Journal of Optimization Theory and Applications (2023) 196:570–589
https://doi.org/10.1007/s10957-022-02150-8

Inverse Tensor Variational Inequalities and Applications

Francesca Anceschi[1] · Annamaria Barbagallo[1] · Serena Guarino Lo Bianco[2]

Received: 3 February 2022 / Accepted: 10 December 2022 / Published online: 24 December 2022

Abstract

The paper aims to introduce inverse tensor variational inequalities and analyze their application to an economic control equilibrium model. More precisely, some existence and uniqueness results are established and the well-posedness analysis is investigated. Moreover, the Tikhonov regularization method is extended to tensor inverse problems to study them when they are ill-posed. Lastly, the policymaker's point of view for the oligopolistic market equilibrium problem is introduced. The equivalence between the equilibrium conditions and a suitable inverse tensor variational inequality is established.

Keywords Tensor variational inequality · Inverse tensor variational inequality · Noncooperative game · Well-posedness · Tikhonov regularization method

Mathematics Subject Classification 49N45 · 49J40 · 49K40 · 90C33 · 91A10

1 Introduction

Since 1980, the variational inequality theory in finite-dimensional spaces has thoroughly been studied and the well-posedness analysis has widely been developed (see also [10, 13]). Such a theory has several applications to applied sciences (e.g., transport

Communicated by Suliman Saleh Al-Homidan.

✉ Annamaria Barbagallo
annamaria.barbagallo@unina.it

Francesca Anceschi
francesca.anceschi@unina.it

Serena Guarino Lo Bianco
serena.guarinolobianco@unimore.it

1 Department of Mathematics and Applications "R. Caccioppoli", University of Naples Federico II, Naples, Italy

2 Department of Physical, Computer and Mathematical Sciences, University of Modena e Reggio Emilia, Modena, Italy

planning, socioeconomic phenomena, game theory), and it is only quite recently that the research has started to focus on its extension to the class of tensors [1, 14]. Some numerical schemes are proposed in [3] to solve the tensor variational inequalities when the uniqueness of solutions is guaranteed.

The topic of our interest is the class of inverse tensor variational inequalities, which is very useful to model some control problems, such as the policymaker's point of view for the general oligopolistic market equilibrium problem (for the Euclidean case see [4]).

In the Euclidean setting, an inverse variational inequality problem consists in finding $x^* \in \mathbb{R}^n$ such that

$$f(x^*) \in \Omega, \quad \langle x^*, f' - f(x^*) \rangle \le 0, \quad \forall f' \in \Omega,$$

where Ω is a nonempty subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ ([7]). Lately, a strict connection between classical variational inequalities and inverse variational inequalities has been pointed out ([6, 15]). This class of inverse problems expresses equilibrium state control problems and can also be viewed as a special case of a classical variational inequality. Indeed, if the inverse function f^{-1} is single-valued, then the inverse variational inequality can be written as a classical variational inequality by setting $u^* = f(x^*)$ and $F(u^*) = f^{-1}(u^*)$. For this reason, we name such problems as inverse variational inequalities. Unfortunately, in some practical applications, $f(x)$ is not available and we only rely on $F(u)$. See [4, 5, 7] for the well-posedness analysis of inverse variational inequalities and to [11, 12] for the Tikhonov regularization method. In [2], the ill-posedness and the stability analysis are studied for tensor variational inequalities.

We aim to introduce inverse tensor variational inequalities and to investigate the existence of solutions and the well-posedness analysis. Moreover, this class of inverse problems is useful to formulate the policymaker's point of view for an oligopolistic market model. In particular, we obtain some well-posedness characterizations for an inverse variational inequality. Also, under suitable assumptions, we prove the equivalence between the well-posedness of an inverse tensor variational inequality and the existence and uniqueness of its solution. Moreover, we show that the well-posedness of an inverse tensor variational inequality is equivalent to the well-posedness of a suitable tensor variational inequality. Finally, we extend the Tikhonov regularization method to the class of inverse variational inequalities we introduce. Lastly, we show how our results can be applied to the study of the policymaker's point of view for the general oligopolistic market equilibrium problem in which the equilibrium conditions are characterized by a suitable inverse variational inequality. The interest in the tensor setting is highlighted in [8], where the authors presented the open question: "**Problem 4:** [...] The TVI problem is an extension of the tensor complementarity problem, which needs to be further studied since it has important applications such as for the multi-person noncooperation games."

The paper is organized as follows. In Sect. 2, some preliminary notations are recalled. In Sect. 3, a class of inverse variational inequalities is introduced and some existence results are proved. Section 4 deals with the well-posedness analysis. On the other hand, the ill-posedness of such inequalities is examined by using a Tikhonov-type regularization method in Sect. 5. Section 6 addresses the policymaker's point of view for the oligopolistic market equilibrium problem. Finally, in Sect. 7 an example is provided.

2 Notations and Preliminaries

Let us recall some definitions and preliminary results about tensors and tensor functions. Given N finite-dimensional vector spaces V_i, $i = 1, \ldots, N$, an N-order tensor is an element of the product space $V_1 \times \cdots \times V_N$, namely a multidimensional array. Indeed, tensors generalize vectors (tensors of order one denoted usually by small letters $v, w, \ldots$) and matrices (tensors of order two denoted usually by capital letters $A, B, \ldots$) in a higher dimension.

Let $\mathbb{T}^{[N,m]}$ denote the set of all N-order m-dimensional tensors. We denote tensors by italic capital letters $\mathcal{A}, \mathcal{B}, \ldots$. The element $(i_1, \ldots, i_N)$ of $\mathcal{A}$ is indicated with $a_{i_1,\ldots,i_N}$. We denote the set of all N-order m-dimensional real tensors by $\mathbb{R}^{[N,m]}$. When it is clear in the context, we use $\mathbb{R}^{[P]}$, where $P = m_1 \ldots m_N$, to denote the class of $N-$order tensors of $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N}$.

Definition 2.1 Let $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{[N,m]}$. We define the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{T}^{[N,m]} \times \mathbb{T}^{[N,m]}$ to $\mathbb{R}$ as follows

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1=1}^{m} \cdots \sum_{i_N=1}^{m} a_{i_1,\ldots,i_N} b_{i_1,\ldots,i_N}.$$

When $n = 2$, $\mathcal{A}$ and $\mathcal{B}$ are matrices, then $\langle \mathcal{A}, \mathcal{B} \rangle = \mathrm{tr}(\mathcal{A}\mathcal{B}^T)$, where $\mathrm{tr}(\cdot)$ and ^{T} denote the trace and the transpose of a matrix, respectively. Therefore, the tensor norm generated by this inner product is

$$\|\mathcal{A}\| = \sqrt{\sum_{i_1=1}^{m} \cdots \sum_{i_N=1}^{m} |a_{i_1,\ldots,i_N}|^2},$$

which is the Frobenius norm. Moreover, the distance between $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{T}^{[N,m]}$ is given by $\|\mathcal{A} - \mathcal{B}\|$.

Now, we introduce some useful concepts for the study of the well-posedness of an inverse tensor variational inequality.

Definition 2.2 Let Ω be a nonempty subset of $\mathbb{T}^{[N,m]}$ and let $\mathcal{A} \in \mathbb{T}^{[N,m]}$. We define the diameter of Ω, the distance between the tensor $\mathcal{A}$ and the set Ω and the projection of $\mathcal{A}$ on the set Ω, as follows:

$$\begin{aligned} \mathrm{diam}\, \Omega &= \sup\left\{\|\mathcal{A} - \mathcal{B}\| :\ \mathcal{A}, \mathcal{B} \in \Omega\right\}, \\ d(\mathcal{A}, \Omega) &= \inf\left\{\|\mathcal{A} - \mathcal{B}\| :\ \mathcal{B} \in \Omega\right\}, \\ P_\Omega(\mathcal{A}) &= \mathrm{argmin}\ \left\{\|\mathcal{A} - \mathcal{B}\| :\ \mathcal{B} \in \Omega\right\}, \end{aligned}$$

respectively.

To prove the well-posedness results, we present a measure of noncompactness, as a generalization of the one introduced by Kuratovski in [9].

Definition 2.3 Let Ω be a nonempty subset of $\mathbb{T}^{[N,m]}$. The noncompactness measure μ of the set Ω is defined by

$$\mu(\Omega) = \inf \left\{ \epsilon > 0 : \ \Omega \subset \cup_{i=1}^{n} \Omega_i, \ \operatorname{diam} \Omega_i < \epsilon, \ i = 1, \dots, n \right\},$$

where every $\{\Omega_i\}_{i=1,\dots,n}$ is a finite covering of the set Ω.

For the reader's convenience, we state the Hausdorff distance.

Definition 2.4 Let Ω_1 and Ω_2 be two nonempty subsets of $\mathbb{T}^{[N,m]}$. We introduce the surplus of Ω_1 over Ω_2 as

$$e(\Omega_1, \Omega_2) = \sup \{d(\mathcal{A}, \Omega_2) : \mathcal{A} \in \Omega_1\}.$$

The Hausdorff distance between Ω_1 and Ω_2 is defined as

$$\mathcal{H}(\Omega_1, \Omega_2) = \max \{e(\Omega_1, \Omega_2), e(\Omega_2, \Omega_1)\}.$$

The next definitions generalize topological and monotonicity properties to tensor mappings.

Definition 2.5 Let K be a nonempty subset of $\mathbb{T}^{[N,m]}$. A mapping $F : K \to \mathbb{T}^{[N,m]}$ is said to be:

- monotone if $\langle F(\mathcal{A}) - F(\mathcal{B}), \mathcal{A} - \mathcal{B} \rangle \geq 0$, for every $\mathcal{A}, \mathcal{B}, \in K$;
- strictly monotone if $\langle F(\mathcal{A}) - F(\mathcal{B}), \mathcal{A} - \mathcal{B} \rangle > 0$, for every $\mathcal{A}, \mathcal{B} \in K$, with $\mathcal{A} \neq \mathcal{B}$;
- strongly monotone if there exists $\nu > 0$ such that $\langle F(\mathcal{A}) - F(\mathcal{B}), \mathcal{A} - \mathcal{B} \rangle \geq \nu \|\mathcal{A} - \mathcal{B}\|^2$, for every $\mathcal{A}, \mathcal{B} \in K$;
- hemicontinuous along line segments if, for every $\mathcal{A}, \mathcal{B}, \in K$, the function $t \mapsto \langle F(\mathcal{A} + t(\mathcal{B} - \mathcal{A})), \mathcal{B} - \mathcal{A} \rangle$, for $t \in [0, 1]$, is continuous at 0^+.

3 Inverse Tensor Variational Inequalities

The class of tensor variational inequalities have been firstly introduced in [1] and [14]. In particular, given a nonempty closed convex subset K of $\mathbb{T}^{[N,m]}$ and a tensor mapping $F : K \to \mathbb{T}^{[N,m]}$, the tensor variational inequality ($TVI(K, F)$) is the problem of finding $\mathcal{A} \in K$ such that

$$\langle F(\mathcal{A}), \mathcal{B} - \mathcal{A} \rangle \geq 0, \quad \forall \mathcal{B} \in K. \tag{1}$$

For this class of variational inequalities, some existence and uniqueness results have been proved in [1]. We now introduce the class of inverse tensor variational inequalities.

Definition 3.1 Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$ and let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be a tensor mapping. The inverse tensor variational inequality ($ITVI(\Omega, f)$) is the problem of finding $\mathcal{A}^* \in \mathbb{T}^{[N,m]}$ such that

$$f(\mathcal{A}^*) \in \Omega, \quad \langle \mathcal{A}^*, \mathcal{F} - f(\mathcal{A}^*)\rangle \leq 0, \quad \forall \mathcal{F} \in \Omega. \tag{2}$$

The following result can be classically shown.

Theorem 3.1 *Let Ω be a nonempty bounded closed convex subset of $\mathbb{T}^{[N,m]}$ and let $f : \Omega \to \Omega$ be an injective continuous open tensor mapping. Then, $ITVI(\Omega, f)$ admits at least one solution.*

Let Ω be a subset of $\mathbb{T}^{[N,m]}$, we consider the following set:

$$\Omega_n = \{\mathcal{B} \in \Omega : \ \|\mathcal{B}\| \leq n\}, \quad \forall n \in \mathbb{N}. \tag{3}$$

For $\epsilon > 0$, let $\{D_\epsilon\}$ be a sequence of subsets of $\mathbb{T}^{[N,m]}$. We define

$$\limsup_{\epsilon \to 0^+} D_\epsilon = \left\{\mathcal{B} \in \mathbb{T}^{[N,m]} : \exists \epsilon_n \to 0^+, \ \mathcal{B}_n \in D_{\epsilon_n}, \forall n \in \mathbb{N}, \text{ such that } \mathcal{B}_n \to \mathcal{B}\right\}. \tag{4}$$

Theorem 3.2 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$ and let $f : \Omega \to \mathbb{T}^{[N,m]}$ be an injective continuous open tensor mapping. If there exists $\overline{n} \in \mathbb{N}$ such that for every $f(\mathcal{A}) \in \Omega \setminus \Omega_{\overline{n}}$ there is $\mathcal{F}_0 \in \Omega$ with $\|\mathcal{F}_0\| < \|f(\mathcal{A})\|$ satisfying $\langle \mathcal{A}, f(\mathcal{A}) - \mathcal{F}_0\rangle \leq 0$, then $ITVI(\Omega, f)$ admits a solution.*

Proof Let us fix $\overline{n} \in \mathbb{N}$ such that the assumption holds true. Let $m > \overline{n}$. We consider Ω_m which is a bounded closed convex subset of $\mathbb{T}^{[N,m]}$. Thus, applying Theorem 3.1 to $ITVI(\Omega_m, f_{|\Omega_m})$, there exists $\mathcal{A}_m \in \mathbb{T}^{[N,m]}$ such that

$$f(\mathcal{A}_m) \in \Omega_m, \quad \langle \mathcal{A}_m, \mathcal{F} - f(\mathcal{A}_m)\rangle \leq 0, \quad \forall \mathcal{F} \in \Omega_m. \tag{5}$$

Firstly, let us suppose $\|f(\mathcal{A}_m)\| = m > \overline{n}$. Hence, there exists $\mathcal{F}_0 \in \Omega$ with $\|\mathcal{F}_0\| < \|f(\mathcal{A}_m)\|$ such that

$$\langle \mathcal{A}_m, f(\mathcal{A}_m) - \mathcal{F}_0\rangle \leq 0. \tag{6}$$

Now, let us fix $\mathcal{F} \in \Omega$. Since $\|\mathcal{F}_0\| < \|f(\mathcal{A}_m)\| = m$, there exists $t \in (0, 1)$ such that $\mathcal{F}_t = t\mathcal{F} + (1-t)\mathcal{F}_0 \in \Omega_m$. Therefore, writing (5) with $\mathcal{F} = \mathcal{F}_t$, we have

$$0 \geq \langle \mathcal{A}_m, \mathcal{F}_t - f(\mathcal{A}_m)\rangle = t\langle \mathcal{A}_m, \mathcal{F} - f(\mathcal{A}_m)\rangle + (1-t)\langle \mathcal{A}_m, \mathcal{F}_0 - f(\mathcal{A}_m)\rangle.$$

Lastly, by using (6), we get $t\langle \mathcal{A}_m, \mathcal{F} - f(\mathcal{A}_m)\rangle \leq 0$. Since $t > 0$ and $\mathcal{F} \in \Omega$ is arbitrary, the claim is achieved.

On the other hand, if $\|f(\mathcal{A}_m)\| < m$, for a fixed tensor $\mathcal{F} \in \Omega$, there exists $t \in (0, 1)$ such that $\mathcal{F}_t = f(\mathcal{A}_m) + t(\mathcal{F} - f(\mathcal{A}_m)) \in \Omega_m$. Then, writing (5) with $\mathcal{F} = \mathcal{F}_t$, we obtain $0 \geq \langle \mathcal{A}_m, \mathcal{F}_t - f(\mathcal{A}_m)\rangle = t\langle \mathcal{A}_m, \mathcal{F} - f(\mathcal{A}_m)\rangle$. Since $t > 0$ and $\mathcal{F} \in \Omega$ is arbitrary, the claim is also reached. □

4 Well-Posedness Results

In this section, we introduce the notion of $\alpha-$well-posedness and generalized $\alpha-$well-posedness. Moreover, we prove some metric characterizations for them. Finally, we provide an equivalence result between the well-posedness of an inverse tensor variational inequality and the well-posedness of a suitable tensor variational inequality.

We start presenting the notion of approximating sequence and $\alpha-$approximating sequence.

Definition 4.1 Let $\alpha > 0$. A sequence $\{\mathcal{A}_n\} \subset \mathbb{T}^{[N,m]}$ is said to be a $\alpha-$approximating sequence for $ITVI(\Omega, f)$ if and only if there exists a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, such that

$$f(\mathcal{A}_n) \in \Omega, \quad \langle \mathcal{A}_n, \mathcal{F} - f(\mathcal{A}_n)\rangle \leq \frac{\alpha}{2} \|\mathcal{F} - f(\mathcal{A}_n)\|^2 + \epsilon_n, \quad \forall \mathcal{F} \in \Omega, \ \forall n \in \mathbb{N}. \tag{7}$$

When $\alpha = 0$, we simply say that $\{\mathcal{A}_n\}$ is an approximating sequence for $ITVI(\Omega, f)$.

Definition 4.2 We say that $ITVI(\Omega, f)$ is $\alpha-$well-posed in the generalized sense if and only if $ITVI(\Omega, f)$ has a nonempty solution set S and every $\alpha-$approximating sequence has some subsequence which converges to a tensor of S. When the solution set S has only one element, we say simply that $ITVI(\Omega, f)$ is $\alpha-$well-posed.

In the sequel, $0-$well-posedness in the generalized sense is said well-posedness in the generalized sense, analogously for the $0-$well-posedness.

We can easily show the following preliminary result.

Lemma 4.1 *Let Ω be a nonempty convex subset of $\mathbb{T}^{[N,m]}$. Let $\alpha \geq 0$ and let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be a tensor mapping. Then, $\mathcal{A}^*$ is a solution to $ITVI(\Omega, f)$ if and only if*

$$\langle \mathcal{A}^*, \mathcal{F} - f(\mathcal{A}^*)\rangle \leq \frac{\alpha}{2} \|\mathcal{F} - f(\mathcal{A}^*)\|^2, \quad \forall \mathcal{F} \in \Omega.$$

Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$. The $\alpha-$approximating solution set $T_\alpha(\epsilon)$ of $ITVI(\Omega, f)$ is defined, for every $\epsilon > 0$, as

$$T_\alpha(\epsilon) = \Big\{\mathcal{A} \in \mathbb{T}^{[N,m]} : f(\mathcal{A}) \in \Omega,$$
$$\langle \mathcal{A}, \mathcal{F} - f(\mathcal{A})\rangle \leq \frac{\alpha}{2}\|\mathcal{F} - f(\mathcal{A})\|^2 + \epsilon, \quad \forall \mathcal{F} \in \Omega\Big\}.$$

First of all, we prove a metric characterization of the $\alpha-$well-posedness of $ITVI(\Omega, f)$ in terms of the diameter of the set $T_\alpha(\epsilon)$.

Theorem 4.1 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$. Let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be a continuous tensor mapping. Then, $ITVI(\Omega, f)$ is $\alpha-$well-posed if and only if*

$$T_\alpha(\epsilon) \neq \emptyset, \ \forall \epsilon > 0, \quad \textit{and} \quad \operatorname{diam} T_\alpha(\epsilon) \to 0, \ \textit{as}\ \epsilon \to 0. \tag{8}$$

Proof Suppose $ITVI(\Omega, f)$ is $\alpha-$well-posed. Hence, the inverse tensor variational inequality has a unique solution $\mathcal{A}^*$ and, in particular, $\mathcal{A}^* \in T_\alpha(\epsilon)$, for every $\epsilon > 0$. By contradiction, if diam $T_\alpha(\epsilon) \not\to 0$, as $\epsilon \to 0^+$, there exist $l > 0$, a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, and $\mathcal{U}_n, \mathcal{V}_n \in T_\alpha(\epsilon_n)$, for every $n \in \mathbb{N}$, such that

$$\|\mathcal{V}_n - \mathcal{U}_n\| > l, \quad \forall n \in \mathbb{N}. \tag{9}$$

Since $\mathcal{U}_n, \mathcal{V}_n \in T_\alpha(\epsilon_n)$, for every $n \in \mathbb{N}$, both $\{\mathcal{U}_n\}$ and $\{\mathcal{V}_n\}$ are $\alpha-$approximating sequences for the inverse tensor variational inequality. Therefore, both two sequences converge to the unique solution $\mathcal{A}^*$ to $ITVI(\Omega, f)$. This contradicts (9).

Vice versa, suppose (8) holds true. Let $\{\mathcal{A}_n\} \subset \mathbb{T}^{[N,m]}$ be an $\alpha-$approximating sequence for $ITVI(\Omega, f)$. Hence, there exists a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, verifying (7). Thus, $\mathcal{A}_n \in T_\alpha(\epsilon_n)$, for every $n \in \mathbb{N}$. By (8), $\{\mathcal{A}_n\}$ is a Cauchy sequence which converges to a tensor $\overline{\mathcal{A}} \in \mathbb{T}^{[N,m]}$. Since f is continuous and Ω is a closed set, we have

$$f(\overline{\mathcal{A}}) \in \Omega, \quad \langle \overline{\mathcal{A}}, \mathcal{F} - f(\overline{\mathcal{A}})\rangle \leq \frac{\alpha}{2}\,\|\mathcal{F} - f(\overline{\mathcal{A}})\|^2, \quad \forall \mathcal{F} \in \Omega.$$

Making use of Lemma 4.1, it follows that $\overline{\mathcal{A}}$ is a solution to $ITVI(\Omega, f)$. Finally, the uniqueness of solution to $ITVI(\Omega, f)$ can be shown with classically arguments. □

Furthermore, the following characterization of the α-well-posedness in generalized sense of $ITVI(\Omega, f)$ in terms of the noncompactness measure holds.

Theorem 4.2 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$. Let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be a continuous mapping. Then, $ITVI(\Omega, f)$ is $\alpha-$well-posed in the generalized sense if and only if*

$$T_\alpha(\epsilon) \neq \emptyset,\ \forall \epsilon > 0, \quad \textit{and} \quad \mu\,(T_\alpha(\epsilon)) \to 0,\ \textit{as}\ \epsilon \to 0^+. \tag{10}$$

Proof First suppose that $ITVI(\Omega, f)$ is $\alpha-$well-posed in the generalized sense. Then, its solution set S is nonempty and compact. Indeed, if $\{\mathcal{A}_n\}$ is any sequence in S, then it is an $\alpha-$approximating sequence of $ITVI(\Omega, f)$. Consequently, $\{\mathcal{A}_n\}$ has a subsequence which converges to a tensor of S. Then, S is compact. In addition, it follows that $\emptyset \neq S \subset T_\alpha(\epsilon)$, for every $\epsilon > 0$. Hence, it results

$$\mathcal{H}(T_\alpha(\epsilon), S) = \max\{e(T_\alpha(\epsilon), S), e(S, T_\alpha(\epsilon))\} = e(T_\alpha(\epsilon), S), \quad \forall \epsilon > 0.$$

Then, we obtain

$$\mu(T_\alpha(\epsilon)) \leq 2\mathcal{H}(T_\alpha(\epsilon), S) + \mu(S) = 2e(T_\alpha(\epsilon), S), \quad \forall \epsilon > 0.$$

We assume by contradiction that $e(T_\alpha(\epsilon), S) \not\to 0$, as $\epsilon \to 0$. Therefore, there exist $l > 0$, a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, and

$\mathcal{A}_n \in T_\alpha(\epsilon_n)$, for every $n \in \mathbb{N}$, such that

$$\mathcal{A}_n \notin S + B(0, l), \quad \forall n \in \mathbb{N}, \tag{11}$$

where $B(0, l)$ is the closed ball centered at the null tensor with radius l in the tensor space $\mathbb{T}^{[N,m]}$. Since $\mathcal{A}_n \in T_\alpha(\epsilon_n)$, for every $n \in \mathbb{N}$, $\{\mathcal{A}_n\}$ is an $\alpha-$approximating sequence for $ITVI(\Omega, f)$. By the $\alpha-$well-posedness in the generalized sense, there exists a subsequence $\{\mathcal{A}_{n_k}\}$ converging to a tensor of S. This contradicts (11), and hence the implication is proved.

Vice versa, let us assume that (10) holds. Since f is continuous and Ω is closed, we have that $T_\alpha(\epsilon)$ is closed and nonempty, for every $\epsilon > 0$. Let us consider the set

$$S' = \cap_{\epsilon>0} T_\alpha(\epsilon) = \left\{\mathcal{A} \in \mathbb{T}^{[N,m]} : \ f(\mathcal{A}) \in \Omega, \ \langle \mathcal{A}, \mathcal{F} - f(\mathcal{A})\rangle \leq \frac{\alpha}{2}\|\mathcal{F} - f(\mathcal{A})\|^2\right\}.$$

Taking into account Lemma 4.1, it results $S' = S$. By using (10) and applying Theorem on p. 412 of [9], we conclude that S is nonempty compact and $e(T_\alpha(\epsilon), S) = \mathcal{H}(T_\alpha(\epsilon), S) \to 0$, as $\epsilon \to 0^+$.

Let $\{\mathcal{U}_n\} \subset \mathbb{T}^{[N,m]}$ be an $\alpha-$approximating sequence for $ITVI(\Omega, f)$. Then, there exists a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, verifying (7).

Therefore, $\mathcal{U}_n \in T_\alpha(\epsilon_n)$, for every $n \in \mathbb{N}$. Then, it results $d(\mathcal{U}_n, S) \leq e(T_\alpha(\epsilon_n), S) \to 0$. Being S compact, there exists $\overline{\mathcal{A}}_n \in S$, for every $n \in \mathbb{N}$, such that $\|\mathcal{U}_n - \overline{\mathcal{A}}_n\| = d(\mathcal{U}_n, S) \to 0$, as $n \to +\infty$. Again, by the compactness of S, we deduce that $\{\overline{\mathcal{A}}_n\}$ has a subsequence which converges to $\overline{\mathcal{A}} \in S$. As a consequence, the corresponding subsequence $\{\mathcal{U}_{n_k}\}$ converges to $\overline{\mathcal{A}}$. Therefore, the claim is completely proved. □

Now we prove that, under suitable assumptions, the well-posedness of an inverse tensor variational inequality is equivalent to the existence and uniqueness of its solutions. To this aim we introduce the following definition.

Definition 4.3 A tensor mapping $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ is said to be anti-monotone if

$$\langle \mathcal{A} - \mathcal{B}, f(\mathcal{A}) - f(\mathcal{B})\rangle \leq 0, \quad \forall \mathcal{A}, \mathcal{B}, \in \mathbb{T}^{[N,m]}.$$

Theorem 4.3 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$. Let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be an hemicontinuous along line segments and anti-monotone tensor mapping. Then, $ITVI(\Omega, f)$ is well-posed if and only if it has a unique solution.*

Proof The necessity holds trivially. For the sufficiency, suppose that $ITVI(\Omega, f)$ has a unique solution $\mathcal{A}^*$; hence, by the anti-monotonicity of f, we get

$$\begin{aligned} 0 &\leq \langle \mathcal{A}^*, f(\mathcal{A}^*) - \mathcal{F}\rangle \\ &= \langle \mathcal{A}^* - \mathcal{A}, f(\mathcal{A}^*) - \mathcal{F}\rangle + \langle \mathcal{A}, f(\mathcal{A}^*) - \mathcal{F}\rangle \\ &\leq \langle \mathcal{A}^* - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, f(\mathcal{A}^*) - \mathcal{F}\rangle, \quad \forall \mathcal{A} \in \mathbb{T}^{[N,m]}, \ \forall \mathcal{F} \in \Omega. \end{aligned} \tag{12}$$

Now, let $\{\mathcal{A}_n\} \subset \mathbb{T}^{[N,m]}$ be an approximating sequence for $ITVI(\Omega, f)$. As a consequence there exits a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, such that

$$f(\mathcal{A}_n) \in \Omega, \quad \langle \mathcal{A}_n, \mathcal{F} - f(\mathcal{A}_n)\rangle \leq \epsilon_n, \quad \forall \mathcal{F} \in \Omega, \ \forall n \in \mathbb{N}. \tag{13}$$

With analogous procedure as in (12), we obtain

$$\langle \mathcal{A}_n - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, f(\mathcal{A}_n) - \mathcal{F}\rangle \geq -\epsilon_n, \quad \forall \mathcal{A} \in \mathbb{T}^{[N,m]}, \ \forall \mathcal{F} \in \Omega. \tag{14}$$

Now, we consider $\mathcal{U}^* = (\mathcal{A}^*, f(\mathcal{A}^*))$ and $\mathcal{U}_n = (\mathcal{A}_n, f(\mathcal{A}_n))$, for every $n \in \mathbb{N}$.

Let us assume that $\{\mathcal{U}_n\}$ is unbounded. Hence, without loss of generality, we may assume that $\|\mathcal{U}_n\| \to +\infty$, as $n \to +\infty$. We set $t_n = \dfrac{1}{\|\mathcal{U}_n - \mathcal{U}^*\|}$ and $\mathcal{W}_n = (\mathcal{Z}_n, \mathcal{G}_n) = (\mathcal{A}^* + t_n(\mathcal{A}_n - \mathcal{A}^*), f(\mathcal{A}^*) + t_n(f(\mathcal{A}_n) - f(\mathcal{A}^*)))$. Thus, without loss of generality, we may assume $t_n \in]0, 1]$ and $\mathcal{W}_n \to \mathcal{W} = (\mathcal{Z}, \mathcal{G}) \neq \mathcal{U}^*$. Furthermore, $\mathcal{G} \in \Omega$ since Ω is closed and convex. Then, for any $\mathcal{F} \in \Omega$ and any $\mathcal{A} \in \mathbb{T}^{[N,m]}$, it follows:

$$\begin{aligned} &\langle \mathcal{Z} - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, \mathcal{G} - \mathcal{F}\rangle \\ &\quad = \langle \mathcal{Z} - \mathcal{Z}_n, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, \mathcal{G} - \mathcal{G}_n\rangle \\ &\qquad + (1 - t_n)\left\{\langle \mathcal{A}^* - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, f(\mathcal{A}^*) - \mathcal{F}\rangle\right\} \\ &\qquad + t_n \left\{\langle \mathcal{A}_n - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, f(\mathcal{A}_n) - \mathcal{F}\rangle\right\}. \end{aligned} \tag{15}$$

Therefore, making use of (12)–(15), it follows, for every $\mathcal{F} \in \Omega$, for every $\mathcal{A} \in \mathbb{T}^{[N,m]}$,

$$\langle \mathcal{Z} - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, \mathcal{G} - \mathcal{F}\rangle \geq \langle \mathcal{Z} - \mathcal{Z}_n, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{G} - \mathcal{G}_n, \mathcal{A}\rangle - t_n\epsilon_n.$$

Letting $n \to +\infty$ in the above inequality, we deduce

$$\langle \mathcal{Z} - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, \mathcal{G} - \mathcal{F}\rangle \geq 0, \quad \forall \mathcal{F} \in \Omega, \ \forall \mathcal{A} \in \mathbb{T}^{[N,m]}. \tag{16}$$

For every $\mathcal{A}' \in \mathbb{T}^{[N,m]}$ and for every $\mathcal{G}' \in \Omega$, we set

$$z(t) = \mathcal{Z} + t\left(\mathcal{A}' - \mathcal{Z}\right) \quad \text{and} \quad g(t) = \mathcal{G} + t\left(\mathcal{G}' - \mathcal{G}\right), \quad \forall t \in [0, 1].$$

By using (16), it follows

$$\langle \mathcal{Z} - z(t), f(z(t)) - g(t)\rangle + \langle z(t), \mathcal{G} - g(t)\rangle \geq 0, \quad \forall t \in [0, 1],$$

which leads to

$$\langle \mathcal{Z} - \mathcal{A}', f(z(t)) - g(t)\rangle + \langle z(t), \mathcal{G} - \mathcal{G}'\rangle \geq 0, \quad \forall \mathcal{A}' \in \mathbb{T}^{[N,m]}, \ \forall \mathcal{G}' \in \Omega.$$

Since f is hemicontinuous along line segments, letting $t \to 0^+$ in the above inequality, we obtain

$$\langle \mathcal{Z} - \mathcal{A}', f(\mathcal{Z}) - \mathcal{G}\rangle + \langle \mathcal{Z}, \mathcal{G} - \mathcal{G}'\rangle \geq 0, \quad \forall \mathcal{A}' \in \mathbb{T}^{[N,m]}, \ \forall \mathcal{G}' \in \Omega. \tag{17}$$

For the arbitrarity of $\mathcal{A}'$, by (17) it follows that

$$; s\langle r, f(\mathcal{Z}) - \mathcal{G}\rangle \leq C, \quad \forall s, r \in \mathbb{R}, \tag{18}$$

where C is a constant. By (18), we get $f(\mathcal{Z}) = \mathcal{G}$, so that taking into account (17), we have

$$\langle \mathcal{Z}, f(\mathcal{Z}) - \mathcal{G}'\rangle \geq 0, \quad \forall \mathcal{G}' \in \Omega. \tag{19}$$

Hence, $\mathcal{Z}$ solves $ITVI(\Omega, f)$ and, then, $\mathcal{Z} = \mathcal{A}^*$ for the uniqueness assumption. This is a contradiction with the fact that $(\mathcal{A}^*, f(\mathcal{A}^*)) \neq (\mathcal{Z}, f(\mathcal{Z}))$.

Therefore, we may assume that $\{\mathcal{U}_n\}$ is bounded. Let $\{\mathcal{U}_{n_k}\}$ be any subsequence of $\{\mathcal{U}_n\}$ such that $\mathcal{U}_{n_k} \to (\overline{\mathcal{A}}, \overline{\mathcal{G}})$, as $k \to +\infty$. By using (14) and letting $k \to +\infty$, it results

$$\langle \overline{\mathcal{A}} - \mathcal{A}, f(\mathcal{A}) - \mathcal{F}\rangle + \langle \mathcal{A}, \overline{\mathcal{G}} - \mathcal{F}\rangle \geq 0, \quad \forall \mathcal{A} \in \mathbb{T}^{[N,m]}, \ \forall \mathcal{F} \in \Omega.$$

By the same arguments as in (16)–(19), we have

$$f(\overline{\mathcal{A}}) = \overline{\mathcal{G}} \in \Omega, \quad \langle \overline{\mathcal{A}}, f(\overline{\mathcal{A}}) - \mathcal{F}\rangle \geq 0, \quad \forall \mathcal{F} \in \Omega.$$

Consequently, $\overline{\mathcal{A}}$ is a solution to $ITVI(\Omega, f)$. Since $ITVI(\Omega, f)$ has a unique solution $\mathcal{A}^*$, it follows $\overline{\mathcal{A}} = \mathcal{A}^*$. Therefore, $\{\mathcal{A}_{n_k}\}$ converges to $\mathcal{A}^*$. Thus, $ITVI(\Omega, f)$ is well-posed. □

4.1 Link with the Well-Posedness of Tensor Variational Inequalities

The purpose of the subsection is to show that the well-posedness of an inverse tensor variational inequality is equivalent to the well-posedness of a suitable tensor variational inequality.

Analogously to the inverse tensor variational inequality problem, a sequence $\{\mathcal{A}_n\} \subset \mathbb{T}^{[N,m]}$ is called $\alpha-$approximating sequence for $TVI(K, F)$ if and only if there exists $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, such that $\langle F(\mathcal{A}_n), \mathcal{A}_n - \mathcal{B}\rangle \leq \epsilon_n$, for every $\mathcal{B} \in K$. Moreover, we say that $TVI(K, F)$ is well-posed if and only if $TVI(K, F)$ has a unique solution and every approximating sequence converges to the unique solution. We say that $TVI(K, F)$ is well-posed in the generalized sense if and only if $TVI(K, F)$ has a nonempty solution set S and every approximating sequence has a subsequence which converges to a tensor of S.

Let Ω be a nonempty subset of $\mathbb{T}^{[N,m]}$. Let us set $K = \mathbb{T}^{[N,m]} \times \Omega$ and consider $F([\mathcal{A}, \mathcal{F}]) = [\mathcal{F} - f(\mathcal{A}), -\mathcal{A}]$, for every $[\mathcal{A}, \mathcal{F}] \in K$, where $[\mathcal{A}, \mathcal{F}]$ is the tensor

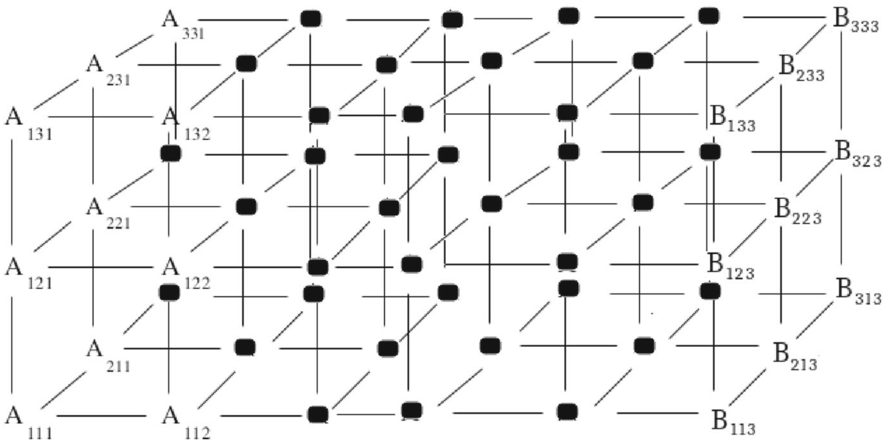

Fig. 1 Example of $[\mathcal{A}, \mathcal{B}]$ for two 3-order tensors

of K obtained by "*gluing*" the tensors $\mathcal{A}$ and $\mathcal{F}$ in a tensorial sense (see Figure 1 for an example of two 3-order tensors). Observe that the space $\left(\mathbb{T}^{[N,m]}\right)^2$ can be identified with $\mathbb{T}^{[N,2m]}$. Hence, the following result holds true.

Theorem 4.4 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$ and let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be a tensor mapping. Then, $\mathcal{A}^* \in \mathbb{T}^{[N,m]}$ is a solution to $ITVI(\Omega, f)$ if and only if $\mathcal{U}^* = \left[\mathcal{A}^*,\ f(\mathcal{A}^*)\right] \in K$ is a solution to $TVI(F, K)$.*

Proof Let $\mathcal{A}^*$ be a solution to $ITVI(\Omega, f)$. It results

$$F(\mathcal{U}^*) = F\left(\left[\mathcal{A}^*,\ f(\mathcal{A}^*)\right]\right) = \left[f(\mathcal{A}^*) - f(\mathcal{A}^*),\ -\mathcal{A}^*\right] = \left[0,\ -\mathcal{A}^*\right].$$

Thus, it follows

$$\langle F(\mathcal{U}^*), [\mathcal{A},\ \mathcal{F}] - \mathcal{U}^*\rangle = \langle\left[0,\ -\mathcal{A}^*\right], \left[\mathcal{A} - \mathcal{A}^*,\ \mathcal{F} - f(\mathcal{A}^*)\right]\rangle \geq 0, \quad \forall [\mathcal{A},\ \mathcal{F}] \in K.$$

Conversely, let $\mathcal{U}^*$ be a solution to $TVI(K, F)$, with analogous computations we deduce that $\mathcal{A}^*$ is a solution to $ITVI(\Omega, f)$. □

Finally, we can establish the following characterizations.

Theorem 4.5 *Let Ω be a closed subset of $\mathbb{T}^{[N,m]}$ and let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be a continuous tensor mapping. Then, the following statements hold:*

1. *$ITVI(\Omega, f)$ is well-posed if and only if $TVI(K, F)$ is well-posed;*
2. *$ITVI(\Omega, f)$ is well-posed in the generalized sense if and only if $TVI(K, F)$ is well-posed in the generalized sense.*

Proof Let us first prove 1. Let $ITVI(\Omega, f)$ be well-posed. Then, $ITVI(\Omega, f)$ has a unique solution $\mathcal{A}^* \in \mathbb{T}^{[N,m]}$. By Theorem 4.4, we have that $\mathcal{U}^* = [\mathcal{A}^*, f(\mathcal{A}^*)]$ is the unique solution to $TVI(K, F)$. Let $\{\mathcal{U}_n\} = \{[\mathcal{A}_n, \mathcal{F}_n]\}$ be an approximating sequence for $TVI(K, F)$. There exists a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, such that $\langle F(\mathcal{U}_n), \mathcal{U}_n - \mathcal{V}\rangle \le \epsilon_n$, $\forall \mathcal{V} = [\mathcal{A}, \mathcal{F}] \in K, \forall n \in \mathbb{N}$. This implies

$$\langle \mathcal{F}_n - f(\mathcal{A}_n), \mathcal{A}_n - \mathcal{A}\rangle \le \langle \mathcal{A}_n, \mathcal{F}_n - \mathcal{F}\rangle + \epsilon_n, \quad \forall \mathcal{A} \in \mathbb{T}^{[N,m]}, \ \forall \mathcal{F} \in \Omega, \ \forall n \in \mathbb{N}.$$

Fix $\mathcal{F} \in \Omega$, $\mathcal{Z} \in \mathbb{T}^{[N,m]}$ and consider $\mathcal{A} = \mathcal{A}_n - s\mathcal{Z}$, then $s\langle \mathcal{F}_n - f(\mathcal{A}_n), \mathcal{Z}\rangle \le C$, where s is arbitrarily chosen and C is a constant. Then, $f(\mathcal{A}_n) = \mathcal{F}_n$ and thus

$$-\langle \mathcal{A}_n, f(\mathcal{A}_n) - \mathcal{F}\rangle = -\langle \mathcal{A}_n, \mathcal{F}_n - \mathcal{F}\rangle \le \epsilon_n, \quad \forall \mathcal{F} \in \Omega, \ \forall n \in \mathbb{N},$$

namely $\{\mathcal{A}_n\} \subset \mathbb{T}^{[N,m]}$ is an approximating sequence for $ITVI(\Omega, f)$. Hence, it results $\mathcal{A}_n \to \mathcal{A}^*$. Therefore, $\mathcal{U}_n = [\mathcal{A}_n, \mathcal{F}_n] \to [\mathcal{A}^*, f(\mathcal{A}^*)]$, as $n \to +\infty$ and, thus, also $TVI(K, F)$ is well-posed.

Conversely, let us assume $TVI(K, F)$ is well-posed. Then, it has a unique solution $\mathcal{U}^* = [\mathcal{A}^*, \mathcal{F}^*]$, with $\mathcal{F}^* = f(\mathcal{A}^*)$. By Theorem 4.4, $\mathcal{A}^*$ is the unique solution to $ITVI(\Omega, f)$. Let $\{\mathcal{A}_n\} \subset \mathbb{T}^{[N,m]}$ be an approximating sequence for $ITVI(\Omega, f)$. Thus, there exists a sequence $\{\epsilon_n\}$, with $\epsilon_n > 0$, for every $n \in \mathbb{N}$, $\epsilon_n \to 0$, as $n \to +\infty$, verifying (13).

Let $\mathcal{F}_n = f(\mathcal{A}_n)$ and $\mathcal{U}_n = [\mathcal{A}_n, \mathcal{F}_n]$, for every $n \in \mathbb{N}$. By (13), it follows that $\mathcal{U}_n \in K$ and

$$\langle F(\mathcal{U}_n), \mathcal{U}_n - \mathcal{V}\rangle = \langle [0, -\mathcal{A}_n], [\mathcal{A}_n - \mathcal{A}, f(\mathcal{A}_n) - \mathcal{F}] \le \epsilon_n,$$
$$\forall \mathcal{V} = [\mathcal{A}, \mathcal{F}] \in K, \ \forall n \in \mathbb{N}.$$

This means that $\{\mathcal{U}_n\}$ is an approximating sequence for $TVI(K, F)$. By its well-posedness, we deduce $\mathcal{U}_n = [\mathcal{A}_n, f(\mathcal{A}_n)] \to [\mathcal{A}^*, f(\mathcal{A}^*)]$, as $n \to +\infty$. Therefore, the sequence $\{\mathcal{A}_n\}$ converges to $\mathcal{A}^*$, as $n \to +\infty$, and so $ITVI(\Omega, f)$ is well-posed.

The second statement follows by adapting the arguments of the previous one. □

5 Tikhonov-Type Regularization Method for Ill-Posed Inverse Tensor Variational Inequalities

This section is devoted to extend the Tikhonov regularization method to the class of inverse tensor variational inequalities. This method allows us to find a solution to the ill-posed $ITVI(\Omega, f)$ as the limit of a sequence of solutions to approximating inverse tensor variational inequalities. Indeed, for $ITVI(\Omega, f)$, we consider the following regularized problem denoted by $ITVI_\epsilon(\Omega, f)$: find $\mathcal{A} \in \mathbb{T}^{[N,m]}$ such that

$$f(\mathcal{A}) \in \Omega, \quad \langle \mathcal{A}, \mathcal{F} - f_\epsilon(\mathcal{A})\rangle \le 0, \quad \forall \mathcal{F} \in \Omega, \tag{20}$$

where $f_\epsilon = f - \epsilon\mathbb{I}$, with $\epsilon > 0$ and $\mathbb{I}$ the identity tensor mapping. From now on, we denote by $S(\Omega, f)$ the solution set of $ITVI(\Omega, f)$ and by $S_\epsilon(\Omega, f)$ the one of $ITVI_\epsilon(\Omega, f)$.

The idea is to find, under which assumptions, the sequence of solutions to $ITVI_\epsilon(\Omega, f)$, for every $\epsilon > 0$, converges to a solution to $ITVI(\Omega, f)$, as $\epsilon \to 0^+$.

To this purpose, we prove that the rather weak coercivity condition assuming in Theorem 3.2 is enough to ensure that the solution set of $ITVI_\epsilon(\Omega, f)$ is nonempty and bounded. Then, we perform the perturbation analysis for the solution set of $ITVI(\Omega, f)$.

Theorem 5.1 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$ and let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be an injective continuous open tensor mapping. Let us suppose that there exists $\overline{n} \in \mathbb{N}$ such that for every $f(\mathcal{A}) \in \Omega \setminus \Omega_{\overline{n}}$ there exists $\mathcal{F}_0 \in \Omega$ with $\|\mathcal{F}_0\| < \|f(\mathcal{A})\|$ satisfying*

$$\langle \mathcal{A}, f(\mathcal{A}) - \mathcal{F}_0 \rangle \leq 0.$$

Then, for every $\epsilon > 0$ we have

1. *$ITVI_\epsilon(\Omega, f)$ has a solution;*
2. *if $f^{-1} : \Omega \to \Omega$ is a bounded tensor mapping, the set $\{S_t(\Omega, f) : t \in (0, \epsilon]\}$ is bounded.*

Proof Let us start proving 1. We aim to show that the assumptions are enough to imply the ones of Theorem 3.2. Indeed,

$$\begin{aligned}\langle \mathcal{A} - \epsilon f(\mathcal{A}), f(\mathcal{A}) - \mathcal{F}_0 \rangle &= \langle \mathcal{A}, f(\mathcal{A}) - \mathcal{F}_0 \rangle - \epsilon \langle f(\mathcal{A}), f(\mathcal{A}) - \mathcal{F}_0 \rangle \\ &= \langle \mathcal{A}, f(\mathcal{A}) - \mathcal{F}_0 \rangle - \epsilon \|f(\mathcal{A})\|^2 + \epsilon \langle f(\mathcal{A}), \mathcal{F}_0 \rangle \\ &\leq -\epsilon \|f(\mathcal{A})\|^2 + \epsilon \|f(\mathcal{A})\| \|\mathcal{F}_0\| \\ &= -\epsilon \|f(\mathcal{A})\| \left(\|f(\mathcal{A})\| - \|\mathcal{F}_0\|\right) \leq 0,\end{aligned}$$

where we have applied the hypothesis and the Cauchy–Schwartz inequality. Then, it follows that $ITVI_\epsilon(\Omega, f)$ has a solution.

Now, let us show 2. Let $t \in (0, \epsilon]$ and $\mathcal{A}_t \in S_t(\Omega, f)$. It is sufficient to show that $f(\mathcal{A}_t) \in \Omega_{\overline{n}}$ since f^{-1} is a bounded mapping. We proceed by contradiction supposing that $f(\mathcal{A}_t) \notin \Omega_{\overline{n}}$ or equivalently $f(\mathcal{A}_t) \in \Omega \setminus \Omega_{\overline{n}}$. Therefore, there exists $\mathcal{F}_0^t \in \Omega$ with $\|\mathcal{F}_0^t\| < \|f(\mathcal{A}_t))\|$ satisfying $\langle \mathcal{A}_t, f(\mathcal{A}_t) - \mathcal{F}_0^t \rangle \leq 0$. Since $\mathcal{A}_t \in S_t(\Omega, f)$ and $\mathcal{F}_0^t \in \Omega$, it results

$$f(\mathcal{A}_t) \in \Omega, \quad \langle \mathcal{A}_t - t f(\mathcal{A}_t), \mathcal{F}_0^t - f(\mathcal{A}_t) \rangle \leq 0.$$

Consequently, it follows

$$t \left[\|f(\mathcal{A}_t)\|^2 - \langle f(\mathcal{A}_t), \mathcal{F}_0^t \rangle \right] \leq \langle \mathcal{A}_t, f(\mathcal{A}_t) - \mathcal{F}_0^t, \rangle \leq 0.$$

Hence, we obtain $\|f(\mathcal{A}_t)\| \leq \|\mathcal{F}_0^t\|$. This is a contradiction. □

Theorem 5.2 *Let Ω be a nonempty closed convex subset of $\mathbb{T}^{[N,m]}$ and let $f : \mathbb{T}^{[N,m]} \to \mathbb{T}^{[N,m]}$ be an injective continuous open tensor mapping. Let us suppose that there exists $\overline{n} \in \mathbb{N}$ such that for every $f(\mathcal{A}) \in \Omega \setminus \Omega_{\overline{n}}$, there exists $\mathcal{F}_0 \in \Omega$ with $\|\mathcal{F}_0\| < \|f(\mathcal{A})\|$ satisfying $\langle \mathcal{A}, f(\mathcal{A}) - \mathcal{F}_0 \rangle \leq 0$. Then, it results*

$$\emptyset \neq \limsup_{\epsilon \to 0^+} S_\epsilon(\Omega, f) \subset S(\Omega, f).$$

Proof By Theorem 5.1, $\emptyset \neq \limsup_{\epsilon \to 0^+} S_\epsilon(\Omega, f)$ follows. Next, we consider $\mathcal{A} \in \limsup_{\epsilon \to 0^+} S_\epsilon(\Omega, f)$ and show that $\mathcal{A} \in S(\Omega, f)$. By definition, there exist a sequence $\{\epsilon_n\}$, with $\epsilon_n \to 0$, and $\mathcal{A}_n \in S_{\epsilon_n}(\Omega, f)$, for every $n \in \mathbb{N}$, such that $\mathcal{A}_n \to \mathcal{A}$, as $n \to +\infty$. This means that

$$f(\mathcal{A}_n) \in \Omega, \quad \langle \mathcal{A}_n - \epsilon_n f(\mathcal{A}_n), \mathcal{F} - f(\mathcal{A}_n) \rangle \leq 0, \quad \forall \mathcal{F} \in \Omega. \tag{21}$$

Since f is continuous, we have that $f(\mathcal{A}_n) \to f(\mathcal{A})$, as $n \to +\infty$. In addition, for the closedness of Ω, it follows $f(\mathcal{A}) \in \Omega$. Hence, for every $\mathcal{F} \in \Omega$, it results

$$\langle \mathcal{A}_n - \epsilon_n f(\mathcal{A}_n), \mathcal{F} - f(\mathcal{A}_n) \rangle = \langle \mathcal{A}_n, \mathcal{F} - f(\mathcal{A}_n) \rangle - \epsilon_n \langle f(\mathcal{A}_n), \mathcal{F} \rangle + \epsilon_n \| f(\mathcal{A}_n) \|^2.$$

Therefore, passing to the limit as $n \to +\infty$, we obtain

$$\lim_{n \to \infty} \langle \mathcal{A}_n - \epsilon_n f(\mathcal{A}_n), \mathcal{F} - f(\mathcal{A}_n) \rangle = \langle \mathcal{A}, \mathcal{F} - f(\mathcal{A}) \rangle.$$

Lastly, making use of (21), we deduce $\langle \mathcal{A}, \mathcal{F} - f(\mathcal{A}) \rangle \leq 0$. Then, the claim is proved. □

6 General Oligopolistic Market Model: the Policymaker's Point of View

The class of inverse tensor variational inequalities we consider has a fundamental role to analyze some economic control equilibrium models. This section aims to show an application of the previous theoretical results to the study of the general oligopolistic market equilibrium problem. More precisely, we first present the firms' point of view and later the policymaker's viewpoint for the problem.

Let us consider m firms P_i, $i = 1, \ldots, m$, and n demand markets Q_j, $j = 1, \ldots, n$, and suppose that every firm produces l different commodities (k denotes a generic commodity). Let x_{ij}^k be the amount of the k-th good that the producer P_i ships to the market Q_j. Let $\mathcal{X} = \left(x_{ij}^k\right)_{ijk} \in \mathbb{R}^{[mnl]}$ be the total shipment strategy. Let p_i^k express the k-th commodity output produced by the firm P_i, $i = 1, \ldots, m$, $k = 1, \ldots, l$. Let q_j^k represent the demand for the k-th commodity of the demand market Q_j, $j = 1, \ldots, n$, $k = 1, \ldots, l$. We assume that both variables x_{ij}^k, p_i^k and q_j^k are nonnegative, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$. Also, we assume that the quantity produced by each firm P_i of the good k must be equal to the commodity shipments of such kind from that firm to all the demand markets. Furthermore, the quantity demanded by each demand market Q_j of the good k must be equal to the commodity shipments of such kind from all the firms to that demand market. Thus, the following feasibility conditions hold:

$$p_i^k = \sum_{j=1}^{n} x_{ij}^k, \quad i = 1, \ldots, m, \ k = 1, \ldots, l,$$

$$q_j^k = \sum_{i=1}^{m} x_{ij}^k, \quad j = 1, \ldots, n, \ k = 1, \ldots, l.$$

Since the transportation vehicles have limited capacity, we need to suppose that there exist two tensors $\underline{\mathcal{X}} = (\underline{x}_{ij}^k) \in \mathbb{R}^{[mnl]}$ and $\overline{\mathcal{X}} = (\overline{x}_{ij}^k) \in \mathbb{R}^{[mnl]}$ such that

$$0 \le \underline{x}_{ij}^k \le x_{ij}^k \le \overline{x}_{ij}^k, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \forall k = 1, \ldots, l.$$

Hence, the feasible set is given by

$$\mathbb{K} = \Big\{ \mathcal{X} \in \mathbb{R}^{[mnl]} : \ 0 \le \underline{x}_{ij}^k \le x_{ij}^k \le \overline{x}_{ij}^k, \\ \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \forall k = 1, \ldots, l \Big\}, \tag{22}$$

which is a bounded closed convex subset of the Hilbert space $\mathbb{R}^{[mnl]}$.

We introduce costs and prizes, as in the list hereafter, assuming that they may depend upon the entire production pattern. More precisely, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$:

- $f_i^k(\mathcal{X})$ is the production cost of the firm P_i for the good k;
- $d_j^k(\mathcal{X})$ is the demand price of the demand market Q_j of the good k;
- $c_{ij}^k(\mathcal{X})$ is the transaction cost between the firm P_i and the demand market Q_j for the good k;
- η_{ij}^k is the resource tax imposed on the firm P_i for the transaction with the demand market Q_j for the good k;
- λ_{ij}^k is the incentive pay imposed on the firm P_i for the transaction with the demand market Q_j for the good k;
- h_{ij}^k is the difference between the supply tax and the incentive pay for the transaction between the firm P_i and the demand market Q_j regarding the good k, namely $h_{ij}^k = \eta_{ij}^k - \lambda_{ij}^k$, hence $\mathcal{H} = (h_{ij}^k) \in \mathbb{R}^{[mnl]}$.

Therefore, the profit v_i of the firm P_i is given by

$$v_i(\mathcal{X}) = \sum_{k=1}^{l} \left[\sum_{j=1}^{n} d_j^k(\mathcal{X}) x_{ij}^k - f_i^k(\mathcal{X}) - \sum_{j=1}^{n} c_{ij}^k(\mathcal{X}) x_{ij}^k - \sum_{j=1}^{n} h_{ij}^k x_{ij}^k, \right], \quad i = 1, \ldots, m,$$

i.e., the difference between the price that each demand market P_i is disposed to pay and the sum of the production costs, the transportation costs and the taxes.

The m firms supply commodities in a noncooperative behavior, i.e., each one tries to maximize its own profit function considered the optimal distribution pattern of the other firms. Thus, a feasible tensor $\mathcal{X}^* \in \mathbb{K}$ is a general oligopolistic market equilibrium distribution from the producers' point of view if and only if, for each $i = 1, \ldots, m$, it results

$$v_i(\mathcal{X}^*) \geq v_i(X_i, \mathcal{X}^*_{-i}), \tag{23}$$

where $\mathcal{X}^*_{-i} = (X^*_1, \ldots, X^*_{i-1}, X^*_{i+1}, \ldots, X^*_m)$ and X_i is a slice of $\mathcal{X}$ of dimension nl. Under suitable assumptions on the profit function, $\mathcal{X}^* \in \mathbb{K}$ is an equilibrium distribution if and only if it satisfies the tensor variational inequality

$$\langle -\nabla_D v(\mathcal{X}^*), \mathcal{X} - \mathcal{X}^* \rangle = -\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \frac{\partial v_i(x^*)}{\partial x^k_{ij}}(x^k_{ij} - x^{k*}_{ij}) \geq 0, \quad \forall x \in \mathbb{K}, \tag{24}$$

([1, Theorem 5.5]).

Now, we change point of view and consider the policymakers' perspective. We first define the optimal regulatory tax $\mathcal{H}^* = (h^{k*}_{ij})$ and, then, we characterize it by means of an inverse tensor variational inequality. As a consequence, the term $\mathcal{H}$ presented above as a fixed parameter, is now considered a variable.

We introduce the feasible state set

$$\Omega = \Big\{ \mathcal{W} \in \mathbb{R}^{[mnl]} : \underline{x}^k_{ij} \leq w^k_{ij} \leq \overline{x}^k_{ij}, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n, \ \forall k = 1, \ldots, l \Big\}, \tag{25}$$

and define the optimal regulatory tax as follows.

Definition 6.1 A tensor $\mathcal{H}^* \in \mathbb{R}^{[mnl]}$ is an optimal regulatory tax if $\mathcal{X}(\mathcal{H}^*) \in \Omega$ and, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, the following conditions hold:

$$x^k_{ij}(\mathcal{H}^*) = \underline{x}^k_{ij} \Rightarrow h^{k*}_{ij} \leq 0, \tag{26}$$

$$\underline{x}^k_{ij} < x^k_{ij}(\mathcal{H}^*) < \overline{x}^k_{ij} \Rightarrow h^{k*}_{ij} = 0, \tag{27}$$

$$x^k_{ij}(\mathcal{H}^*) = \overline{x}^k_{ij} \Rightarrow h^{k*}_{ij} \geq 0. \tag{28}$$

This definition must be interpreted as follows: first of all, the optimal regulatory tax $\mathcal{H}^*$ is such that the corresponding state $\mathcal{X}(\mathcal{H}^*)$ has to satisfy the capacity constraints, namely $\underline{x}^k_{ij} \leq x^k_{ij}(\mathcal{H}^*) \leq \overline{x}^k_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$. Moreover, if $x^k_{ij}(\mathcal{H}^*) = \underline{x}^k_{ij}$, then the exportations have to be encouraged; thus, taxes must be less than or equal to the incentive pays. If $x^k_{ij}(\mathcal{H}^*) = \overline{x}^k_{ij}$, then the exportations have to be reduced; hence, taxes must be greater than or equal to the incentive pays. Finally, if $\underline{x}^k_{ij} < x^k_{ij}(\mathcal{H}^*) < \overline{x}^k_{ij}$, taxes have to be equal to the incentive pays.

Theorem 6.1 *A regulatory tax* $\mathcal{H}^* = (h_{ij}^{k*}) \in \mathbb{R}^{[mnl]}$ *is an optimal regulatory tax if and only if*

$$\mathcal{X}(\mathcal{H}^*) \in \Omega, \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left(w_{ij}^k - x_{ij}^k(\mathcal{H}^*)\right) h_{ij}^{k*} \leq 0, \quad \forall \mathcal{W} \in \Omega. \tag{29}$$

Proof Let $\mathcal{H}^*$ be an optimal regulatory tax. Let us fix $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, and $\mathcal{W} \in \Omega$. In particular $\underline{x}_{ij}^k \leq w_{ij}^k \leq \overline{x}_{ij}^k$. If $x_{ij}^k(\mathcal{H}^*) = \underline{x}_{ij}^k$, by (26) we have $(w_{ij}^k - \underline{x}_{ij}^k(\mathcal{H}^*))(h_{ij}^{k*}) \leq 0$. If $\underline{x}_{ij}^k < x_{ij}^k(\mathcal{H}^*) < \overline{x}_{ij}^k$, by applying (27) we deduce $(w_{ij}^k - x_{ij}^k(\mathcal{H}^*))(h_{ij}^{k*}) = 0$. If $x_{ij}^k(\mathcal{H}^*) = \overline{x}_{ij}^k$ by (28) it follows $(w_{ij}^k - \overline{x}_{ij}^k(\mathcal{H}^*))(h_{ij}^{k*}) \leq 0$. Therefore, for every $i = 1, \ldots, m$, $j = 1, \ldots, n, k = 1, \ldots, l$ and for every $\mathcal{W} \in \Omega$ we obtain $(w_{ij}^k - x_{ij}^k(\mathcal{H}^*))h_{ij}^{k*} \leq 0$. By summing over $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $k = 1, \ldots, l$, we reach (29).

On the other hand, if we assume there exists $\mathcal{H}^* \in \mathbb{R}^{[mnl]}$ such that (29) holds true, then for every $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $k = 1, \ldots, l$ we deduce

$$\left(w_{ij}^k - x_{ij}^k(\mathcal{H}^*)\right) h_{ij}^{k*} \leq 0, \quad \forall \mathcal{W} \in \Omega. \tag{30}$$

Now, let us fix $i = 1, \ldots, m$, $j = 1, \ldots, n, k = 1, \ldots, l$ and $\mathcal{W} \in \Omega$. Hence, $\underline{x}_{ij}^k \leq w_{ij}^k \leq \overline{x}_{ij}^k$. We first show that if $x_{ij}^k(\mathcal{H}^*) = \underline{x}_{ij}^k$ we have $h_{ij}^{k*} \leq 0$. By contradiction, we suppose $h_{ij}^{k*} > 0$. Then, choosing $w_{ij}^k = \overline{x}_{ij}^k$ in (30), it results $(\overline{x}_{ij}^k - \underline{x}_{ij}^k)h_{ij}^{k*} > 0$, which contradicts (30). Secondly, if $x_{ij}^k(\mathcal{H}^*) = \overline{x}_{ij}^k$ we want to prove $h_{ij}^{k*} \geq 0$. By contradiction, we assume $h_{ij}^{k*} < 0$. Hence, choosing $w_{ij}^k = \underline{x}_{ij}^k$ in (30), it follows $(\underline{x}_{ij}^k - \overline{x}_{ij}^k)h_{ij}^{k*} > 0$, which is in contradiction with (30). Lastly, if $\underline{x}_{ij}^k < x_{ij}^k(\mathcal{H}^*) < \overline{x}_{ij}^k$ we can show, by employing the same techniques as the two previous cases, that h_{ij}^{k*} can neither be positive nor negative. □

We denote by $\mathbb{W} = \mathbb{R}^{[mnl]} \times \Omega$ whose elements are the tensors $\mathcal{Z} = [\mathcal{H}, \mathcal{W}]$, where $\mathcal{H} \in \mathbb{R}^{[mnl]}$ and $\mathcal{W} \in \Omega$. We define the tensor mapping $F(\mathcal{Z}) = [\mathcal{W} - \mathcal{X}(\mathcal{H}), -\mathcal{H}]$, for every $\mathcal{Z} \in \mathbb{W}$. Making use of Theorem 4.4, the inverse tensor variational inequality (29) can be equivalently expressed as a classical tensor variational inequality. Precisely, the following result holds.

Theorem 6.2 *The inverse tensor variational inequality (29) is equivalent to the following tensor variational inequality:*

$$\mathcal{Z}^* \in \mathbb{W} : \quad \langle F(\mathcal{Z}^*), \mathcal{Z} - \mathcal{Z}^* \rangle \geq 0, \quad \forall \mathcal{Z} \in \mathbb{W}. \tag{31}$$

7 Numerical Example

In this section, we provide a simple numerical example of the theoretical achievements presented. We consider two firms and two demand markets. Each firm produces two different goods. The feasible set is

$$\mathbb{K} = \left\{ \mathcal{X} \in \mathbb{R}^{[8]} : \ 0 \le x_{ij}^k \le 100, \quad i = 1, 2, \ j = 1, 2, \ k = 1, 2 \right\}.$$

The profit functions of two firms are given by

$$\begin{aligned} v_1(\mathcal{X}) &= (x_{11}^1)^2 - x_{11}^1 x_{21}^2 - x_{11}^1 h_{12}^2 + 2x_{11}^1 + 2h_{11}^1 x_{12}^1 + (x_{12}^1)^2 - 5x_{12}^1, \\ v_2(\mathcal{X}) &= (x_{21}^2)^2 - x_{21}^2 h_{22}^2 + 3x_{21}^2 x_{11}^1 - (x_{22}^2)^2 + x_{22}^2 x_{12}^1 + x_{22}^2 h_{12}^1. \end{aligned}$$

The components of the tensor mapping $\nabla_D v$ different from zero are

$$\begin{aligned} \frac{\partial v_1}{x_{11}^1} &= 2x_{11}^1 - x_{21}^2 - h_{12}^2 + 2, & \frac{\partial v_2}{x_{21}^2} &= 3x_{11}^1 + 2x_{21}^2 - h_{22}^2, \\ \frac{\partial v_1}{x_{12}^1} &= 2h_{11}^1 + 2x_{12}^1 - 5 + x_{22}^2, & \frac{\partial v_2}{x_{22}^2} &= x_{12}^1 - 2x_{22}^2 + h_{12}^1. \end{aligned}$$

It is possible to prove that if the solution to the system

$$\begin{cases} 2x_{11}^1 - x_{21}^2 - h_{12}^2 + 2 = 0 \\ 2h_{11}^1 + 2x_{12}^1 - 5 + x_{22}^2 = 0 \\ 3x_{11}^1 + 2x_{21}^2 - h_{22}^2 = 0 \\ x_{12}^1 - 2x_{22}^2 + h_{12}^1 = 0, \end{cases}$$

belongs to the interior of the feasible set $\mathbb{K}$, then it solves (24). Hence, we get

$$\begin{aligned} x_{11}^{1*} &= \frac{h_{22}^2 + 6h_{12}^2 - 12}{10}, & x_{21}^{2*} &= \frac{h_{12}^2 + h_{22}^2 - 2}{5}, \\ x_{12}^{1*} &= \frac{10 - 4h_{11}^1 + h_{12}^1}{3}, & x_{22}^{2*} &= \frac{5 - 2h_{11}^1 + 2h_{12}^1}{3}. \end{aligned}$$

For the inverse problem, we first consider the set of feasible states

$$\Omega = \left\{ \mathcal{W} \in \mathbb{R}^{[8]} : \ 0 \le w_{ij}^k \le 100, \quad i = 1, 2, \ j = 1, 2, \ k = 1, 2. \right\}.$$

We would like to find a solution to (29). Therefore, set $y_{ij}^k = \omega_{ij}^{k*}, \ i = 1, 2, \ j = 1, 2, \ k = 1, 2,$ we can solve the system

$$\begin{cases} 10\omega_{11}^{1*} - h_{22}^{2*} - 6h_{12}^{2*} + 12 = 0 \\ 3\omega_{12}^{1*} - 10 + 4h_{11}^{1*} - h_{12}^{1*} = 0 \\ 5\omega_{21}^{2*} - h_{12}^{2*} - h_{22}^{2*} + 2 = 0 \\ 3\omega_{22}^{2*} - 5 + 2h_{11}^{1*} - 2h_{12}^{1*} = 0. \end{cases}$$

Then, for a given ω_{ij}^{k*}, we obtain

$$\begin{aligned} h_{11}^{1*} &= \frac{9\omega_{22}^{2*} + 5 - 6\omega_{12}^{1*}}{2}, & h_{12}^{2*} &= 2 - \omega_{21}^{2*} + 2\omega_{11}^{1*}, \\ h_{12}^{1*} &= 6\omega_{22}^{2*} - 3\omega_{12}^{1*}, & h_{22}^{2*} &= 6\omega_{21}^{2*} - 2\omega_{11}^{1*}. \end{aligned}$$

In order to solve (29), we have to consider different cases in which ω_{ij}^{k*} assumes maximal or minimal values. Let us deal here with the case

$$\omega_{i2}^{k*} = 0 \quad \text{and} \quad \omega_{i1}^{k*} = 100, \qquad \text{for } i, k = 1, 2.$$

We obtain that (components different from zero of) the optimal regulatory tax and (components different from zero of) the optimal commodity distribution are

$$\begin{aligned} h_{11}^{1*} &= \frac{5}{2} & & x_{11}^{1*} = \frac{194}{5} \\ h_{12}^{2*} &= 102 & \text{and} \quad & x_{12}^{1*} = \frac{25}{6} \\ h_{22}^{2*} &= 400 & & x_{21}^{2*} = 100 \\ h_{12}^{1*} &= 0 & & x_{22}^{2*} = 0, \end{aligned}$$

respectively. The other cases, obtained varying ω_{ij}^{k*}, $i = 1, 2$, $j = 1, 2$, $k = 1, 2$, and h_{ij}^{k*}, $i = 1, 2$, $j = 1, 2$, $k = 1, 2$, between the capacity constraints, can be treated analogously. Therefore, we find the optimal regulatory tax and the optimal commodity distribution in any case.

8 Conclusions

In this paper, we introduced inverse tensor variational inequalities and analyzed their application to an economic control equilibrium model. We proved some existence and uniqueness results. Moreover, we investigated the well-posedness analysis proving that, under suitable assumptions, the well-posedness of an inverse tensor variational inequality is equivalent to the existence and uniqueness of its solution. We extended also the Tikhonov regularization method to this class of inverse problems when they are ill-posed. Finally, we analyzed the policymaker's point of view for the general oligopolistic market equilibrium problem showing the equivalence between the equilibrium conditions and a suitable inverse tensor variational inequality.

Acknowledgements The first author is funded by the research grant PRIN2017 2017AYM8XW "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods." The second and third authors were partially supported by PRIN2017 2017AYM8XW "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods." The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM.

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