to C_4, namely the graphs of the three involutions $C_4 \to C_4$ obtained by lifting to C_4 the hyperelliptic involution of C_2. The cyclic group $\langle \xi_{xy} \rangle$ acts transitively on the set of these curves, whereas σ acts on each of them as the corresponding involution, which has six fixed points. So Z is a smooth, irreducible curve of genus 1 contained in S, such that

$$ZR = (\beta^* E).R = E.(\beta_* R) = ED_B = 6. \tag{2.6}$$

On the other hand, S does not contain any rational curve. Otherwise, such a curve would map would map onto E via $\beta : S \to B$, impossible because we have seen that $\beta^* E$ is smooth of genus 1.

Since the double cover $u : T \to S$ is branched over the curve D_S, it follows that D_S is 2-divisible in $\mathrm{Pic}(S)$ and moreover

$$24 = K_T^2 = 2\left(K_S + \frac{1}{2} D_S\right)^2.$$

Using $(D_S)^2 = -4$ and $K_S D_S = 6$, we find $K_S^2 = 7$. Since S does not contain any rational curve and $K_S^2 > 0$, we deduce that S is a minimal surface of general type with ample canonical class.

Now, as $K_B = \pi^* K_A + E = E$, the Riemann–Hurwitz formula yields

$$K_S = \beta^* K_B + R = Z + R, \tag{2.7}$$

and this allows us to compute Z^2. In fact, using (2.6) and (2.7), we can write

$$7 = K_S^2 = Z^2 + 2ZR + R^2 = Z^2 + 10,$$

that is $Z^2 = -3$.

Next, denoting by χ_{top} the topological Euler number, we have

$$\begin{aligned}\chi_{\mathrm{top}}(S - D_S - R) &= \frac{1}{2}\chi_{\mathrm{top}}(T - \Sigma_0 - \Sigma_1 - \Sigma_2) \\ &= \frac{1}{2}(c_2(T) - \chi_{\mathrm{top}}(\Sigma_0) - \chi_{\mathrm{top}}(\Sigma_1) - \chi_{\mathrm{top}}(\Sigma_2)) = \frac{1}{2}(12 - 3(-2)) = 9,\end{aligned}$$

so

$$c_2(S) = \chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(S - D_S - R) + \chi_{\mathrm{top}}(D_S) + \chi_{\mathrm{top}}(R) = 9 - 2 - 2 = 5.$$

Therefore Noether's formula yields $\chi(\mathcal{O}_S) = 1$, that is $p_g(S) = q(S)$.

The existence of the surjective morphism $\alpha : S \to A$ implies $q \geq 2$, and since minimal surfaces of general type with $p_g = q \geq 3$ have either $K^2 = 6$ or $K^2 = 8$ (see for instance [2]), we deduce $p_g(S) = q(S) = 2$.

The morphism $\beta : S \to B$ is a non-Galois triple cover, because G is a non-normal subgroup of index 3 in H. Since $t : C_4 \times C_4 \to T$ is étale and $u : T \to S$ is branched over D_S, by Lemma 2.2 it follows that $\beta : S \to B$ is simply ramified over R, and hence simply branched over $\beta(R) = D_B$. □

Remark 2.6 The existence of surfaces S was first established in [7], using a computer-aided construction based on `Magma` computations. The present paper provides the first computer-free description of them. Actually, S is a *semi-isogenous mixed surface*, namely a quotient of type $(C \times C)/G$, where C is a smooth curve and G is a finite subgroup of $\mathrm{Aut}(C \times C)$, such that the subgroup G^0 of the automorphisms preserving both factors has index 2 and acts freely. In fact, with our previous notation

$$C = C_4, \quad G = \langle \xi_{xy}, \sigma \rangle, \quad G^0 = \langle \xi_{xy} \rangle.$$

The paper [7] provides a detailed study of semi-isogenous mixed surfaces, showing, among other things, that they are smooth and how to compute their invariants. For instance, [7, Proposition 2.6] allows us to prove the equality $q(S) = 2$ without exploiting the classification of surfaces with $p_g = q \geq 3$.

Let us now identify the blow-up morphism $\pi : B \to A$ with the Abel–Jacobi map

$$\mathrm{Sym}^2(C_2) \longrightarrow J(C_2).$$

If Θ is the class of a theta divisor in $\mathrm{NS}(A)$, let us define the class $\Theta_B := \pi^*\Theta$ in $\mathrm{NS}(B)$. Moreover, let us write x for the class in $\mathrm{NS}(B)$ given by the image of the map

$$C_2 \longrightarrow \mathrm{Sym}^2(C_2), \quad p \longmapsto p_0 + p$$

where $p_0 \in C_2$ is fixed (such a class does not depend on p_0). Then we can prove the following

Lemma 2.7 *The equality* $\pi_* D_B = 4\Theta$ *holds in* $\mathrm{NS}(A)$.

Proof. This is a consequence of general results on g-fold symmetric products of curves of genus g. For instance, [18, Equations (1) and (5)] give in our case the relations

$$2E + D_B = 4x, \quad \Theta_B = E + x$$

in $\mathrm{NS}(B)$, and these in turn imply $D_B = 4\Theta_B - 6E$. So the result follows by applying the push-forward map $\pi_* : \mathrm{NS}(B) \to \mathrm{NS}(A)$. □

The next step consists in describing the Albanese morphism of S.

Proposition 2.8 *The abelian surface A is isomorphic to* $\mathrm{Alb}(S)$ *and, up to automorphisms of A, the generically finite triple cover $\alpha = \pi \circ \beta : S \to A$ coincides with the Albanese morphism of S. Furthermore, the only curve contracted by α is Z. Finally, α is branched over a divisor D_A numerically equivalent to* 4Θ*, having an ordinary sextuple point and no other singularities.*

Proof. By the universal property of the Albanese variety ([4, Chapter V]), the morphism $\alpha : S \to A$ must factor through the Albanese morphism of S; but α is surjective and generically of degree 3, so it must actually coincide with the Albanese morphism of S up to automorphisms of A. Since β is a finite morphism, α only contracts the preimage of E in S, which is Z. The branch locus D_A of α is equal to the image of the diagonal D_B via $\pi : B \to A$; since D_B is smooth and intersects E transversally at six points, it follows that D_A has an ordinary sextuple point and no other singularities. Finally, the fact that D_A is numerically equivalent to 4Θ follows from Lemma 2.7. □

The situation is summarized in Figure 1 below.

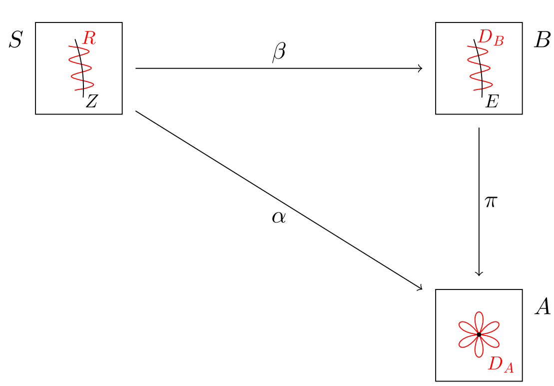

Fig. 1 The triple covers α and β.

Furthermore, the Stein factorization of $\alpha : S \to A$ is described in the diagram

$$\begin{array}{ccc} S & \xrightarrow{c_Z} & \tilde{S} \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \tilde{\alpha}} \\ & & A, \end{array}$$

where $c_Z : S \to \tilde{S}$ is the birational morphism given by the contraction of the elliptic curve Z. Since $Z^2 = -3$, the normal surface $\tilde{S}$ has a Gorenstein elliptic singularity of type $\tilde{E}_6$, see [15, Theorem 7.6.4].

Recall that an *irrational pencil* (or *irrational fibration*) on a smooth, projective surface is a surjective morphism with connected fibres over a curve of positive genus.

Proposition 2.9 *The general surface S contains no irrational pencils.*

Proof. Assume that $\phi : S \to W$ is an irrational pencil on S. Since $q(S) = 2$, we have either $g(W) = 1$ or $g(W) = 2$. On the other hand, using the embedding $W \hookrightarrow J(W)$ and the universal property of the Albanese map, we obtain a morphism $A \to J(W)$ whose image is isomorphic to the curve W. This rules out the case $g(W) = 2$, hence W is an elliptic curve and so A is a non-simple abelian surface. The proof is now complete, because A is isomorphic to the Jacobian variety $J(C_2)$, which is known to be simple for a general choice of C_2 ([17, Theorem 3.1]). □

3 The moduli space

A projective variety X is called *of maximal Albanese dimension* if its Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is generically finite onto its image. For surfaces of general type with irregularity at least 2, this is actually a topological property, as shown by the result below.

Proposition 3.1 *Let S be a minimal surface of general type with $q(S) \geq 2$. If S is of maximal Albanese dimension, then the same holds for any surface which is homeomorphic to S. Furthermore, in the case $q(S) = 2$ the degree of the Albanese map $\alpha : S \to A$ is a topological invariant.*

Proof. This follows by the results of [8], see for instance [24, Proposition 3.1]. □

Proposition 3.1 allows us to study the deformations of S by relating them to those of the flat triple cover $\beta : S \to B$. Since the trace map provides a splitting of the short exact sequence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \beta_* \mathcal{O}_S \longrightarrow \mathcal{E}_\beta \longrightarrow 0,$$

we obtain a direct sum decomposition

$$\beta_* \mathcal{O}_S = \mathcal{O}_B \oplus \mathcal{E}_\beta, \tag{3.1}$$

where $\mathcal{E}_\beta$ is a vector bundle of rank 2 on B which satisfies

$$h^0(B,\, \mathcal{E}_\beta) = 0, \quad h^1(B,\, \mathcal{E}_\beta) = 0, \quad h^2(B,\, \mathcal{E}_\beta) = 1 \tag{3.2}$$

and that, according to [20], is called the *Tschirnhausen bundle* of β.

As in [24, Section 3] we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_S & \xrightarrow{d\beta} & \beta^* T_B & \longrightarrow & \mathcal{N}_\beta & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_S & \xrightarrow{d\alpha} & \alpha^* T_A & \longrightarrow & \mathcal{N}_\alpha & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_Z(-Z) & = & \mathcal{O}_Z(-Z) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.3}$$

whose central column is the pullback via $\beta : S \longrightarrow B$ of the sequence

$$0 \longrightarrow T_B \xrightarrow{d\pi} \pi^* T_A \longrightarrow \mathcal{O}_E(-E) \longrightarrow 0, \tag{3.4}$$

see [30, p. 73]. The normal sheaf $\mathcal{N}_\alpha$ of $\alpha : S \to A$ is supported on the set of critical points of α, namely on the reducible divisor $R + Z$. Analogously, the normal sheaf $\mathcal{N}_\beta$ of $\beta : S \to B$ is supported on the set of critical points of β, namely on R.

Lemma 3.2 *We have*

$$\mathcal{N}_\beta = (N_{R/S})^{\otimes 2} = \mathcal{O}_R(2R). \tag{3.5}$$

Hence all first-order deformations of $\beta : S \to B$ leaving B fixed are trivial.

P r o o f . Since R is smooth, the first statement is a consequence of [29, Lemma 3.2]. Furthermore, we observe that $R^2 = -2$ implies that the line bundle $\mathcal{N}_\beta$ has negative degree on R, hence $H^0(R, \mathcal{N}_\beta) = 0$. By [30, Corollary 3.4.9], this shows that $\beta : S \to B$ is rigid as a morphism with fixed target. □

Note the the last statement of Lemma 3.2 agrees with the fact that the branch locus D_B of $\beta : S \to B$ is a rigid divisor in B.

Lemma 3.3 *We have*

$$h^1(S, T_S) = h^0(R + Z, \mathcal{N}_\alpha) + 1 \geq 3.$$

P r o o f . Let us write down the cohomology exact sequence associated with the short exact sequence in the central row of (3.3), recalling first that S is a surface of general type and therefore $h^0(S, T_S) = 0$:

$$0 \longrightarrow H^0(S, \alpha^* T_A) \cong \mathbb{C}^2 \longrightarrow H^0(R + Z, \mathcal{N}_\alpha) \longrightarrow H^1(S, T_S) \xrightarrow{\varepsilon} H^1(S, \alpha^* T_A).$$

Then the claim will follow if we show that $\mathrm{rank}(\varepsilon) = 3$, and this can be done by using the same argument as in [24, Section 3].

More precisely, since T_A is trivial and the Albanese map induces an isomorphism $H^1(S, \mathcal{O}_S) \cong H^1(A, \mathcal{O}_A)$, then $H^1(S, \alpha^* T_A) \cong H^1(A, T_A)$ and we can see ε as the map $H^1(S, T_S) \to H^1(A, T_A)$ induced on first-order deformations by the Albanese map. By Remark 2.3 the first-order deformations of S dominate the first-order algebraic deformations of A, so $\mathrm{rank}(\varepsilon) \geq 3$; on the other hand, the Albanese variety of every deformation of S has to remain algebraic, so $\mathrm{rank}(\varepsilon) \leq 3$ and we are done. □

Thus, in order to understand the first-order deformations of S, we can study $\mathcal{N}_\alpha$.

Lemma 3.4 *The sheaf $\mathcal{N}_\alpha$ is locally free of rank 1 on the reducible curve $R + Z$.*

P r o o f . By a standard application of Nakayama's lemma (see for instance [16, Corollary 5.3.4]), it suffices to check that the $\mathbb{C}$-vector space $\mathcal{N}_{\alpha, x}/\mathfrak{m}_x \mathcal{N}_{\alpha, x}$ has dimension 1 for all $x \in R + Z$, where $\mathfrak{m}_x \subset \mathcal{O}_{R+Z, x}$ is the maximal ideal. Equivalently, we will check that the vector bundle map $d\alpha : T_S \to \alpha^* T_A$ has rank 1 at each point $x \in R + Z$. Let us distinguish three cases.

- If $x \in R \setminus Z$, then α is locally of the form $(u, v) \mapsto (u^2, v)$, with $x = (0, 0)$ and R given by $u = 0$. Then $d\alpha$ is the linear map associated with the matrix

  $$\begin{pmatrix} 2u & 0 \\ 0 & 1 \end{pmatrix},$$

  which has rank 1 at the point x.
- If $x \in Z \setminus R$, then α is locally a smooth blow-up, hence of the form $(u, v) \mapsto (uv, v)$, where $x = (0, 0)$ and Z corresponds to the exceptional divisor, whose equation is $v = 0$. Then $d\alpha$ is the linear map associated with the matrix

  $$\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix},$$

  which has rank 1 at the point x.

- Finally, if $x \in R \cap Z$ then α is locally the composition of the two maps above, so of the form $(u, v) \mapsto (u^2 v, v)$, where $x = (0, 0)$, the curve R corresponds to the locus $u = 0$ and the curve Z to the locus $v = 0$. Then $d\alpha$ is the linear map associated with the matrix

$$\begin{pmatrix} 2uv & u^2 \\ 0 & 1 \end{pmatrix},$$

which has rank 1 at the point x.

This completes the proof. □

We can be more precise and compute the restrictions of $\mathcal{N}_\alpha$ to both curves R and Z.

Lemma 3.5 *We have*

$$\mathcal{N}_{\alpha|Z} = \mathcal{O}_Z(-Z), \quad \mathcal{N}_{\alpha|R} = \mathcal{O}_R(2R + Z) = \mathcal{O}_R(K_R).$$

Proof. Let us first apply the functor $\otimes_{\mathcal{O}_{R+Z}} \mathcal{O}_Z$ to the exact sequence forming the last column of diagram (3.3); using (3.5), we get

$$\mathcal{O}_R(2R) \otimes \mathcal{O}_Z \xrightarrow{\zeta} \mathcal{N}_{\alpha|Z} \longrightarrow \mathcal{O}_Z(-Z) \longrightarrow 0.$$

By Lemma 3.4, the sheaf $\mathcal{N}_{\alpha|Z}$ is locally free on Z; on the other hand, $\mathcal{O}_R(2R) \otimes \mathcal{O}_Z$ is a torsion sheaf, hence ζ is the zero map and so $\mathcal{N}_{\alpha|Z} \cong \mathcal{O}_Z(-Z)$.

Next, we apply to the same exact sequence the functor $\otimes_{\mathcal{O}_{R+Z}} \mathcal{O}_R$, obtaining

$$\mathcal{T} \xrightarrow{\tau} \mathcal{O}_R(2R) \longrightarrow \mathcal{N}_{\alpha|R} \longrightarrow \mathcal{O}_Z(-Z) \otimes \mathcal{O}_R \longrightarrow 0. \tag{3.6}$$

Since $\mathcal{T} := \mathrm{Tor}^1_{\mathcal{O}_{R+Z}}(\mathcal{O}_Z(-Z), \mathcal{O}_R)$ is a torsion sheaf (supported on $R \cap Z$) and $\mathcal{O}_R(2R)$ is locally free on R, we deduce that τ is the zero map and so (3.6) becomes

$$0 \longrightarrow \mathcal{O}_R(2R) \longrightarrow \mathcal{N}_{\alpha|R} \longrightarrow \mathcal{O}_Z(-Z) \otimes \mathcal{O}_R \longrightarrow 0. \tag{3.7}$$

On the other hand, the curves R and Z intersect transversally at the six Weierstrass points $p_1, \dots, p_6$ of R, so we infer

$$\mathcal{O}_Z(-Z) \otimes \mathcal{O}_R = \mathcal{O}_Z \otimes \mathcal{O}_R = \bigoplus_{i=1}^{6} \mathcal{O}_{p_i}. \tag{3.8}$$

Hence (3.7) and (3.8) yield

$$0 \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{N}_{\alpha|R}(-2R) \longrightarrow \bigoplus_{1}^{6} \mathcal{O}_{p_i} \longrightarrow 0,$$

that is the invertible sheaf $\mathcal{N}_{\alpha|R}(-2R)$ has a global section whose divisor is $\sum p_i$. This means $\mathcal{N}_{\alpha|R} \cong \mathcal{O}_R\big(2R + \sum p_i\big) = \mathcal{O}_R(2R + Z)$. Finally, Equation (2.7) shows that $R + Z$ is a canonical divisor on S, so by using adjunction formula we obtain

$$\mathcal{O}_R(2R + Z) = \mathcal{O}_S(K_S + R) \otimes \mathcal{O}_R = \mathcal{O}_R(K_R).$$

□

We can finally prove

Proposition 3.6 *All surfaces S constructed in Section 2 satisfy*

$$h^1(S, T_S) = 3.$$

Proof. By Lemma 3.3 it suffices to show the inequality $h^0(R + Z, \mathcal{N}_\alpha) \le 2$. By [1, p. 62] we have a "decomposition sequence"

$$0 \longrightarrow \mathcal{O}_Z(-R) \longrightarrow \mathcal{O}_{R+Z} \longrightarrow \mathcal{O}_R \longrightarrow 0,$$

which gives, tensoring with $\mathcal{N}_\alpha$ and using Lemma 3.5,

$$0 \longrightarrow \mathcal{O}_Z(-R-Z) \longrightarrow \mathcal{N}_\alpha \longrightarrow \mathcal{O}_R(K_R) \longrightarrow 0.$$

Since $Z(-R-Z) = -3 < 0$, we deduce $H^0(Z, \mathcal{O}_Z(-R-Z)) = 0$. So $H^0(R+Z, \mathcal{N}_\alpha)$ injects into $H^0(R, K_R) = \mathbb{C}^2$ and we are done. □

The moduli space of principally polarized abelian surfaces has dimension 3; moreover,the rigidity of the curve D_B in B implies that the curve D_A has only trivial deformations in A. So our surfaces S provide a 3-dimensional subset $\mathcal{M}$ of the moduli space $\mathcal{M}^{\text{can}}_{2,2,7}$ of (canonical models of) minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$. Because of Proposition 3.6, the corresponding Kuranishi family is smooth; this implies that $\mathcal{M}$ has at most quotient singularities, so it is a normal (and hence generically smooth) open subset of $\mathcal{M}^{\text{can}}_{2,2,7}$. In particular, $\mathcal{M}$ provides a dense open set of a generically smooth, irreducible component of this moduli space.

Summing up, we have proven the Main Theorem stated in the introduction, namely

Theorem 3.7 *There exists a 3-dimensional family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ such that, for all elements $S \in \mathcal{M}$, the canonical class K_S is ample and the Albanese map $\alpha : S \to A$ is a generically finite triple cover of a principally polarized abelian surface (A, Θ), simply branched over a curve D_A numerically equivalent to 4Θ having an ordinary sextuple point and no other singularities. The family $\mathcal{M}$ provides a generically smooth, irreducible, open and normal subset of the Gieseker moduli space $\mathcal{M}^{\text{can}}_{2,2,7}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$.*

Remark 3.8 By Proposition 3.1 the degree of the Albanese map is in our case a topological invariant, so it follows that the surfaces in $\mathcal{M}$ lie in a different connected component of $\mathcal{M}^{\text{can}}_{2,2,7}$ than the only other known example with the same invariants, namely the surface with $p_g = q = 2$ and $K^2 = 7$ constructed in [28], whose Albanese map is a generically finite *double* cover of an abelian surface with polarization of type $(1, 2)$. Hence the family $\mathcal{M}$ provides a substantially new piece in the fine classification of minimal surfaces of general type with $p_g = q = 2$.

For every surface S whose isomorphism class $[S]$ belongs to $\mathcal{M}$, the normalization of the branching curve D_A of $\alpha : S \longrightarrow A$ is isomorphic to C_2, hence we obtain a morphism

$$\varsigma : \mathcal{M} \longrightarrow \mathcal{M}_2, \quad \varsigma([S]) := [C_2],$$

where $\mathcal{M}_2$ denotes as usual the coarse moduli space of curves of genus 2. Note that such a morphism is surjective by Proposition 2.1. Correspondingly, we have a morphism of deformation functors, namely

$$\delta_S : \text{Def}_S \longrightarrow \text{Def}_{C_2}.$$

The next result clarifies the relation between the deformations of S and those of the curve C_2.

Proposition 3.9 *The following hold:*

(1) $\delta_S : \text{Def}_S \to \text{Def}_{C_2}$ *is an isomorphism of functors.*
(2) $\varsigma : \mathcal{M} \to \mathcal{M}_2$ *is a quasi-finite morphism of degree 40.*

Proof. **(1)** Since $\dim \mathcal{M} = \dim H^1(S, T_S) = 3$, the functor Def_S is unobstructed; moreover, the functor Def_{C_2} is clearly unobstructed, too. Proposition 2.1 implies that the first-order deformations of S dominate the first-order deformations of C_2, so the differential map

$$d\delta_S : H^1(S, T_S) \longrightarrow H^1(C_2, T_{C_2}) \tag{3.9}$$

is surjective, and hence it is an isomorphism because $H^1(S, T_S)$ and $H^1(C_2, T_{C_2})$ have the same dimension. Since Def_S and Def_{C_2} are both unobstructed, this shows that δ_S is an isomorphism of functors, see [30, Corollary 2.3.7 and Remark 2.3.8].

(2) We have to show that, for each $[C_2] \in \mathcal{M}_2$, the cardinality of $\varsigma^{-1}([C_2])$ is at most 40 and that it is exactly 40 for a general choice of C_2.

Remark that, once C_2 is fixed, the étale $\mathbb{Z}/3\mathbb{Z}$-cover $c : C_4 \to C_2$ completely determines S. Conversely, we claim that, starting from S, it is possible to reconstruct the étale morphism $c : C_4 \to C_2$ up to automorphisms of

C_2 and C_4. In fact, the subgroup ξ_{xy} is normal in H and the quotient $H/\langle \xi_{xy} \rangle$ is isomorphic to S_3, hence looking at diagram (2.5) we see that the map

$$v \circ \gamma : T \longrightarrow B = \mathrm{Sym}^2(C_2)$$

yields the Galois closure of the triple cover $\beta : S \to B$. This shows that S determines the quasi-bundle $T = (C_4 \times C_4)/\langle \xi_{xy} \rangle$. On the other hand, since the action of $\langle \xi_{xy} \rangle$ on $C_4 \times C_4$ is faithful, if we know T we can reconstruct C_4 and the the étale $\mathbb{Z}/3\mathbb{Z}$-cover $c : C_4 \to C_2$ up to automorphisms by using the rigidity result for minimal realizations of surfaces isogenous to a product proven in [9, Proposition 3.13].

Summing up, the cardinality of $\varsigma^{-1}([C_2])$ equals the number of Galois étale triple covers $c : C_4 \longrightarrow C_2$ up to equivalence. Here by "equivalence of covers" we intend commutative diagrams of the form

$$\begin{array}{ccc} C_4 & \xrightarrow{\varphi_4} & C_4 \\ \downarrow & & \downarrow \\ C_2 & \xrightarrow{\varphi_2} & C_2 \end{array}$$

where the horizontal arrows are automorphisms of the corresponding curves. In particular, as explained for instance in [22], if $\varphi_2 = \mathrm{id}_{C_2}$ then the number of equivalence classes of Galois triple covers $c : C_4 \to C_2$ coincides with the number of distinct subgroups of order 3 in $\mathrm{Pic}^0(C_2)$, i.e. with half the number of non-trivial 3-torsion points, that is $\big(3^4 - 1\big)/2 = 40$.

On the other hand, if C_2 is a general curve of genus 2 its unique non-trivial automorphism is the hyperelliptic involution, which acts as the multiplication by -1 on the group $\mathrm{Pic}^0(C_2)$ and hence trivially on the set of its 40 subgroups of order 3. Thus, for a general choice of C_2, the fibre $\varsigma^{-1}([C_2])$ consists of exactly 40 distinct points. □

Remark 3.10 Let us denote by $\mathcal{A}_2$ the coarse moduli space of principally polarized abelian surfaces. It is well-known that the Torelli map $\tau_2 : \mathcal{M}_2 \to \mathcal{A}_2$, sending every curve to its polarized Jacobian, is an immersion, see [21]. Thus, composing τ_2 with ς, we obtain a generically finite dominant morphism $\tau_2 \circ \varsigma : \mathcal{M} \to \mathcal{A}_2$ of degree 40, which is the one induced by the deformations of the Albanese map $\alpha : S \to \mathrm{Alb}(S)$. Observe that such a morphism is not surjective, because its image does not contain the products of elliptic curves that are not isomorphic to Jacobians.

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