Numerical Simulation of a Multi-group Age-of-infection Model

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Introduction

We present a dynamically consistent numerical method for the approximation of an intregro–differential epidemic model with heterogeneous mixing.

The Continuous-time Model

For a closed population of d sub-groups $\mathcal{P}_i$ of sizes N_i, Kermack and McKendrick age–of–infection model [1, 2] reads

$$S_i'(t) = -a_i S_i(t) \sum_{j=1}^{d} \frac{p_{ij}}{N_j} \varphi_j(t),$$
$$\varphi_i(t) = \varphi_{0i}(t) - \int_0^t A_i(s) S_i'(t-s)\, ds, \qquad 1 \le i \le d,$$

where $t \ge 0,\ 0 \le S_i(t) \le S_i(0) = S_i^0$ and :

- a_i contact rate for members of $\mathcal{P}_i$;
- p_{ij} fraction of contacts made by a member of $\mathcal{P}_i$ with a member of $\mathcal{P}_j$;
- $0 \le A_i(s) \in L^1(\mathbb{R}^+)$ mean infectivity of members of $\mathcal{P}_i$ with infection age s;
- $\varphi_{0i}(t) \in L^1(\mathbb{R}^+)$ infectivity, at time t, of members of $\mathcal{P}_i$ who were infected before $t = 0$.

Define $Z_i(t) = S_i(t)/S_i^0$. A more convenient formulation of the model is

$$\log(Z_i(t)) = -a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} \int_0^t S_j^0 A_j(t-s)(1 - Z_j(s))\, ds$$
$$\underbrace{-a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} \int_0^t \varphi_{0j}(s)\, ds,}_{G_i(t)} \qquad 1 \le i \le d. \qquad (1)$$

Final Size Relation

The solution to (1) is component–wise **positive**, **bounded** ($0 < Z_i(t) \le 1$) and **non-increasing** [2]. Furthermore

$$Z_i(\infty) = \lim_{t \to +\infty} Z_i(t),$$

satisfies, for $1 \le i \le d$, the **final size relation**

$$\Phi_i(Z(\infty)) = \log(Z_i(\infty)) - G_i(\infty)$$
$$+ a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} S_j^0 (1 - Z_j(\infty)) \int_0^{+\infty} A_j(s)\, ds = 0,$$

with $G_i(\infty) = \lim_{t \to +\infty} G_i(t)$.
Proportionate mixing ($p_{ij} = p_j$, for all $1 \le i, j \le d$), yields $Z_i(\infty) = \sigma^{a_i}$, where

$$\log(\sigma) = -\sum_{j=1}^{d} \frac{p_j}{N_j} \int_0^{+\infty} \varphi_{0j}(s)\, ds$$
$$- \sum_{j=1}^{d} \frac{p_j}{N_j} S_j^0 (1 - \sigma^{a_j}) \int_0^{+\infty} A_j(s)\, ds.$$

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High order And Positive Scheme

The discretization of (1) by n_0 steps DQ with Gregory convolution weights [3] reads, for $n \ge n_0$ and $1 \le i \le d$,

$$\log(Z_i^n) = G_i^n$$
$$- h a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} S_j^0 \sum_{k=n_0}^{n} \omega_{n-k} A_j(t_{n-k})(1 - Z_j^k),$$
$$G_i^n = G_i(t_n) \qquad (2)$$
$$- h a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} S_j^0 \sum_{k=0}^{n_0-1} w_{nk} A_j(t_{n-k})(1 - Z_j^k).$$

$Z_i^n = Z_i^n(h) \approx Z_i(t_n)$, with $t_n = nh$, $Z_i^0, Z_i^1, \dots, Z_i^{n_0-1} \in (0, 1]$ given starting values, w_{nj} and ω_j weights. When $n_0 = 1$, $\omega_0 = 0$, $\omega_j = 1$ for $j > 0$, eq. (2) corresponds to the discrete-time model introduced by O. Diekman et al. in [4].

Theorem 1.

For all $h > 0$, equation (2) has a unique **positive, bounded solution** with

$$\bar{Z}_i(h) \le Z_i^n \le 1,$$

and $\bar{Z}_i(h) > 0$. Furthermore if $n_0 = 1$ and $A_j(0) = 0$, for $1 \le j \le d$, then $\{Z_i^n\}_{n \ge n_0}$ is **non-increasing** as the continuous solution.

Theorem 2.

If the starting errors $\{\eta^j\}_{0 \le j \le n_0 - 1}$ satisfy $\|\eta^j(h)\| = \mathcal{O}(h^{n_0})$, and $A_j(t)$ and $\varphi_{0j}(t)$, $j = 1, \dots, d$, are smooth enough, then the method is **convergent of order $n_0 + 1$**.

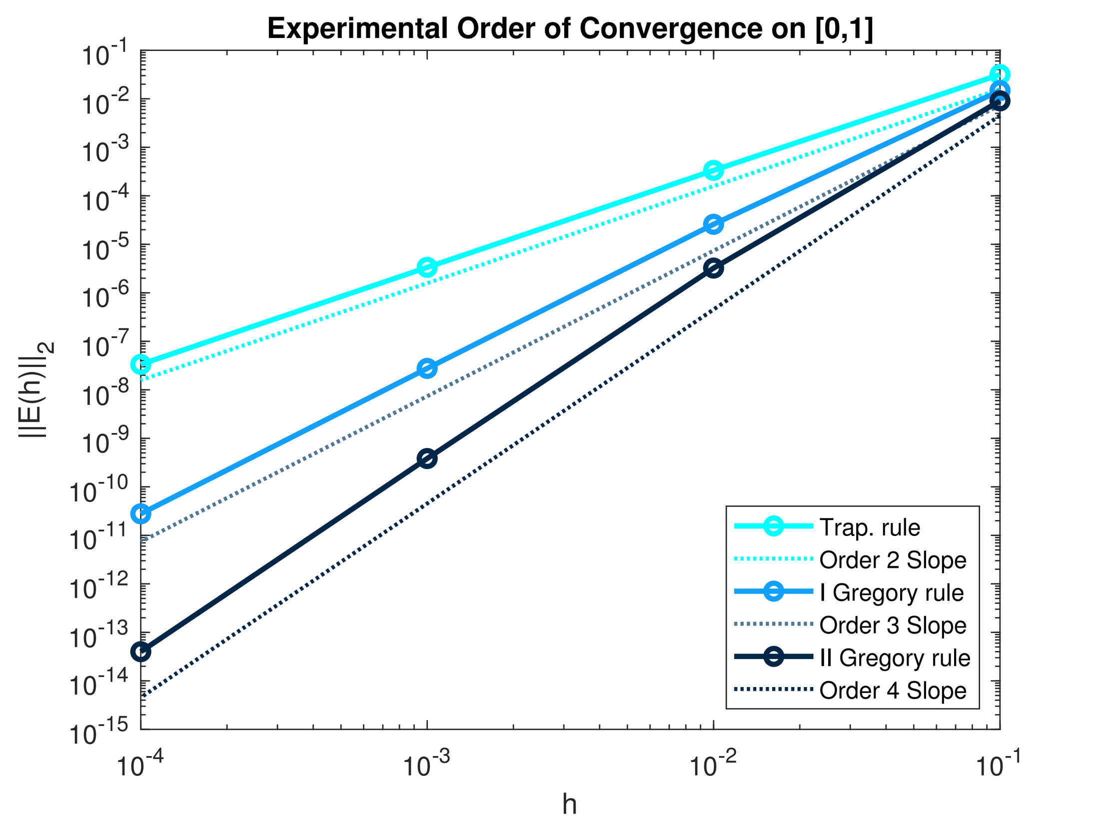

Figure 1: Example with $d = 4$, $i \cdot N_i = 1.2 \cdot 10^3$, $S_i^0 = N_i - 100$, $3a_i = i$, $p_{ij} = 0.25$, $A_1(t) = 5/(1+t^2)$, $A_2(t) = e^{-2t}$, $A_3(t) = te^{-0.2t}$, $A_4(t) = (0.2+t)^{-2}$, $\varphi_{0j}(t) = (N_j - S_j^0)A_j(t)$ and $1 \le i, j \le 4$.

Numerical Asymptotic Properties

Assume $h \sum_{n=0}^{+\infty} \omega_n A_j(t_n) < \infty$, for $1 \le j \le d$ and $h > 0$.

Theorem 3.

For the unique solution Z^n of (2) it is

$$Z^\infty(h) = \lim_{n \to \infty} Z^n,$$

and it satisfies for $1 \le i \le d$, the **numerical final size relation**

$$\tilde{\Phi}_i(Z^\infty(h), h) = \log(Z_i^\infty(h)) - G_i(\infty)$$
$$+ a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} S_j^0 (1 - Z_j^\infty(h))\, h \sum_{n=0}^{+\infty} \omega_n A_j(t_n) = 0.$$

Theorem 4.

If $A_j(t)$ is such that

$$\lim_{h \to 0} h \sum_{n=0}^{+\infty} \omega_n A_j(t_n) = \int_0^{+\infty} A_j(s)\, ds,$$

for each $1 \le j \le d$, then

$$\lim_{h \to 0} \lim_{n \to +\infty} Z^n(h) = Z(\infty).$$

Proportionate mixing yields $Z_i^\infty(h) = \sigma(h)^{a_i}$, where

$$\log(\sigma(h)) = -\sum_{j=1}^{d} \frac{p_j}{N_j} \int_0^{+\infty} \varphi_{0j}(s)\, ds$$
$$- \sum_{j=1}^{d} \frac{p_j}{N_j} S_j^0 (1 - \sigma(h)^{a_j}) h \sum_{n=0}^{+\infty} \omega_n A_j(t_n).$$

In Table 1, $Z^M(h)$, $Mh = T >> 0$, is the **truncated numerical final size**, and $r_i(h;T)$ is the *residual*, where

$$r_i(h;T) = \log(Z_i^M(h)) - G_i(\infty)$$
$$+ a_i \sum_{j=1}^{d} \frac{p_{ij}}{N_j} S_j^0 (1 - Z_j^M(h))\, h \sum_{n=0}^{M} \omega_n A_j(t_n).$$

DQ in (1)	h	$\|Z(\infty) - Z^M(h)\|_2$	$\|r(h;T)\|_2$
Trap. rule	10^{0}	$6.242 \cdot 10^{-3}$	$6.128 \cdot 10^{-17}$
	10^{-1}	$6.571 \cdot 10^{-5}$	$1.595 \cdot 10^{-16}$
	10^{-2}	$6.574 \cdot 10^{-7}$	$1.392 \cdot 10^{-16}$
	10^{-3}	$6.574 \cdot 10^{-9}$	$2.115 \cdot 10^{-16}$
I Greg. rule	10^{0}	$2.717 \cdot 10^{-3}$	$8.064 \cdot 10^{-17}$
	10^{-1}	$6.827 \cdot 10^{-6}$	$2.892 \cdot 10^{-16}$
	10^{-2}	$7.458 \cdot 10^{-9}$	$1.170 \cdot 10^{-17}$
	10^{-3}	$7.524 \cdot 10^{-12}$	$5.929 \cdot 10^{-16}$
II Greg. rule	10^{0}	$1.701 \cdot 10^{-3}$	$5.400 \cdot 10^{-17}$
	10^{-1}	$1.104 \cdot 10^{-6}$	$1.552 \cdot 10^{-16}$
	10^{-2}	$1.247 \cdot 10^{-10}$	$3.199 \cdot 10^{-16}$

Table 1: Example with $d = 3$, $T = 10^2$, $N_i = i \cdot 10^3$, $S_i^0 = 0.9N_i$, $a_i = 0.6/i$, $p_{ij} = 1/3$, $A_1(t) = 2e^{-2t}$, $A_2(t) = \sqrt{2/\pi}e^{-t^2/2}$, $A_3(t) = 2te^{-t^2}$, $\varphi_{0j}(t) = (N_j - S_j^0)A_j(t)$ and $1 \le i, j \le 3$.

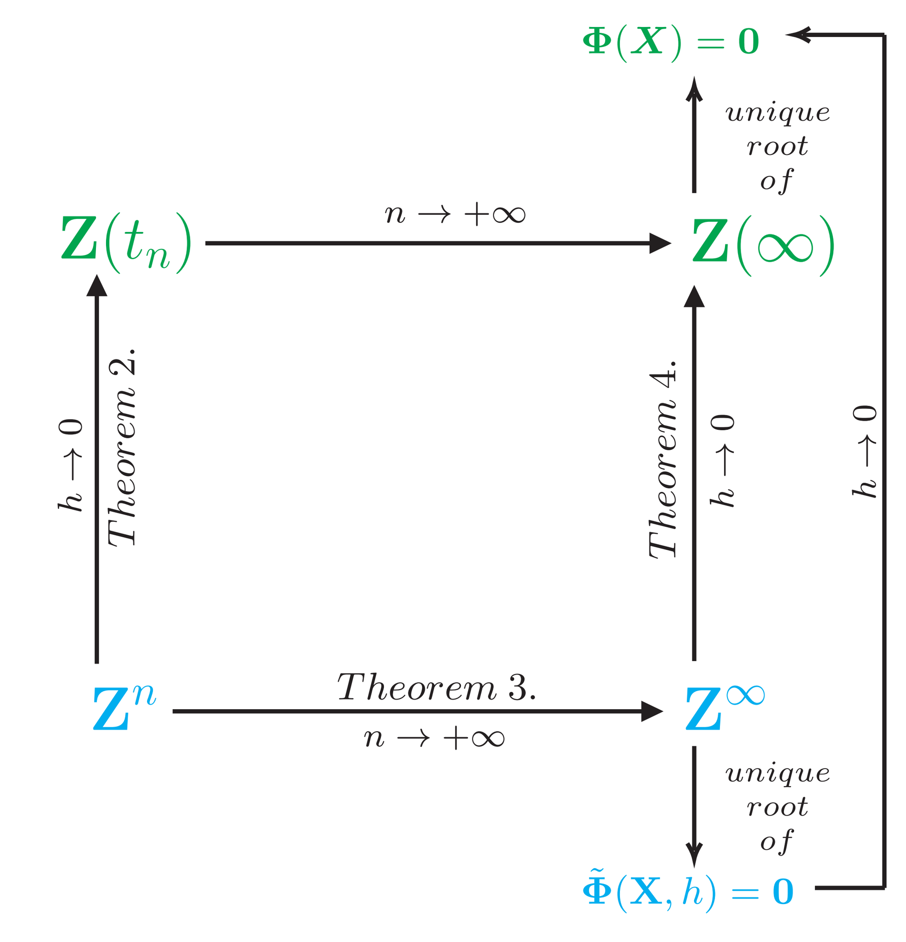

Figure 2: Diagram of theoretical results (**green**=continuous, **cyan**=discrete).

Example: Age Structured Influenza Model

Consider, for $1 \le j \le 5$, the infectivity function

$$t_1 = 1.0\ days, \quad t_2 = 2.0\ days, \quad t_3 = 2.4\ days, \quad t_4 = 3.0\ days,$$

$$A_j(t) = \begin{cases} \frac{1}{T_I} \frac{t - t_1}{t_2 - t_1} & if\ t_1 \le t \le t_2, \\ \frac{1}{T_I} & if\ t_2 \le t \le t_3, \\ \frac{1}{T_I} \frac{t_4 - t}{t_4 - t_3} & if\ t_3 \le t \le t_4, \\ 0 & elsewhere, \end{cases}$$

setting the **latent period** to $T_L = (t_1 + t_2)/2 = 1.5$ days and the **infectious period** to $T_I = (t_4 + t_3 - t_2 - t_1)/2 = 1.2$ days.

i	$\mathcal{P}_i$ **(years)**	N_i	p_{i1}	p_{i2}	p_{i3}	p_{i4}	p_{i5}	a_i
1	**00 – 19**	10745563	0.578	0.186	0.176	0.048	0.012	2.557
2	**20 – 39**	13113139	0.158	0.394	0.326	0.107	0.015	2.464
3	**40 – 59**	18445702	0.165	0.360	0.322	0.129	0.025	1.589
4	**60 – 79**	13215981	0.100	0.265	0.288	0.283	0.064	0.991
5	**80+**	4296288	0.120	0.175	0.268	0.310	0.128	0.631

Table 2: Age structured groups for 2018 Italian population (total number of members: 59816673). Sources: Istat for demographic data and [5] for contact matrix.

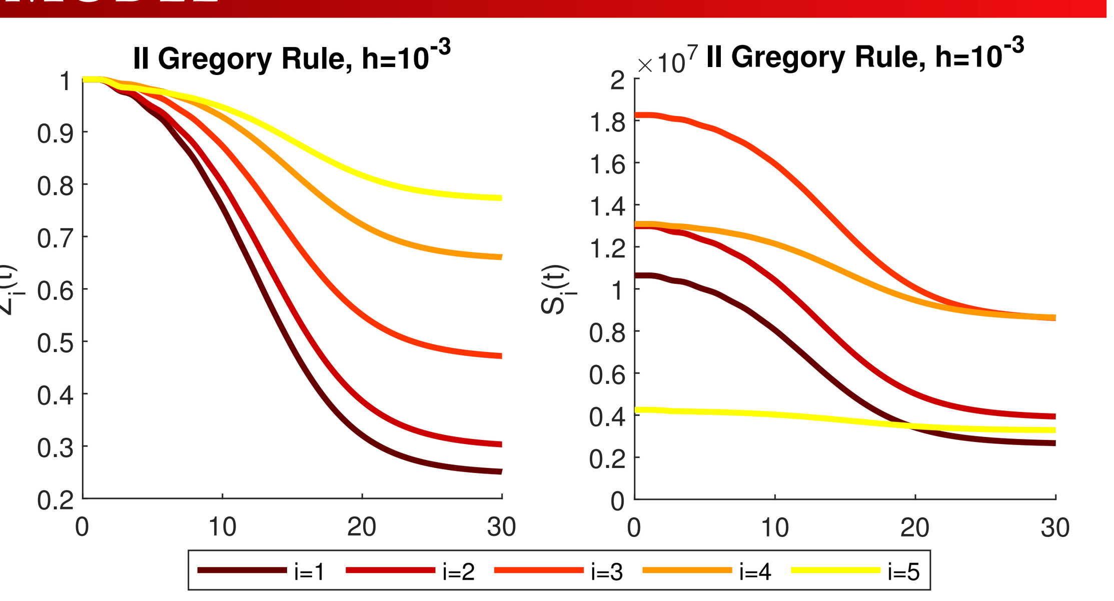

Figure 3: Example with $S_i^0 = 0.99N_i$, $i = 1, \dots, 5$.