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Estimates for Robin p-Laplacian eigenvalues of convex sets with prescribed perimeter

Vincenzo Amato [a], Andrea Gentile [b,*], Alba Lia Masiello [a]

[a] *Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Universitario Monte S. Angelo, via Cintia, 80126 Napoli, Italy*
[b] *Mathematical and Physical Sciences for Advanced Materials and Technologies, Scuola Superiore Meridionale, Largo San Marcellino 10, 80126 Napoli, Italy*

A R T I C L E I N F O

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A B S T R A C T

In this paper, we prove an upper bound for the first Robin eigenvalue of the p-Laplacian with a positive boundary parameter and a quantitative version of the reverse Faber-Krahn type inequality for the first Robin eigenvalue of the p-Laplacian with negative boundary parameter, among convex sets with prescribed perimeter. The proofs are based on a comparison argument obtained by means of inner sets, introduced by Payne, Weinberger [32] and Pólya [33].
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1. Introduction

Let Ω be a bounded, open and convex set in $\mathbb{R}^n$. We consider the following problem

$$\begin{cases} -\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{p,\beta}(\Omega)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\beta \in \mathbb{R}$, $1 < p < +\infty$.

The fundamental eigenvalue of the Robin Laplacian on Ω is defined by

$$\lambda_{p,\beta}(\Omega) = \min_{\substack{w \in W^{1,p}(\Omega) \\ w \neq 0}} \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |w|^p \, dx} \tag{1.2}$$

* Corresponding author.
E-mail addresses: vincenzo.amato@unina.it (V. Amato), andrea.gentile2@unina.it (A. Gentile), albalia.masiello@unina.it (A.L. Masiello).

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and a minimizer u in (1.2) satisfies the equation (1.1) in the weak form if

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx + \beta \int_{\partial\Omega} |u|^{p-2}u\varphi\, d\mathcal{H}^{n-1} = \lambda_{p,\beta}(\Omega)\int_{\Omega} |u|^{p-2}u\varphi\, dx, \quad \forall\varphi \in W^{1,p}(\Omega).$$

It is well-known (see for instance [29,6] for the Dirichlet case) that the first eigenvalue (1.2) is simple and that in the case of a ball, the corresponding eigenfunction is radially symmetric (see [12, Remark 3.1]).

In this paper, we want to study the different behavior of eigenvalues in the case $\beta > 0$ and $\beta < 0$. For the sake of completeness, we recall that, in the case of $\beta = 0$, we recover the Neumann boundary condition, for which the first eigenvalue is zero and the associated eigenfunctions are constants.

In the case $\beta > 0$, Bossel [7] and Daners [18] proved a Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian in two and higher dimensional case, respectively. In particular, they proved that among sets of given volume, the one which minimizes the first Robin-eigenvalue is the ball, i.e.

$$\lambda_{2,\beta}(\Omega^\sharp) \le \lambda_{2,\beta}(\Omega), \tag{1.3}$$

where $\Omega^\sharp$ is the ball, centered at the origin, having the same volume as Ω. This result was generalized by Bucur, Daners, Giacomini and Trebeschi in [10,11,14–16,21] to the eigenvalues of the p-Laplacian with Robin boundary conditions.

This Faber-Krahn inequality for fixed volume and the following rescaling property (see [13])

$$\lambda_{p,\beta}(t\Omega) \le \frac{1}{t}\lambda_{p,\beta}(\Omega) \le \lambda_{p,\beta}(\Omega), \quad \forall t > 1, \tag{1.4}$$

give a Faber-Krahn inequality for fixed perimeters, i.e.

$$\lambda_{p,\beta}(\Omega^\star) \le \lambda_{p,\beta}(\Omega),$$

where $\Omega^\star$ is the ball having the same perimeter as Ω.

Our aim is to give a continuity bound, in terms of the isoperimetric deficit, to the ratio

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},$$

indeed, we prove

Theorem 1.1. *Let β be a positive parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $P(\Omega) = P(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the p-Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)} \le C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right), \tag{1.5}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}$.

It is possible to give a uniform bound to the constant in (1.5) from above with a constant independent of the parameter β and the perimeter, indeed it holds

$$C(n,p,\beta,\rho) \le C(n,p) := \frac{\|v_\infty\|_\infty^p |\Omega^\star|}{\|v_\infty\|_p^p}, \tag{1.6}$$

where v_∞ is the first Dirichlet eigenfunction. Thanks to the rescaling property of Dirichlet eigenvalues and eigenfunctions, $C(n,p)$ does not depend on the perimeter. For more details see Remark 3.1.

We observe that this result can be seen as a generalization to the Robin case of the result in [8], which holds true in the case of Dirichlet eigenvalues of the p-Laplacian. Incidentally, let us mention that the anisotropic case is studied in [19].

When β is a negative parameter, the authors in [12] proved a reverse Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian among convex sets of given perimeter. In particular, they proved that among convex sets of given perimeter the ball $\Omega^\star$ maximizes the first Robin eigenvalue of the p-Laplacian, i.e.

$$\lambda_{p,\beta}(\Omega) \le \lambda_{p,\beta}(\Omega^\star). \tag{1.7}$$

For completeness' sake, we quote that in [4], the authors already proved that the disc maximizes the first eigenvalue under a perimeter constraint, among C^2 domains in $\mathbb{R}^2$, while the question remained open in arbitrary dimension.

This question is related to the conjecture of Bareket (see [5]) claiming that the ball maximizes $\lambda_{p,\beta}(\Omega)$ among all Lipschitz sets with given volume. Freitas and Krejčiřík in [23] proved that the conjecture is false, giving a counter-example based on the asymptotic behavior of the eigenvalues on a disc and an annulus of the same area when $\beta \to -\infty$. They also proved that among sets of area equal to 1, the conjecture is true, provided β is close to 0. For more details see [28].

In [17] the authors proved a quantitative version of the reverse Faber-Krahn (1.7) in the case $p = 2$ among convex sets of fixed perimeter following the Fuglede's approach introduced in [24].

In this paper, we recover the result in [17] obtaining a quantitative version of (1.7) for all $p \in (1,+\infty)$, but using a different approach. Indeed, following the method introduced by Payne and Weinberger in [32] and Pólya in [33], which has proved to be adaptable to many problems such as in [8,12,20,31], we establish a comparison using the so-called parallel coordinates method. In particular, we firstly prove a lower bound in terms of perimeter and measure of Ω, that is

Theorem 1.2. *Let β be a negative parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $P(\Omega) = P(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the p-Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \ge C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}} \omega_n^{\frac{1}{n-1}} |\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right), \tag{1.8}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{v_m^p |\Omega^\star|}{\|v\|_p^p}$ with $v_m = \min_{\Omega^\star} v$.

In this case, the constant $C(n,p,\beta,\rho)$ cannot be replaced by a constant independent of β and the perimeter, as it is shown in Remark 3.2.

Then we prove the quantitative result as in [17].

Theorem 1.3. *Let $n \ge 2$, $\rho > 0$ and $\beta < 0$. Then, there exist two positive constants $C(n,p,\beta,\rho) > 0$ and $\delta_0(n,p,\beta,\rho) > 0$, such that, for all $\Omega \subset \mathbb{R}^n$ bounded and convex with $P(\Omega) = \rho$ and $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \le \delta_0$, it holds*

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n,p,\beta,\rho) g(\mathcal{A}^\star_{\mathcal{H}}(\Omega)) \tag{1.9}$$

where $\Omega^\star$ is a ball with the same perimeter of Ω, $\mathcal{A}^\star_{\mathcal{H}}$ is the Hausdorff asymmetry defined in (2.3) and g is defined in (2.7).

The paper is organized as follows: in Section 2 we recall some preliminary results and useful tools for our aim; in Section 3 we provide the proof of our main results.

2. Notations and preliminaries

Throughout this article, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$, while $\cdot$ is the standard Euclidean scalar product for $n \geq 2$. By $\mathcal{H}^k(\cdot)$, for $k \in [0,n)$, we denote the k-dimensional Hausdorff measure in $\mathbb{R}^n$.

The measure and the perimeter of Ω in $\mathbb{R}^n$ will be denoted by $|\Omega|$ and $P(\Omega)$, respectively, and, if $P(\Omega) < \infty$, we say that Ω is a set of finite perimeter. In our case, Ω is a bounded, open and convex set; this ensures us that Ω is a set of finite perimeter and that $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$. Moreover, if Ω is an open set with Lipschitz boundary, it holds

Theorem 2.1 *(Coarea formula). Let $f : \Omega \to \mathbb{R}$ be a Lipschitz function and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then,*

$$\int_\Omega u(x)|\nabla f(x)|\, dx = \int_{\mathbb{R}} dt \int_{(\Omega\cap f^{-1}(t))} u(y)\, d\mathcal{H}^{n-1}(y). \tag{2.1}$$

Some references for results relative to the sets of finite perimeter and the coarea formula are, for instance, [2,30].

2.1. Asymmetry index $\mathcal{A}^\star(E)$

In the case $\beta < 0$, we want to prove a quantitative result, so we need a geometric quantity that gives us information about the shape of Ω. We will consider the Hausdorff asymmetry index, as already done in [17], so we recall some basic notions about the Hausdorff distance.

If E, F are any two convex sets in $\mathbb{R}^n$ the Hausdorff distance between E and F is defined as

$$d_{\mathcal{H}}(E,F) := \inf\left\{\varepsilon > 0 \,:\, E \subset F + \varepsilon B,\, F \subset E + \varepsilon B\right\},$$

where B is the unitary ball centered at the origin and $F + \varepsilon B$ is the well-known Minkowski sum. For any measurable set E of finite perimeter, we define two isoperimetric deficits

$$\mathcal{D}(E) := P(E) - P(E^\sharp), \quad \mathcal{M}(E) := |E^\star| - |E|, \tag{2.2}$$

where $E^\sharp$ and $E^\star$ are the balls with the same measure and the same perimeter of E, respectively.

Moreover, we will consider the following Hausdorff asymmetry indices

$$\mathcal{A}^\star_{\mathcal{H}}(E) = \min_{x\in\mathbb{R}^n}\left\{d_{\mathcal{H}}(E, B_r(x)),\, P(\Omega) = P(B_r(x))\right\}, \tag{2.3}$$

and

$$\mathcal{A}^\sharp_{\mathcal{H}}(E) = \min_{x\in\mathbb{R}^n}\left\{d_{\mathcal{H}}(E, B_r(x)),\, |\Omega| = |B_r(x)|\right\}. \tag{2.4}$$

Lemma 2.9 in [27] tells us how these two indices are related one to the other.

Lemma 2.2. *Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be a bounded, open and convex set with $\mathcal{D}(E) \leq \delta$, then*

$$\mathcal{A}_H^\star(E) \leq C(n)\mathcal{A}_H^\sharp(E). \tag{2.5}$$

With these definitions, we can recall the quantitative isoperimetric inequality proved in [24,25].

Theorem 2.3 *(Fuglede). Let $n \geq 2$, and let E be a bounded open and convex set with $|E| = \omega_n$. There exists δ_F, C, depending only on n, such that if $\mathcal{D}(E) \leq \delta_F$ then*

$$\mathcal{D}(E) \geq Cg\big(\mathcal{A}_{\mathcal{H}}^\sharp(E)\big), \tag{2.6}$$

where g is defined by

$$g(s) = \begin{cases} s^2 & \text{if } n = 2 \\ f^{-1}(s^2) & \text{if } n = 3 \\ s^{\frac{n+1}{2}} & \text{if } n \geq 4 \end{cases} \tag{2.7}$$

and $f(t) = \sqrt{t \log(\frac{1}{t})}$ for $0 < t < e^{-1}$.

We are interested in a modified version of this theorem, in terms of $\mathcal{M}(E)$, so we have

Lemma 2.4. *Let $E \subset \mathbb{R}^n$ be a bounded, open and convex set and let $E^\star$ be the ball satisfying $P(E) = P(E^\star) = \rho$. Then, there exist δ, C, depending only on n and ρ, such that, if*

$$\mathcal{M}(E) = |E^\star| - |E| \leq \delta, \tag{2.8}$$

then

$$\mathcal{M}(E) \geq Cg\big(\mathcal{A}_{\mathcal{H}}^\star(E)\big),$$

where g is the function defined in (2.7).

Proof. Let us start by setting $\delta < \dfrac{|E^\star|}{2}$, so

$$|E| > \frac{|E^\star|}{2}. \tag{2.9}$$

Let us divide the proof in two steps. First we assume that $|E| = \omega_n$. By the differentiability of the function $h(t) = t^{\frac{n-1}{n}}$, there exists $\xi \in (|E|, |E^\star|)$ such that

$$|E^\star|^{\frac{n-1}{n}} - |E|^{\frac{n-1}{n}} = h'(\xi)\left(|E^\star| - |E|\right) \leq \frac{n-1}{n\omega_n^{\frac{1}{n}}}\left(|E^\star| - |E|\right). \tag{2.10}$$

Moreover,

$$|E^\star|^{\frac{n-1}{n}} - |E|^{\frac{n-1}{n}} = \frac{P(E)}{n\omega_n^{\frac{1}{n}}} - |E|^{\frac{n-1}{n}} = \frac{P(E)}{n\omega_n^{\frac{1}{n}}} - \frac{P(E^\sharp)}{n\omega_n^{\frac{1}{n}}} = \frac{\mathcal{D}(E)}{n\omega^{\frac{1}{n}}}. \tag{2.11}$$

Hence, by (2.10) and (2.11), we get

$$\mathcal{D}(E) \leq (n-1)\mathcal{M}(E).$$

So, choosing $\delta \leq \dfrac{\delta_F(n)}{n-1}$, where $\delta_F(n)$ is the one provided by Theorem 2.3, we can apply the aforementioned Theorem to write

$$P(E) \geq n\omega_n\Big(1+\gamma(n)g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big)\Big),$$

obtaining

$$\begin{aligned}|E^\star| - |E| &= \frac{P(E)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}} - |E| \geq \omega_n\left(1+\gamma(n)g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big)\right)^{\frac{n}{n-1}} - \omega_n \\ &\geq \frac{n\omega_n\gamma(n)}{n-1}g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big),\end{aligned}$$

where in the last step we used Bernoulli's inequality

$$(1+x)^r \geq 1+rx \qquad x \geq -1,\, r \geq 1.$$

Applying Lemma 2.2, we eventually have

$$|\Omega^\star| - |\Omega| \geq C(n)g\big(\mathcal{A}^\star_{\mathcal{H}}(\Omega)\big).$$

Now we treat the general case. Let us rescale the set E as

$$E_1 = \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E,$$

so we have

$$\begin{aligned}\mathcal{M}(E_1) = |E_1^\star| - |E_1| &= \frac{\omega_n}{|E|}|E^\star| - \omega_n \\ &= \frac{\omega_n}{|E|}\mathcal{M}(E) \leq \frac{2\omega_n}{|E^\star|}\mathcal{M}(E),\end{aligned}$$

where the last inequality follows from (2.9).

In order to apply the previous case to the set E_1, we need

$$\mathcal{M}(E_1) \leq \frac{\delta_F}{n-1},$$

which is guaranteed if we choose

$$\mathcal{M}(E) \leq \frac{|E^\star|}{2(n-1)\omega_n}\,\delta_F = \delta(n,\rho). \quad \square$$

2.2. Quermassintegrals

Let us recall some basic facts about convex sets. For the content of this section, we will refer to [34]. Let $K \subset \mathbb{R}^n$ be a non-empty, bounded, convex set, let B be the unitary ball centered at the origin and $\rho > 0$. We can write the Steiner formula for the Minkowski sum $K+\rho B$ as

$$|K+\rho B| = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i. \tag{2.12}$$

The coefficients $W_i(K)$ are known in the literature as quermassintegrals of K. In particular, $W_0(K) = |K|, nW_1(K) = P(K)$ and $W_n(K) = \omega_n$ where ω_n is the measure of B.

Incidentally let us mention that if K has C^2 boundary, the quermassintegrals can be written in terms of principal curvatures of K.

Formula (2.12) can be generalized also to the other quermassintegrals, obtaining

$$W_j(K+\rho B) = \sum_{i=0}^{n-j} \binom{n-j}{i} W_{j+i}(K)\rho^i \qquad j = 0, \ldots, n-1. \tag{2.13}$$

For $j = 1$ we have

$$\begin{aligned} P(K+\rho B_1) &= n\sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(K)\rho^i \\ &= P(K) + n(n-1)W_2(K)\rho + \ldots + nW_n(K)\rho^n, \end{aligned}$$

from which follows

$$\lim_{\rho\to 0} \frac{P(K+\rho B) - P(K)}{\rho} = n(n-1)W_2(K). \tag{2.14}$$

Furthermore, Aleksandrov-Fenchel inequalities hold true

$$\left(\frac{W_j(K)}{\omega_n}\right)^{\frac{1}{n-j}} \geq \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{1}{n-i}} \qquad 0 \leq i < j \leq n-1, \tag{2.15}$$

where equality holds if and only if K is a ball. When $i = 0$ and $j = 1$, formula (2.15) reduces to the classical isoperimetric inequality, i.e.

$$P(K) \geq n\omega_n^{\frac{1}{n}} |K|^{\frac{n-1}{n}}.$$

In the following, we will use (2.15) for $i = 1$ and $j = 2$, that is

$$W_2(K) \geq n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} P(K)^{\frac{n-2}{n-1}}. \tag{2.16}$$

2.3. Some useful lemmas

Let $\Omega \subset \mathbb{R}^n$ be a convex set, for $t \in [0, r_\Omega]$ we denote by

$$\Omega_t = \{\, x \in \Omega : d(x) > t \,\},$$

where $d(x)$ is the distance of $x \in \Omega$ from the boundary of Ω and r_Ω is the inradius of Ω, that is to say, the radius of the biggest ball contained in Ω. By the Brunn-Minkowski Theorem ([34, Theorem 7.4.5]) and the concavity of the distance function, the map

$$t \mapsto P(\Omega_t)^{\frac{1}{n-1}},$$

is concave in $[0, r_\Omega]$, hence absolutely continuous in $(0, r_\Omega)$. Moreover, there exists its right derivative at 0 and it is negative, since $P(\Omega_t)^{\frac{1}{n-1}}$ is strictly monotone decreasing, hence almost everywhere differentiable. The following two results are contained in [8], we report them here for the reader's convenience.

Lemma 2.5. *Let Ω be a bounded, convex, open set in $\mathbb{R}^n$. Then for almost every $t \in (0, r_\Omega)$*

$$-\frac{d}{dt}P(\Omega_t) \geq n(n-1)W_2(\Omega_t), \tag{2.17}$$

and equality holds if Ω is a ball.

Proof. For every $s \in (0,t)$ it holds

$$\Omega_t + sB_1 \subset \Omega_{t-s},$$

and if Ω is a ball, the two sets coincide. The monotonicity of the perimeter with respect to the inclusion of convex sets and formula (2.14) give, for almost every $t \in (0, r_\Omega)$,

$$\begin{aligned} -\frac{d}{dt}P(\Omega_t) &= \lim_{s\to 0^+} \frac{P(\Omega_{t-s}) - P(\Omega_t)}{s} \\ &\geq \lim_{s\to 0^+} \frac{P(\Omega_t + sB_1) - P(\Omega_t)}{s} = n(n-1)W_2(\Omega_t). \quad \square \end{aligned}$$

Combining the chain rule, the previous lemma and the fact that $|\nabla d(x)| = 1$ almost everywhere, we obtain

Lemma 2.6. *Let $f\colon [0, +\infty) \to [0, +\infty)$ be a strictly increasing C^1 function with $f(0) = 0$. Set $u(x) = f(d(x))$ and*

$$E_t = \{\, x \in \Omega \,:\, u(x) > t \,\} = \Omega_{f^{-1}(t)},$$

then

$$-\frac{d}{dt}P(E_t) \geq (n-1)\frac{W_2(E_t)}{|\nabla u|_{u=t}}. \tag{2.18}$$

In particular, we observe that the map $t \mapsto P(E_t)$ is absolutely continuous since it is obtained by composing the map $t \mapsto f^{-1}(t)$ which is a C^1 function and the map $s \mapsto P(\Omega_s)$ which is absolutely continuous.

2.4. Monotonicity of eigenfunctions

It is clear from the definition of the eigenvalue (1.2) that for fixed $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$, the map

$$\beta \in \mathbb{R} \mapsto \lambda_{p,\beta}(\Omega)$$

is increasing and it holds (see for instance [3])

$$\lambda_{p,N} < \lambda_{p,\beta} < \lambda_{p,D}, \qquad \forall \beta \in (0, +\infty),$$

and

$$\lim_{\beta\to 0^+} \lambda_{p,\beta} = \lambda_{p,N} = 0, \qquad \lim_{\beta\to +\infty} \lambda_{p,\beta} = \lambda_{p,D},$$

where $\lambda_{p,N}$ and $\lambda_{p,D}$ are the first eigenvalues of the Neumann p-Laplacian and Dirichlet p-Laplacian respectively.

Now we want to observe that, if we consider a ball B_R and we fix the value of the eigenfunction in the origin, the map

$$\beta \to v_{p,\beta}, \quad \text{such that } v_{p,\beta(0)} = C,$$

is decreasing, in the sense that

$$\text{if } \beta_1 < \beta_2 \Rightarrow v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x), \qquad \forall x \in B_R. \tag{2.19}$$

With this aim, let us recall that the first eigenvalue of p-Laplacian is simple and that the corresponding eigenfunction is radially symmetric, i.e. there exists $h\colon [0,R] \to \mathbb{R}$ such that $v_{p,\beta}(x) = h(|x|)$. In the following, we will denote with $v_{p,\beta}$ both the eigenfunction and the function h.

In order to prove the monotonicity in (2.19), we first prove an intermediate lemma. This result is not new in the literature (see for instance [10, Proposition 4.2]), but it is not usually stated as proposition or lemma.

Lemma 2.7. *Let $0 < r < R$, $1 < p < +\infty$, $\beta \in \mathbb{R}$. Let us denote by $\lambda_{p,\beta}$ the first eigenvalue of the p-Laplacian on the ball B_R defined in (1.2) with boundary parameter β and let $v_{p,\beta}$ be the corresponding eigenfunction. Then, $v_{p,\beta}|_{B_r}$ is the first eigenfunction of the p-Laplacian on the ball B_r with boundary parameter*

$$\gamma = -\frac{|v'_{p,\beta}|^{p-2}(r)v'_{p,\beta}(r)}{v_{p,\beta}^{p-1}(r)}.$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_\beta := \lambda_{2,\beta}$ and $v_\beta := v_{2,\beta}$.

By the radiality of v_β, we can infer that it is a solution to

$$\begin{cases} -\Delta v_\beta = \lambda_\beta v_\beta & \text{in } B_r \\ \dfrac{\partial v_\beta}{\partial \nu} + \gamma v_\beta = 0 & \text{on } \partial B_r. \end{cases}$$

Let us suppose by contradiction that λ_β is not the first eigenvalue of the Robin Laplacian with boundary parameter γ. So we can choose

$$\lambda_\gamma < \lambda_\beta, \tag{2.20}$$

the first Robin eigenvalue with boundary parameter γ and w_γ the associated eigenfunction, that is

$$\lambda_\gamma = \frac{\displaystyle\int_{B_r} |\nabla w_\gamma|^2\,dx + \gamma \int_{\partial B_r} w_\gamma^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{B_r} w_\gamma^2\,dx} = \min_{\psi \in W^{1,2}(B_r)} \frac{\displaystyle\int_{B_r} |\nabla \psi|^2\,dx + \gamma \int_{\partial B_r} \psi^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{B_r} \psi^2\,dx}. \tag{2.21}$$

We know that w_γ is unique up to a multiplicative constant, so we can choose the constant such that $v_\beta = w_\gamma$ on ∂B_r.

Let us consider the function

$$f(x) = \begin{cases} w_\gamma(x) & \text{if } x \in B_r \\ v_\beta(x) & \text{if } x \in B_R \setminus B_r, \end{cases}$$

we can use it as a test function in the definition of λ_β. In particular, we have

$$\lambda_\beta \le \frac{\displaystyle\int_{B_R} |\nabla f|^2 + \beta \int_{\partial B_r} f^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_{B_R} f^2 \, dx}$$

$$= \frac{\displaystyle\int_{B_r} |\nabla w_\gamma|^2 + \int_{B_R \setminus B_r} |\nabla v_\beta|^2 + \beta \int_{\partial B_R} v_\beta^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}.$$

If we add and subtract $\gamma \displaystyle\int_{\partial B_r} w_\gamma^2 \, d\mathcal{H}^{n-1}$ and $\displaystyle\int_{B_r} |\nabla v_\beta|^2 \, dx$, and we recall that v_β achieves (1.2) on B_R with value λ_β and w_γ achieves (1.2) on B_r with value λ_γ, we have

$$\lambda_\beta \le \frac{\displaystyle\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R} v_\beta^2 \, dx - \gamma \int_{\partial B_r} w_\gamma^2 \, d\mathcal{H}^{n-1} - \int_{B_r} |\nabla v_\beta|^2 \, dx}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}.$$

Since $v_\beta = w_\gamma$ on ∂B_r and v_β is an eigenfunction on B_r with eigenvalue λ_β, we get

$$\lambda_\beta \le \frac{\displaystyle\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R} v_\beta^2 \, dx - \lambda_\beta \int_{B_r} v_\beta^2 \, dx}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}$$

$$= \frac{\displaystyle\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R \setminus B_r} v_\beta^2 \, dx}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R \setminus B_r} v_\beta^2 \, dx}$$

$$< \lambda_\beta \frac{\displaystyle\int_{B_r} w^2 \, dx + \int_{B_R \setminus B_r} w^2 \, dx}{\displaystyle\int_{B_r} w^2 \, dx + \int_{B_R \setminus B_r} w^2 \, dx} = \lambda_\beta,$$

where in the last formula we use (2.20), and we get a contradiction. □

Now we are in position to prove (2.19).

Proposition 2.8. *Let $R > 0$, $1 < p < +\infty$ and $\beta_1 < \beta_2$. Let us denote by λ_{p,β_1} and λ_{p,β_2} the eigenvalues defined in (1.2) and let v_{p,β_1} and v_{p,β_2} be the corresponding eigenfunctions normalized such that $v_{p,\beta_1}(0) = v_{p,\beta_2}(0) > 0$, then*

$$v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x) \qquad \forall x \in B_R. \tag{2.22}$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_{\beta_i} := \lambda_{2,\beta_i}$ and $v_{\beta_i} := v_{2,\beta_i}$.

Since both v_{β_1} and v_{β_2} are radial, we can write the Laplacian in polar coordinates, that is

$$r^{n-1}\Delta u(r) = \left(r^{n-1}u'(r)\right)' \qquad r \in [0, R]. \tag{2.23}$$

Therefore function u_{β_i}, for $i = 1, 2$, satisfies

$$v'_{\beta_i}(r) = -\frac{1}{r^{n-1}}\left[\int_0^r s^{n-1}\lambda_{\beta_i} v_{\beta_i}\, ds\right]. \tag{2.24}$$

Since $\lambda_{\beta_1} < \lambda_{\beta_2}$ and $v_{\beta_1}(0) = v_{\beta_2}(0)$, by continuity there exists $\sigma > 0$ such that

$$\lambda_{\beta_1} v_{\beta_1}(s) < \lambda_{\beta_2} v_{\beta_2}(s) \qquad \forall s \in (0, \sigma),$$

and by (2.24)

$$v'_{\beta_1}(s) > v'_{\beta_2}(s) \qquad s \in (0, \sigma).$$

Hence, integrating the previous inequality and recalling $v_{\beta_1}(0) = v_{\beta_2}(0)$, we obtain

$$v_{\beta_1}(s) > v_{\beta_2}(s) \qquad s \in (0, \sigma). \tag{2.25}$$

Let us define

$$A = \{r > \sigma \,:\, v_{\beta_1}(r) = v_{\beta_2}(r)\},$$

we want to prove that A is empty, so by (2.25) we get the claim. Let us suppose by contradiction that $A \neq \varnothing$, hence there exists

$$t = \inf A.$$

By continuity $v_{\beta_1}(t) = v_{\beta_2}(t)$, and this, combined with (2.25), leads to

$$v'_{\beta_1}(t) < v'_{\beta_2}(t). \tag{2.26}$$

Let us set

$$\gamma_i = -\frac{v'_{\beta_i}(t)}{v_{\beta_i}(t)},$$

by (2.26) we have $\gamma_1 > \gamma_2$.

By Lemma 2.7, v_{β_1} and v_{β_2} are the first eigenfunctions of the Robin Laplacian in B_t with eigenvalue λ_{γ_1} and λ_{γ_2} with parameter γ_1 and γ_2 respectively. Therefore by monotonicity of eigenvalues with respect to the boundary parameter, we get

$$\lambda_{\beta_1} = \lambda_{\gamma_1} > \lambda_{\gamma_2} = \lambda_{\beta_2}$$

which is a contradiction. $\square$

Remark 2.1. We highlight that, if we fix the value $v_{p,\beta}(0)$, (2.22) implies that

$$\beta \mapsto \|v_{p,\beta}\|_p$$

is non-increasing, while the map

$$\beta \mapsto C(n,p,\beta,\rho) = \frac{v_{p,\beta}(0)|\Omega^\star|}{\|v_{p,\beta}\|_p^p}$$

is non-decreasing, for all $\beta \in \mathbb{R}$.

3. Proofs of main results

Let us prove the continuity estimate in the case of a positive boundary parameter β.

Proof of Theorem 1.1. The quantity

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},$$

is bounded from above by 1, so inequality (1.5) is trivial when

$$\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right) \geq \frac{1}{C(n,p,\beta,\rho)} = \frac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}.$$

We can assume

$$\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right) < \frac{1}{C(n,p,\beta,\rho)}. \tag{3.1}$$

Let v be the solution to

$$\begin{cases} -\Delta_p v = \lambda_{p,\beta}(\Omega^\star)|v|^{p-2}v & \text{in } \Omega^\star \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta|v|^{p-2}v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{3.2}$$

as we have already remarked, v is positive and radially symmetric.

So the function

$$g(t) = |\nabla v|_{v=t}$$

is well defined for all $t \in (v_m, v_M)$, where $v_m = \min\limits_{\Omega^\star} v$ and $v_M = \max\limits_{\Omega^\star} v$. Let us define the following function

$$G^{-1}(t) = \int_{v_m}^{t} \frac{1}{g(s)}\, ds. \qquad v_m < t < v_M. \tag{3.3}$$

Since G^{-1} is strictly increasing, its inverse G is well defined and it is an increasing function satisfying $G(0) = v_m$. So we can construct $u(x) = G(d(x))$ for $x \in \Omega$. By construction, $u \in W^{1,p}(\Omega)$ and

- $\min_{\Omega} u = G(0) = v_m$, being G increasing;
- $\|u\|_\infty \le \|G\|_\infty \le v_M$, by formula (3.3);
- the chain rule formula implies

$$|\nabla u|_{u=t} = |G'(d(x))|_{u=t} = \left| g\Big(G\big(d(x)\big)\Big)\right|_{u=t} = g(t) = |\nabla v|_{v=t}.$$

Let us set

$$E_t = \{x \in \Omega : \ u(x) > t\}, \ \ B_t = \{x \in \Omega^\star : \ v(x) > t\}.$$

By Lemma 2.6 and formula (2.16), for almost every $t \in [0, \|u\|_\infty]$, we have

$$-\frac{d}{dt}P(E_t) \ge (n-1)\frac{W_2(E_t)}{g(t)} \ge (n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(E_t))^{\frac{n-2}{n-1}}}{g(t)},$$

while for v it holds

$$-\frac{d}{dt}P(B_t) = (n-1)\frac{W_2(B_t)}{g(t)} = (n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(B_t))^{\frac{n-2}{n-1}}}{g(t)},$$

and $P(E_0) = P(B_0)$. By direct computations, it holds

$$-\frac{1}{P(B_t)^{\frac{n-2}{n-1}}}\frac{d}{dt}P(B_t) \le -\frac{1}{P(E_t)^{\frac{n-2}{n-1}}}\frac{d}{dt}P(E_t) \qquad \text{a.e. } t \in [0, \|u\|_\infty],$$

therefore, integrating and recalling that $P(E_t)$ and $P(B_t)$ are absolutely continuous functions, we get

$$P(E_t) \le P(B_t), \quad 0 \le t \le \|u\|_\infty. \tag{3.4}$$

Denoting by $\mu(t) = |E_t|$ and by $\nu(t) = |B_t|$, the coarea formula (2.1) ensures us that

$$\begin{aligned} -\mu'(t) &= \int_{u=t} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} = \frac{P(E_t)}{g(t)} \le \frac{P(B_t)}{g(t)} \\ &= \int_{v=t} \frac{1}{|\nabla v|}\, d\mathcal{H}^{n-1} = -\nu'(t), \end{aligned}$$

for almost every $t \in (0, \|u\|_\infty)$. The first equality holds true since $|\nabla u| \ne 0$ in $\{\, v_m < u < \|u\|_\infty \,\}$ (see [9]). So the function $\nu - \mu$ is decreasing in $[0, \|u\|_\infty]$, and

$$\int_\Omega u^p\, dx = \int_0^{\|u\|_\infty} pt^{p-1}\mu(t)\, dt = \int_0^{v_M} pt^{p-1}\mu(t)\, dt$$

$$= \int_0^{v_M} pt^{p-1}\nu(t)\,dt - \int_0^{v_M} pt^{p-1}(\nu(t)-\mu(t))\,dt \geq \int_{\Omega^\star} v^p\,dx - v_M^p(|\Omega^\star|-|\Omega|).$$

Moreover, by (3.4), we get

$$\int_{u=t} |\nabla u|^{p-1}\,d\mathcal{H}^{n-1} = g(t)^{p-1}P(E_t) \leq g(t)^{p-1}P(B_t) = \int_{v=t} |\nabla v|^{p-1}\,d\mathcal{H}^{n-1},$$

so, if we integrate over Ω,

$$\int_\Omega |\nabla u|^p\,dx \leq \int_0^{\|u\|_\infty}\int_{v=t} |\nabla v|^{p-1}\,d\mathcal{H}^{n-1}\,dt \leq \int_{\Omega^\star} |\nabla v|^p.$$

We observe that by construction both u and v are constant on $\partial\Omega$, so

$$\beta\int_{\partial\Omega} u^p\,d\mathcal{H}^{n-1} = \beta v_m^p P(\Omega) = \beta\int_{\partial\Omega^\star} v^p\,d\mathcal{H}^{n-1}.$$

We eventually get

$$\begin{aligned}\lambda_{p,\beta}(\Omega) &\leq \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx + \beta\int_{\partial\Omega} u^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p\,dx} \leq \frac{\displaystyle\int_{\Omega^\star} |\nabla v|^p\,dx + \beta\int_{\partial\Omega^\star} v^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\star} v^p\,dx - v_M^p(|\Omega^\star|-|\Omega|)} \\ &= \lambda_{p,\beta}(\Omega^\star)\frac{1}{1-C(n,p,\beta,\rho)\left(1-\dfrac{|\Omega|}{|\Omega^\star|}\right)}.\end{aligned} \tag{3.5}$$

The claim follows from (3.1) as the quantity

$$1-C(n,p,\beta,\rho)\left(1-\frac{|\Omega|}{|\Omega^\star|}\right),$$

is non-negative. □

Remark 3.1. The constant $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}$ depends on the perimeter of the set Ω and on β. Thanks to Proposition 2.8 and Remark 2.1, it is possible to bound the constant $C(n,p,\beta,\rho)$ from above with a constant independent of the perimeter and β. Indeed, $\forall\beta>0$, if we denote by $v_{p,\infty}$ the first Dirichlet eigenfunction normalized in such a way $v_{p,\beta}(0) = v_{p,\infty}(0)$, we have

$$C(n,p,\beta,\rho) \leq \frac{v_{p,\infty}(0)|\Omega^\star|}{\|v_{p,\infty}\|_p^p} =: C(n,p)$$

that is independent of the perimeter thanks to the rescaling properties of the Dirichlet p-Laplacian eigenfunction.

Let us now move to the case of a negative boundary parameter β and let us prove the quantitative estimate with respect to the isoperimetric deficit.

Proof of Theorem 1.2. Let v be a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then v is a p sub-harmonic function as $\lambda_{p,\beta}(\Omega^\star) < 0$. We denote by $v_m = v(0) = \min_{\Omega^\star} v$ and by $v_M = \max_{\Omega^\star} v$.

Let us consider the function $\tilde{v} = v_M - v$, which is a positive function with zero trace. The function

$$g(t) = |\nabla \tilde{v}|_{\tilde{v}=t},$$

is well-defined for all $t \in (0, v_M - v_m)$. Let us define the following function

$$G^{-1}(t) = \int_0^t \frac{1}{g(s)}\, ds. \qquad 0 < t < v_M - v_m. \tag{3.6}$$

Since G^{-1} is strictly increasing, its inverse G is well defined and it is an increasing function satisfying $G(0) = 0$. So we can construct $\tilde{u}(x) = G(d(x))$ for $x \in \Omega$ and $u = v_M - \tilde{u}$. By construction, $\tilde{u}, u \in W^{1,p}(\Omega)$ and

- $u_M = \max_\Omega u = \max_\Omega (v_M - \tilde{u}) = v_M$, since $\tilde{u}$ is a positive function in $W_0^{1,p}(\Omega)$;
- by formula (3.6), it holds

$$u_m = \min_\Omega u = v_M - \max_\Omega \tilde{u} \geq v_M - \max_{\Omega^\star} \tilde{v} = \min_{\Omega^\star} v = v_m$$

- the chain rule formula implies, for $u_m < t < u_M$

$$|\nabla u|_{u=t} = |\nabla \tilde{u}|_{\tilde{u}=v_M-t} = \left| g\Big(G\big(d(x)\big)\Big)\right|_{\tilde{u}=v_M-t} = |\nabla \tilde{v}|_{\tilde{v}=v_M-t} = |\nabla v|_{v=t} = g(t).$$

Let us set

$$\begin{aligned} &\widetilde{E}_t = \{x \in \Omega \,:\, \tilde{u}(x) > t\}, && \widetilde{B}_t = \{x \in \Omega^\star \,:\, \tilde{v}(x) > t\}, \\ &E_t = \{x \in \Omega \,:\, u(x) > t\} = \Omega \setminus \overline{\widetilde{E}}_{v_M-t}, && B_t = \{x \in \Omega^\star \,:\, v(x) > t\} = \Omega^\star \setminus \overline{\widetilde{B}}_{v_M-t}. \end{aligned} \tag{3.7}$$

By Lemma 2.6 and formula (2.16) we have

$$-\frac{d}{dt} P(\widetilde{E}_t) \geq (n-1)\frac{W_2(\widetilde{E}_t)}{g(t)} \geq (n-1) n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \frac{(P(\widetilde{E}_t))^{\frac{n-2}{n-1}}}{g(t)},$$

while for $\tilde{v}$ it holds

$$-\frac{d}{dt} P(\widetilde{B}_t) = (n-1)\frac{W_2(\widetilde{B}_t)}{g(t)} = (n-1) n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \frac{(P(\widetilde{B}_t))^{\frac{n-2}{n-1}}}{g(t)},$$

and $P(\widetilde{E}_0) = P(\widetilde{B}_0)$. By direct computations, it holds

$$-\frac{1}{P(\widetilde{B}_t)^{\frac{n-2}{n-1}}} \frac{d}{dt} P(\widetilde{B}_t) \leq -\frac{1}{P(\widetilde{E}_t)^{\frac{n-2}{n-1}}} \frac{d}{dt} P(\widetilde{E}_t) \qquad \text{a.e. } t \in [0, v_M - v_m],$$

therefore, integrating and recalling that $P(\tilde{E}_t)$ and $P(\tilde{B}_t)$ are absolutely continuous functions, we have

$$P(\widetilde{E}_t) \leq P(\widetilde{B}_t), \quad 0 \leq t \leq v_M - v_m. \tag{3.8}$$

Hence

$$P(E_t) = P(\widetilde{E}_{v_M - t}) \le P(\widetilde{B}_{v_M - t}) = P(B_t), \quad v_m \le t \le v_M. \tag{3.9}$$

Denoting by $\tilde{\mu}(t) = |\widetilde{E}_t|$ and $\tilde{\nu}(t) = |\widetilde{B}_t|$, the coarea formula (2.1) ensures us that

$$\begin{aligned} -\tilde{\mu}'(t) &= \int\limits_{\tilde{u}=t} \frac{1}{|\nabla \tilde{u}|} \, d\mathcal{H}^{n-1} = \frac{P(\widetilde{E}_t)}{g(t)} \le \frac{P(\widetilde{B}_t)}{g(t)} \\ &= \int\limits_{\tilde{v}=t} \frac{1}{|\nabla \tilde{v}|} \, d\mathcal{H}^{n-1} = -\tilde{\nu}'(t), \qquad 0 \le t < v_M - v_m. \end{aligned}$$

Moreover, setting $\mu(t) = |E_t| = |\Omega| - \tilde{\mu}(v_M - t)$ and by $\nu(t) = |B_t| = |\Omega^\star| - \tilde{\nu}(v_M - t)$, we have $-\mu'(t) \le -\nu'(t)$ in $[v_m, v_M]$. So the function $\nu - \mu$ is decreasing in $[v_m, v_M]$, and

$$\begin{aligned} \int\limits_\Omega u^p \, dx &= \int\limits_0^{u_M} pt^{p-1} \mu(t) \, dt = \int\limits_0^{v_M} pt^{p-1} \nu(t) \, dt - \int\limits_0^{v_M} pt^{p-1} (\nu(t) - \mu(t)) \, dt \\ &= \int\limits_0^{v_M} pt^{p-1} \nu(t) \, dt - \int\limits_0^{v_m} pt^{p-1} (\nu(t) - \mu(t)) \, dt - \int\limits_{v_m}^{v_M} pt^{p-1} (\nu(t) - \mu(t)) \, dt \\ &\le \int\limits_{\Omega^\star} v^p \, dx - v_m^p (|\Omega^\star| - |\Omega|) = \int\limits_{\Omega^\star} v^p \, dx \left[1 - \frac{v_m^p}{\|v\|_p^p} (|\Omega^\star| - |\Omega|) \right]. \end{aligned}$$

By (3.9), we get

$$\int\limits_{\tilde{u}=t} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1} = g(t)^{p-1} P(\widetilde{E}_t) \le g(t)^{p-1} P(\widetilde{B}_t) = \int\limits_{\tilde{v}=t} |\nabla \tilde{v}|^{p-1} \, d\mathcal{H}^{n-1},$$

so, if we integrate over Ω,

$$\begin{aligned} \int\limits_\Omega |\nabla u|^p \, dx = \int\limits_\Omega |\nabla \tilde{u}|^p \, dx &= \int\limits_0^{\|\tilde{u}\|_\infty} \int\limits_{\tilde{u}=t} |\nabla \tilde{u}|^{p-1} \, d\mathcal{H}^{n-1} \, dt \\ &\le \int\limits_0^{\|\tilde{u}\|_\infty} \int\limits_{\tilde{v}=t} |\nabla \tilde{v}|^{p-1} \, d\mathcal{H}^{n-1} \, dt = \int\limits_{\Omega^\star} |\nabla \tilde{v}|^p = \int\limits_{\Omega^\star} |\nabla v|^p. \end{aligned}$$

We also observe that by construction both u and v are constant on $\partial\Omega$, so

$$\beta \int\limits_{\partial\Omega} u^p \, d\mathcal{H}^{n-1} = \beta u_M^p P(\Omega) = \beta v_M^p P(\Omega) = \beta \int\limits_{\partial\Omega^\star} v^p \, d\mathcal{H}^{n-1}.$$

We eventually get

$$\begin{aligned}\lambda_{p,\beta}(\Omega) &\leq \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx + \beta\int_{\partial\Omega} u^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p\,dx} \leq \frac{\displaystyle\int_{\Omega^\star} |\nabla v|^p\,dx + \beta\int_{\partial\Omega^\star} v^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\star} v^p\,dx\Big[1 - \frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|)\Big]} \\ &= \lambda_{p,\beta}(\Omega^\star)\,\frac{1}{1 - \dfrac{v_m^p}{\|v\|_p^p}\,(|\Omega^\star| - |\Omega|)}.\end{aligned} \tag{3.10}$$

Hence, by direct calculation

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \geq \frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|). \quad \square \tag{3.11}$$

Remark 3.2. Unlike the constant in Theorem 1.1, the constant $C(n,p,\beta,\rho) = \dfrac{v_m^p|\Omega^\star|}{\|v\|_p^p}$ cannot be bounded from below with a constant independent of the perimeter and β. Indeed, for example if $n = p = 2$ and $P(\Omega) = 2\pi$, we have that

$$v_\beta(x) = I_0\left(\sqrt{-\lambda_\beta(B_1)}|x|\right),$$

where I_0 is the modified Bessel function.

We recall that for z sufficiently large (see [1, Section 9.7]), we have

$$I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 + \frac{1}{8z} + \frac{9}{2!(8z)^2} + \dots\right]$$

therefore

$$\|v_\beta(x)\|_2 \sim \frac{e^{-\beta}}{\beta^2} \xrightarrow{\beta\to-\infty} +\infty$$

and

$$C(2,2,\beta,2\pi) = \frac{(v_\beta)_m^2|B_1|}{\|v_\beta\|_2^2} \sim \frac{\beta^2}{e^{-\beta}} \xrightarrow{\beta\to-\infty} 0.$$

Remark 3.3. We want to highlight that the constant C depends actually just on n, p and $\rho^{\frac{p-1}{n-1}}\beta$. Indeed, for all $\Omega \subset \mathbb{R}^n$ bounded and convex set with $P(\Omega) = \rho$, we can consider

$$\Omega_1 = \left(\frac{n\omega_n}{\rho}\right)^{\frac{1}{n-1}}\Omega \qquad \Omega_1^* = \left(\frac{n\omega_n}{\rho}\right)^{\frac{1}{n-1}}\Omega^\star \qquad t := \left(\frac{\rho}{n\omega_n}\right)^{\frac{1}{n-1}}$$

so $P(\Omega_1) = P(\Omega_1^\star) = n\omega_n$ and we have

$$\begin{aligned}\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{-\lambda_{p,\beta}(\Omega)} &= \frac{\lambda_{p,\beta}(t\Omega_1^\star) - \lambda_{p,\beta}(t\Omega_1)}{-\lambda_{p,\beta}(t\Omega_1)} \\ &= \frac{\lambda_{p,t^{p-1}\beta}(\Omega_1^*) - \lambda_{p,t^{p-1}\beta}(\Omega_1)}{\lambda_{p,t^{p-1}\beta}(\Omega_1)} \\ &\geq C\big(n,p,n\omega_n,t^{p-1}\beta\big)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\right)\end{aligned}$$

$$= C\Big(n, p, n\omega_n, \Big(\frac{\rho}{n\omega_n}\Big)^{\frac{p-1}{n-1}}\beta\Big)\Bigg(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\Bigg)$$

$$= C\big(n, p, \rho^{\frac{p-1}{n-1}}\beta\big)\Bigg(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}}\Bigg)$$

Eventually, let us prove the quantitative estimate in the spirit of [17].

Proof of Theorem 1.3. If we choose $w = 1$ in the definition of $\lambda_{p,\beta}(\Omega)$ (1.2), we get

$$|\lambda_{p,\beta}(\Omega)| \geq |\beta|\frac{P(\Omega)}{|\Omega|},$$

and if we combine it with (3.11) we have

$$\begin{aligned}
\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) &\geq |\lambda_{p,\beta}(\Omega)|\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|) \\
&\geq |\beta|\frac{P(\Omega)}{|\Omega|}\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|) \\
&\geq |\beta|\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}}{\rho^{\frac{1}{n-1}}}\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star| - |\Omega|).
\end{aligned} \tag{3.12}$$

Now, if we suppose that $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$, by (3.12) we have

$$|\Omega^\star| - |\Omega| \leq K(n, \rho, p, \beta)\delta_0.$$

So by Lemma 2.4 we conclude

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n, \rho, p, \beta)\, g(\mathcal{A}^\star_{\mathcal{H}}(\Omega)). \quad \square$$

Remark 3.4. Our result, Theorem 1.3, applies only when

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0.$$

It is possible to remove this hypothesis, obtaining a weaker result.

In order to obtain it, we need the quantitative version of the isoperimetric inequality proved in [26]

$$P(\Omega) \geq P(\Omega^\sharp)\big(1 + \gamma(n)\alpha(\Omega)^2\big) \qquad \text{where } \alpha(\Omega) = \min\left\{ \frac{|\Omega \triangle B_r|}{|\Omega|} \;\middle|\; |B_r| = |\Omega| \right\}, \tag{3.13}$$

and the following result [22, Lemma 4.2]: there exists a constant $C(n)$ such that if C, W are open and convex sets such that $|C| = |W|$ and $|C \triangle W| < \frac{|C|}{2}$, it holds

$$d_{\mathcal{H}}(C, W) \leq C(n)\big[\operatorname{diam}(C) + \operatorname{diam}(W)\big]\left(\frac{|C \triangle W|}{|C|}\right)^{\frac{1}{n}}. \tag{3.14}$$

Moreover, we have to recall that

$$|\Omega^\star| = \frac{P(\Omega)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}} \qquad \text{and} \qquad P(\Omega^\sharp) = n\omega_n^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}},$$

so we have

$$
\begin{aligned}
1-\frac{|\Omega|}{|\Omega^\star|} &= 1-\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)}\\
&\geq 1-\frac{1}{\big(1+\gamma(n)\alpha^2(\Omega)\big)^{\frac{n}{n-1}}}\\
&\geq 1-\frac{1}{\big(1+\gamma(n)\alpha^2(\Omega)\big)}\\
&=\frac{\gamma(n)\alpha^2(\Omega)}{1+\gamma(n)\alpha^2(\Omega)},
\end{aligned}
$$

where we used Bernoulli's inequality

$$(1+x)^r \geq 1+rx \qquad \forall x\geq -1,\, \forall r\geq 0.$$

Since $0<\alpha^2(\Omega)<4$ we have

$$1-\frac{|\Omega|}{|\Omega^\star|}\geq \frac{\gamma(n)}{1+4\gamma(n)}\,\alpha^2(\Omega)=C(n)\alpha^2(\Omega).$$

If $\Omega^\sharp$ is a ball that realizes the minimum in (3.13), then using (3.14) we obtain

$$C(n)\alpha^2(\Omega)\geq C(n)\,\frac{d_{\mathcal{H}}(\Omega,\Omega^\sharp)^{2n}}{\big[\operatorname{diam}(\Omega)+\operatorname{diam}(\Omega^\sharp)\big]^{2n}},$$

and so

$$\frac{\lambda_{p,\beta}(\Omega^\star)-\lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|}\geq C(n,p,\beta\rho)C(n)\frac{d_{\mathcal{H}}(\Omega,\Omega^\sharp)^{2n}}{\big[\operatorname{diam}(\Omega)+\operatorname{diam}(\Omega^\sharp)\big]^{2n}}.$$

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