Research Article

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Linearization in magnetoelasticity

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Abstract: Starting from a model of nonlinear magnetoelasticity where magnetization is defined in the Eulerian configuration while elastic deformation is in the Lagrangian one, we rigorously derive a linearized model that coincides with the standard one that already appeared in the literature and where the zero-stress strain is quadratic in the magnetization. The relation of the nonlinear and linear model is stated in terms of the Γ-convergence and convergence of minimizers.

Keywords: Linearization, magnetoelasticity

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1 Introduction

Magnetoelasticity is a characteristic observed in certain solids wherein there is a significant interdependence between their mechanical and magnetic properties. This coupling manifests as reversible mechanical deformations that can be induced by applying an external magnetic field. This phenomenon holds considerable practical value, especially in the design of sensors, actuators, and various innovative functional-material devices; see e.g. [30].

The origin of magnetoelasticity can be traced to the intricate interplay between the crystallographic structure of the material, where different crystals exhibit distinct easy axes of magnetization, and magnetic domains [33]. In the absence of external magnetic fields, these magnetic domains align to minimize long-range dipolar effects, leading to a generally minimal or even negligible magnetization of the medium. Upon the application of an external magnetic field, the magnetic domains undergo a reorientation towards it. This reorientation leads to the emergence of a macroscopic deformation since magnetizations are linked to specific stress-free reference strains. As the magnetic field's intensity increases, an increasing number of magnetic domains align themselves so that their principal axes of anisotropy align with the magnetic field in each region, eventually reaching saturation. For a deeper exploration of the foundational aspects of magnetoelasticity, we refer to [16, 25, 33].

Magnetoelasticity is a dynamic area of mathematical research which reflects modeling in linear and nonlinear regimes [21, 42], analysis in the static and quasistatic setting [6, 11, 13] while also being coupled with thermal and other phenomena, e.g. [39]. Moreover, dimension-reduction problems covering lower-dimensional structures in various scaling regimes were recently as well analyzed; for instance, see [11, 20].

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The aim of our contribution is to establish a relationship between nonlinear and linear models of magnetoelasticity in the static case. To our best knowledge, it is the first situation where this program is undertaken in the combined Eulerian and Lagrangian description.

Let Ω be a (connected) bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with Lipschitz boundary $\partial\Omega$, representing a reference configuration occupied by an elastic body. We fix an open subset Γ of $\partial\Omega$ such that $\mathcal{H}^{d-1}(\Gamma) > 0$, where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. We consider the magnetoelastic energy of the following form [13, 16, 34, 41]:

$$\mathcal{G}(y, m) := \int_\Omega W(\nabla y, m \circ y)\,\mathrm{d}x + \frac{1}{2}\int_{y(\Omega)} |\nabla m|^2\,\mathrm{d}z + \frac{\mu_0}{2}\int_{\mathbb{R}^d} |\nabla v_m|^2\,\mathrm{d}z \tag{1.1}$$

for a deformation $y \in W^{1,p}(\Omega;\mathbb{R}^d)$ with $p > d$, $\det\nabla y > 0$ a.e., y is injective a.e. in Ω, and $y = y_0$ on Γ, where $y_0 \in W^{1,p}(\Omega;\mathbb{R}^d)$ is a given boundary datum, and a magnetization $m \in W^{1,2}(y(\Omega)\setminus y(\partial\Omega);\mathbb{R}^d)$. In (1.1), the first term represents the elastic energy, the second term is the exchange energy, and the last term stands for magnetostatic energy; μ_0 is the permittivity of void and v_m is the magnetostatic potential generated by m. In particular, v_m is a solution to the Maxwell equation $\nabla\cdot(-\mu_0\nabla v_m + \chi_{y(\Omega)}m) = 0$ in $\mathbb{R}^d$ with $\chi_{y(\Omega)}$ the characteristic function of $y(\Omega)$. Furthermore, we can assume that the specimen temperature is fixed and therefore the Heisenberg normalized constraint

$$|m \circ y| \det\nabla y = 1 \quad \text{a.e. in } \Omega \tag{1.2}$$

holds for all deformations [16, 34].

Let us point out some peculiarities caused by the mixed Eulerian–Lagrangian formulation. First, to correctly define the composition $m \circ y$, we must identify y with its continuous representative, due to Sobolev embeddings theorems [36, Section 1.4.5]. Second, as it is conventional in magnetoelasticity [13, 15, 41], we define $m \in W^{1,2}(y(\Omega)\setminus y(\partial\Omega);\mathbb{R}^d)$, since $y(\Omega)\setminus y(\partial\Omega)$ is an open set and it differs from $y(\Omega)$ just on a set of measure zero, i.e. $|y(\Omega)| = |y(\Omega)\setminus y(\partial\Omega)|$; see for example [27] or [13, Lemma 2.1].

As in [26, 43], we suppose the deformation gradient F can be multiplicatively decomposed into magnetic and elastic parts $F = F_\mathrm{m}F_\mathrm{el}$ with the magnetic part F_m being volume-preserving, i.e., $\det F_\mathrm{m} = 1$. In our setting, we assume $F_\mathrm{m} := \exp(-E(M))$, i.e., $F_\mathrm{m}^{-1} = \exp(E(M))$, where $E(M)$ is defined as

$$E(M) := -M \otimes M + \frac{1}{d}I.$$

Here, $M \in \mathbb{R}^d$ is a placeholder for the Lagrangian magnetization whose length $|M|$ satisfies $M \cdot M = |M|^2 = 1$, and I is the identity matrix in $\mathbb{R}^{d\times d}$. Note that $\operatorname{tr} E(M) = 0$, so that, indeed, $\det F_\mathrm{m}^{-1} = 1$. In view of the Heisenberg constraint (1.2),

$$M(x) = m(y(x)) \det\nabla y(x) \tag{1.3}$$

satisfies $|M(x)| = 1$ for $x \in \Omega$. Consequently, using the change-of-variables formula for Sobolev mappings [31] (considering that y is a.e. injective and satisfies the Lusin (N)-condition) for $\omega \subset \Omega$, it holds

$$\int_\omega M(x)\,\mathrm{d}x = \int_{y(\omega)} m(z)\,\mathrm{d}z.$$

Therefore, M and m transform as volume densities in the reference (Lagrangian) and the deformed (Eulerian) configurations, respectively.

We define $e\colon \mathbb{R}^{d\times d} \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ as

$$e(F, m) = -(\det F)^2 m \otimes m + \frac{1}{d}I.$$

Whenever m and M are two magnetic fields related by (1.3), then $e(\nabla y, m \circ y) = E(M)$ in Ω.

In the case $F = I$, we simplify the notation by setting $e(m) := e(I, m) = -m \otimes m + \frac{1}{d}I$. Next, we assume that the elastic energy density is of the form $W(F, m) = \Phi(\exp(e(F, m))F) = \Phi(F_\mathrm{el})$. We recall that the matrix exponential $\exp(A) = \sum_{k=0}^\infty A^k/k!$ is defined for every $A \in \mathbb{R}^{d\times d}$, setting also $A^0 = I$. Here, Φ is a nonnegative function that is minimized precisely at the set of proper rotations. This form of the energy density is inspired by energy expressions in the papers [1–3, 22] on nematic elastomers.

In the *small-strain regime*, we set $y_\varepsilon = \mathrm{id} + \varepsilon u$ and assume that the influence of the magnetization on the elastic deformation of the specimen vanishes along $\varepsilon \to 0$ in the sense that $F_{\mathrm{m},\varepsilon} = \exp(-\varepsilon E(M))$ and $\lim_{\varepsilon\to 0} F_{\mathrm{m},\varepsilon} = I$. Then we consider the rescaled energy

$$\begin{aligned}\mathcal{G}_\varepsilon(u,m) &:= \frac{1}{\varepsilon^2}\int_\Omega W(Id + \varepsilon\nabla u(x), m(x+\varepsilon u(x)))\,\mathrm{d}x + \frac{1}{2}\int_{y_\varepsilon(\Omega)} |\nabla m(z)|^2\,\mathrm{d}z + \frac{\mu_0}{2}\int_{\mathbb{R}^d} |\nabla v_m(z)|^2\,\mathrm{d}z \\ &= \frac{1}{\varepsilon^2}\int_\Omega \Phi\big(\exp(\varepsilon E(M))(I+\varepsilon\nabla u(x))\big)\,\mathrm{d}x + \frac{1}{2}\int_{y_\varepsilon(\Omega)} |\nabla m(z)|^2\,\mathrm{d}z + \frac{\mu_0}{2}\int_{\mathbb{R}^d} |\nabla v_m(z)|^2\,\mathrm{d}z.\end{aligned}$$

We formally get the expression for the elastic strain

$$F_{\mathrm{el},\varepsilon} = F_{m,\varepsilon}^{-1}\nabla y_\varepsilon = \exp(\varepsilon E(M))\cdot(I+\varepsilon\nabla u) \approx I + \varepsilon(\nabla u + e(m)).$$

Moreover, a formal Taylor expansion shows that the limit energy is, as in [21], of the form

$$\mathcal{G}_0(u,m) := \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\epsilon}(u) + e(m)) : (\boldsymbol{\epsilon}(u) + e(m))\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla m|^2\,\mathrm{d}x + \frac{\mu_0}{2}\int_{\mathbb{R}^d} |\nabla v_m|^2\,\mathrm{d}z, \tag{1.4}$$

where

$$e(u) := \frac{1}{2}(\nabla u^T + \nabla u)$$

is the symmetric part of the displacement gradient and $\mathbb{C} := D^2\Phi(I)$ is the elasticity tensor. Note that (1.4) suggests a decomposition of the total strain $\boldsymbol{\epsilon}(u)$ into the magnetic part $-e(m) = m\otimes m - I/d$ and the elastic part $\boldsymbol{\epsilon}(u) + e(m)$. This decomposition has already been used, e.g., in [21, 35, 37]. The aim of this contribution is to derive (1.4) rigorously by means of Γ-convergence.

We use the following assumptions for $p > d$:

(a) Φ is frame-indifferent;
(b) Φ belongs to C^2 in some neighborhood of $\mathrm{SO}(d)$;
(c) $\Phi(F) = 0$ if $F \in \mathrm{SO}(d)$;
(d) $\Phi(F) \geq g_p(\mathrm{dist}(F, \mathrm{SO}(d)))$ for some p, where

$$g_p(t) := \begin{cases} \frac{t^2}{2} & \text{if } 0 \leq t \leq 1, \\ \frac{t^p}{p} + \frac{1}{2} - \frac{1}{p} & \text{if } t > 1; \end{cases} \tag{1.5}$$

(e) there exist $a > 1$ and $C > 0$ such that

$$\Phi(F) \geq C\Big(\frac{1}{(\det F)^a} - 1\Big) \quad \text{for every } F \in \mathbb{R}^{d\times d}.$$

Let us notice that g_p satisfies the following properties: g_p is convex,

$$g_p(t) \leq \frac{1}{p}\min\{t^p, t^2\}, \quad g_p(s+t) \leq C(g_p(s) + t^2) \tag{1.6}$$

for some constant $C > 0$ depending on p, and for any $C_1 > 0$, there exists a constant $C_2 > 0$, depending only on p and on C_1, such that

$$g_p(C_1 t) \leq C_2 g_p(t). \tag{1.7}$$

Frame-indifference (a) implies that $W(F,m) = W(F,-m) = W(QF, Qm)$ for all $F \in \mathbb{R}^{d\times d}$ all $Q \in \mathrm{SO}(d)$, and all $m \in \mathbb{R}^d$. Moreover, for every $F \in \mathbb{R}^{d\times d}$,

$$D^2\Phi(I)[F,F] = D^2\Phi(I)[F_{\mathrm{sym}}, F_{\mathrm{sym}}].$$

Together with (d), this implies that $D^2\Phi(I)[F,F] = 0$ if $F = -F^\top$ and

$$D^2\Phi(I)[F_{\mathrm{sym}}, F_{\mathrm{sym}}] \geq |F_{\mathrm{sym}}|^2.$$

Above, $F_{\rm sym}$ stands for the symmetric part of F. Condition (b) together with $\Phi(I) = D\Phi(I) = 0$ leads to

$$\Phi(I+F) \geq \frac{1}{2}D^2\Phi(I)[F,F] - \eta(|F|)|F|^2,$$

where η is an increasing nonnegative function such that $\eta(t) \leq C_R t$ for $|t| \leq R$ and $R > 0$.

In what follows, the mechanical loading is modeled by a continuous linear functional $\mathscr{L}\colon L^2(\Omega;\mathbb{R}^d) \to \mathbb{R}$ and the magnetic loading is described by a functional $\mathscr{M}\colon L^2(\Omega;\mathbb{R}^d) \times L^2(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R}$,

$$\mathscr{L}(y) := \int_\Omega f\cdot y \,\mathrm{d}x \quad \text{and} \quad \mathscr{M}(y,m) := \int_{y(\Omega)} h\cdot m\,\mathrm{d}z,$$

where $f \in L^2(\Omega;\mathbb{R}^d)$ is a body force and $h \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ represents an external magnetic field, while $\mathscr{M}$ is called the Zeeman energy.

If $y \in W^{1,p}(\Omega;\mathbb{R}^d)$ represents the deformation of the elastic body, the stable equilibria of the elastic body are obtained by minimizing the functional $\int_\Omega W(\nabla y, m\circ y)\,\mathrm{d}x - \mathscr{L}(y) - \mathscr{M}(y,m)$, under the prescribed boundary conditions. The area of interest of this paper is the case where the load has the form $\varepsilon\mathscr{L}$, and to study the behavior of the solution as ε tends to zero. We write $y_\varepsilon = \mathrm{id} + \varepsilon u$ and assume the Dirichlet boundary condition of the form $y_\varepsilon = \mathrm{id} + \varepsilon w$, $\mathcal{H}^{d-1}$-a.e. on Γ, with a given function $w \in W^{2,\infty}(\Omega;\mathbb{R}^d)$. Denoting a set of functions satisfying boundary conditions as

$$W^{1,p}_w(\Omega;\mathbb{R}^d) := \{u \in W^{1,p}(\Omega;\mathbb{R}^d) : u = w,\ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma\},$$
$$W^{2,\infty}_w(\Omega;\mathbb{R}^d) := \{u \in W^{2,\infty}(\Omega;\mathbb{R}^d) : u = w,\ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma\},$$

the corresponding minimum problem for (u,m) becomes

$$\min\Big\{\int_\Omega W(I+\varepsilon\nabla u(x), m(x+\varepsilon u(x)))\,\mathrm{d}x - \varepsilon\mathscr{L}(\varepsilon u) - \mathscr{M}(\mathrm{id}+\varepsilon u, m) : \\ (u,m) \in W^{1,p}_w(\Omega;\mathbb{R}^d) \times W^{1,2}(y_\varepsilon(\Omega)\setminus y_\varepsilon(\partial\Omega);\mathbb{R}^d)\Big\}.$$

The class of admissible deformations is given by

$$\mathcal{Y} := \{y \in W^{1,p}(\Omega;\mathbb{R}^d) : \det\nabla y > 0 \text{ a.e. in } \Omega,\ y \text{ is a.e. injective in } \Omega\}.$$

For $w \in W^{2,\infty}(\Omega;\mathbb{R}^d)$ and $\varepsilon > 0$, define the class of admissible states as

$$\mathcal{A}^w_\varepsilon := \{(u,m) \in W^{1,2}_w(\Omega;\mathbb{R}^d) \times L^2(\mathbb{R}^d;\mathbb{R}^d) : y_\varepsilon := \mathrm{id}+\varepsilon u \in \mathcal{Y},\ m \in W^{1,2}(y_\varepsilon(\Omega)\setminus y_\varepsilon(\partial\Omega);\mathbb{R}^d), \\ |m\circ y_\varepsilon|\det\nabla y_\varepsilon = 1 \text{ a.e. in } \Omega\},$$

and

$$\mathcal{A}^w_0 := W^{1,2}_w(\Omega;\mathbb{R}^d) \times W^{1,2}(\Omega;\mathbb{S}^{d-1}).$$

The main result of this work is the following Γ-convergence statement.

Theorem 1.1. *Let $W\colon \mathbb{R}^{d\times d}\times\mathbb{R}^d \to [0,\infty]$ satisfy conditions* (a)–(e) *for some $p > d$, and let $w \in W^{2,\infty}(\Omega;\mathbb{R}^d)$. For every $\varepsilon_j \to 0$, we have that*

$$\mathcal{G}_{\varepsilon_j} \xrightarrow{\Gamma} \mathcal{G}_0 \quad \text{as } j\to\infty$$

in the weak topology of $W^{1,2}(\Omega;\mathbb{R}^d)\times L^2(\mathbb{R}^d;\mathbb{R}^d)$.

Remark 1.2. In view of the compactness properties stated in Proposition 3.1 and in order to state the Γ-convergence in $W^{1,2}(\Omega;\mathbb{R}^d)\times L^2(\mathbb{R}^d;\mathbb{R}^d)$, in Theorem 1.1, we identify m_ε with $\chi_{y_\varepsilon(\Omega)}m_\varepsilon$ for $(u_\varepsilon,m_\varepsilon) \in \mathcal{A}^w_\varepsilon$ and $y_\varepsilon = \mathrm{id}+\varepsilon u_\varepsilon$. Similarly, in the limit as $\varepsilon \to 0$, we identify the magnetization m with $\chi_\Omega m$.

Theorem 1.1 is completed by the following theorem, stating the convergence of minima and minimizers. Here, we consider loading terms as well. In particular, for every $\varepsilon > 0$ and every $(u,m) \in \mathcal{A}^w_\varepsilon$, we set

$$\mathcal{F}_\varepsilon(u,m) := \mathcal{G}_\varepsilon(u,m) - \mathscr{L}(u) - \mathscr{M}(\mathrm{id}+\varepsilon u, m).$$

Theorem 1.3. *Under the hypotheses of Theorem 1.1, for every $\varepsilon > 0$, define $s_\varepsilon := \inf\{\mathcal{F}_\varepsilon(u, m) : (u, m) \in \mathcal{A}^w_\varepsilon\}$ and let $(u_\varepsilon, m_\varepsilon) \in \mathcal{A}^w_\varepsilon$ be a sequence such that*

$$\mathcal{F}_\varepsilon(u_\varepsilon, m_\varepsilon) = s_\varepsilon + o(1) \quad \text{as } \varepsilon \to 0. \tag{1.8}$$

Then $u_\varepsilon \rightharpoonup u_0$ weakly in $W^{1,2}(\Omega; \mathbb{R}^d)$, $\chi_{y_\varepsilon(\Omega)} m_\varepsilon \to \chi_\Omega m_0$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$, and $\chi_{y_\varepsilon(\Omega)} \nabla m_\varepsilon \rightharpoonup \chi_\Omega \nabla m_0$ weakly in $L^2(\mathbb{R}^d; \mathbb{R}^{d\times d})$, where $(u_0, m_0) \in \mathcal{A}^w_0$ is the solution to the minimum problem

$$s_0 := \min\{\mathcal{F}_0(u, m) : (u, m) \in \mathcal{A}^w_0\},$$

where $\mathcal{F}_0(u, m) := \mathcal{G}_0(u, m) - \mathscr{L}(u) - \mathscr{M}(\mathrm{id}, m)$ and $\mathcal{G}_0$ is defined by (1.4). *Moreover, $s_\varepsilon \to s_0$.*

Theorem 1.4. *Under the hypotheses of Theorems 1.1 and 1.3, let $(u_\varepsilon, m_\varepsilon) \in \mathcal{A}^w_\varepsilon$ satisfy* (1.8). *Then, up to a subsequence, $(u_\varepsilon, \chi_{y_\varepsilon(\Omega)} m_\varepsilon) \to (u_0, \chi_\Omega m_0)$ strongly in $W^{1,2}(\Omega; \mathbb{R}^d) \times L^2(\mathbb{R}^d; \mathbb{R}^d)$.*

The proof of Theorems 1.1–1.4 is postponed to Section 4. Compactness issues are discussed in Section 3.

Remark 1.5 (Discussion for the case $p \le d$). Let us notice that the p-integrability of deformations' gradients plays an essential role only for the compactness. The result of this paper can be extended for the case $p = d$ using the means of quasiconformal analysis; see details in, e.g., [32]. In this case, an admissible deformation $y \in \mathcal{Y}$ is continuous and differentiable a.e., and a continuous representative satisfies the Lusin (N)- and $(N)^{-1}$-conditions. Thus, the composition $m \circ y$ is well defined, and the change-of-variables formula is valid for the chosen representative. Second, instead of $y(\Omega) \setminus y(\partial\Omega)$, one needs to consider the interior of $y(\Omega)$ to define m properly; see e.g. [5]. Finally, there is no uniform convergence $y_\varepsilon \to y$, only locally uniform convergence. Therefore, the proof of compactness in Proposition 3.1 needs more refined estimates.

There are several options in the literature coming from [38], which offer ways to deal with the case $p > d - 1$. However, a stronger coercivity condition is needed to guarantee weak convergence of determinants. For instance, one may use the approach prohibiting cavitations as in [6, 12]. Another option is to consider deformations with finite surface energy as in [6], (INV)-condition, or weak limits of homeomorphisms [10, 23]. We refer an interested reader to the paper [14], which studies magnetoelasticity in the case $p > d - 1$. For the case $p = d - 1$, admissible deformations may be considered within the limits of homeomorphisms [24] or mappings with finite surface energy [7, 8], although this case appears to be much more difficult.

2 Auxiliary results

The homogeneous Sobolev space $L^{1,2}(\mathbb{R}^d)$ is defined as the space of $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\nabla\varphi \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. We equip the space $L^{1,2}(\mathbb{R}^d)$ with a seminorm $\|\nabla\varphi\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$. It can be as well supplied with a norm

$$\|\varphi\|_{L^{1,2}(\mathbb{R}^d)} := \|\nabla\varphi\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} + \|\varphi\|_{L^2(\omega)},$$

where $\omega \subset \mathbb{R}^d$ is an arbitrary bounded open nonempty set. For more details, the interested reader is referred to [36, Chapter 1]. The magnetic potential $v_m \in L^{1,2}(\mathbb{R}^d)$ generated by magnetization m exists, unique up to a constant, and continuously depends on m.

Proposition 2.1. *For every $f \in L^2(\mathbb{R}^d; \mathbb{R}^d)$, there exists $v_f \in L^{1,2}(\mathbb{R}^d)$, which is unique up to additive constants, such that*

$$\int_{\mathbb{R}^d} (-\nabla v_f + f) \cdot \nabla\varphi \,\mathrm{d}z = 0 \quad \text{for all } \varphi \in L^{1,2}(\mathbb{R}^d) \quad \text{and} \quad \|\nabla v_f\|_{L^2} \le \|f\|_{L^2}.$$

Moreover, if $f_j \to f$ strongly in L^2, then $\nabla v_{f_j} \to \nabla v_f$ strongly in $L^2(\mathbb{R}^d, \mathbb{R}^d)$.

Proof. This proposition is classical; see for example [6, Proposition 8.8 & Theorem 8.9] or [41, proof of Theorem 4.1]. □

Let us now recall sufficient conditions for a deformation to be injective if the gradient of the displacement is small enough in the operator norm, i.e., $|B|_O = \sup_{v\in\mathbb{R}^d} \frac{|Bv|}{|v|}$.

Theorem 2.2 ([17, Theorem 5.5-1]). *Let $\Omega \subset \mathbb{R}^d$ be an open set.*
(1) *Let $y = \mathrm{id} + u\colon \Omega \to \mathbb{R}^d$ be differentiable in $x \in \Omega$. Then $|\nabla u(x)|_O < 1 \Rightarrow \det \nabla y(x) > 0$.*
(2) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then there exists a constant $c(\Omega)$ such that any $y = \mathrm{id} + u \in C^1(\bar{\Omega}; \mathbb{R}^d)$ with $\sup_{x\in\bar{\Omega}} |\nabla u(x)|_O < c(\Omega)$ is injective.*

To establish compactness, we need the following geometric rigidity, proven in [28, Theorem 3.1] for $p = 2$ and in [18, Section 2.4] for $p \in (1, +\infty)$.

Theorem 2.3. *There exists a constant $C = C(\Omega, p) > 0$ such that, for every $v \in W^{1,p}(\Omega; \mathbb{R}^d)$, there exists a constant rotation $R \in \mathrm{SO}(d)$ satisfying*

$$\int_\Omega |\nabla v - R|^p \,\mathrm{d}x \le C \int_\Omega g_p(\mathrm{dist}(\nabla v, \mathrm{SO}(d)))\,\mathrm{d}x,$$

where g_p is defined by (1.5)*. In particular, we may choose R such that*

$$\Big|R - \frac{1}{\mathcal{L}^d(\Omega)} \int_\Omega \nabla v \,\mathrm{d}x\Big| = \mathrm{dist}\Big(\frac{1}{\mathcal{L}^d(\Omega)} \int_\Omega \nabla v \,\mathrm{d}x, \mathrm{SO}(d)\Big). \tag{2.1}$$

Proof. It is enough to resort to the classical rigidity estimates for $p \in (1, +\infty)$ and notice that

$$\frac{1}{2p}(t^p + t^2) \le \frac{1}{p}\max\{t^p, t^2\} \le g_p(t) \le \frac{1}{2}\max\{t^p, t^2\} \le \frac{1}{2}(t^p + t^2).$$

The particular choice of R as in (2.1) can be ensured arguing, e.g., as in [4, Theorem 3.1]. □

3 Compactness

This section is devoted to the proof of the following compactness result for sequences with bounded energy $\mathcal{G}_\varepsilon$.

Proposition 3.1 (Compactness). *Under the assumptions of Theorem 1.3, let $(u_\varepsilon, m_\varepsilon) \in \mathcal{A}_\varepsilon^w$ be such that*

$$\sup_{\varepsilon>0} \mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) < +\infty. \tag{3.1}$$

Then there exists $(u_0, m_0) \in W_w^{1,2}(\Omega; \mathbb{R}^d) \times W^{1,2}(\Omega; \mathbb{S}^{d-1})$ such that, up to a subsequence,

$$\begin{aligned}
y_\varepsilon := \mathrm{id} + \varepsilon u_\varepsilon &\to \mathrm{id} && \text{strongly in } W^{1,p}(\Omega; \mathbb{R}^d),\\
u_\varepsilon &\rightharpoonup u_0 && \text{weakly in } W^{1,2}(\Omega; \mathbb{R}^d),\\
\chi_{y_\varepsilon(\Omega)} m_\varepsilon &\to \chi_\Omega m_0 && \text{strongly in } L^2(\mathbb{R}^d; \mathbb{R}^d),\\
\chi_{y_\varepsilon(\Omega)} \nabla m_\varepsilon &\rightharpoonup \chi_\Omega \nabla m_0 && \text{weakly in } L^2(\mathbb{R}^d; \mathbb{R}^{d\times d}),\\
m_\varepsilon \circ y_\varepsilon &\to m_0 && \text{strongly in } L^1(\Omega; \mathbb{R}^d),\\
m_\varepsilon \circ y_\varepsilon \det \nabla y_\varepsilon &\to m_0 && \text{strongly in } L^r(\Omega; \mathbb{R}^d) \text{ for every } 1 \le r < \infty.
\end{aligned}$$

Before proving Proposition 3.1, we show the following lemmas, which provide boundedness of the displacement variable in terms of the energy $\mathcal{G}_\varepsilon$.

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exist positive constants $C_1 = C_1(\Omega, d, p) > 0$ and $C_2 = C_2(\Omega, d, p) > 0$ such that, for every $\varepsilon \in (0, 1)$ and every $(u, m) \in \mathcal{A}_\varepsilon^w$, it holds*

$$\varepsilon^2 \mathcal{G}_\varepsilon(u, m) \ge C_1 \int_\Omega g_p(\mathrm{dist}(\nabla y_\varepsilon, \mathrm{SO}(d))\,\mathrm{d}x - C_2 \varepsilon^2 \tag{3.2}$$

for $y_\varepsilon = \mathrm{id} + \varepsilon u$.

Proof. Let $(u, m) \in \mathcal{A}_\varepsilon^w$ and $\varepsilon \in (0, 1)$. By assumption (d), we can estimate

$$\varepsilon^2 \mathcal{G}_\varepsilon(u, m) \ge C \int_\Omega g_p(\mathrm{dist}(\exp(\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\nabla y_\varepsilon; \mathrm{SO}(d)))\,\mathrm{d}x. \tag{3.3}$$

By definition of $e(\nabla y_\varepsilon, m \circ y_\varepsilon)$, by the properties of the exponential map, and by the fact that $\det \nabla y_\varepsilon |m \circ y_\varepsilon| = 1$ in Ω, we have that, for $x \in \Omega$ and $R \in SO(d)$,

$$\begin{aligned}\operatorname{dist}(\nabla y_\varepsilon; \exp(-\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\, SO(d)) &\le |\nabla y_\varepsilon - \exp(-\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))R| \\ &\le |\exp(-\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))||\exp(\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\nabla y_\varepsilon - R| \\ &\le C|\exp(\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\nabla y_\varepsilon - R|.\end{aligned}$$

This implies that, for a.e. $x \in \Omega$,

$$\operatorname{dist}(\nabla y_\varepsilon; \exp(-\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\, SO(d)) \le C \operatorname{dist}(\exp(\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\nabla y_\varepsilon; SO(d)).$$

Again by the properties of the exponential, we further estimate

$$\operatorname{dist}(\nabla y_\varepsilon; SO(d)) \le C \operatorname{dist}(\nabla y_\varepsilon; \exp(-\varepsilon e(\nabla y_\varepsilon, m \circ y_\varepsilon))\, SO(d)) + C\varepsilon \quad \text{a.e. in } \Omega. \tag{3.4}$$

Combining (3.3)–(3.4) and the properties of g_p (1.6)–(1.7), we get (3.2). ☐

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exists a positive constant $C = C(\Omega, d, p) > 0$ such that, for every $\varepsilon > 0$, every $(u, m) \in \mathcal{A}^w_\varepsilon$, and every $\varepsilon > 0$, the following estimates hold:*

$$\int_\Omega |\nabla u|^2 \,dx \le C\Big(\mathcal{G}_\varepsilon(u, m) + \int_\Gamma |w|^2 \,d\mathcal{H}^{d-1} + 1\Big), \tag{3.5}$$

$$\int_\Omega |\varepsilon \nabla u|^p \,dx \le C\varepsilon^2\Big(\mathcal{G}_\varepsilon(u, m) + \int_\Gamma |w|^2 \,d\mathcal{H}^{d-1} + 1\Big). \tag{3.6}$$

Proof. Along the proof, we denote by C a generic positive constant depending only on Ω, d, and p, but not on ε and on $(u, m) \in \mathcal{A}^w_\varepsilon$.

Let us prove (3.5). For $(u, m) \in \mathcal{A}^w_\varepsilon$, we set $y_\varepsilon := \mathrm{id} + \varepsilon u$. Applying the Geometric Rigidity Theorem 2.3 and Lemma 3.2, we deduce that there exists $R_\varepsilon \in SO(d)$ such that

$$\int_\Omega |\nabla y_\varepsilon - R_\varepsilon|^2 \,dx \le C \int_\Omega g_p(\operatorname{dist}(\nabla y_\varepsilon; SO(d))) \,dx \le C\varepsilon^2(\mathcal{G}_\varepsilon(u, m) + 1).$$

By triangle inequality, we continue with

$$\int_\Omega |\nabla y_\varepsilon - I|^2 \,dx \le C(\varepsilon^2(\mathcal{G}_\varepsilon(u, m) + 1) + |R_\varepsilon - I|^2). \tag{3.7}$$

Arguing as in [19, Proposition 3.4], we get that

$$|R_\varepsilon - I|^2 \le C\varepsilon^2\Big(\mathcal{G}_\varepsilon(u, m) + \int_\Gamma |w|^2 \,d\mathcal{H}^{d-1} + 1\Big), \tag{3.8}$$

which, together with (3.7), implies (3.5).

As for (3.6), we notice that, by Theorem 2.3 and Lemma 3.2, for the same rotation R_ε, it holds

$$\int_\Omega |\nabla y_\varepsilon - R_\varepsilon|^p \,dx \le C \int_\Omega g_p(\operatorname{dist}(\nabla y_\varepsilon; SO(d))) \,dx + C\varepsilon^2 \le C\varepsilon^2(\mathcal{G}_\varepsilon(u, m) + 1).$$

Hence, recalling (3.8), we deduce that

$$\begin{aligned}\int_\Omega |\varepsilon \nabla u|^p \,dx &\le 2^{p-1}\Big(\int_\Omega |\nabla y_\varepsilon - R_\varepsilon|^p \,dx + |R_\varepsilon - I|^p\Big) \le C(\varepsilon^2 \mathcal{G}_\varepsilon(u, m) + \varepsilon^2 + |R_\varepsilon - I|^p) \\ &\le C(\varepsilon^2 \mathcal{G}_\varepsilon(u, m) + \varepsilon^2 + |R_\varepsilon - I|^2) \le C\varepsilon^2\Big(\mathcal{G}_\varepsilon(u, m) + \int_\Gamma |w|^2 \,d\mathcal{H}^{d-1} + 1\Big),\end{aligned}$$

where we have used the fact that $p > d \ge 2$ and that $R_\varepsilon \in SO(d)$, so that $|R_\varepsilon - I| \le 2\sqrt{d}$. This concludes the proof of the proposition. ☐

We are now ready to prove the compactness result of Proposition 3.1. We closely follow the arguments of [11, Proposition 4.3] (see also [15, Proposition 3.7]).

Proof of Proposition 3.1. Let us notice that the Poincaré inequality and Lemma 3.3 guarantee, up to a subsequence, weak convergence $u_\varepsilon \rightharpoonup u_0$ in $W^{1,2}(\Omega, \mathbb{R}^d)$. Moreover, (3.6) implies that

$$\|\nabla y_\varepsilon - I\|_{L^p} \le C\varepsilon^{\frac{2}{p}} \tag{3.9}$$

for some positive constant C independent of ε. And thus, strong convergence $y_\varepsilon \to \mathrm{id}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$.

For simplicity of notation, let us set $\Lambda_\varepsilon := \exp(\varepsilon e(\nabla y_\varepsilon, m_\varepsilon \circ y_\varepsilon))\nabla y_\varepsilon$. Then, by the assumptions on Φ, we have that

$$\int_\Omega \frac{1}{(\det \nabla y_\varepsilon)^a}\,\mathrm{d}x = \int_\Omega \frac{1}{(\det \Lambda_\varepsilon)^a}\,\mathrm{d}x \le C\int_\Omega W(\nabla y_\varepsilon, m_\varepsilon \circ y_\varepsilon)\,\mathrm{d}x + C\mathcal{L}^d(\Omega) \le C\varepsilon^2 \mathcal{G}_\varepsilon(y_\varepsilon, m_\varepsilon) + C\mathcal{L}^d(\Omega). \tag{3.10}$$

Thus, $1/\det \nabla y_\varepsilon$ is equi-integrable in Ω.

For $\delta > 0$, let us set

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\} \quad \text{and} \quad \mu_\varepsilon^\delta := \chi_{\Omega_\delta} m_\varepsilon.$$

Without loss of generality, we may assume that Ω_δ is a Lipschitz set. Since $y_\varepsilon \to \mathrm{id}$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and $p > d$, we have that $y_\varepsilon \to \mathrm{id}$ in $C^0(\overline{\Omega}; \mathbb{R}^d)$. Hence, for $\varepsilon > 0$ small enough, we may assume that $\Omega_\delta \subseteq y_\varepsilon(\Omega)$. Without going into details about the degree theory, see [27], let us provide rather a standard argument to prove this fact. For y_ε, let us define the topological degree $\deg(x, y_\varepsilon, \Omega)$ as a topological degree of its continuous representative. Notice that $h_t(x) := t y_\varepsilon(x) + (1-t)x$ is a homotopy between y_ε and id, and by the uniform convergence $y_\varepsilon \to \mathrm{id}$, for any $x \in \Omega_\delta$, it holds that $x \notin y_\varepsilon(\partial\Omega)$ for any small enough ε. Thus, $\deg(x, y_\varepsilon, \Omega) = \deg(x, \mathrm{id}, \Omega) > 0$; in other words, $x \in y_\varepsilon(\Omega)$.

Next, by change of variables, using the equality $|m_\varepsilon \circ y_\varepsilon| \det \nabla y_\varepsilon = 1$ a.e. in Ω, we get that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mu_\varepsilon^\delta|^2\,\mathrm{d}x &= \int_{\Omega_\delta} |m_\varepsilon|^2\,\mathrm{d}x \\ &\le \int_{y_\varepsilon(\Omega)} |m_\varepsilon|^2\,\mathrm{d}x = \int_\Omega |m_\varepsilon \circ y_\varepsilon|^2 \det \nabla y_\varepsilon\,\mathrm{d}x = \int_\Omega \frac{1}{\det \nabla y_\varepsilon}\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

Moreover, by definition of μ_ε^δ, it holds

$$\int_{\Omega_\delta} |\nabla \mu_\varepsilon^\delta|^2\,\mathrm{d}x \le \int_{y_\varepsilon(\Omega)} |\nabla m_\varepsilon|^2\,\mathrm{d}x \le \mathcal{G}_\varepsilon(y_\varepsilon, m_\varepsilon). \tag{3.12}$$

Therefore, we have that μ_ε^δ is bounded in $W^{1,2}(\Omega_\delta; \mathbb{R}^d)$ uniformly for all $\delta > 0$. By a diagonal argument, we find $m_0 \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ such that $\mu_\varepsilon^\delta \rightharpoonup m_0$ weakly in $W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$. While the Sobolev embedding theorems provide that

$$m_\varepsilon(x) = \mu_\varepsilon^\delta(x) \to m_0(x) \quad \text{for a.e. } x \in \Omega_\delta. \tag{3.13}$$

We further deduce that $m_0 \in W^{1,2}(\Omega; \mathbb{R}^d)$, passing to the liminf as $\varepsilon \to 0$ and then to the limit as $\delta \to 0$ in (3.11) and (3.12). Note further that the second line in (3.11) ensures that $\chi_{y_\varepsilon(\Omega)} m_\varepsilon$ is bounded in $L^2(\mathbb{R}^d; \mathbb{R}^d)$, and thus converges weakly to some $\mu \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. Using pointwise convergence (3.13), we conclude $\chi_{y_\varepsilon(\Omega)} m_\varepsilon(x) \to \chi_\Omega m_0(x)$ a.e. in $\mathbb{R}^d$. Therefore, $\mu = \chi_\Omega m_0$ a.e. and, up to a subsequence,

$$\chi_{y_\varepsilon(\Omega)} m_\varepsilon \rightharpoonup \chi_\Omega m_0 \quad \text{weakly in } L^2(\mathbb{R}^d; \mathbb{R}^d). \tag{3.14}$$

Now it is not difficult to see that $|\chi_{y_\varepsilon(\Omega)} m_\varepsilon|^2$ converges in measure and, moreover, is equi-integrable by (3.10) and (3.11). Thus, applying the Lebesgue–Vitali Convergence Theorem (see [9, Theorem 4.5.4] or [40, Chapter 6, Exercise 10]) to $|\chi_{y_\varepsilon(\Omega)} m_\varepsilon|^2$, we obtain convergence of norms $\|\chi_{y_\varepsilon(\Omega)} m_\varepsilon\|_{L^2} \to \|\chi_\Omega m_0\|_{L^2}$. This, together with weak convergence (3.14) guarantees strong convergence $\chi_{y_\varepsilon(\Omega)} m_\varepsilon \to \chi_\Omega m_0$ in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. Since the weak convergence in $L^2(\Omega; \mathbb{R}^d)$ of $m_\varepsilon \chi_{y_\varepsilon(\Omega)}$ to m_0 has been already shown, we deduce the strong L^2-convergence.

By (3.1), we know that $\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon$ is bounded in $L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})$. For every test $\varphi \in L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})$, we consider

$$\int_{\mathbb{R}^d} (\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon - \chi_\Omega \nabla m_0) : \varphi \,\mathrm{d}x = \int_{\Omega_\delta} (\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon - \chi_\Omega \nabla m_0) : \varphi \,\mathrm{d}x + \int_{\mathbb{R}^d\setminus\Omega_\delta} (\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon - \chi_\Omega \nabla m_0) : \varphi \,\mathrm{d}x.$$

Then the first integral tends to 0 since $\Omega_\delta \subseteq y_\varepsilon(\Omega)\cap\Omega$, $m_\varepsilon = \mu_\varepsilon^\delta$ in Ω_δ, and $\mu_\varepsilon^\delta \rightharpoonup m_0$ weakly in $W^{1,2}(\Omega_\delta;\mathbb{R}^d)$. As for the second integral, we have that

$$\left|\int_{\mathbb{R}^d\setminus\Omega_\delta} (\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon - \chi_\Omega \nabla m_0) : \varphi \,\mathrm{d}x\right| \le C\|\varphi\|_{L^2(C_\delta)},$$

where $C_\delta = \{x \in \mathbb{R}^d : \mathrm{dist}(x,\partial\Omega) < \delta\}$. Since $\mathcal{L}^d(C_\delta) \to 0$ as $\delta \to 0$, we infer that $\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon$ converges to $\chi_\Omega\nabla m_0$ weakly in $L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})$.

We now show that $m_\varepsilon \circ y_\varepsilon$ converges to m_0 in $L^1(\Omega;\mathbb{R}^d)$. For $\delta > 0$ and $\varepsilon \ll 1$, we write

$$\int_{\Omega_\delta} |m_\varepsilon \circ y_\varepsilon - m_0|\,\mathrm{d}x \le \int_{\Omega_\delta} |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x + \int_{\Omega_\delta} |\mu_\varepsilon^\delta - m_0|\,\mathrm{d}x. \tag{3.15}$$

For the second integral on the right-hand side of (3.15), we simply have that $\mu_\varepsilon^\delta \to m_0$ in $L^2(\Omega_\delta;\mathbb{R}^d)$. As for the first term on the right-hand side of (3.15), we notice that $m_\varepsilon \circ y_\varepsilon$ and μ_ε^δ are equi-integrable in Ω_δ. Thus, for $\eta > 0$, there exists $\rho > 0$ such that, for every $A \subseteq \Omega_\delta$ measurable,

$$\mathcal{L}^d(A) < \rho \implies \limsup_{\varepsilon\to 0} \int_A |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x < \eta.$$

Let us consider $\overline{\delta} \in (0,\delta)$ such that $\mathcal{L}^d(\Omega_{\overline{\delta}} \setminus \Omega_\delta) < \rho$ and set $A_{\delta,\varepsilon} := y_\varepsilon^{-1}(\Omega_\delta)$. For $\varepsilon > 0$ small enough, we have that $A_{\delta,\varepsilon} \subseteq \Omega_{\overline{\delta}}$. We rewrite

$$\int_{\Omega_\delta} |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x = \int_{\Omega_\delta\cap A_{\delta,\varepsilon}} |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x + \int_{\Omega_\delta\setminus A_{\delta,\varepsilon}} |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x. \tag{3.16}$$

We notice that, by change of variable,

$$\mathcal{L}^d(\Omega_\delta) = \mathcal{L}^d(y_\varepsilon(A_{\delta,\varepsilon})) = \mathcal{L}^d(A_{\delta,\varepsilon}) + \int_{A_{\delta,\varepsilon}} (\det\nabla y_\varepsilon - 1)\,\mathrm{d}x.$$

Since $\det\nabla y_\varepsilon \to 1$ in $L^1(\Omega)$, $A_{\delta,\varepsilon} \subseteq \Omega_{\overline{\delta}}$, and $\mathcal{L}^d(\Omega_{\overline{\delta}} \setminus \Omega_\delta) < \rho$, we conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{L}^d(\Omega_\delta \setminus A_{\delta,\varepsilon}) \le \limsup_{\varepsilon\to 0} (\mathcal{L}^d(\Omega_{\overline{\delta}}) - \mathcal{L}^d(A_{\delta,\varepsilon})) = \limsup_{\varepsilon\to 0} \Big(\mathcal{L}^d(\Omega_{\overline{\delta}} \setminus \Omega_\delta) - \int_{A_{\delta,\varepsilon}} (\det\nabla y_\varepsilon - 1)\,\mathrm{d}x\Big) < \rho.$$

In turn, this implies that

$$\limsup_{\varepsilon\to 0} \int_{\Omega_\delta\setminus A_{\delta,\varepsilon}} |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x < \eta. \tag{3.17}$$

To estimate the integral on $\Omega_\delta \cap A_{\delta,\varepsilon}$, we fix $\lambda \in (-1/p, 0)$, extend μ_ε^δ from Ω_δ to a map $M_\varepsilon^\delta \in W^{1,2}(\mathbb{R}^d;\mathbb{R}^d)$ with $\|M_\varepsilon^\delta\|_{W^{1,2}(\mathbb{R}^d)} \le C(\Omega_\delta)\|\mu_\varepsilon^\delta\|^2_{W^{1,2}(\Omega_\delta)}$, and use the Lusin approximation of Sobolev functions: for every $\varepsilon > 0$, there exists a set $G_{\delta,\varepsilon}$ such that M_ε^δ is ε^λ-Lipschitz on $G_{\delta,\varepsilon}$ and

$$\mathcal{L}^d(\mathbb{R}^d \setminus G_{\delta,\varepsilon}) \le \varepsilon^{-2\lambda} \int_{\mathbb{R}^d} |\nabla M_\varepsilon^\delta|^2\,\mathrm{d}x \le C(\Omega_\delta)\varepsilon^{-2\lambda}\|\mu_\varepsilon^\delta\|^2_{W^{1,2}(\Omega_\delta)}. \tag{3.18}$$

Setting $X_{\delta,\varepsilon} := y_\varepsilon^{-1}(G_{\delta,\varepsilon})$ and noticing that $m_\varepsilon = M_\varepsilon^\delta$ on $\Omega_\delta \cap G_{\delta,\varepsilon}$, we have that

$$\int_{\Omega_\delta\cap A_{\delta,\varepsilon}\cap X_{\delta,\varepsilon}\cap G_{\delta,\varepsilon}} |m_\varepsilon \circ y_\varepsilon - \mu_\varepsilon^\delta|\,\mathrm{d}x = \int_{\Omega_\delta\cap A_{\delta,\varepsilon}\cap X_{\delta,\varepsilon}\cap G_{\delta,\varepsilon}} |M_\varepsilon^\delta \circ y_\varepsilon - M_\varepsilon^\delta|\,\mathrm{d}x \le \varepsilon^\lambda \mathcal{L}^d(\Omega)\|y_\varepsilon - \mathrm{id}\|_{C^0} \le \varepsilon^{\frac{2}{p}+\lambda}\mathcal{L}^d(\Omega).$$

Combining (3.18), the convergence rate (3.9), and a change of variable, we infer that

$$\lim_{\varepsilon\to 0}\mathcal{L}^d(\Omega_\delta\setminus G_{\delta,\varepsilon})=0\quad\text{and}\quad\lim_{\varepsilon\to 0}\mathcal{L}^d(\Omega_\delta\setminus X_{\delta,\varepsilon})=0.$$

Thus,

$$\limsup_{\varepsilon\to 0}\int\limits_{\Omega_\delta\cap A_{\delta,\varepsilon}\setminus(X_{\delta,\varepsilon}\cap G_{\delta,\varepsilon})}|m^\delta_\varepsilon\circ y_\varepsilon-\mu^\delta_\varepsilon|\,dx<\eta. \tag{3.19}$$

Combining (3.15), (3.16), (3.17), and (3.19) yields

$$\limsup_{\varepsilon\to 0}\int\limits_{\Omega_\delta}|m_\varepsilon\circ y_\varepsilon-m_0|\,dx<3\eta.$$

Since $\eta>0$ is arbitrary, this implies that $m_\varepsilon\circ y_\varepsilon\to m_0$ in $L^1(\Omega_\delta;\mathbb{R}^d)$ for every $\delta>0$. By the equi-integrability of $m_\varepsilon\circ y_\varepsilon$, this entails the convergence in $L^1(\Omega;\mathbb{R}^d)$. Furthermore, since strong L^1-convergence implies convergence a.e., we have $m_\varepsilon\circ y_\varepsilon\det\nabla y_\varepsilon\to m_0$ a.e. and $|m_\varepsilon\circ y_\varepsilon|\det\nabla y_\varepsilon=|m_0|=1$ a.e. in Ω. Thus, $m\in W^{1,2}(\Omega;\mathbb{S}^{d-1})$ and $m_\varepsilon\circ y_\varepsilon\det\nabla y_\varepsilon\to m_0$ in $L^r(\Omega;\mathbb{R}^d)$ for any $r\in[1,+\infty)$. □

4 Proof of Γ-convergence

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof in lower and upper bound for the Γ-convergence.

Γ-liminf inequality. We start with the lower bound. Let $(u_\varepsilon,m_\varepsilon)\in\mathcal{A}^w_\varepsilon$ be such that $\sup_{\varepsilon>0}\mathcal{G}_\varepsilon(u_\varepsilon,m_\varepsilon)<+\infty$. By Proposition 3.1, there exist $u\in W^{1,2}_w(\Omega;\mathbb{R}^d)$ and $m\in W^{1,2}(\Omega;\mathbb{S}^{d-1})$ such that, up to a subsequence,

$$\begin{aligned}
u_\varepsilon &\rightharpoonup u && \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^d),\\
m_\varepsilon\circ y_\varepsilon\det\nabla y_\varepsilon &\to m && \text{in } L^r(\Omega;\mathbb{R}^d)\text{ for any }1\le r<\infty,\\
m_\varepsilon\circ y_\varepsilon &\to m && \text{in } L^1(\Omega;\mathbb{R}^d),\\
\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon &\rightharpoonup \chi_\Omega\nabla m && \text{weakly in } L^2(\mathbb{R}^d;\mathbb{R}^{d\times d}),\\
\chi_{y_\varepsilon(\Omega)}m_\varepsilon &\to \chi_\Omega m && \text{in } L^2(\mathbb{R}^d;\mathbb{R}^d).
\end{aligned}$$

Then, by Proposition 2.1, we have that

$$\int\limits_\Omega|\nabla m|^2\,dx+\int\limits_{\mathbb{R}^d}|\nabla v_m|^2\,dx\le\liminf_{\varepsilon\to 0}\int\limits_{y_\varepsilon(\Omega)}|\nabla m_\varepsilon|^2\,dz+\int\limits_{\mathbb{R}^d}|\nabla v_{m_\varepsilon}|^2\,dz.$$

For a fixed $0<\alpha<1$, define the sets $E_\varepsilon:=\{x\in\Omega:|\nabla u_\varepsilon(x)|<\varepsilon^{-\alpha}\}$. Then, by the Chebyshev inequality,

$$\mathcal{L}^d(\Omega\setminus E_\varepsilon)=\mathcal{L}^d(\{x\in\Omega:|\nabla u_\varepsilon(x)|\ge\varepsilon^{-\alpha}\})\le\frac{1}{\varepsilon^{-\alpha}}\int\limits_{\Omega\setminus E_\varepsilon}|\nabla u_\varepsilon(x)|\,dx\le\varepsilon^\alpha|\Omega|^{1/2}\|\nabla u_\varepsilon\|_{L^2}\le C\varepsilon^\alpha.$$

Thus, $\chi_{E_\varepsilon}\to 1$ in measure and $\chi_{E_\varepsilon}\nabla u_\varepsilon\rightharpoonup\nabla u$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. Indeed, for $1\le r<2$, we may estimate

$$\int\limits_{\Omega\setminus E_\varepsilon}|\nabla u_\varepsilon-\nabla u|^r\,dx\le\|\nabla u_\varepsilon-\nabla u\|_{L^2}^{\frac{1}{r}}\cdot\mathcal{L}^d(\Omega\setminus E_\varepsilon)^{\frac{2-r}{2r}}\le C\varepsilon^{\frac{(2-r)\alpha}{2r}}\to 0.$$

Therefore, $\chi_{E_\varepsilon}\nabla u_\varepsilon=\nabla u_\varepsilon-\chi_{\Omega\setminus E_\varepsilon}\nabla u_\varepsilon\rightharpoonup\nabla u$ weakly in $L^r(\Omega;\mathbb{R}^{d\times d})$. Since the weak limit is unique, we get that $\chi_{E_\varepsilon}\nabla u_\varepsilon\rightharpoonup\nabla u$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$.

Moreover, since $m_\varepsilon\circ y_\varepsilon\det\nabla y_\varepsilon\to m$ strongly in $L^r(\Omega;\mathbb{R}^d)$ for every $1\le r<\infty$, then

$$(\det\nabla y_\varepsilon)^2 m_\varepsilon\circ y_\varepsilon\otimes m_\varepsilon\circ y_\varepsilon\to m\otimes m\quad\text{strongly in } L^r(\Omega;\mathbb{R}^d).$$

Thus, we have that $e(\nabla y_\varepsilon,m_\varepsilon\circ y_\varepsilon)\to e(m)$ and $\chi_{E_\varepsilon}e(\nabla y_\varepsilon,m_\varepsilon\circ y_\varepsilon)\to e(m)$ strongly in $L^2(\Omega;\mathbb{R}^d)$.

Let us set

$$\beta(\varepsilon) := \big(\exp(\varepsilon e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) - I - \varepsilon e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)\big)\nabla y_\varepsilon + \varepsilon^2 e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)\nabla u_\varepsilon.$$

Recalling that $\|e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)\|_\infty$ and $\|\exp(\varepsilon e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon))\|_\infty$ are uniformly bounded and $\Phi(I) = D\Phi(I) = 0$, by Taylor expansion, we have that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_\Omega W(I+\varepsilon\nabla u_\varepsilon, m_\varepsilon\circ y_\varepsilon)\,\mathrm{d}x\\
&\quad= \frac{1}{\varepsilon^2}\int_{E_\varepsilon} W(I+\varepsilon\nabla u_\varepsilon, m_\varepsilon\circ y_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_{\Omega\setminus E_\varepsilon} W(I+\varepsilon\nabla u_\varepsilon, m_\varepsilon\circ y_\varepsilon)\,\mathrm{d}x\\
&\quad\geq \frac{1}{\varepsilon^2}\int_{E_\varepsilon} W(I+\varepsilon\nabla u_\varepsilon, m_\varepsilon\circ y_\varepsilon)\,\mathrm{d}x = \frac{1}{\varepsilon^2}\int_{E_\varepsilon} \Phi\big(\exp(\varepsilon e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon))(I+\varepsilon\nabla u_\varepsilon)\big)\,\mathrm{d}x\\
&\quad= \frac{1}{\varepsilon^2}\int_{E_\varepsilon} \Phi\big(I+\varepsilon(\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) + \beta(\varepsilon)\big)\,\mathrm{d}x\\
&\quad\geq \frac{1}{2\varepsilon^2}\int_{E_\varepsilon} \mathbb{C}\big(\varepsilon(\boldsymbol{\epsilon}(u_\varepsilon) + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) + \beta(\varepsilon)\big) : \big(\varepsilon(\boldsymbol{\epsilon}(u_\varepsilon) + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) + \beta(\varepsilon)\big)\,\mathrm{d}x\\
&\qquad\quad - \int_{E_\varepsilon} \frac{\eta\big(|\varepsilon(\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) + \beta(\varepsilon)|\big)|\varepsilon(\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) + \beta(\varepsilon)|^2}{\varepsilon^2}\,\mathrm{d}x\\
&\quad= \frac{1}{2}\int_{E_\varepsilon} \mathbb{C}(\boldsymbol{\epsilon}(u_\varepsilon) + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) : (\boldsymbol{\epsilon}(u_\varepsilon) + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon))\,\mathrm{d}x\\
&\qquad\quad - 2\int_{E_\varepsilon} \eta\big(|\varepsilon(\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon) + \beta(\varepsilon))|\big)\big(|\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)|^2 + \varepsilon^{-2}|\beta(\varepsilon)|^2\big)\,\mathrm{d}x\\
&\qquad\quad - \frac{1}{2\varepsilon^2}\int_{E_\varepsilon} \mathbb{C}\beta(\varepsilon) : \beta(\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{E_\varepsilon} \mathbb{C}(\boldsymbol{\epsilon}(u_\varepsilon) + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)) : \beta(\varepsilon)\,\mathrm{d}x.
\end{aligned}\tag{4.1}$$

Since $|\nabla u_\varepsilon| \leq \varepsilon^{-\alpha}$ in E_ε and $\alpha \in (0,1)$, we have that $\|\varepsilon\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon)\|_{L^\infty(E_\varepsilon)}$ is bounded. Hence, we estimate

$$\|\beta(\varepsilon)\|_{L^\infty(E_\varepsilon)} \leq C\varepsilon^2(1+\varepsilon^{-\alpha}) \tag{4.2}$$

for some $C > 0$ independent of ε. In particular, also $\|\beta(\varepsilon)\|_{L^\infty(E_\varepsilon)}$ is bounded. Thus, up to a redefinition of C, we have that

$$\big\|\eta\big(|\varepsilon(\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon) + \beta(\varepsilon))|\big)\big\|_{L^\infty(E_\varepsilon)} \leq C\|\varepsilon(\nabla u_\varepsilon + e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon) + \beta(\varepsilon))\|_{L^\infty(E_\varepsilon)} \leq C\varepsilon^{1-\alpha} \tag{4.3}$$

for $\varepsilon \in (0,1)$. Combining (4.2) and (4.3) and recalling that $e(\nabla y_\varepsilon, m_\varepsilon\circ y_\varepsilon) \to e(m)$ in $L^2(\Omega;\mathbb{R}^d)$ (see Proposition 3.1), passing to the liminf in (4.1), we infer that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega W(I+\varepsilon\nabla u_\varepsilon, m_\varepsilon\circ y_\varepsilon)\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\epsilon}(u) + e(m)) : (\boldsymbol{\epsilon}(u) + e(m))\,\mathrm{d}x.$$

Γ-lim sup inequality. We now construct a recovery sequence. Let $u_0 \in W^{2,\infty}_w(\Omega;\mathbb{R}^d)$ and $m_0 \in W^{1,2}(\Omega;\mathbb{S}^{d-1})$. Define

$$\begin{aligned} y_\varepsilon(x) &= x + \varepsilon u_0(x),\\ m_\varepsilon(x+\varepsilon u_0(x))\det\nabla y_\varepsilon(x) &= m_0(x). \end{aligned}\tag{4.4}$$

We want to prove that

$$y_\varepsilon \to \mathrm{id} \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^d), \tag{4.5}$$

$$u_\varepsilon = u_0 \to u_0 \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^d), \tag{4.6}$$

$$\chi_{y_\varepsilon(\Omega)} m_\varepsilon \to \chi_\Omega m_0 \quad \text{in } L^2(\mathbb{R}^d;\mathbb{R}^d), \tag{4.7}$$

$$\chi_{y_\varepsilon(\Omega)} \nabla m_\varepsilon \to \chi_\Omega \nabla m_0 \quad \text{in } L^2(\mathbb{R}^d;\mathbb{R}^{d\times d}), \tag{4.8}$$

and

$$\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) \leq \mathcal{G}_0(u_0, m_0). \tag{4.9}$$

First, we notice that $y_\varepsilon \to \mathrm{id}$ in $W^{2,\infty}(\Omega;\mathbb{R}^d)$, since $\|y_\varepsilon - \mathrm{id}\|_{W^{2,\infty}} \le \varepsilon\|u_0\|_{W^{2,\infty}}$. This implies the convergences (4.5) and (4.6).

The injectivity of y_ε and the sign condition $\det\nabla y_\varepsilon > 0$ follow from Theorem 2.2 [17, Theorem 5.5-1], since $\varepsilon\|\nabla u_0\|_\infty \le c(\Omega)$ for small enough ε. Moreover, y_ε is bi-Lipschitz for small enough ε. Indeed, there exist constants $0 < l < L < +\infty$ (that may depend on u_0 but not on x and ε) such that, for $\varepsilon > 0$ small enough, it holds that $\sup_{x\in\Omega}|\nabla y_\varepsilon| = \sup_{x\in\Omega}|I + \varepsilon\nabla u_0(x)| \le 1 + \varepsilon\sup_{x\in\Omega}|\nabla u_0(x)| =: L_\varepsilon < L$ with $L_\varepsilon \to 1$ if $\varepsilon \to 0$. At the same time, $\inf_{x\in\Omega}\det\nabla y_\varepsilon = \inf_{x\in\Omega}\det(I + \varepsilon\nabla u_0) = 1 + \varepsilon\inf_{x\in\Omega}\mathrm{tr}\,\nabla u_0(x) + o(\varepsilon) =: l_\varepsilon > l > 0$ and $l_\varepsilon \to 1$ if $\varepsilon \to 0$. By the inverse function theorem, y_ε^{-1} is differentiable and $\nabla y_\varepsilon^{-1}(y_\varepsilon(x)) = (\nabla y_\varepsilon(x))^{-1} \approx I - \varepsilon\nabla u_0(x)$. Thus, m_ε defined in (4.4) can be explicitly written as

$$m_\varepsilon(z) = \frac{1}{\det\nabla y_\varepsilon} m_0 \circ y_\varepsilon^{-1}(z) \quad \text{for } z \in y_\varepsilon(\Omega).$$

Hence, we have that

$$|m_\varepsilon(z)| = \left|\frac{m_0(y_\varepsilon^{-1}(z))}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))}\right| \le \frac{1}{l_\varepsilon}.$$

By direct computation, we have that

$$\nabla\Big(\frac{m_0}{\det\nabla y_\varepsilon}\Big) = \frac{1}{\det\nabla y_\varepsilon}\nabla m_0 + m_0 \otimes \nabla\Big(\frac{1}{\det\nabla y_\varepsilon}\Big) = \frac{1}{\det\nabla y_\varepsilon}\nabla m_0 - \frac{1}{(\det\nabla y_\varepsilon)^2} m_0 \otimes \nabla(\det\nabla y_\varepsilon).$$

This implies that

$$\Big\|\nabla\Big(\frac{m_0}{\det\nabla y_\varepsilon}\Big)\Big\|_{L^2} \le \frac{\|\nabla m_0\|_{L^2}}{l_\varepsilon} + C\frac{\|\nabla\det\nabla y_\varepsilon\|_{L^\infty}}{l_\varepsilon}$$

for some positive constant C only depending on Ω. This inequality, together with the fact that $|\nabla y_\varepsilon^{-1}(z)| \le L_\varepsilon^{d-1}/l_\varepsilon$, provides

$$\begin{aligned}\int_{y_\varepsilon(\Omega)} |\nabla m_\varepsilon(z)|^2\,\mathrm{d}z &= \int_{y_\varepsilon(\Omega)} \Big|\nabla\Big(\frac{m_0(y_\varepsilon^{-1}(z))}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))}\Big)\cdot\nabla y_\varepsilon^{-1}(z)\Big|^2 \det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))\,\mathrm{d}z \\ &\le \int_\Omega \Big|\nabla\Big(\frac{m_0(x)}{\det\nabla y_\varepsilon(x)}\Big)\Big|^2 |\nabla y_\varepsilon^{-1}(y_\varepsilon(x))|^2 \det\nabla y_\varepsilon(x)\,\mathrm{d}x \\ &\le \frac{2L_\varepsilon^{3d-2}}{l_\varepsilon^4}\|\nabla m_0(x)\|_{L^2}^2 + C\frac{\|\nabla\det\nabla y_\varepsilon\|_{L^\infty}^2}{l_\varepsilon^2}.\end{aligned} \tag{4.10}$$

Thus, $m_\varepsilon \in W^{1,2}(y_\varepsilon(\Omega);\mathbb{R}^d)$.

Let us define $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \delta\}$. In particular, $\Omega_\delta \subset \Omega \cap y_\varepsilon(\Omega)$ for $\varepsilon > 0$ small enough, due to the uniform convergence of y_ε to the identity. For such ε and for $z \in \Omega_\delta$, we estimate

$$\begin{aligned}|m_\varepsilon(z) - m_0(z)| &\le \left|\frac{m_0(y_\varepsilon^{-1}(z))}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))} - \frac{m_0(z)}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))}\right| + \left|\frac{m_0(z)}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))} - m_0(z)\right| \\ &\le \frac{1}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))}|m_0(y_\varepsilon^{-1}(z)) - m_0(z)| + |m_0(z)|\cdot\left|\frac{1}{\det\nabla y_\varepsilon(y_\varepsilon^{-1}(z))} - 1\right|.\end{aligned}$$

Therefore, taking the squares, integrating over Ω_δ, and letting $\varepsilon \to 0$, we infer that $m_\varepsilon \to m_0$ in $L^2(\Omega_\delta;\mathbb{R}^d)$, for every $\delta > 0$. Arguing as in the proof of Proposition 3.1, it is not difficult to see that $\chi_{y_\varepsilon(\Omega)} m_\varepsilon \to \chi_\Omega m_0$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$ and $\chi_{y_\varepsilon(\Omega)}\nabla m_\varepsilon \to \chi_\Omega\nabla m_0$ weakly in $L^2(\mathbb{R}^d;\mathbb{R}^{d\times d})$. Moreover, by (4.10), we can estimate the exchange energy as

$$\limsup_{\varepsilon\to 0}\int_{y_\varepsilon(\Omega)} |\nabla m_\varepsilon(y)|^2\,\mathrm{d}z \le \limsup_{\varepsilon\to 0}\int_\Omega \frac{L_\varepsilon^{3d-2}}{l_\varepsilon^4}|\nabla m_0(x)|^2\,\mathrm{d}x + C\frac{\|\nabla\det\nabla y_\varepsilon\|_{L^\infty}^2}{l_\varepsilon^2} = \int_\Omega |\nabla m_0(x)|^2\,\mathrm{d}x.$$

Thus, (4.7) and (4.8) hold.

For a.e. $x \in \Omega$, the elastic energy has the following form by Taylor's expansion and by definition of m_ε:

$$\frac{1}{\varepsilon^2} W(I + \varepsilon\nabla u_0, m_\varepsilon \circ y_\varepsilon) = \frac{1}{\varepsilon^2}\Phi\big(\exp(\varepsilon e(\nabla y_\varepsilon, m_\varepsilon \circ y_\varepsilon))(I + \varepsilon\nabla u_0)\big)$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^2}\Phi\big(\exp(\varepsilon e(m_0))(I+\varepsilon\nabla u_0)\big)\\
&= \frac{1}{\varepsilon^2}\Phi\big(I+\varepsilon(\nabla u_0 + e(m_0)) + o(\varepsilon)\big)\\
&= \frac{1}{2}\mathbb{C}(\boldsymbol{\epsilon}(u_0)+e(m_0)) : (\boldsymbol{\epsilon}(u_0)+e(m_0)) + O(1)
\end{aligned}
$$

with $O(1) \to 0$ as $\varepsilon \to 0$ uniformly in Ω. Then (d) and the Dominated Convergence Theorem give

$$
\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega W(I+\varepsilon\nabla u_0, m_\varepsilon\circ y_\varepsilon)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\epsilon}(u_0)+e(m_0)) : (\boldsymbol{\epsilon}(u_0)+e(m_0))\,\mathrm{d}x.
$$

We obtain (4.9), using

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) &= \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega W(I+\varepsilon\nabla u_0, m_0)\,\mathrm{d}x + \frac{1}{2}\int_{y_\varepsilon(\Omega)} |\nabla m_\varepsilon|^2\,\mathrm{d}z + \frac{\mu_0}{2}\int_{\mathbb{R}^d} |\nabla v_{m_\varepsilon}|^2\,\mathrm{d}z\\
&\le \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\epsilon}(u_0)+e(m_0)) : (\boldsymbol{\epsilon}(u_0)+e(m_0))\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla m_0|^2\,\mathrm{d}x + \frac{\mu_0}{2}\int_{\mathbb{R}^d} |\nabla v_{m_0}|^2\,\mathrm{d}z\\
&= \mathcal{G}_0(u_0, m_0),
\end{aligned}
$$

where, for the convergence of ∇v_{m_ε}, we have used Proposition 2.1.

For a general $u_0 \in W^{1,2}_w(\Omega;\mathbb{R}^d)$, we conclude by a standard diagonal argument. □

We conclude this section with the proof of convergence of minimizers of $\mathcal{F}_\varepsilon$ to minimizers of $\mathcal{F}_0$, namely, Theorems 1.3 and 1.4. We start with the coercivity of $\mathcal{F}_\varepsilon$.

Proposition 4.1. *Let $(u_\varepsilon, m_\varepsilon) \in \mathcal{A}^w_\varepsilon$ and $K > 0$ be such that $\sup_{\varepsilon>0}\mathcal{F}_\varepsilon(u_\varepsilon, m_\varepsilon) \le K$. Then there exists $C(K) > 0$ such that $\sup_{\varepsilon>0}\mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) \le C(K)$.*

Proof. Recall that $\mathcal{F}_\varepsilon(u_\varepsilon, m_\varepsilon) := \mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) - \mathscr{L}(u_\varepsilon) - \mathscr{M}(y_\varepsilon, m_\varepsilon)$, with

$$
|\mathscr{L}(u_\varepsilon)| \le C_L\|u_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \quad \text{and} \quad |\mathscr{M}(y_\varepsilon, m_\varepsilon)| \le C_M\|m_\varepsilon\|_{L^2(y_\varepsilon(\Omega);\mathbb{R}^d)}.
$$

Moreover, by Lemma 3.3, we have

$$
\|u_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^d)} \le C\Big(\mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) + \int_\Gamma |w|^2\,\mathrm{d}\mathcal{H}^{d-1} + 1\Big),
$$

while arguing as in (3.11), we deduce from assumption (e) that

$$
\|m_\varepsilon\|^2_{L^2(y_\varepsilon(\Omega);\mathbb{R}^d)} = \int_\Omega \frac{1}{\det\nabla y_\varepsilon}\,\mathrm{d}x \le C(\mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) + 1).
$$

Summing up the estimates above, we obtain

$$
\mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon) = \mathcal{F}_\varepsilon(u_\varepsilon, m_\varepsilon) + \mathscr{L}(u_\varepsilon) + \mathscr{M}(y_\varepsilon, m_\varepsilon) \le K + C\sqrt{\mathcal{G}_\varepsilon(u_\varepsilon, m_\varepsilon)} + C.
$$

This concludes the proof of the proposition. □

We are now able to prove Theorem 1.3.

Proof of Theorem 1.3. Consider a sequence $\varepsilon_k \to 0$ and recall that

$$
s_\varepsilon := \inf\{\mathcal{F}_\varepsilon(u,m) : (u,m) \in \mathcal{A}^w_\varepsilon\} \quad \text{and} \quad s_0 := \inf\{\mathcal{F}_0(u,m) : (u,m) \in \mathcal{A}^w_0\}.
$$

It is standard to show that $\mathcal{F}_0$ has a minimizer $(u_0, m_0) \in W^{1,2}_w(\Omega;\mathbb{R}^d) \times W^{1,2}(\Omega;\mathbb{S}^{d-1})$ on $\mathcal{A}^w_0$. It is also straightforward to check that $\inf \mathcal{F}_{\varepsilon_k}(u,m)$ is bounded with respect to ε. Since the functionals $\mathscr{L}$ and $\mathscr{M}$ do not depend on ε, from Γ-convergence of $\mathcal{G}_\varepsilon$, we deduce that $s_{\varepsilon_k} \to s_0$ and thus $\mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}, m_{\varepsilon_k}) \to \mathcal{F}_0(u_0, m_0)$. Propositions 4.1 and 3.1 imply that $u_{\varepsilon_k} \rightharpoonup u_0$ weakly in $W^{1,2}_w(\Omega;\mathbb{R}^d)$, $\chi_{y_{\varepsilon_k}(\Omega)} m_{\varepsilon_k} \to \chi_\Omega m_0$ strongly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$, and $\chi_{y_{\varepsilon_k}(\Omega)}\nabla m_{\varepsilon_k} \rightharpoonup \chi_\Omega \nabla m_0$ weakly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$, and $\mathcal{G}_{\varepsilon_k}(u_{\varepsilon_k}, m_{\varepsilon_k}) \to \mathcal{G}(u_0, m_0)$. □

Proof of Theorem 1.4. Convergence of the sequence u_ε can be improved to a strong convergence in $W^{1,2}(\Omega;\mathbb{R}^d)$ arguing as in [29, Subsection 7.2] (see also [15, Subsection 3.2]). □

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