

From Quasi-Dominions to Progress Measures

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We revisit the approaches to the solution of *parity games* based on *progress measures* and show how the notion of *quasi dominions* can be integrated with those approaches. The idea is that, while progress measure based techniques typically focus on one of the two players, little information is gathered on the other player during the solution process. Adding quasi dominions provides additional information on this player that can be leveraged to greatly accelerate convergence to a progress measure. To accommodate quasi dominions, however, non trivial refinements of the approach are necessary. In particular, we need to introduce a novel notion of measure and a new method to prove correctness of the resulting solution technique.

1. Introduction

Parity games are two-player infinite-duration games on graphs, which play a crucial role in various fields of theoretical computer science. These are games played on graphs, whose nodes, called *positions*, are labelled with natural numbers, called *priorities*, and controlled by one of two players: player 0 and player 1. Each player can choose edges, called *moves*, when the game is at one of its positions. The goal of player 1 is to force a play π , namely an infinite path in the underlying graph, such that the maximal priority occurring infinitely often along π is of odd parity. If such a play cannot be enforced by player 1, player 0 wins the game. In this case, player 0 can indeed force a play whose maximal recurring priority is even.

Finding efficient algorithms to solve these games in practice is a core problem in formal verification and reactive synthesis, as it leads to efficient solutions of the model-checking and satisfiability problems of expressive

temporal logics. These algorithms can, indeed, be used as back-ends of satisfiability and model-checking procedures [1–3]. In particular, the solution problem for these games has been proved linear-time interreducible with the model-checking problem for the *modal μ Calculus* [2] and it is closely related to other games of infinite duration, such as *mean payoff* [4,5], *discounted payoff* [6], *simple stochastic* [7], and *energy* [8] games. Parity games are also central to several techniques employed in automata theory [9–12]. Besides the practical importance, parity games are also interesting from a computational complexity point of view, since their solution problem is one of the few inhabitants of the $\text{UPTIME} \cap \text{COUPTIME}$ class [13]. That result improves the $\text{NPTIME} \cap \text{CONPTIME}$ membership [2], which easily follows from the property of *memoryless determinacy* [10,14].

A number of quite different approaches to solve parity games have been proposed in the literature that exhibit quite different characteristics. Typically, the most efficient ones in practice are those based on game decomposition, such as the Recursive Algorithm [15,16], Priority Promotion [17–19], and Tangle Learning [20], which, however, usually suffer from poor worst-case bounds. On the other hand, the approaches based on progress measures [21,22] often lead to good worst-case behaviours [23,24], but typically perform very poorly in practice. The main reason for this inefficiency resides in the fact that those algorithms iteratively explore a space of functions assigning some value, called *measure*, to each position in the game. At each iteration, the measures of some of the positions may increase and when they become stationary for all the positions, a fixpoint is reached and a solution can be extracted from the resulting measures. In order to guarantee correctness, measures are allowed to increase very slowly, which often leads to slow convergence to a solution and makes these approaches less appealing in practical contexts. The slow growth is the result of a uniform measure update policy for one of the two players, specifically player 1, which only allows for a minimal measure increase for each of its positions that must be updated.

In this chapter we show that the update policy can be considerably improved upon, without sacrificing correctness. Instead of relying on the minimal increase policy to ensure soundness of measure updates, we propose an approach that brings *quasi dominions* into the equation. Note that the same idea has been implemented for Mean Payoff Games in [25] where the resulting algorithm proved to be very efficient. Informally, a quasi dominion is a set of positions from which one of the players, say player 0, can win the game as long as the opponent, player 1, chooses not to escape from

that set. As such, the notion is not new and is at the very heart of the Priority Promotion algorithms [26]. The idea is to leverage quasi dominions to justify a larger, but still sound, increase in the measure for positions controlled by player 1. The crucial observation is that player 1 surely loses from any position of a quasi dominion for player 0, unless it can escape that set by taking some exiting move. Therefore, player 1 can safely increase the measure of the escaping position according to the exiting moves chosen, regardless of the fact that the increase may not be minimal. In this way, we are able to skip lower measures and jump directly to measures that would be reached anyway, albeit with a number of iterations that is usually much higher.

The integration of progress measures and quasi dominions, however, requires (i) a richer form of measure, able to encode additional information that allows us to identify quasi dominions in the game, and (ii) a new update algorithm that takes this information into account when increasing the measures.

The main contributions of this chapter can be summarised as follows: (a) a novel solution algorithm for parity games based on the integration of progress measures and quasi dominions; (b) the experimental results showing an improvement on the performance of orders of magnitude *w.r.t.* the classic and quasi-polynomial small progress measure algorithm; (c) the present approach can pave the way to practically efficient quasi-polynomial algorithms based on the integration of succinct progress measures, such as those in [23, 24].

2. Preliminaries

A two-player turn-based *arena* is a tuple $\mathcal{A} = \langle \text{Ps}_0, \text{Ps}_1, Mv \rangle$, with $\text{Ps}_0 \cap \text{Ps}_1 = \emptyset$ and $\text{Ps} \triangleq \text{Ps}_0 \cup \text{Ps}_1$, such that $\langle \text{Ps}, Mv \rangle$ is a finite directed graph without sinks. Ps_0 (*resp.*, Ps_1) is the set of positions of player 0 (*resp.*, 1) and $Mv \subseteq \text{Ps} \times \text{Ps}$ is a left-total relation describing all possible moves. A *path* in $V \subseteq \text{Ps}$ is a finite (possible empty) or infinite sequence $\pi \in \text{Pth}(V)$ of positions in V compatible with the move relation, *i.e.*, $(\pi_i, \pi_{i+1}) \in Mv$, for all $0 \leq i < |\pi| - 1$. The set $\text{FPth}(v)$ contains all the finite (possible empty) paths originating at the position v . For a finite path π , with $\text{lst}(\pi)$ we denote the last position of π . Finally, $\text{SPth}(v, V)$ is the set of simple paths in $\text{FPth}(v)$ that are completely composed of positions in V . Notice that, if v does not belong to V , then we have that $\text{SPth}(v, V) = \{\varepsilon\}$. A positional *strategy* for player $\alpha \in \{0, 1\}$ on $V \subseteq \text{Ps}$ is a function $\sigma_\alpha \in \text{Str}_\alpha(V) \triangleq (V \cap \text{Ps}_\alpha) \rightarrow \text{Ps}$, mapping

each α -position v in V to a position $\sigma_\alpha(v)$ compatible with the move relation, *i.e.*, $(v, \sigma_\alpha(v)) \in Mv$. With $\text{Str}_\alpha(V)$ we denote the set of all α -strategies on V . Given an α -strategy $\sigma_\alpha \in \text{Str}_\alpha(V)$ and a set of positions $U \subseteq \text{Ps}$, the operator $\sigma_\alpha \downarrow U$ restricts σ_α to the positions in $V \cap U$. A *play* in $V \subseteq \text{Ps}$ from a position $v \in V$ *w.r.t.* a pair of strategies $(\sigma_0, \sigma_1) \in \text{Str}_0(V) \times \text{Str}_1(V)$, called $((\sigma_0, \sigma_1), v)$ -*play*, is a path $\pi \in \text{Pth}(V)$ such that $(\pi)_0 = v$ and, for all $0 \leq i < |\pi| - 1$, if $(\pi)_i \in \text{Ps}_0$ then $(\pi)_{i+1} = \sigma_0((\pi)_i)$ else $(\pi)_{i+1} = \sigma_1((\pi)_i)$. The *play function* $\text{play} : (\text{Str}_0(V) \times \text{Str}_1(V)) \times V \rightarrow \text{Pth}(V)$ returns, for each position $v \in V$ and pair of strategies $(\sigma_0, \sigma_1) \in \text{Str}_0(V) \times \text{Str}_1(V)$, the maximal $((\sigma_0, \sigma_1), v)$ -play $\text{play}((\sigma_0, \sigma_1), v)$. A path $\pi \in \text{Pth}(v)$ is called a (σ_α, v) -*play in V*, if $\pi = \text{play}((\sigma_0, \sigma_1), v)$, for some $\sigma_\alpha \in \text{Str}_\alpha(V)$.

A *parity game* is a tuple $\mathcal{D} = \langle \mathcal{A}, \text{Pr}, \text{pr} \rangle \in \text{PG}$, where \mathcal{A} is an arena, $\text{Pr} \subset \mathbb{N}$ is a finite set of priorities, and $\text{pr} : \text{Ps} \rightarrow \text{Pr}$ is a *priority function* assigning a priority to each position. The priority function can be naturally extended to games and paths as follows: $\text{pr}(\mathcal{D}) \triangleq \max_{v \in \text{Ps}} \text{pr}(v)$; for a path $\pi \in \text{Pth}$, we set $\text{pr}(\pi) \triangleq \max_{0 \leq i < |\pi|} \text{pr}((\pi)_i)$, if π is finite, and $\text{pr}(\pi) \triangleq \limsup_{i \in \mathbb{N}} \text{pr}((\pi)_i)$, otherwise. A set of positions $V \subseteq \text{Ps}$ is an α -*dominion*, with $\alpha \in \{0, 1\}$, if there exists an α -strategy $\sigma_\alpha \in \text{Str}_\alpha(V)$ such that, for all $\bar{\alpha}$ -strategies $\sigma_{\bar{\alpha}} \in \text{Str}_{\bar{\alpha}}(V)$ and positions $v \in V$, the induced play $\pi = \text{play}((\sigma_0, \sigma_1), v)$ is infinite and $\text{pr}(\pi) \equiv_2 \alpha$. In other words, σ_α only induces on V infinite plays whose maximal priority visited infinitely often has parity α . By $\mathcal{D} \setminus V$ we denote the maximal subgame of \mathcal{D} with set of positions Ps' contained in $\text{Ps} \setminus V$ and move relation Mv' equal to the restriction of Mv to Ps' .

3. Solving Parity Games via Progress Measures

The abstract notion of *progress measure* [27] has been introduced as a way to encode *global properties* on paths of a graph by means of simpler *local properties* of adjacent vertexes, *i.e.*, of edges. In particular, this notion has been successfully employed in the literature, *e.g.*, for the solution of automata theory [28–32] and program verification [33, 34] problems.

In the context of parity games [21], the graph property of interest, called *parity property*, asserts that, along every path in the graph, the maximal priority occurring infinitely often is of odd parity. More precisely, in game theoretic terms, a *parity progress measure* witnesses the existence of a strategy σ for one of the two players, from now on player 1, such that each path in the graph induced by fixing σ satisfies the desired property, where the graph induced by that strategy is obtained from the game arena by removing all the moves exiting from position owned by player 1, except

those specified by σ itself. A parity progress measure associates with each vertex of the underlying graph a value, called *parity measure* (or simply *measure*, for short), taken from some totally-ordered set. Measures are thus associated with positions in the game and the measure η of a position v can intuitively be interpreted as a local assessment of *how far* v is from satisfying the parity property, with the maximal value $\eta = \top$ denoting failure in the satisfaction of the property for v . More precisely, a progress measure implicitly identifies a strategy σ with the following characteristic: in the graph induced by σ , along every path, measures cannot increase and they strictly decrease when passing through an even-priority position. This ensures that every path eventually gets trapped into a cycle whose maximal priority is odd.

In order to obtain a progress measure, we start from some well-behaved assignment of measures to positions of the game. The local information encoded by these measures is, then, propagated back along the edges of the underlying graph so as to associate with each position the information on the priorities occurring along the plays of some finite length starting at that position. The propagation process is performed by means of a low-level measure-update operator, called *stretch* operator $+$. The operator computes the contribution that a given position v would provide to a given measure η . Consider, for instance, a position v that has an adjacent position u with measure η . Then $\eta + v$ is the measure that v would obtain by choosing the move leading to u , namely, the one obtained by augmenting η with the contribution of (the priority of) position v . When v is an even-priority position, the augmented measure $\eta + v$ strictly increases, moving further away from the parity property.

The process described above terminates when no position can be pushed further away from the property. More specifically, each even position has to strictly dominate the measures obtainable through all, respectively one of, its adjacent positions, depending on whether that position belongs to player 0 or to player 1, respectively. Similarly, each odd position must have measures no lower than those obtainable through all, respectively one of, its adjacent moves, again depending on the owner of the position. When this happens, the positions with measure \top are the ones from which player 0 wins the parity game, while the remaining ones are those from which the opponent can win, by simply forcing plays of non-increasing measures. The measures currently associated with this second set of positions form a progress measure for the game.

Different notions of parity measures have been proposed in the literature,

see, *e.g.*, [21–24]. In this section we introduce an abstract concept of measure space and progress measure. All the progress measure based approaches in the literature, including the one presented in this chapter, can be viewed as instantiations of this abstract schema.

3.1. Measure-Function Spaces

As mentioned above, techniques based on progress measures rely on attaching, at each step of the computation, suitable information to all positions in the game and updating it until a fixpoint is reached. The piece of information associated with every single position is called the current *measure* of that position, whereas the set of all possible measures is called a *measure space*. Such a space is a totally ordered set, with minimum and maximum elements, and provides the two special operations of *truncation* and *stretch* that evaluate and update the measure of a given position *w.r.t.* another position. Intuitively, the truncation operator \downarrow_v disregards the contribution to a given measure that is due to positions with priority lower than that of v along the explored finite plays. The stretch operator $+$, introduced in the previous paragraph, propagates the contribution that the position would provide to a given measure.

These two operators essentially embed the semantics of the parity property into the propagation operation that sits at the basis of the computation of a progress measure. At the abstract level, canonical instances of these operators can be any functions that preserve the maximum element and the order, except for possibly mapping different measures onto the same one. All these requirements are formalised by the following definition.

Definition 3.1: (Measure Space) A *measure space* is a mathematical structure $\mathcal{M} \triangleq \langle \text{Ms}, <, \perp, \top, \downarrow, + \rangle$, whose components enjoy the following properties:

- (1) $\langle \text{Ms}, <, \perp, \top \rangle$ is a *strict total order* with *minimum* and *maximum* on elements referred to as *measures*;
- (2) the function $\downarrow: \text{Ms} \times \text{Ps} \rightarrow \text{Ms}$, called *truncation operator*, maps a measure $\eta \in \text{Ms}$ and a position $v \in \text{Ps}$ to a measure $\eta \downarrow_v \in \text{Ms}$; this operator is *canonical* whenever (i) $\eta = \top$ iff $\eta \downarrow_v = \top$ and (ii) if $\eta \leq \eta^*$ then $\eta \downarrow_v \leq \eta^* \downarrow_v$, for all $\eta^* \in \text{Ms}$;
- (3) the function $+: \text{Ms} \times \text{Ps} \rightarrow \text{Ms}$, called *stretch operator*, maps a measure $\eta \in \text{Ms}$ and a position $v \in \text{Ps}$ to a measure $\eta + v \in \text{Ms}$; this operator is *canonical* whenever (i) $\eta = \top$ iff $\eta + v = \top$ and

(ii) if $\eta \leq \eta^*$ then $\eta + v \leq \eta^* + v$, for all $\eta^* \in \text{Ms}$.

Notice that both operators are canonical if they are *monotone* in their first argument and preserve the distinction between the measure \top and the other measures, in the sense that \top cannot be obtained by truncating or stretching a non-top measure and, *vice versa*, no non- \top measure is derivable by truncating or stretching \top .^a

Given a measure space \mathcal{M} , a *measure function* μ formalises the intuitive association discussed above by mapping each position v in the game to a measure $\mu(v)$ in \mathcal{M} . In addition, the order relation $<$ between measures declared in \mathcal{M} induces a pointwise partial order \sqsubseteq on the set measure functions MF defined in the usual way. This set together with its induced order form what we call a *measure-function space*.

Definition 3.2: (Measure-Function Space) The *measure-function space* induced by a given measure space \mathcal{M} is the *partial order* $\mathcal{F} \triangleq \langle \text{MF}, \sqsubseteq \rangle$, whose components are defined as prescribed in the following:

- (1) $\text{MF} \triangleq \text{Ps} \rightarrow \text{Ms}$ is the set of all functions $\mu \in \text{MF}$, named *measure functions*, mapping each position $v \in \text{Ps}$ to a measure $\mu(v) \in \text{Ms}$;
- (2) for all $\mu_1, \mu_2 \in \text{MF}$, it holds that $\mu_1 \sqsubseteq \mu_2$ *iff* $\mu_1(v) \leq \mu_2(v)$, for all positions $v \in \text{Ps}$.

By taking μ_\perp as the measure function associating measure \perp with every position, the following property of measure-function spaces immediately follows.

Proposition 3.3: *Every measure-function space \mathcal{F} contains a unique minimal element $\mu_\perp \in \text{MF}$.*

^aReaders familiar with the research published in [21, 23] might find it interesting to observe that both the small progress measure [21] and the succinct progress measure [23] algorithms make implicit use of a measure space with a canonical truncation operator, but a non-canonical stretch operator. In more detail, the *d/2-tuples* associated with the positions during an execution form a totally ordered set with minimum and maximum, once extended with the value \top and where the value \perp is identified with the unique all-zero *d/2-tuple*. Moreover, the truncation operator is represented by the function that zeros all components of a *d/2-tuple* with indexes smaller than the priority of the position given as second argument. Finally, the maps over *d/2-tuples* induced by the ternary functions $\text{Prog}(\cdot, \cdot, \cdot)$ [21] and $\text{lift}(\cdot, \cdot, \cdot)$ [23], used in the definition of the function $\text{Lift}(\cdot, \cdot)$ at the core of the algorithms, implement the corresponding stretch operators. Such operators are, however, not canonical, since they can map some of the non- \top measures to \top , failing so to satisfy the *if* direction of Condition 3i of Definition 3.1.

The 0 -denotation (resp., 1 -denotation) of a measure function $\mu \in \text{MF}$ is the set $\|\mu\|_0 \triangleq \{v \in \text{Ps} : \mu(v) \upharpoonright_v = \top\}$ (resp., $\|\mu\|_1 \triangleq \overline{\|\mu\|_0}$) of all positions having maximal (resp., non-maximal) measures associated with them in μ , once truncated. Similarly, the \perp -denotation (resp., $+$ -denotation) of μ is the set $\|\mu\|_\perp \triangleq \{v \in \text{Ps} : \mu(v) \upharpoonright_v = \perp\}$ (resp., $\|\mu\|_+ \triangleq \overline{\|\mu\|_\perp}$) of all positions having minimal (resp., non-minimal) measures.

According to the intuition reported at the beginning of this section, the measure associated with a given position v is meant to encode information about the priorities encountered along the plays starting at that positions. More specifically, each measure contains the information gathered along some finite path and can be obtained by repeatedly applying the stretch operator backwards from the last position of that path. To formalise this intuition, we introduce the notion of measure $\eta(\pi)$ of a finite path π , including the empty one ε , that can be recursively computed via the function $\eta: \text{FPth} \rightarrow \text{MF}$ as follows:

$$\eta(\pi) \triangleq \begin{cases} \perp, & \text{if } \pi = \varepsilon; \\ \eta(\pi') + v, & \text{otherwise.} \end{cases}$$

Where $\pi = v \cdot \pi'$, for some unique $v \in \text{Ps}$ and $\pi' \in \text{FPth}$.

At this point, we can constrain a measure function μ , by requiring the measure $\mu(v)$ of a position v to be witnessed by some finite path π starting at v , i.e., $\mu(v) = \eta(\pi)$. By doing this, we obtain a *ground measure function*.

Definition 3.4: (Ground Measure-Function Space) The *ground measure-function space* induced by a given measure space \mathcal{M} is the *subspace* $\langle \text{GM}, \sqsubseteq \rangle$ of the measure-function space \mathcal{F} , where $\text{GM} \triangleq \{\mu \in \text{MF} : \forall v \in \text{Ps}. \mu(v) \in \text{GMs}(v)\}$ with $\text{GMs}(v) \triangleq \{\eta(\pi) : \pi \in \text{FPth}(v)\}$.

All progress-measure approaches proposed in the literature implicitly work by updating ground measure functions only. The same holds true for the algorithm proposed in the current work, which actually runs on the even more restricted set of *simple measure functions* introduced in Sec. 4.

3.2. (Progress-Measure Functions)

The following definition, which tightly connects the truncation and stretch operators, formalises the essential semantic features of a measure space that are required to provide a meaningful notion of progress measure, as proven in Theorem 3.7.

Definition 3.5: Progress Measure Space A measure space \mathcal{M} is a *progress*

measure space if the following properties hold true, for each measure $\eta \in \text{Ms}$ and position $v \in \text{Ps}$:

- (1) $\eta \upharpoonright_v < (\eta + v) \upharpoonright_v$, if $\eta \upharpoonright_v < \top$ and $\text{pr}(v)$ is even;
- (2) $\eta \upharpoonright_u \leq (\eta + v) \upharpoonright_u$, for all even-priority positions u with $\text{pr}(v) \leq \text{pr}(u)$.

The first condition requires that the contribution to a measure due to an even position v cannot be cancelled out by truncating at v itself and that such contribution is always meaningful, namely strictly increasing. This matches the intuition that even priorities tend to move away from the parity property, therefore increasing the associated measure. The second property, instead, ensures that no lower-priority position v can overcome a higher even priority position u , in the sense that the contribution of v to the measure cannot move closer to the parity property, once the stretched measure is analysed *w.r.t.* u . Technically, this means that the stretch forced by a lower priority position is always viewed as a non-strict improvement by any even position with higher priority, regardless of the parity of the first one.

We can now turn our attention to the notion of *progress measure*. Intuitively, a measure function over a progress measure space is a progress measure if it guarantees that every position with a non- \top measure can progress toward the parity property, namely toward lower measures. In other words, this establishes a type of stability property on the positions of a game according to the following intuition, which takes into account the opposite attitude of the two players. While player 0 tries to push toward higher measures, the opponent will try to keep the measure as low as possible. If player 0 cannot increase the measures of its positions and the opponent is not forced to increase the measures of its own positions, then player 0 cannot prevent player 1 from winning the game from all those positions whose measure did not reach value \top . This corresponds to requiring that player 0 cannot increase the measure of its positions by stretching the measure of any adjacent move, while the opponent can always choose a move whose corresponding stretch does not force the increment of the measure. The following definition precisely formalises this intuitive explanation.

Definition 3.6: (Progress Measure) A measure function $\mu \in \text{MF}$ is a *progress measure* if the following conditions hold, for all positions $v \in \text{Ps}$:

- (1) $\mu(w) + v \leq \mu(v)$, for all adjacents $w \in Mv(v)$ of v , if $v \in \text{Ps}_0$;
- (2) $\mu(w) + v \leq \mu(v)$, for some adjacent $w \in Mv(v)$ of v , if $v \in \text{Ps}_1$.

A 1-strategy $\sigma \in \text{Str}_1$ is μ -coherent if $\mu(\sigma(v)) + v \leq \mu(v)$, for all 1-positions

$v \in \text{Ps}_1$.

Assuming a progress measure space with a canonical truncation operator, the notion of progress measure actually ensures that any play satisfying this progress condition will eventually be trapped in a cycle in which the maximal priority is odd, thereby witnessing a win for player 1. This is established by the following theorem.

Theorem 3.7: (*Progress Measure*) Let $\mu \in \text{MF}$ be a progress measure w.r.t. a progress measure space \mathcal{M} with a canonical truncation operator. Then, $\|\mu\|_1$ is a 1-dominion for which all μ -coherent 1-strategies are 1-winning.

Proof: Let us consider an arbitrary μ -coherent 1-strategy $\sigma_1 \in \text{Str}_1$. All measures $\mu(v)$ of positions $v \in \|\mu\|_1 \cap \text{Ps}_1$ are a progress for v w.r.t. the measures $\mu(\sigma_1(v))$ of their adjacents $\sigma_1(v)$, i.e., formally, $\mu(\sigma_1(v)) + v \leq \mu(v)$. The existence of such a coherent strategy is ensured by the fact that μ is a progress measure. Indeed, by Condition 2 of Definition 3.6, there necessarily exists an adjacent $w^* \in Mv(v)$ of v such that $\mu(w^*) + v \leq \mu(v)$.

It can be shown that σ_1 is a winning strategy for player 1 from all the positions in $\|\mu\|_1$, which implies that $\|\mu\|_1 \subseteq \text{Win}_1$. To do this, consider a 0-strategy $\sigma_0 \in \text{Str}_0$ and the associated play $\pi = \text{play}((\sigma_0, \sigma_1), v)$ starting at a position $v \in \|\mu\|_1$. Assume, by contradiction, that π is won by player 0. Since the game \mathcal{G} is finite and the strategies are memoryless, π must contain a finite simple cycle, and so a finite simple path, and the maximal priority seen infinitely often along it needs to be even. In other words, there exist two natural numbers $h \in \mathbb{N}$ and $k \in \mathbb{N}_+$, such that $(\pi)_h = (\pi)_{h+k}$ and $\text{pr}(\rho) \equiv_2 0$, where $\rho \triangleq ((\pi)_{\geq h})_{< h+k}$ is the simple path named above. Moreover, one can choose the value of the index h in such a way that $\text{pr}((\pi)_h) \geq \text{pr}((\pi)_i)$, for all $i \in \mathbb{N}$ with $h < i < h+k$. Recall that $((\pi)_i, (\pi)_{i+1}) \in Mv$, for all indexes $i \in \mathbb{N}$. Thanks to the two conditions of Definition 3.6 and the notion of play, it holds that

$$\mu((\pi)_{i+1}) + (\pi)_i \leq \mu((\pi)_i).$$

By Item 2ii of Definition 3.1, for all indexes $h \leq i < h+k$, it immediately follows that

$$(\mu((\pi)_{i+1}) + (\pi)_i) \upharpoonright_{(\pi)_h} \leq \mu((\pi)_i) \upharpoonright_{(\pi)_h}. \quad (*)$$

At this point, it is important to observe that μ cannot associate the maximum value \top with position in the play, in particular when restricted to $(\pi)_h$, which means that the entire path is contained into $\|\mu\|_1$. The first

element of the play trivially satisfies such a constraint, as $\mu((\pi)_0) = \mu(v) \neq \top$, since $v \in \|\mu\|_1$. Hence, $\mu((\pi)_0) \upharpoonright_{(\pi)_h} \neq \top$, by Item 2i of Definition 3.1. Now, suppose by contradiction that $\mu((\pi)_{i+1}) \upharpoonright_{(\pi)_h} = \top$, for some index $i \in \mathbb{N}_+$. By Inequality (*) and Item 2 of Definition 3.5, it follows that

$$\top = \mu((\pi)_{i+1}) \upharpoonright_{(\pi)_h} \leq (\mu((\pi)_{i+1}) + (\pi)_i) \upharpoonright_{(\pi)_h} \leq \mu((\pi)_i) \upharpoonright_{(\pi)_h},$$

being $\text{pr}((\pi)_h)$ the maximal priority along the path ρ that is also even. Due to the maximality of \top ensured by Item 1 of Definition 3.1, it obviously follows that $\mu((\pi)_i) \upharpoonright_{(\pi)_h} = \top$ as well. Therefore, by iterating this process until index 0 is reached, one would obtain $\mu((\pi)_0) \upharpoonright_{(\pi)_h} = \top$, which is impossible, as previously observed.

To complete the proof, we can exploit the properties of the progress measure space. Recall that $\mu((\pi)_{h+1}) \upharpoonright_{(\pi)_h} < \top$, thanks to the above observation. Thus, by Item 1 of Definition 3.5 and the fact that $\text{pr}((\pi)_h)$ is an even priority, one can derive that

$$\mu((\pi)_{h+1}) \upharpoonright_{(\pi)_h} < (\mu((\pi)_{h+1}) + (\pi)_h) \upharpoonright_{(\pi)_h}. \quad (<)$$

Moreover, by applying again Item 2 of Definition 3.5 and due to the fact that $\text{pr}((\pi)_h)$ is the maximal priority in the cycle, for all indexes $h < i < h+k$, it holds that

$$\mu((\pi)_{i+1}) \upharpoonright_{(\pi)_h} \leq (\mu((\pi)_{i+1}) + (\pi)_i) \upharpoonright_{(\pi)_h}. \quad (\leq)$$

As a consequence of the transitivity of the order relation between measures, by putting together Inequalities (*), (<), and (\leq), one would therefore obtain

$$\mu((\pi)_{h+k}) \upharpoonright_{(\pi)_h} < \mu((\pi)_h) \upharpoonright_{(\pi)_h}.$$

However, $(\pi)_{h+k} = (\pi)_h$, leading to $\mu((\pi)_h) \upharpoonright_{(\pi)_h} < \mu((\pi)_h) \upharpoonright_{(\pi)_h}$, which is clearly impossible, being $<$ an irreflexive relation. \square

4. Solving Parity Games via Quasi-Dominion Measures

The framework set forth in the previous section is already sufficient to define a sound and complete approach for the solution of parity games, as shown in [21]. Here, however, we shall further refine the measure space in order to accommodate the notion of quasi dominion into the measure functions.

4.1. Quasi-Dominion-Measure Functions

The notion of *quasi dominion* was originally introduced for parity games in [17, 26] and, in a slightly different form, in [35]. Here we provide a somewhat more general version that can be easily integrated with measure functions.

Definition 4.1: (Quasi Dominion) A set of positions $Q \subseteq Ps$ is a *weak quasi 0-dominion* if there exists a 0-strategy $\sigma_0 \in \text{Str}_0(Q)$, called *0-witness for Q*, such that, for all 1-strategies $\sigma_1 \in \text{Str}_1(Q)$ and positions $v \in Q$, if the induced play $\pi = \text{play}((\sigma_0, \sigma_1), v)$ is infinite then $\text{pr}(\pi)$ is even. If the even-parity condition holds also for finite plays, then Q is called *quasi 0-dominion*.

Essentially, a quasi 0-dominion consists in a set Q of positions, starting from which player 0 can force plays whose maximal prefixes contained in Q have even maximal priority. Observe that, in case the maximal prefix contained in Q is infinite, then the play is actually winning for player 0. When the condition holds only for infinite plays, the set is called weak quasi 0-dominion. Clearly, any quasi 0-dominion is also a weak quasi 0-dominion. Moreover, the latter are closed under subsets, while the former are not. It is an immediate consequence of the definition above that all infinite plays induced by the 0-witness, if any, are winning for player 0. This also entails that any subset Q^* of a weak quasi 0-dominion Q , in which the player 0 can remain and from where the opponent cannot escape, is a 0-dominion. Indeed, in such a set of positions, player 0 always has moves that remain in Q^* , while the opponent can only choose moves remaining in Q^* . Hence, any play compatible with the 0-witness for Q that starts in Q^* is infinite and entirely contained in Q^* . We have, so, the following result.

Corollary 4.2: (Quasi Dominion) Let $Q \subseteq Ps$ be a weak quasi 0-dominion, $\sigma_0 \in \text{Str}_0(Q)$ one of its 0-witnesses, and $Q^* \subseteq Q$ a subset such that for all positions $v \in Q^* \cap Ps_0$ it holds $\sigma_0(v) \in Q^*$ and for all positions $v \in Q^* \cap Ps_1$ it holds $Mv(v) \subseteq Q^*$. Then, Q^* is a 0-dominion.

The notion of progress measure introduced in the previous section basically gives us positions that do satisfy the parity property and, thus, are winning for player 1, namely the non- \top positions. This is done by enforcing on the measure space the conditions of Definition 3.5 that formally captures the idea that even priority positions push further away from the parity property. However, the progress measure is an asymmetric notion, centred

around one of the two players, specifically player 1, and does not provide any meaningful information on the other player. More specifically, no estimation on how far player 0 is from winning the game starting in a given position, *i.e.*, from satisfying the dual/even parity property, can be extracted from it.

Quasi dominions, however, are precisely intended to encode the dual information for player 0. In this case, the odd priority positions are those that push further away from satisfying the dual/even parity property. A natural way to embed information about quasi dominions into the measures is, thus, to enforce the dual conditions of Definition 3.5, which leads us to the notion of *regress measure*. Here we constrain the behaviour of the truncation and stretch operators *w.r.t.* the odd positions, instead of the even ones.

Definition 4.3: (Regress Measure Space) A measure space \mathcal{M} is a *regress space* if the following properties hold true, for each measure $\eta \in \text{Ms}$ and position $v \in \text{Ps}$:

- (1) $(\eta + v)\downarrow_v < \eta\downarrow_v$, if $\perp < \eta\downarrow_v < \top$ and $\text{pr}(v)$ is odd;
- (2) $(\eta + v)\downarrow_u \leq \eta\downarrow_u$, for all odd-priority positions u with $\text{pr}(v) \leq \text{pr}(u)$.

We can now define the notion of *regress measure* as the dual of the progress measure.

Definition 4.4: (Regress Measure) A measure function $\mu \in \text{MF}$ is a *regress measure* if the following conditions hold, for all positions $v \in \|\mu\|_+ \setminus \|\mu\|_0$:

- (1) $\mu(v) \leq \mu(w) + v$, for some adjacent $w \in Mv(v)$ of v , if $v \in \text{Ps}_0$;
- (2) $\mu(v) \leq \mu(w) + v$, for all adjacents $w \in Mv(v)$ of v , if $v \in \text{Ps}_1$.

A 0-strategy $\sigma \in \text{Str}_0$ is μ -coherent if $\mu(v) \leq \mu(\sigma(v)) + v$, for all 0-positions $v \in \text{Ps}_0$.

Regress measures are meant to ensure that all the positions whose truncation is neither \perp nor \top form a weak quasi dominion for player 0, as established by the following theorem.

Theorem 4.5: (Regress Measure) Let $\mu \in \text{MF}$ be a regress measure *w.r.t.* a regress measure space \mathcal{M} with a canonical truncation operator. Then, $\|\mu\|_+ \setminus \|\mu\|_0$ is a weak quasi 0-dominion for which all μ -coherent 0-strategies are 0-witnesses, once restricted to $\|\mu\|_+ \setminus \|\mu\|_0$.

Proof: *Mutatis mutandis*, the proof proceeds similarly to the one previously presented for Theorem 3.7.

First of all, let us consider an arbitrary μ -coherent 0-strategy $\sigma_0 \in \text{Str}_0$. All measures $\mu(v)$ of positions $v \in \|\mu\|_0 \cap \text{Ps}_0$ are a regress for v w.r.t. the measures $\mu(\sigma_0(v))$ of their adjacents $\sigma_0(v)$, i.e., formally, $\mu(v) \leq \mu(\sigma_0(v)) + v$. The existence of such a coherent strategy is ensured by the fact that μ is a regress measure. Indeed, by Condition 1 of Definition 4.4, there necessarily exists an adjacent $w^* \in Mv(v)$ of v such that $\mu(v) \leq \mu(w^*) + v$.

To prove that $\mathbb{Q} \triangleq \|\mu\|_+ \setminus \|\mu\|_0$ is a weak quasi 0-dominion with $\sigma_0 \downarrow \mathbb{Q} \in \text{Str}_0(\mathbb{Q})$ as 0-witness, consider a 1-strategy $\sigma_1 \in \text{Str}_1(\mathbb{Q})$ such that the associated play $\pi = \text{play}((\sigma_0 \downarrow \mathbb{Q}, \sigma_1), v)$ starting at a position $v \in \mathbb{Q}$ is infinite. Now, one needs to show that $\text{pr}(\pi)$ is even. Assume by contradiction that this condition on the parity of the priority does not hold. Since the game \mathcal{G} is finite and the strategies are memoryless, π must contain a finite simple cycle, and so a finite simple path, and the maximal priority seen infinitely often along it needs to be odd. In other words, there exist two natural numbers $h \in \mathbb{N}$ and $k \in \mathbb{N}_+$, such that $(\pi)_h = (\pi)_{h+k}$ and $\text{pr}(\rho) \equiv_2 1$, where $\rho \triangleq ((\pi)_{\geq h})_{< h+k}$ is the simple path named above. Moreover, one can choose the value of the index h in such a way that $\text{pr}((\pi)_h) \geq \text{pr}((\pi)_i)$, for all $i \in \mathbb{N}$ with $h < i < h+k$. Recall that $((\pi)_i, (\pi)_{i+1}) \in Mv$, for all indexes $i \in \mathbb{N}$. Thanks to the two conditions of Definition 4.4 and the notion of play, it holds that

$$\mu((\pi)_i) \leq \mu((\pi)_{i+1}) + (\pi)_i.$$

By Item 2ii of Definition 3.1, for all indexes $h \leq i < h+k$, it immediately follows that

$$\mu((\pi)_i) \upharpoonright_{(\pi)_h} \leq (\mu((\pi)_{i+1}) + (\pi)_i) \upharpoonright_{(\pi)_h}. \quad (*)$$

At this point, we can exploit the properties of the regress measure space. By construction, $\pi \in \text{Pth}(\mathbb{Q})$, where we recall that $\mathbb{Q} \triangleq \|\mu\|_+ \setminus \|\mu\|_0$. Thus, observe that both $\mu((\pi)_i) \upharpoonright_{(\pi)_i} \neq \perp$ and $\mu((\pi)_i) \upharpoonright_{(\pi)_i} \neq \top$ hold, for all $i \in \mathbb{N}$. This implies that $\perp < \mu((\pi)_i) \upharpoonright_{(\pi)_i}$ and $\mu((\pi)_i) \upharpoonright_{(\pi)_h} < \top$, thanks to Item 2i of Definition 3.1. By Item 2 of Definition 4.3 and the fact that $\text{pr}((\pi)_h)$ is an odd priority, one can derive that

$$(\mu((\pi)_{h+1}) + (\pi)_h) \upharpoonright_{(\pi)_h} \leq \mu((\pi)_{h+1}) \upharpoonright_{(\pi)_h}.$$

Hence, by Inequality (*) and the above observations, it follows that

$$\perp < \mu((\pi)_h) \upharpoonright_{(\pi)_h} \leq (\mu((\pi)_{h+1}) + (\pi)_h) \upharpoonright_{(\pi)_h} \leq \mu((\pi)_{h+1}) \upharpoonright_{(\pi)_h} < \top,$$

which in turn implies

$$\perp < \mu((\pi)_{h+1}) \upharpoonright_{(\pi)_h} < \top.$$

Now, by Item 1 of Definition 4.3, one can obtain that

$$(\mu((\pi)_{h+1}) + (\pi)_h) \upharpoonright_{(\pi)_h} < \mu((\pi)_{h+1}) \upharpoonright_{(\pi)_h}. \quad (<)$$

Moreover, by applying again Item 2 of Definition 4.3 and due to the fact that $\text{pr}((\pi)_h)$ is the maximal priority in the cycle, for all indexes $h < i < h + k$, it holds that

$$(\mu((\pi)_{i+1}) + (\pi)_i) \upharpoonright_{(\pi)_h} \leq \mu((\pi)_{i+1}) \upharpoonright_{(\pi)_h}. \quad (\leq)$$

As a consequence of the transitivity of the order relation between measures, by putting together Inequalities (*), (<), and (\leq), one would derive

$$\mu((\pi)_h) \upharpoonright_{(\pi)_h} < \mu((\pi)_{h+k}) \upharpoonright_{(\pi)_h}.$$

However, $(\pi)_{h+k} = (\pi)_h$, leading to $\mu((\pi)_h) \upharpoonright_{(\pi)_h} < \mu((\pi)_h) \upharpoonright_{(\pi)_h}$, which is obviously impossible, being $<$ an irreflexive relation. \square

A *quasi-dominion measure* is, then, defined as a regress measure with the additional property that all the positions with measure \top form a 0-dominion, *i.e.*, are indeed winning for player 0.

Definition 4.6: (Quasi-Dominion Measure) A measure function $\mu \in \text{MF}$ is a *quasi-dominion measure* (QDM, for short) if it is a regress measure for which $\|\mu\|_0$ is a 0-dominion. QDM denotes the set of all QDMs.

The following theorem establishes the main property of quasi-dominion measures, namely that the set of non- \perp positions always forms a weak quasi 0-dominion.

Theorem 4.7: (Quasi-Dominion Measure I) Let $\mu \in \text{MF}$ be a quasi-dominion measure w.r.t. a regress measure space \mathcal{M} with a canonical truncation operator. Then, $\|\mu\|_+$ is a weak quasi 0-dominion for which all μ -coherent 0-strategies that are winning on $\|\mu\|_0$ are 0-witnesses, once restricted to $\|\mu\|_+$.

Proof: Let $Q \triangleq \|\mu\|_+$ and $\sigma_0 \in \text{Str}_0$ be an arbitrary μ -coherent 0-strategy that is winning on $\|\mu\|_0$. To prove that Q is a weak quasi 0-dominion with $\sigma_0 \downarrow Q \in \text{Str}_0(Q)$ as 0-witness, consider any 1-strategy $\sigma_1 \in \text{Str}_1(Q)$ such that the play $\pi = \text{play}((\sigma_0 \downarrow Q, \sigma_1), v)$ from position $v \in Q$ is infinite. We need to show that $\text{pr}(\pi)$ is even. The following two different cases may arise, where $H \triangleq \|\mu\|_+ \setminus \|\mu\|_0 \subseteq Q$:

- $[\pi \in \text{Pth}(\mathbb{H})]$. By Theorem 4.5, \mathbb{H} is a weak quasi 0-dominion with $\sigma_0 \downarrow \mathbb{H} = (\sigma_0 \downarrow \mathbb{Q}) \downarrow \mathbb{H}$ as 0-witness. Moreover, π is a $(\sigma_0 \downarrow \mathbb{H}, v)$ -play in \mathbb{H} . Hence, the thesis immediately follows from the definition of weak quasi 0-dominion.
- $[\pi \notin \text{Pth}(\mathbb{H})]$. Since $\pi \notin \text{Pth}(\mathbb{H})$, there clearly exists an index $i \in \mathbb{N}$ such that $(\pi)_{\geq i} \in \text{Pth}(\|\mu\|_0)$. This follows from the fact that $\|\mu\|_0$ is a 0-dominion with σ_0 as a 0-winning strategy, since every play compatible with σ_0 gets necessarily trapped in $\|\mu\|_0$. Moreover, $(\pi)_{\geq i}$ is a $(\sigma_0 \downarrow \|\mu\|_0, (\pi)_i)$ -play in $\|\mu\|_0$. Hence, we immediately obtain the thesis from the definition of 0-dominion, since $\text{pr}(\pi) = \text{pr}((\pi)_{\geq i})$. \square

4.2. Simple-Measure Functions

While the solution of a parity game involves checking the parity property along infinite plays, as the solution algorithm proceeds, the measure of a position v encodes a finite horizon approximation of that condition for v . This approximation is progressively refined during the computation, by exploring longer and longer finite prefixes of the possible plays starting from v . Clearly, any play that contains a cycle is either winning for player 0 or for the opponent. In a sense, the shortest prefix of the play that ends with a repetition of some position already provides all the necessary information to assess the winning player of the entire infinite play. This observation suggests that measures need only encode information of finite simple paths in the game, since those are the only prefixes that need to be extended to obtain finer approximations.

From now on, we shall fix a measure space \mathcal{M} and, thus, the induced measure-function space \mathcal{F} . Given a position $v \in \text{Ps}$ and a set of positions $X \subseteq \text{Ps}$, we introduce the set

$$\text{SMs}(v, X) \triangleq \{\eta(\pi) \in \text{Ms} : \pi \in \text{SPth}(v, X)\} \cup \{\top\}$$

of the *simple measures* of v w.r.t. X . This set contains, besides the measure \top , only those measures induced by the finite simple paths originating at v and composed only of positions in X . It is immediate to observe that $\text{SMs}(v, X) = \{\top, \perp\}$, whenever $v \notin \text{Ps}$, since $\text{SPth}(v, X) = \{\varepsilon\}$.

Proposition 4.8: $\langle \text{SMs}(v, X), <, \perp, \top \rangle$ is a finite strict total order with minimum and maximum, for all positions $v \in \text{Ps}$ and sets of positions $X \subseteq \text{Ps}$.

Following the observations above, *simple measure functions* restrict the possible measures of each position to those induced by finite simple paths contained in the quasi 0-dominion $\|\mu\|_+$.

Definition 4.9: (Simple Measure Function) A measure function $\mu \in \text{MF}$ is a *simple measure* (SM, for short) if $\mu(v) \in \text{SMs}(v, \|\mu\|_+)$, for all positions $v \in \text{Ps}$. SM denotes the set of all SMS.

The next proposition shows that, for simple measures μ , if the truncation of the measure $\mu(v)$ w.r.t. the position v itself is \perp , then the original measure $\mu(v)$ needs to be \perp as well, since (i) $\text{SMs}(v, \|\mu\|_+) = \{\top, \perp\}$ and (ii) $\top \upharpoonright_v = \top$, due to Item 2i of Definition 3.1.

Proposition 4.10: For every simple measure $\mu \in \text{SM}$ and position $v \in \|\mu\|_+$, it holds that $\mu(v) = \perp$.

The measure function μ_\perp is clearly the minimal element w.r.t. \sqsubseteq in the set SM of simple measures.

Proposition 4.11: $\langle \text{SM}, \sqsubseteq \rangle$ is a finite partial order with $\mu_\perp \in \text{SM}$ as unique minimal element.

Putting together Definitions 4.6 and 4.9, we obtain *simple quasi-dominion measures*, which enjoy a stronger property than the one stated in Theorem 4.7.

Theorem 4.12: (Quasi-Dominion Measure II) Let $\mu \in \text{MF}$ be a simple quasi-dominion measure w.r.t. a regress measure space \mathcal{M} with a canonical truncation operator satisfying the equality $\perp \upharpoonright_v = \perp$, for all odd priority positions v . Then, $\|\mu\|_+$ is a quasi 0-dominion for which all μ -coherent 0-strategies that are winning on $\|\mu\|_0$ are 0-witnesses, once restricted to $\|\mu\|_+$.

Proof: By Theorem 4.7, $\mathbb{Q} \triangleq \|\mu\|_+$ is a weak quasi 0-dominion. Therefore, to prove that it is a quasi 0-dominion with $\sigma_0 \downarrow \mathbb{Q} \in \text{Str}_0(\mathbb{Q})$ as 0-witness, for some arbitrary μ -coherent 0-strategy $\sigma_0 \in \text{Str}_0$ that is winning on $\|\mu\|_0$, consider a 1-strategy $\sigma_1 \in \text{Str}_1(\mathbb{Q})$ for which the associated play $\pi = \text{play}((\sigma_0 \downarrow \mathbb{Q}, \sigma_1), v)$ starting at a position $v \in \mathbb{Q}$ is finite. Now, one needs to show that $\text{pr}(\pi)$ is even.

Suppose by contradiction that $\text{pr}(\pi)$ is odd. Then, there exists an index $h \in \mathbb{N}$ with $0 \leq h \leq n \triangleq |\pi| - 1$, such that $\text{pr}((\pi)_h) \equiv_2 1$ and $\text{pr}((\pi)_h) \geq \text{pr}((\pi)_i)$, for all $i \in \mathbb{N}$ with $h \leq i \leq n$. Thanks to Definition 4.4,

Items 2i and 2ii of Definition 3.1, Item 2 of Definition 4.3, and the fact that $\mu((\pi)_h) \upharpoonright_{(\pi)_h} \neq \perp$ and $\text{pr}((\pi)_h)$ is the maximal priority along the finite path $(\pi)_{\geq h}$ that is also odd, one can obtain the following inequality, by applying the same inductive reasoning employed in the second half of the proof of Theorem 4.5:

$$\perp < \mu((\pi)_h) \upharpoonright_{(\pi)_h} \leq \mu((\pi)_n) \upharpoonright_{(\pi)_h}. \quad (*)$$

Since π is finite, either one of the two strategies σ_0 and σ_1 have to terminate in $(\pi)_n$, *i.e.*, we necessarily have that $w^* \in \|\mu\|_{\perp}$, where $w^* \triangleq \sigma_0((\pi)_n)$, if $(\pi)_n \in \text{Ps}_0$, and $w^* \triangleq \sigma_1((\pi)_n)$, otherwise. Hence, by Proposition 4.10, it holds that $\mu(w^*) = \perp$, due to the fact that μ is a simple measure. Moreover, again by Definition 4.4, it holds that

$$\mu((\pi)_n) \leq \mu(w^*) + (\pi)_n = \perp + (\pi)_n,$$

from which, by Item 2ii of Definition 3.1, it follows that

$$\mu((\pi)_n) \upharpoonright_{(\pi)_h} \leq (\perp + (\pi)_n) \upharpoonright_{(\pi)_h}. \quad (\diamond)$$

At this point, by Item 2 of Definition 4.3 and the equality $\perp \upharpoonright_v = \perp$, for the odd-priority position v , one can obtain that

$$(\perp + (\pi)_n) \upharpoonright_{(\pi)_h} \leq \perp \upharpoonright_{(\pi)_h} = \perp. \quad (\perp)$$

Thus, as an immediate consequence of Inequalities (*), (\diamond), and (\perp), one would derive $\perp < \perp$, which is obviously impossible, being $<$ an irreflexive relation. \square

Since μ_{\perp} is the measure function induced by the empty path on each position, the following property is immediate.

Proposition 4.13: *The minimal measure function $\mu_{\perp} \in \text{MF}$ is a simple quasi-dominion measure.*

5. A Concrete Algorithm

This section describes an algorithm that solves any parity game by maintaining and updating a simple quasi-dominion measure function, until it reaches a fixpoint that is both a progress and a quasi-dominion measure. At that point, the results in the previous sections ensure that the winning positions for both players are determined and easily recovered from the final measure by computing the 0- and 1-denotations.

5.1. A Concrete Measure Space

The measures used by the concrete algorithm associate a non-negative integer with each priority in the game, in other words, they are sequences of naturals, one for each priority. For technical reasons, however, we introduce a more general class of concrete measures $\text{CMs} \triangleq \text{Pr} \rightarrow \mathbb{Z}$, whose range also includes the negative integers. This allows us to provide algebraic operations on measures, such as addition and subtraction, which will prove instrumental in the implementation of the basic operators used by the algorithm. More specifically: (i) the null element $\mathbf{0} \in \text{CMs}$ is the distinguished measure such that $(\mathbf{0})(k) \triangleq 0$, for all indexes $k \in \text{Pr}$; (ii) the opposite of a measure and the sum of two measures are defined point-wise: $(-\phi)(k) \triangleq -\phi(k)$ and $(\phi_1 + \phi_2)(k) \triangleq \phi_1(k) + \phi_2(k)$, for all $k \in \text{Pr}$. Recall that, according to Definition 3.1, measures need to be totally ordered. For our concrete measures, we employ an *alternate lexicographic order*, that is, if we interpret a measure as a sequence of integers, with decreasing indexes from left to right, values that are later in the sequence are less important than those that come earlier: as in the standard numeric representation, the left-most integer is the most-significant digit, while the right-most integer is the least-significant one. Moreover, values with even indexes are ordered in the natural way, namely by increasing magnitude, whereas those with odd indexes are ordered in the opposite fashion, *i.e.*, by decreasing magnitude. Formally, $< \subseteq \text{CMs} \times \text{CMs}$ is the strict total order defined as follows: $\phi_1 < \phi_2$ if there exists an index $k \in \text{Pr}$ such that (i) k is the greatest index for which $\phi_1(k) \neq \phi_2(k)$ and (ii) $\phi_1(k) < \phi_2(k)$, if k is even, and $\phi_2(k) < \phi_1(k)$, otherwise. A special family of measures is given by the Kronecker delta $\delta: \text{Pr} \rightarrow (\text{Pr} \rightarrow \{0, 1\})$, where $\delta_i(i) \triangleq 1$, and $\delta_i(j) \triangleq 0$, for all $j \neq i$. Obviously, $\delta_i \in \text{CMs}$, for every index $i \in \text{Pr}$. It is quite immediate to show the following property.

Proposition 5.1: *The structure $\mathcal{C} \triangleq \langle \text{CMs}, <, \mathbf{0}, -, + \rangle$ is a totally-ordered Abelian group.*

We can now define the measure space used by the concrete algorithm. It suffices to restrict the measures to only assign non-negative values to the existing priorities in the game, and then define the appropriate canonical truncation and stretch operators. Let CMs^+ denote the set of all the concrete measures $\phi \in \text{CMs}$ with the following two properties: (i) the highest priority k for which $\phi(k) > 0$ is even; (ii) $\phi(k) \geq 0$, for all $k \in \text{Pr}$.

Definition 5.2: (Concrete Measure Structure) The *concrete measure structure* is the tuple $\mathcal{M} \triangleq \langle \text{Ms}, <, \perp, \top, \uparrow, + \rangle$, whose components are defined as

follows:

- (1) $\text{Ms} \triangleq \text{CMs}^+ \cup \{\top\}$, where \top is a distinguished fresh element and $\perp \triangleq \mathbf{0}$;
- (2) $< \subseteq \text{Ms} \times \text{Ms}$ is the order on CMs^+ extended with $\eta < \top$, for every measure $\eta \in \text{CMs}^+$;
- (3) \upharpoonright : $\text{Ms} \times \text{Ps} \rightarrow \text{Ms}$ is the operator such that, for all positions $v \in \text{Ps}$, the following holds: (i) $\top \upharpoonright_v \triangleq \top$; (ii) $(\eta \upharpoonright_v)(p) \triangleq \eta(p)$, if $p \geq \text{pr}(v)$, and $(\eta \upharpoonright_v)(p) \triangleq 0$, otherwise, for all $\eta \in \text{CMs}^+$ and $p \in \text{Pr}$;
- (4) $+$: $\text{Ms} \times \text{Ps} \rightarrow \text{Ms}$ is the operator such that $\eta + v \triangleq \max\{0, \eta + \delta_{\text{pr}(v)}\}$, for all positions $v \in \text{Ps}$.

Truncating a measure *w.r.t.* a position v consists in setting to zero the value associated with all priorities smaller than the priority of v . Stretching a measure *w.r.t.* a position v , instead, means incrementing the value associated with the priority of v , unless the result is lower than 0, in which case the stretch is set to 0, so as to enforce it to belong to CMs^+ . For example, consider a game with priorities from 0 to 4 and the three measures $\eta_1 \triangleq (0, 0, 1, 0, 1)$, $\eta_2 \triangleq (0, 0, 1, 1, 0)$, and $\eta_3 \triangleq (0, 0, 1, 1, 1)$. Then, we have $\eta_2 < \eta_3 < \eta_1$. Indeed, 0 is the greatest priority in which η_2 and η_3 differ, it is even, and $\eta_2(0) = 0 < 1 = \eta_3(0)$. Moreover, 1 is the greatest priority in which both η_2 and η_3 differ from η_1 , it is odd, and $\eta_2(1) = \eta_3(1) = 1 > 0 = \eta_1(1)$. By truncating the three measures at a position v with priority 1, we obtain $\eta_2 \upharpoonright_v = \eta_3 \upharpoonright_v = (0, 0, 1, 1, 0) < \eta_1 \upharpoonright_v = (0, 0, 1, 0, 0)$. We also have $\eta_1 + u = \eta_2 + u = \eta_3 + u = \perp$, if u is a position with priority 3 and $(\eta_2 + w) + u = (1, 1, 1, 1, 0) < (1, 0, 1, 2, 0) = (\eta_2 + w) + v$, if w is a position with priority 4.

The following straightforward result states that the concrete measure structure satisfies all the desired properties introduced in the previous sections, plus some additional ones. Specifically, the truncation operator preserves the \perp measure, the truncation of a non- \perp stretch cannot lead to the \perp measure, and the stretch operator commutes with the addition on measures.

Proposition 5.3: *The concrete measure structure is a progress and regress measure space, whose restriction and stretch operators are canonical. Moreover, the following properties hold true, for all positions $v \in \text{Ps}$, measures $\eta \in \text{Ms}$, and evaluations $\phi \in \text{CMs}$: (i) $\perp \upharpoonright_v = \perp$; (ii) if $\eta + v \neq \perp$ then $(\eta + v) \upharpoonright_v \neq \perp$; (iii) if $\eta + v \neq \perp$ then $(\eta + v) + \phi = (\eta + \phi) + v$.*

5.2. The Solution Algorithm

The algorithm we now propose, which makes implicit use of the concrete measure structure just introduced, is based on repeatedly applying two *progress operators*, prg_+ and prg_\perp , to an initial quasi-dominion measure function μ , until a fixpoint μ^* is reached. At that point, the positions whose measure is \top in μ^* are winning for player 0, while all other positions are winning for player 1.

Recall that a function f on an ordered set is *inflationary* if for all elements x of its domain it holds $x \leq f(x)$. We show in the following that both progress operators are inflationary, so that the fixpoint μ^* is the limit of the ascending sequence of measures obtained by repeated application of those progress operators (*a.k.a.* the *inflationary fixpoint*). We denote by $\text{sol}(\mu)$ such limit, as a partial mapping from MF to MF, when starting from measure μ :

$$\text{sol} \triangleq \text{ifp } \mu . \text{prg}_+(\text{prg}_\perp(\mu)) : \text{MF} \rightarrow \text{MF}.$$

Intuitively, all positions with a non- \perp measure in μ , *i.e.*, $Q \triangleq \|\mu\|_+$, form a quasi 0-dominion and the prg_+ (*resp.*, prg_\perp) operator is responsible for enforcing the progress condition on the positions inside (*resp.*, outside) Q that do not satisfy the proper inequalities between the measures along the moves. This is done in such a way to preserve the properties of quasi-dominion measure function and represents the main point where the classic progress-measure approaches and the proposed technique diverge.

Both operators prg_+ and prg_\perp internally employ a *lift operator* $\text{lift} : \text{MF} \times 2^{\text{Ps}} \times 2^{\text{Ps}} \rightarrow \text{MF}$, that adjusts a measure function so that it locally satisfies the conditions of both the progress and regress measures (Definitions 3.6 and 4.4). The two operators need to selectively adjust the measure of specific sets of positions. For this reason, besides the current measure function, the lift operator carries two additional arguments: (i) the set of positions S whose measure we want to update, and (ii) the set of successor positions T that the update must be based on. Formally, we obtain the following definition:

$$\text{lift}(\mu, S, T)(v) \triangleq \begin{cases} \max\{\mu(w) + v : w \in Mv(v) \cap T\}, & \text{if } v \in S \cap \text{Ps}_0; \\ \min\{\mu(w) + v : w \in Mv(v) \cap T\}, & \text{if } v \in S \cap \text{Ps}_1; \\ \mu(v), & \text{otherwise.} \end{cases}$$

The prg_\perp operator is tasked with adjusting the measure of the positions that currently have the minimal measure \perp . After the update, some of them will acquire a positive measure, thus entering into the current quasi

0-dominion. From an operational viewpoint, prg_\perp consists of a single call to lift:

$$\text{prg}_\perp(\mu) \triangleq \text{lift}(\mu, \|\mu\|_\perp, \text{Ps}) : \text{MF} \rightarrow \text{MF}.$$

Applying the definitions, one can easily see that prg_\perp raises the measure of those positions such that: (i) they have the minimal measure \perp , (ii) they either belong to player 0 and have an adjacent with positive measure or belong to player 1 and have all adjacents with positive measures, and (iii) the stretch of the adjacent measures is greater than \perp .

The following lemma states the main properties of interest for the prg_\perp operator, where we assume $\text{MF}_\perp \triangleq \{\mu \in \text{MF} : \forall v \in \|\mu\|_\perp. \mu(v) = \perp\}$ and $\text{QDM}_\perp \triangleq \text{QDM} \cap \text{MF}_\perp$. Observe that, by Proposition 4.10, it holds that $\text{SM} \subseteq \text{MF}_\perp$.

Lemma 5.4: *The progress operator prg_\perp enjoys the following properties: (i) it is an inflationary function from MF_\perp to MF_\perp ; (ii) it maps QDM_\perp into QDM_\perp ; (iii) it maps SM into SM ; (iv) every fixpoint $\mu \in \text{MF}$ of prg_\perp is a progress measure over $\|\mu\|_\perp$.*

Proof: We analyse the four properties separately, where we recall that $\text{prg}_\perp(\mu) = \text{lift}(\mu, \|\mu\|_\perp, \text{Ps})$.

- **[i].** Let v be a position such that $\mu^*(v) \neq \mu(v)$, where $\mu^* \triangleq \text{prg}_\perp(\mu)$. By definition of the lift operator, it holds that $v \in \|\mu\|_\perp$. Obviously, $\mu(v) = \perp$, since $\mu \in \text{MF}_\perp$. Thus, by Item 1 of Definition 3.1, it follows that $\mu(v) = \perp < \mu^*(v)$, being \perp the minimal measure. Hence, $\mu \sqsubseteq \mu^*$, due to the arbitrary choice of v , which means that prg_\perp is inflationary on MF_\perp , as required by the lemma statement. In addition, it holds that $\mu^* \in \text{MF}_\perp$. Indeed, again by definition of the lift operator, there exists an adjacent $w \in Mv(v)$ of v such that $\mu^*(v) = \mu(w) + v$. If v has even priority, then either $\mu(w)\upharpoonright_v = \top$ and, so, $\mu^*(v)\upharpoonright_v = \top$, by Item 1 of Definition 3.1 and Item 2 of Definition 3.5, or $\mu(w)\upharpoonright_v < \top$ and $\mu^*(v)\upharpoonright_v = (\mu(w) + v)\upharpoonright_v > \mu(w)\upharpoonright_v$, by Item 1 of Definition 3.5. If v has odd priority, instead, by Item ii of Proposition 5.3, it holds that if $\mu(w) + v \neq \perp$ then $(\mu(w) + v)\upharpoonright_v \neq \perp$, from which it follows that $\mu^*(v)\upharpoonright_v = (\mu(w) + v)\upharpoonright_v \neq \perp$, since $\mu(w) + v = \mu^*(v) \neq \mu(v) = \perp$. Thus, in all cases, $\mu^*(v)\upharpoonright_v \neq \perp$, i.e., $v \notin \|\mu^*\|_\perp$, which vacuously satisfies the definitional requirement of MF_\perp .

- **[ii]**. Let μ be a QDM in MF_\perp . Thanks to the above item, it suffices to prove that $\mu^* \triangleq \text{prg}_\perp(\mu)$ is a QDM as well. To do this, one first needs to show that μ^* is a regress measure. Consider an arbitrary position $v \in \|\mu^*\|_+ \setminus \|\mu^*\|_0$. Then, two cases may arise.
 - $[v \in \|\mu\|_+]$. $\mu^*(v) = \mu(v)$, so, $v \in \|\mu\|_+ \setminus \|\mu\|_0$. Since μ is a quasi-dominion measure and, therefore, also a regress measure, it satisfies both conditions of Definition 4.4 at v . It is quite immediate to prove that the same holds for μ^* at the same position v as well, thanks to Item 3ii of Definition 3.1, since $\mu \sqsubseteq \mu^*$ as proved in Item i above.
 - $[v \notin \|\mu\|_+]$. In this case, one needs to analyse the following two subcases. If $v \in \text{Ps}_0$, it holds that

$$\begin{aligned} \mu^*(v) &= \max\{\mu(w) + v : w \in Mv(v)\} \\ &\leq \max\{\mu^*(w) + v : w \in Mv(v)\} = \mu^*(w) + v \end{aligned}$$

for some $w \in Mv(v)$ adjacent to v . Thus, Condition 1 of Definition 4.4 is satisfied for μ^* . If $v \in \text{Ps}_1$, instead, it holds that $\mu^*(v) = \min\{\mu(w) + v : w \in Mv(v)\} \leq \min\{\mu^*(w) + v : w \in Mv(v)\} \leq \mu^*(w) + v$, for all $w \in Mv(v)$ adjacent to v . Hence, also Condition 2 of Definition 4.4 is satisfied for μ^* . Observe that, to prove both conditions, we applied again Item 3ii of Definition 3.1 and the fact that $\mu \sqsubseteq \mu^*$, as proved in Item i above.

At this point, it only remains to prove that $\|\mu^*\|_0$ is a 0-dominion. By hypothesis, it is known that $\|\mu\|_0$ is a 0-dominion, being μ a QDM. Therefore, let us consider a position $v \in \|\mu^*\|_0 \setminus \|\mu\|_0$. We can show that, again by definition of the lift operator, there exists an adjacent $w \in Mv(v)$ of v , such that $w \in \|\mu\|_0$, if $v \in \text{Ps}_0$, and all adjacents $w \in Mv(v)$ of v satisfy $w \in \|\mu\|_0$, otherwise. Indeed, if $v \in \text{Ps}_0$, there exists an adjacent $w \in Mv(v)$ of v , such that $\mu(w) + v = \mu^*(v) = \top$. This implies that $\mu(w) = \top$, due to Item 3i of Definition 3.1. Hence, $w \in \|\mu\|_0$. Similarly, if $v \in \text{Ps}_1$, all adjacents $w \in Mv(v)$ of v satisfy the equality $\mu(w) + v = \mu^*(v) = \top$. Thus, again by Item 3i of Definition 3.1, it holds that $\mu(w) = \top$, which means that $w \in \|\mu\|_0$. As an immediate consequence, every play starting at v and compatible with the 0-winning strategy on $\|\mu\|_0$, suitably extended to $\|\mu^*\|_0$, is won by player 0. Therefore, $\|\mu^*\|_0$ is necessarily a 0-dominion, as required by the definition of QDM.

- **[iii]**. Let μ be a SM and $\mu^* \triangleq \text{prg}_\perp(\mu)$ the result of the prg_\perp operator. One needs to prove that the latter is a SM too. To do this, let us focus on a position v , such that $\mu^*(v) \neq \mu(v)$. If $\mu^*(v) = \top$, there is nothing more to show, as $\top \in \text{SMs}(v, \|\mu^*\|_+)$, as required by Definition 4.9. Therefore, assume $\mu^*(v) \neq \top$. By definition of the lift operator, there exists an adjacent $w \in Mv(v)$ of v , such that $\mu^*(v) = \mu(w) + v$. Now, by Item 3i of Definition 3.1, it follows that $\mu(w) \neq \top$. Thus, thanks to the fact that μ is a SM, it holds that $\mu(w) \in \text{SMs}(w, \|\mu\|_+)$, which means that there exists a simple path $\pi \in \text{SPth}(w, \|\mu\|_+)$, such that $\mu(w) = \eta(\pi)$. Obviously, $\mu^*(v) = \mu(w) + v = \eta(\pi) + v = \eta(v \cdot \pi)$. Moreover, $v \cdot \pi$ is a simple path passing through positions in $\{v\} \cup \|\mu\|_+$, *i.e.*, $v \cdot \pi \in \text{SPth}(v, \{v\} \cup \|\mu\|_+)$, since $v \notin \|\mu\|_+$. As shown at the end of the proof of the first item of this lemma, $\mu^*(v) \upharpoonright_v \neq \perp$, so, $v \in \|\mu^*\|_+$. Thus, as an immediate consequence, one obtains that $v \cdot \pi \in \text{SPth}(v, \|\mu^*\|_+)$, being $\{v\} \cup \|\mu\|_+ \subseteq \|\mu^*\|_+$, which implies that $\mu^*(v) \in \text{SMs}(v, \|\mu^*\|_+)$. Hence, μ^* is a SM.
- **[iv]**. Let μ be a fixpoint of prg_\perp , *i.e.*, $\mu = \text{lift}(\mu, \|\mu\|_\perp, \text{Ps})$, and $v \in \|\mu\|_\perp$ an arbitrary position. If $v \in \text{Ps}_0$, it holds that $\mu(w) + v \leq \max\{\mu(w) + v : w \in Mv(v)\} = \mu(v)$, for all adjacent $w \in Mv(v)$ of v . Thus, Condition 1 of Definition 3.6 is satisfied on $\|\mu\|_\perp$. If $v \in \text{Ps}_1$, instead, it holds that $\mu(w) + v = \min\{\mu(w) + v : w \in Mv(v)\} = \mu(v)$, for some adjacent $w \in Mv(v)$ of v . Hence, Condition 2 of Definition 3.6 is satisfied on $\|\mu\|_\perp$ as well. \square

Algorithm 1: Operator prg_+

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signature  $\text{prg}_+ : \text{MF} \rightarrow \text{MF}$ 
function  $\text{prg}_+(\mu)$ 
1   Q  $\leftarrow \|\mu\|_+$ 
2   while  $\text{esc}(\mu, Q) \neq \emptyset$  do
3     E  $\leftarrow \text{bep}(\mu, Q)$ 
4      $\mu \leftarrow \text{lift}(\mu, E, \overline{Q})$ 
5     Q  $\leftarrow Q \setminus E$ 
6    $\mu \leftarrow \mu[Q \mapsto \top]$ 
7   return  $\mu$ 

```

We now turn our attention to the progress operator prg_+ , whose pseudo-code is reported in Algorithm 1. Besides the lift function, this operator employs two other functions, $\text{esc} : \text{MF} \times 2^{\text{Ps}} \rightarrow 2^{\text{Ps}}$ and $\text{bep} : \text{MF} \times 2^{\text{Ps}} \times \text{Ps} \rightarrow$

CMs, called, respectively, *escape function* and *best-escape position function*. Given a set of positions Q , the escape function collects the subset of positions in Q from which their owner wants or is forced to exit from Q , according to their objective. Specifically, those are (i) the 1-positions having a successor outside Q and (ii) the 0-positions v , such that none of their successors w belonging to Q support the measure of v in the current measure function. Formally:

$$\begin{aligned} \text{esc}(\mu, Q) \triangleq & \{v \in Q \cap \text{Ps}_1 : Mv(v) \setminus Q \neq \emptyset\} \\ & \cup \\ & \{v \in Q \cap \text{Ps}_0 : \forall w \in Mv(v) \cap Q. \mu(w) + v < \mu(v)\}. \end{aligned}$$

All positions belonging to Q must be lifted during the execution of prg_+ . However, they must be lifted in the appropriate order: namely, the first positions to be lifted are those whose measure will rise the least. This is the role of the bep function and its supporting *best escape forfeit function* bef . The bep simply collects the positions v having minimal *forfeit* $\text{bef}(\mu, Q, v)$ (defined below).

$$\text{bep}(\mu, Q) \triangleq \underset{v \in Q}{\text{argmin}} \text{bef}(\mu, Q, v) : \text{MF} \times 2^{\text{Ps}} \rightarrow 2^{\text{Ps}}.$$

Assuming that v is a position in the escape set $\text{esc}(\mu, Q)$, its forfeit is the difference between the measure that v would acquire if lifted and its current measure. In the following, we use the difference operation $\phi_1 - \phi_2 \triangleq \phi_1 + (-\phi_2)$ defined as usual.

$$\text{bef}(\mu, Q, v) \triangleq \begin{cases} \max\{(\mu(w) + v) - \mu(v) : w \in Mv(v) \setminus Q\}, & \text{if } v \in \text{esc}(\mu, Q) \cap \text{Ps}_0; \\ \min\{(\mu(w) + v) - \mu(v) : w \in Mv(v) \setminus Q\}, & \text{if } v \in \text{esc}(\mu, Q) \cap \text{Ps}_1; \\ \top, & \text{otherwise.} \end{cases}$$

Notice that the following inclusions are an immediate consequence of the definitions: $\text{bep}(\mu, Q) \subseteq \text{esc}(\mu, Q) \subseteq Q$.

We can now describe Algorithm 1, which computes operator prg_+ . The purpose of this operator is to enforce the progress condition on the positions in Q , by lifting their value. Line 1 identifies Q as the set of positions with non-minimal measures. If Q is closed (condition at Line 2), namely the escape set of Q is empty, then every position in Q is lifted to value \top (Line 6). Indeed, by Theorem 4.7, Q is a weak quasi 0-dominion and any strategy

compatible with μ is a possible 0-witness. The emptiness of $\text{esc}(\mu, Q)$ implies that player 1 cannot exit from Q and player 0 has a strategy compatible with μ to remain in the set. Hence, by Corollary 4.2, Q is a dominion for player 0. Otherwise, if Q is not closed, Line 3 collects the best escape forfeit set of positions in E . This step ensures that the positions are lifted in increasing order of forfeit. Once the positions in E have been lifted, Lines 4 and 5 remove them from Q , and the function iterates Steps 2-5 until Q is either empty or closed. The following lemma is the core result of this section and proves the key properties of the progress operator prg_+ .

Lemma 5.5: *The progress operator prg_+ enjoys the following properties: (i) it maps MF_\perp into MF_\perp ; (ii) it is an inflationary function from QDM to QDM; (iii) it maps SM into SM; (iv) every fixpoint $\mu \in \text{MF}$ of prg_+ is a progress measure over $\|\mu\|_+$.*

Proof: Let us assume $\|\mu\|_+ \neq \emptyset$, since there is nothing to prove, otherwise, being $\mu^* \triangleq \text{prg}_+(\mu) = \mu$, and consider the three (potentially) infinite sequences $Q_0, Q_1, \dots, E_0, E_1, \dots$, and μ_0, μ_1, \dots generated by Algorithm 1, which explicitly implements the progress operator prg_+ . These sequences are defined as follows: (i) $Q_0 \triangleq \|\mu\|_+$ and $\mu_0 \triangleq \mu$; (ii) $Q_{i+1} \triangleq Q_i \setminus E_i$ and $\mu_{i+1} = \text{lift}(\mu_i, E_i, \overline{Q_i})$, where $E_i \triangleq \text{bep}(\mu_i, Q_i) \subseteq \text{esc}(\mu_i, Q_i)$, for all $i \in \mathbb{N}$. Since $|Q_0| < \infty$ and $Q_{i+1} \subseteq Q_i$, there necessarily exists an index $k \in \mathbb{N}$, such that $Q_{k+1} = Q_k$, $\mu_{k+1} = \mu_k$, $E_k = \emptyset$, and $E_j \neq \emptyset$, for all $j < k$. Moreover, observe that $\mu^* = \mu_k[Q_k \mapsto \top]$. At this point, we analyse the four properties separately.

- **[i].** To prove that $\mu^* \in \text{MF}_\perp$, whenever $\mu \in \text{MF}_\perp$, one can focus on those positions v that changed their measure from μ to μ^* , *i.e.*, such that $\mu^*(v) \neq \mu(v)$. If $v \in \|\mu^*\|_+$, there is nothing to prove, as v vacuously satisfies the definitional requirement of MF_\perp . If $v \notin \|\mu^*\|_+$, instead, it holds that $\mu^*(v) \upharpoonright_v = \perp$. Due to the fact that the position changed its measure, there is an index $i \in [0, k]$, such that $v \in E_i$ and, so, $\mu^*(v) = \mu_{i+1}(v)$. Therefore, by definition of the lift operator, there exists an adjacent $w \in Mv(v) \setminus Q_i$, such that $\mu_{i+1}(v) = \mu_i(w) + v$. At this point, we can observe that v has odd priority. Indeed, if, by contradiction, v has even priority, by Item 1 of Definition 3.1 and Item 2 of Definition 3.5, one would have $\mu_i(w) \upharpoonright_v = \perp \neq \top$, since $\mu_i(w) \upharpoonright_v \leq (\mu_i(w) + v) \upharpoonright_v = \mu^*(v) \upharpoonright_v = \perp$, which would in turn imply $\perp = \mu_i(w) \upharpoonright_v < (\mu_i(w) + v) \upharpoonright_v = \mu^*(v) \upharpoonright_v = \perp$, due to Item 1 of the same definition, which is obviously impossible,

being $<$ an irreflexive relation. Thus, as a consequence of Item **ii** of Proposition 5.3, $\mu_i(w) + v = \perp$, since $(\mu_i(w) + v)\upharpoonright_v = \perp$, which means that $\mu^*(v) = \perp$, as required by the definition of the set MF_\perp . Hence, $\mu^* \in \text{MF}_\perp$.

- **[ii]**. We first prove the inflationary property of the progress operator, assuming $\mu = \mu_0 \in \text{QDM}$ is a quasi-dominion measure, hence also a regress measure. To do this, consider the sequence of forfeit values $\phi_0, \dots, \phi_{k-1} \in \text{CMs}$ defined as follows: $\phi_i \triangleq \min_{v \in \text{Q}_i} \text{bef}(\mu_i, \text{Q}_i, v)$, for all indexes $i \in [0, k)$. In addition, let $\iota: \|\mu\|_+ \setminus \text{Q}_k \rightarrow [0, k)$ be the function associating each position $v \in \|\mu\|_+ \setminus \text{Q}_k$ with the index $\iota(v) \in [0, k)$, such that $v \in \text{E}_{\iota(v)}$. Due to the way the sequence of measure functions μ_0, μ_1, \dots is constructed, it is immediate to observe that, for all positions $v \in \text{Ps}$ and indexes $i \in [0, k]$, it holds that

$$\begin{aligned} \text{if } v \in \|\mu\|_+ \setminus \text{Q}_k \text{ and } \iota(v) < i \text{ then } \mu_i(v) &= \mu_{\iota(v)}(v) + \phi_{\iota(v)} \\ &\text{else } \mu_i(v) = \mu(v). \end{aligned} \quad (*)$$

At this point, by induction on the index $i \in [0, k)$ and in that specific order, we can prove the following four auxiliary properties: (a) $\mu \sqsubseteq \mu_i$; (b) for each position $v \in \|\mu\|_+ \setminus \text{Q}_{i+1}$, there exists an adjacent $w \in Mv(v) \setminus \text{Q}_i$, such that $\mu_{i+1}(v) = \mu_i(w) + v \neq \top$; (c) if $i > 0$ then $\phi_{i-1} \leq \phi_i$; (d) $\mathbf{0} \leq \phi_i$.

- **[a]**. If $i = 0$ then $\mu \sqsubseteq \mu_i$, since $\mu_0 = \mu$. If $i > 0$, instead, by the Inductive Hypotheses **a** and **d**, it holds that $\mu \sqsubseteq \mu_{i-1}$ and $\mathbf{0} \leq \phi_{i-1}$. Moreover, by the previous Observation (*), it follows that $\mu_i(v) \neq \mu_{i-1}(v)$ only if $v \in \text{E}_{i-1}$ and, in this case, $\mu_i(v) = \mu_{i-1}(v) + \phi_{i-1}$, since $\iota(v) = i - 1$. As a consequence, if $\mu_i(v) \neq \mu_{i-1}(v)$ then $\mu_i(v) > \mu_{i-1}(v)$, thanks to Proposition 5.1, which implies, in general, that $\mu_i(v) \geq \mu_{i-1}(v) \geq \mu(v)$. Hence, $\mu \sqsubseteq \mu_i$ holds true.
- **[b]**. Let $v \in \|\mu\|_+ \setminus \text{Q}_{i+1}$. If $v \in \|\mu\|_+ \setminus \text{Q}_i$ then $i > 0$, since $\text{Q}_0 = \|\mu\|_+$. By the Inductive Hypothesis **b**, there exists an adjacent $w \in Mv(v) \setminus \text{Q}_{i-1}$, such that $\mu_i(v) = \mu_{i-1}(w) + v \neq \top$. Thanks to Observation (*), we have that $\mu_{i+1}(v) = \mu_i(v)$ and $\mu_i(w) = \mu_{i-1}(w)$, since $\iota(v) < i$ and $\iota(w) < i - 1$. Thus, $\mu_{i+1}(v) = \mu_i(w) + v \neq \top$, as required. If $v \notin \|\mu\|_+ \setminus \text{Q}_i$, instead, it holds that $v \in \text{E}_i$. As a consequence, by definition of the lift operator, there exists an adjacent $w \in Mv(v) \setminus \text{Q}_i$, such that $\mu_{i+1}(v) = \mu_i(w) + v$. Notice now that $\mu_i(w) \neq \top$.

Indeed, if $w \in \|\mu\|_+$ then $\mu_i(w) \neq \top$ directly follows from the Inductive Hypothesis **b**. Otherwise, $w \in \|\mu\|_\perp$, which implies that $\mu(w) \neq \top$, thanks to Item **2i** of Definition **3.1**. Moreover, $\mu_i(w) = \mu(w)$, by Observation (*). To conclude, $\mu_{i+1}(v) = \mu_i(w) + v \neq \top$, due to Item **3i** of Definition **3.1**.

– **[c]**. Let $i > 0$ and $v \in E_i$. Thanks to Observation (*), we have that $\mu_{i+1}(v) = \mu_i(v) + \phi_i$ and, so, $\phi_i = \mu_{i+1}(v) - \mu_i(v)$, due to Proposition **5.1**. To continue, we need to consider the following case analysis in three parts, which is partially based on ownership of the positions v .

* $[v \in \text{Ps}_0 \cap \text{esc}(\mu_{i-1}, Q_{i-1})]$. Since $v \in \text{Ps}_0 \cap \text{esc}(\mu_{i-1}, Q_{i-1})$, it holds that $\mu_{i-1}(u) + v < \mu_{i-1}(v)$, for all adjacents $u \in Mv(v) \cap Q_{i-1}$. However, by Condition **1** of Definition **4.4**, there exists an adjacent $w \in Mv(v)$, such that $\mu(v) \leq \mu(w) + v$. By Observation (*), $\mu_{i-1}(v) = \mu(v)$, as $\iota(v) = i$. Thus, $\mu_{i-1}(v) = \mu(v) \leq \mu(w) + v \leq \mu_{i-1}(w) + v$, thanks to the Inductive Hypothesis **a** and Item **3ii** of Definition **3.1**. As a consequence, $w \notin Q_{i-1}$ and, so, $w \notin Q_i$, since all adjacents of v inside Q_{i-1} falsify the inequality, as shown before. Moreover, $v \notin E_{i-1}$, as $v \in E_i$. Hence, $\phi_{i-1} < \text{bef}(\mu_{i-1}, Q_{i-1}, v)$. At this point, the following inequalities hold:

$$\begin{aligned}
\phi_{i-1} &< \text{bef}(\mu_{i-1}, Q_{i-1}, v) \\
&= \min\{(\mu_{i-1}(u) + v) - \mu_{i-1}(v) : u \in Mv(v) \setminus Q_{i-1}\} \\
&\leq (\mu_{i-1}(w) + v) - \mu_{i-1}(v) \\
&= (\mu_i(w) + v) - \mu_i(v) \\
&\leq \max\{\mu_i(u) + v : u \in Mv(v) \setminus Q_i\} - \mu_i(v) \\
&= \text{lift}(\mu_i, E_i, \overline{Q_i})(v) - \mu_i(v) \\
&= \mu_{i+1}(v) - \mu_i(v) \\
&= \phi_i.
\end{aligned}$$

Notice that the first equality follows from the definition of the best escape forfeit function, while the second one is due to Observation (*). Indeed, $\mu_{i-1}(v) = \mu_i(v)$ and $\mu_{i-1}(w) = \mu_i(w)$, since $\iota(v) = i$ and $\iota(w) < i - 1$. Finally, from the last inequality onward, we applied the definition of the lift operator.

* $[v \in \text{Ps}_0 \setminus \text{esc}(\mu_{i-1}, \mathbb{Q}_{i-1})]$. Since $v \in \text{Ps}_0 \setminus \text{esc}(\mu_{i-1}, \mathbb{Q}_{i-1})$, there exists an adjacent $w \in Mv(v) \cap \mathbb{Q}_{i-1}$, such that $\mu_{i-1}(v) \leq \mu_{i-1}(w) + v$. Moreover, by Observation (*), $\mu_{i-1}(v) = \mu_i(v)$, as $\iota(v) = i$, from which we derive $\mu_i(v) \leq \mu_{i-1}(w) + v$ and, so, $(\mu_{i-1}(w) + v) - \mu_i(v) \geq \mathbf{0}$, due to Proposition 5.1. Obviously, $\iota(w) \geq i - 1$, due to the definition of the function ι , since the membership of w in \mathbb{Q}_{i-1} implies $w \notin E_j$, for all indexes $j < i - 1$. Actually, it holds true that the value $\iota(w)$ is precisely $i - 1$. Indeed, suppose, by contradiction, that $\iota(w) > i - 1$. Then, we would have $\mu_{i-1}(w) = \mu_i(w)$ and, thus, $\mu_i(v) \leq \mu_{i-1}(w) + v = \mu_i(w) + v$, contradicting the fact that $v \in E_i \subseteq \text{esc}(\mu_i, \mathbb{Q}_i)$. Now, the equality $\iota(w) = i - 1$ implies $w \in E_{i-1}$ and, so, $w \notin \mathbb{Q}_i$. Moreover, $\mu_i(w) = \mu_{i-1}(w) + \phi_{i-1}$, again due to Observation (*). At this point, the following holds:

$$\begin{aligned}
\phi_{i-1} &\leq ((\mu_{i-1}(w) + v) - \mu_i(v)) + \phi_{i-1} \\
&= ((\mu_{i-1}(w) + \phi_{i-1}) + v) - \mu_i(v) \\
&= (\mu_i(w) + v) - \mu_i(v) \\
&\leq \max\{\mu_i(u) + v : u \in Mv(v) \setminus \mathbb{Q}_i\} - \mu_i(v) \\
&= \text{lift}(\mu_i, E_i, \overline{\mathbb{Q}_i})(v) - \mu_i(v) \\
&= \mu_{i+1}(v) - \mu_i(v) \\
&= \phi_i.
\end{aligned}$$

Notice that the first two derivation steps follow from the Abelian group properties of the evaluation structure stated in Proposition 5.1 and from Item iii of Proposition 5.3. Moreover, from the last inequality onward, we applied the definitions of the lift operator and forfeit values.

* $[v \in \text{Ps}_1]$. Since $v \in E_i$, it holds that $v \in \|\mu\|_+ \setminus \mathbb{Q}_{i+1}$. By the Inductive Hypothesis **b**, there exists an adjacent $w \in Mv(v) \setminus \mathbb{Q}_i$, such that $\mu_{i+1}(v) = \mu_i(w) + v$, which implies $\phi_i = (\mu_i(w) + v) - \mu_i(v)$. Now, consider the nested case analysis on the membership of the position w *w.r.t.* E_{i-1} .

· $[w \notin E_{i-1}]$. Since $w \notin E_{i-1}$ and $w \notin \mathbb{Q}_i$, it holds that $w \notin \mathbb{Q}_{i-1}$. This means that $w \in Mv(v) \setminus \mathbb{Q}_{i-1} \neq$

\emptyset , which in turn implies that $v \in \text{esc}(\mu_{i-1}, Q_{i-1})$, since $v \in \text{Ps}_1$. Moreover, $v \notin E_{i-1}$, as $v \in E_i$. Thus, $\phi_{i-1} < \text{bef}(\mu_{i-1}, Q_{i-1}, v)$. At this point, the following inequalities hold:

$$\begin{aligned} \phi_{i-1} &< \text{bef}(\mu_{i-1}, Q_{i-1}, v) \\ &= \min\{(\mu_{i-1}(u) + v) - \mu_{i-1}(v) : u \in Mv(v) \setminus Q_{i-1}\} \\ &\leq (\mu_{i-1}(w) + v) - \mu_{i-1}(v) \\ &= (\mu_i(w) + v) - \mu_i(v) \\ &= \phi_i. \end{aligned}$$

Notice that the first equality follows from the definition of the best escape forfeit function, while the second one is due to Observation (*). Indeed, $\mu_{i-1}(v) = \mu_i(v)$ and $\mu_{i-1}(w) = \mu_i(w)$, since $\iota(v) = i$ and $\iota(w) < i - 1$. [$w \in E_{i-1}$]. Since $w \in E_{i-1}$, by Observation (*), it follows that $\mu_i(w) = \mu_{i-1}(w) + \phi_{i-1}$, as $\iota(w) = i - 1$. Similarly, $\mu_i(v) = \mu(v)$, as $\iota(v) = i$. Now, by Condition 2 of Definition 4.4, $\mu_i(v) = \mu(v) \leq \mu(u) + v \leq \mu_{i-1}(u) + v$, for all adjacents $u \in Mv(v)$, thanks to the Inductive Hypothesis a and Item 3ii of Definition 3.1. As an immediate consequence, $\mu_i(v) \leq \mu_{i-1}(w) + v$, i.e., $(\mu_{i-1}(w) + v) - \mu_i(v) \geq \mathbf{0}$. At this point, the following holds:

$$\begin{aligned} \phi_{i-1} &\leq ((\mu_{i-1}(w) + v) - \mu_i(v)) + \phi_{i-1} \\ &= ((\mu_{i-1}(w) + \phi_{i-1}) + v) - \mu_i(v) \\ &= (\mu_i(w) + v) - \mu_i(v) \\ &= \phi_i. \end{aligned}$$

Notice that the first two derivation steps follow from the Abelian group properties of the evaluation structure stated in Proposition 5.1 and from Item iii of Proposition 5.3.

Summing up, in all cases we have $\phi_{i-1} \leq \phi_i$.

- [d]. If $i > 0$, by the Inductive Hypotheses d and c, it holds that $\mathbf{0} \leq \phi_{i-1}$ and $\phi_{i-1} \leq \phi_i$. Hence, $\mathbf{0} \leq \phi_i$. If $i = 0$, instead, let $v \in E_0$. Thanks to Observation (*), one has that $\mu_1(v) = \mu_0(v) + \phi_0$ and, so, $\phi_0 = \mu_1(v) - \mu_0(v)$, due to

Proposition 5.1. To continue, we need to consider the following case analysis on the ownership of the position v .

- * [$v \in \text{Ps}_0$]. By Condition 1 of Definition 4.4, there exists an adjacent $w \in Mv(v)$, such that $\mu_o(v) \leq \mu_o(w) + v$, as $\mu_o = \mu$. Since $v \in E_o$, thanks to the definitions of both the best escape forfeit and escape functions, it holds that $\mu_o(u) + v < \mu_o(v)$, for all adjacents $u \in Mv(v) \cap Q_o$. Hence, as an immediate consequence, $w \notin Q_o$. Now, by definition of the lift operator, $\mu_1(v) = \max\{\mu_o(u) + v : u \in Mv(v) \setminus Q_o\} \geq \mu_o(w) + v \geq \mu_o(v)$, from which it follows that $\phi_o = \mu_1(v) - \mu_o(v) \geq \mathbf{0}$, due to Proposition 5.1.
- * [$v \in \text{Ps}_1$]. By the Inductive Hypothesis b, there exists an adjacent $w \in Mv(v) \setminus Q_o$, such that $\mu_1(v) = \mu_o(w) + v$, which implies $\phi_o = (\mu_o(w) + v) - \mu_o(v)$. Moreover, by Condition 2 of Definition 4.4, $\mu_o(v) \leq \mu_o(u) + v$, for all adjacents $u \in Mv(v)$, since $\mu_o = \mu$. Thus, as an obvious consequence, $\mu_o(w) + v \geq \mu_o(v)$, which immediately implies $\phi_o = (\mu_o(w) + v) - \mu_o(v) \geq \mathbf{0}$, as required, again due to Proposition 5.1.

Summing up, in both cases we have $\mathbf{0} \leq \phi_o$.

We now have the necessary tool to show that the progress operator is inflationary. Indeed, by Property a, $\mu \sqsubseteq \mu_k$. Moreover, μ^* and μ_k differ only on positions $v \in \|\mu^*\|_+$, such that $\mu^*(v) = \top$. Hence, it easily follows that $\mu \sqsubseteq \mu^*$, thanks to Item 1 of Definition 3.1. Observe also that $\|\mu\|_+ \subseteq \|\mu_k\|_+ \subseteq \|\mu^*\|_+$ and $\|\mu\|_0 \subseteq \|\mu_k\|_0 \subseteq \|\mu^*\|_0$, due to Item 2ii of the same definition.

It remains to prove that μ^* is a QDM. First notice that, for any position $v \in \text{Ps}$, if $\mu_k(v) \neq \mu(v)$ then $v \in \|\mu\|_+ \setminus Q_k$, due to Observation (*). Thus, by Property b, for such a position v , there exists an adjacent $w \in Mv(v) \setminus Q_{k-1} \subseteq Mv(v) \setminus Q_k$, such that $\mu_k(v) = \mu_{k-1}(w) + v \neq \top$. Since $w \notin Q_{k-1}$, it holds that $\iota(w) < k - 1$, so, $\mu_k(w) = \mu_{k-1}(w)$, again due to Observation (*). Hence, for all positions $v \in \text{Ps}$, either one of the following two possibilities holds:

$$\left. \begin{array}{l} v \notin \|\mu\|_+ \setminus Q_k \text{ and } \mu_k(v) = \mu(v); \\ v \in \|\mu\|_+ \setminus Q_k \text{ and there exists an adjacent } w \in Mv(v) \setminus Q_k \\ \text{such that } \mu_k(v) = \mu_k(w) + v \neq \top. \end{array} \right\} \quad (\#)$$

From this, we easily derive that $\|\mu\|_+ = \|\mu_k\|_+ = \|\mu^*\|_+$ and $\|\mu\|_0 = \|\mu_k\|_0 \subseteq \|\mu^*\|_0 = Q_k$. Indeed, the only positions that can change their measure are those in $\|\mu\|_+$ and they are not set to \top in μ_k . Moreover, $\|\mu\|_0 \subseteq \|\mu\|_+$, so, $\|\mu_k\|_0 \subseteq \|\mu\|_+$. Now, $(\|\mu\|_+ \setminus Q_k) \cap \|\mu_k\|_0 = \emptyset$, hence, $\|\mu_k\|_0 \subseteq Q_k$. Finally, $\|\mu^*\|_0 = Q_k$, since, by construction, $\|\mu^*\|_0 = \|\mu_k\|_0 \cup Q_k$.

We can now show that μ^* is a regress measure, *i.e.*, that every position $v \in \|\mu^*\|_+ \setminus \|\mu^*\|_0$ satisfies the suitable condition of Definition 4.4. We do this, via a case analysis on the ownership of v .

- $[v \in \text{Ps}_0]$. Since $\|\mu^*\|_+ \setminus \|\mu^*\|_0 = \|\mu\|_+ \setminus Q_k$, by Observation (#), there exists an adjacent $w \in Mv(v) \setminus Q_k$, such that $\mu_k(v) = \mu_k(w) + v$. Obviously, $\mu^*(v) = \mu_k(v)$ and $\mu^*(w) = \mu_k(w)$, since $v, w \notin Q_k$. Thus, $\mu^*(v) = \mu^*(w) + v$, as required by Condition 1.
- $[v \in \text{Ps}_1]$. Suppose by contradiction that μ^* does not satisfy Condition 2 on v . Then, there exists one of its adjacents $w \in Mv(v)$, such that $\mu^*(w) + v < \mu^*(v)$. Obviously, $\mu^*(w) + v \neq \top$, so, $\mu^*(w) \neq \top$, due to Item 3i of Definition 3.1, which in turn implies that $w \notin Q_k$. As a consequence, $v \notin Q_k$ too, since we would have had, otherwise, $Mv(v) \subseteq Q_k$, as $\text{esc}(\mu_k, Q_k) = \emptyset$. Thus, $v \in \|\mu\|_+ \setminus Q_k$, from which it follows that $\mu^*(v) = \mu_k(v) = \mu_{\iota(v)+1}(v) = \mu_{\iota(v)}(v) + \phi_{\iota(v)} = \mu(v) + \phi_{\iota(v)}$, due to Observation (*). To proceed, we now need the following nested case analysis, which allows to prove that $w \notin Q_{\iota(v)}$ and $\mu^*(w) = \mu_{\iota(v)}(w)$.
 - * $[w \in \|\mu^*\|_{\perp}]$. Notice that $w \notin Q_0 = \|\mu\|_+ = \|\mu^*\|_+$ and, so, $w \notin Q_{\iota(v)}$, as $Q_{\iota(v)} \subseteq Q_0$. Moreover, $\mu^*(w) = \mu_k(w) = \mu_{\iota(v)}(w)$, due to Observations (*).
 - * $[w \in \|\mu^*\|_+]$. By Observations (*), it holds that $\mu^*(w) = \mu_k(w) = \mu_{\iota(w)}(w) + \phi_{\iota(w)} = \mu(w) + \phi_{\iota(w)}$. Now, by substituting in $\mu^*(w) + v < \mu^*(v)$ both $\mu^*(w)$ and $\mu^*(v)$ with $\mu(w) + \phi_{\iota(w)}$ and $\mu(v) + \phi_{\iota(v)}$, respectively, and exploiting Proposition 5.1 and Item iii of Proposition 5.3, one can obtain $(\mu(w) + v) - \mu(v) < \phi_{\iota(v)} - \phi_{\iota(w)}$. Since μ is a QDM and, so, a regress measure, we have that it satisfies Condition 2 on v , *i.e.*, $\mu(v) \leq \mu(w) + v$, which implies $(\mu(w) + v) - \mu(v) \geq \mathbf{0}$. Hence, $\phi_{\iota(v)} - \phi_{\iota(w)} > \mathbf{0}$ and, so,

$\iota(v) > \iota(w)$, thanks to Property **c**. From this we can derive that $w \notin Q_{\iota(v)}$ and $\mu^*(w) = \mu_k(w) = \mu_{\iota(v)}(w)$, due to Observations (*).

At this point, the following impossible inequality chain should hold:

$$\begin{aligned}
\mu^*(v) &= \mu_{\iota(v)+1}(v) \\
&= \text{lift}(\mu_{\iota(v)}, E_{\iota(v)}, \overline{Q_{\iota(v)}})(v) \\
&= \min\{\mu_{\iota(v)}(u) + v : u \in Mv(v) \setminus Q_{\iota(v)}\} \\
&\leq \mu_{\iota(v)}(w) + v \\
&= \mu^*(w) + v \\
&< \mu^*(v).
\end{aligned}$$

Finally, we can conclude the proof of this item by showing that $\|\mu^*\|_0$, which we now know to be equal to Q_k , is a 0-dominion. Let $\sigma_0 \in \text{Str}_0$ be a μ -coherent 0-strategy winning on $\|\mu\|_0$, such that $\sigma_0(v) \in Q_k$, for all 0-positions $v \in Q_k \cap \text{Ps}_0$. Such a strategy surely exists, since μ is, by hypothesis, a QDM and Q_k is closed, *i.e.*, $\text{esc}(\mu, Q_k) = \text{esc}(\mu_k, Q_k) = \emptyset$. To state the first equality we exploited the fact that $\mu_k(v) = \mu(v)$, for all positions $v \in Q_k$, due to Observation (#). Now, by Theorem 4.7, $\|\mu\|_+$ is a weak quasi-0-dominion, for which $\sigma_0 \downarrow \|\mu\|_+ \in \text{Str}_0(\|\mu\|_+)$ is a 0-witness. Hence, Q_k and, so, $\|\mu^*\|_0$, is a 0-dominion, thanks to Corollary 4.2.

- **[iii]**. To show that μ^* is a SM, whenever μ is a SM, we first prove the following statement: $\mu_i(v) \in \text{SMs}(v, \|\mu\|_+ \setminus Q_i) \setminus \{\top\}$, for all indexes $i \in [0, k]$ and positions $v \in \text{Ps} \setminus Q_i$.

The proof proceeds by induction on i . The base case $i = 0$ trivially follows from the hypothesis, since $\mu_0 = \mu$. Indeed, it holds that $\text{Ps} \setminus Q_0 = \|\mu\|_{\perp}$ and $\mu_i(v) = \perp$, for all positions $v \in \|\mu\|_{\perp}$, due to Proposition 4.10. For the inductive case $i > 0$, let $v \in \text{Ps} \setminus Q_i$ and assume that $\mu_{i-1}(w) \in \text{SMs}(w, \|\mu\|_+ \setminus Q_{i-1}) \setminus \{\top\}$, for all positions $w \in \text{Ps} \setminus Q_{i-1}$. If $v \in \text{Ps} \setminus Q_{i-1}$, there is nothing more to prove, since $\mu_i(v) = \mu_{i-1}(v)$ and, by inductive hypothesis, $\mu_{i-1}(v) \in \text{SMs}(v, \|\mu\|_+ \setminus Q_{i-1}) \setminus \{\top\} \subseteq \text{SMs}(v, \|\mu\|_+ \setminus Q_i) \setminus \{\top\}$, being $\|\mu\|_+ \setminus Q_{i-1} \subseteq \|\mu\|_+ \setminus Q_i$. Therefore, consider the case $v \notin \text{Ps} \setminus Q_{i-1}$, which implies that $v \in E_{i-1} \subseteq Q_{i-1}$. By definition of the lift operator, there exists an adjacent $w \in Mv(v) \setminus Q_{i-1} \subseteq \text{Ps} \setminus Q_{i-1}$, such that $\mu_i(v) = \mu_{i-1}(w) + v$. By the inductive hypothesis, $\mu_{i-1}(w) \neq \top$. Thus, by Item 3i of Definition 3.1, it holds that $\mu_i(v) \neq \top$. Moreover,

there exists a simple path $\pi \in \text{SPth}(w, \|\mu\|_+ \setminus Q_{i-1})$ such that $\mu_{i-1}(w) = \eta(\pi)$. Obviously, $\mu_i(v) = \mu_{i-1}(w) + v = \eta(\pi) + v = \eta(v \cdot \pi) \neq \top$. Now, it is quite clear that $v \cdot \pi$ is a simple path passing through positions in $\{v\} \cup \|\mu\|_+ \setminus Q_{i-1}$, *i.e.*, $v \cdot \pi \in \text{SPth}(v, \|\mu\|_+ \setminus Q_i)$, since $v \in Q_{i-1}$. Hence, $\mu_i(v) \in \text{SMs}(v, \|\mu\|_+ \setminus Q_i) \setminus \{\top\}$. This concludes the inductive proof.

At this point, it immediately follows, from what we have just proved, that μ_k is SM. In addition, μ^* potentially differs from μ_k only on positions $v \in \|\mu^*\|_+$ such that $\mu^*(v) = \top$. Consequently, μ^* is a SM as well.

- **[iv]**. By hypothesis, $\mu^* = \mu$, which implies that $\mu_i = \mu$, for all $i \in \mathbb{N}$, due to the way the sequence μ_0, μ_1, \dots is constructed. Now, let us consider an arbitrary position $v \in \|\mu\|_+$. Due to the definition of the sequence Q_0, Q_1, \dots , it obviously holds that either $v \in Q_k$ or there is a unique index $i \in [0, k)$, such that $v \in Q_i \setminus Q_{i+1}$, *i.e.*, $v \in E_i$. In the first case, we have $\mu(v) = \top$, due to the assignment $\mu^* = \mu_k[Q_k \mapsto \top]$. Therefore, v is a progress position, *i.e.*, it satisfies both conditions of Definition 3.6. In the other case, the proof proceeds by a case analysis on the ownership of the position v itself.

- $[v \in \text{Ps}_0]$. First recall that $E_i = \text{bep}(\mu, Q_i) \subseteq \text{esc}(\mu, Q_i)$. Thus, due to the definition of the escape function, we have that $\mu(w) + v < \mu(v)$, for all positions $w \in Mv(v) \cap Q_i$. Now, by definition of the lift operator, we have that $\mu(w) + v \leq \max\{\mu(w) + v : w \in Mv(v) \cap \overline{Q_i}\} = \mu(v)$, for all adjacents $w \in Mv(v) \cap \overline{Q_i}$ of v . Thus, $\mu(w) + v \leq \mu(v)$, for all positions $w \in Mv(v)$, as required by Condition 1 of Definition 3.6 on $\|\mu\|_+$.
- $[v \in \text{Ps}_1]$. Again by definition of the lift operator, we have that $\mu(w) + v \leq \min\{\mu(w) + v : w \in Mv(v) \cap \overline{Q_i}\} = \mu(v)$, for some adjacent $w \in Mv(v) \cap \overline{Q_i} \subseteq Mv(v)$ of v . Hence, Condition 2 of Definition 3.6 is satisfied on $\|\mu\|_+$ as well. \square

As an example, consider the simple game \mathcal{D} in Fig. 1. Starting from the minimal measure function μ_\perp , the solution algorithm first computes $\mu_1 = \text{prg}_\perp(\mu_\perp)$, by lifting the four even-priority positions **a**, **b**, **c**, and **h** to their respective measures $\eta_a = \eta_h \triangleq (0, 0, 0, 0, 1, 0, 0)$, $\eta_b \triangleq (1, 0, 0, 0, 0, 0, 0)$, and $\eta_c \triangleq (0, 0, 1, 0, 0, 0, 0)$. Fig. 1.1 reports the situation after this initial phase, where the blue (*resp.*, dashed red) edges indicate the moves satis-

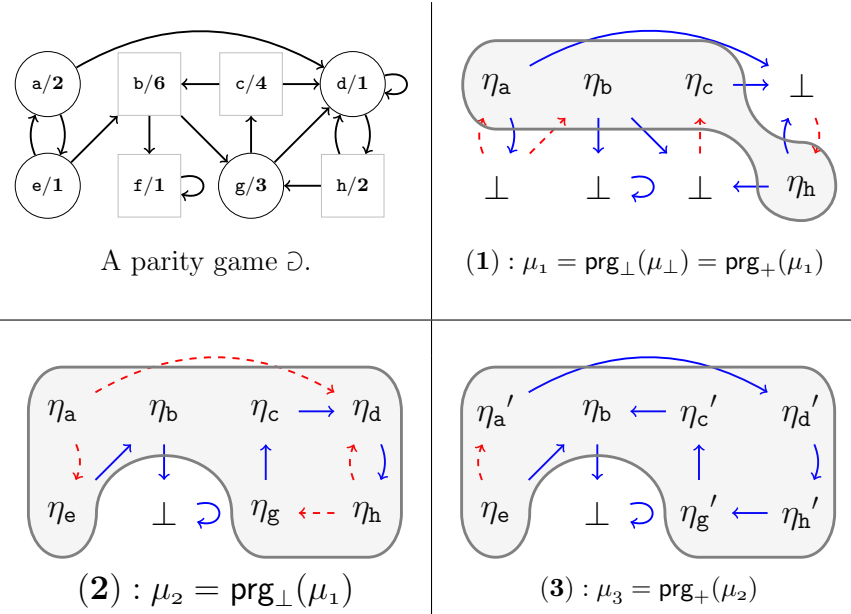


Figure 1. Simulating the first steps of the concrete algorithm on a simple game. Positions of player 0 are circles and positions of player 1 are squares. The label inside each position indicates its name and priority.

fying (*resp.*, not satisfying) the progress condition. Since inside the quasi 0-dominion $\|\mu_1\|_+ = \{a, b, c, h\}$, identified by the grey area, all positions are in progress, the prg_+ operator does not change their measures, *i.e.*, $\mu_1 = \text{prg}_+(\mu_1)$. The three odd-priority positions d , e , and g outside $\|\mu_1\|_+$ do not satisfy the progress condition, so the prg_\perp operator lifts their measure to $\eta_d \triangleq (0, 0, 0, 0, 1, 1, 0)$, $\eta_e \triangleq (1, 0, 0, 0, 0, 1, 0)$, and $\eta_g \triangleq (0, 0, 1, 1, 0, 0, 0)$, as reported in Fig. 1.2. Now, positions a and h inside the quasi 0-dominion do not satisfy the progress condition anymore. Therefore, the prg_+ operator tries to recover the condition as follows. It starts by identifying the escape positions $\text{esc}(\mu_2, Q_0) = \{b\}$ of $Q_0 \triangleq \|\mu_2\|_+ = \{a, b, c, d, e, g, h\}$. Since b has a progress move exiting from Q_0 , its measure remains unchanged. Now, $\text{esc}(\mu_2, Q_1) = \{c, e\}$, where $Q_1 \triangleq Q_0 \setminus \{b\}$. Since e has a progress move exiting from Q_1 , while c can escape from Q_1 only by increasing its measure, we have $\text{bep}(\mu_2, Q_1) = \{e\}$ and $Q_2 \triangleq Q_1 \setminus \{e\}$. Also in this case, e does not change its measure. The process continues by extracting and lifting the measures of all the remaining positions in Q_2 in the following order: (i) c with $\eta_c' \triangleq (1, 0, 1, 0, 0, 0, 0)$; (ii) g with $\eta_g' \triangleq (1, 0, 1, 1, 0, 0, 0)$; (iii) h with $\eta_h' \triangleq (1, 0, 1, 1, 1, 0, 0)$; (iv) d with $\eta_d' \triangleq (1, 0, 1, 1, 1, 1, 0)$; (v) a with

$\eta_{\mathbf{a}'} \triangleq (1, 0, 1, 1, 2, 1, 0)$. Note that player 0 positions \mathbf{g} and \mathbf{d} are forced to exit the quasi 0-dominion, since their internal moves (\mathbf{g}, \mathbf{d}) and (\mathbf{d}, \mathbf{d}) do not satisfy the regress condition, as $\eta_{\mathbf{a}} + \mathbf{g} = \perp < \eta_{\mathbf{g}}$ and $\eta_{\mathbf{a}} + \mathbf{d} = (0, 0, 0, 0, 1, 2, 0) < \eta_{\mathbf{a}}$; these moves would form, indeed, odd cycles. Fig. 1.3 reports the situation after the complete execution of prg_+ , where position \mathbf{e} has the non-progress move (\mathbf{e}, \mathbf{a}) . Another application of prg_+ modifies the measure of \mathbf{e} to $\eta_{\mathbf{e}'} \triangleq (1, 0, 1, 1, 2, 2, 0)$, triggering the non-progress move (\mathbf{a}, \mathbf{e}) . After a final application of prg_+ , positions \mathbf{a} and \mathbf{e} are lifted to \top and the algorithm reaches its fixpoint. All positions, except \mathbf{a} and \mathbf{e} , satisfy the progress conditions and are, thus, winning for player 1; \mathbf{a} and \mathbf{e} are won by player 0.

We can prove that the solver operator is well-defined and that, when it is applied to a simple measure, it converges in a finite number of iterations, at most equal to the depth of the finite partial order $\langle \text{SM}, \sqsubseteq \rangle$. A very coarse upper bound on this depth, for a game with n positions, is given by $(n+1)!$, since every non- \top position is associated with the measure of a simple path of length less than n and there are at most $n!$ such paths. In the following, we use $\text{SQDM} \triangleq \text{QDM} \cap \text{SM}$.

Theorem 5.6: (*Termination*) *The solver operator $\text{sol}: \text{SQDM} \rightarrow \text{SQDM}$ is a well-defined function. Moreover, for every $\mu \in \text{SQDM}$, there exists an index $k \leq d$, such that $\text{sol}(\mu) = (\text{ifp}_k \nu . \text{prg}_+(\text{prg}_\perp(\nu)))(\mu)$, where $d \in \mathbb{N}$ is the depth of the finite partial order $\langle \text{SM}, \sqsubseteq \rangle$.*

Proof: By Items i-iii of Lemma 5.4 and Items i-iii of Lemma 5.5, we have that prg_\perp and prg_+ are inflationary total functions on SQDM, which implies that their composition $\text{prg}_+ \circ \text{prg}_\perp$ is both inflationary and total on SQDM as well. Consider now the infinite sequence μ_0, μ_1, \dots of measure functions recursively derived from an arbitrary input element $\mu \in \text{SQDM}$ as follows: $\mu_0 \triangleq (\text{ifp}_0 \nu . \text{prg}_+(\text{prg}_\perp(\nu)))(\mu) = \mu$ and $\mu_{i+1} \triangleq (\text{ifp}_{i+1} \nu . \text{prg}_+(\text{prg}_\perp(\nu)))(\mu) = \text{prg}_+(\text{prg}_\perp(\mu_i))$, for all $i \in \mathbb{N}$. Obviously, $\mu_i \sqsubseteq \mu_{i+1}$. Moreover, each element μ_i is a SM. Since every strict chain in $\langle \text{SM}, \sqsubseteq \rangle$ can be composed of at most d elements, there necessarily exists an index $k \leq d$ such that $\mu_{k+1} = \mu_k$, as required by the theorem statement. \square

The next theorem stating the soundness and completeness of the solution algorithm is a simple consequence of the properties of the prg_\perp and prg_+ operators, combined with the general results about the measure-function

spaces discussed in the previous sections.

Theorem 5.7: (*Solution*) $\text{Win}_0 = \|\text{sol}(\mu_\perp)\|_0$ and $\text{Win}_1 = \|\text{sol}(\mu_\perp)\|_1$.

Proof: Let $\mu^* \triangleq \text{sol}(\mu_\perp)$ be the result of the application of the solver operator to the minimal measure function $\mu_\perp \in \text{SQDM}$. By the notion of inflationary fixpoint, μ^* is a fixpoint of the composition of the two progress operators, *i.e.*, $\mu^* = \text{prg}_+(\text{prg}_\perp(\mu^*))$, which are inflationary functions on SQDM, due to Items **i** and **ii** of Lemma 5.4 and Items **i** and **ii** of Lemma 5.5. Therefore, it holds that $\mu^* \sqsubseteq \text{prg}_\perp(\mu^*) \sqsubseteq \text{prg}_+(\text{prg}_\perp(\mu^*)) = \mu^*$, which implies that $\text{prg}_\perp(\mu^*) = \mu^*$ and, so, $\text{prg}_+(\mu^*) = \mu^*$. As a consequence of Item **iv** of Lemma 5.4 and Item **iv** of Lemma 5.5, it holds that μ^* is a progress measure. Hence, $\|\mu^*\|_1 \subseteq \text{Win}_1$ follows from Theorem 3.7. By Theorem 5.6, it holds that $\mu^* \in \text{SQDM}$, which implies that $\|\mu^*\|_0 \subseteq \text{Win}_0$, due to Definition 4.6. Hence, the thesis follows, since $\|\mu^*\|_0$ and $\|\mu^*\|_1$ partition the set of positions. \square

6. Experimental Evaluation

The algorithm proposed in this chapter has been implemented in OINK [36], a C++ framework supporting different parity game solvers^b and providing tools to compare their performance on various worst-case families. The solvers considered in the experiments include the original priority promotion algorithm *PP* [26] and the progress measure version presented in this chapter *QDPM*, the optimised version [37] of the Recursive algorithm *Rec* [16], the optimised version of the Small Progress Measure algorithm *SPM* [21] and its quasi-polynomial version *SSPM* [23], the quasi-polynomial algorithm *QPT* [24], the Tangle Learning algorithm *TL* [20], and the Distraction-based Fixpoint Iteration algorithm with justifications *FPJ* [38]. The benchmarks include worst-case games for the considered solvers and clustered random games generated with the PGSolver framework [39]. The latter are games that exhibit a complex structure *w.r.t.* the class of randomly generated games. Indeed, while most of the random games consist in a single *strongly connected component (SCC)* and are easily solved by any attractor-based approach, clustered games rely, instead, on a tree-like structure with multiple SCCs.

^bExperiments were carried out on a 64-bit 1.6GHz INTEL quad-core machine, with i5-8250U processor and 8GB of RAM, running UBUNTU 18.04.5 with LINUX kernel version 3.28.2. OINK was compiled with gcc version 7.4.

Benchmarks	Exponential						Quasi Polynomial	
	Rec	PP	TL	FPJ	SPM	QDPM	SSPM	QPT
Two Counters [40]	20	107	11	26	7	85	4	5
QPT [24]	[0s]	[0s]	[0s]	[0s]	[0s]	[0s]	[0.41s]	30
Gazda’s wc [41]	35	35	[0.01s]	37	380	[0.67s]	22	29
DP [18]	36	23	[0.19s]	38	20	[0s]	[0.17s]	28
Divide&Impera [42]	17	91	[27.75s]	16	7	[7.45s]	173	4

Figure 2. Biggest (index of the) instance of the worst-case families solved within 30s. If the 1000th instance is solved, its approximated solution time is reported in brackets.

The table in Fig. 2 displays the results on the following worst-case families^c: the family for TL “Two Counters” [40], the one for QPT algorithm “QPT” [24], the family for Zielonka’s Optimised algorithm “Gazda’s wc” [41], the family for the Delayed Priority Promotion algorithm “DP” [18], and the Robust Worst Case for Divide-et-Impera Algorithms “Divide&Impera” [42]. Each row reports the biggest instance each solver could solve within the time limit of 30 seconds. The “Two Counters” family proved to be very demanding for all the solvers, as none of them could solve the 108th instance within the time limit. On the contrary, the “QPT” family can easily be solved by all the solvers except QPT. The proposed solver QDPM performs extremely well on all the families, being able to solve the 1000th instance faster than the competitors, except for the “Two Counters”, on which it is outperformed only by PP, and Gazda’s family, where TL is slightly better.

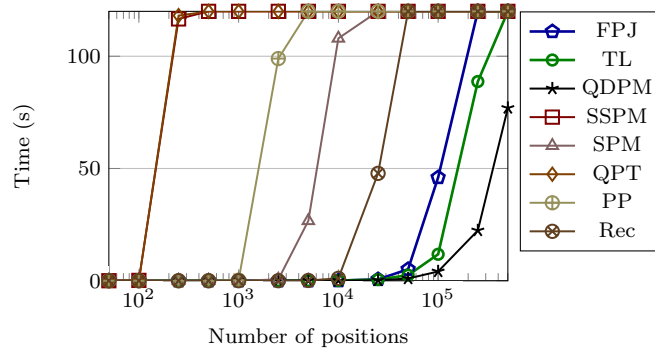


Figure 3. Time on clustered random games with 2 moves per position.

^cThe instances were generated by issuing the following OINK commands: tc+ n; counter_qpt n; counter_m n; counter_dp n. The Robust Worst Case has been implemented according to [42].

Fig. 3 compares the running times on 1300 random clustered games of size ranging from 50 to $5 \cdot 10^5$ positions and 2 outgoing moves per position^d. We set the time-out at 120 seconds. Each point in the graph shows the average time over a cluster of 100 different games of the same size shown on a logarithmic scale. For a game of size n , we set the number of priorities to $k = n/10$. The performance of the quasi-polynomial solvers (QPT and SSPM) reaches the time-out already for the smaller instances: for games with 250 positions they could solve less than 20% of the instances. Almost all of the quasi-dominion-based algorithms, namely TL, FPJ, and QDPM, instead, scale quite well. Their behaviour start to differentiate for games with at least 10^5 positions. On the biggest instances ($5 \cdot 10^5$), QDPM is the only algorithm to terminate within 2 minutes, with an average solution time of 76 seconds and only 14% of time-outs.

7. Discussion

We propose a revisited progress measures-based algorithm for parity games that integrates progress measures and quasi-dominions. This integration requires a novel notion of measure to encode the additional information needed to identify quasi-dominions and a new update policy that takes advantage of quasi-dominions and often allows to skip intermediate measures and reach a progress measure much more quickly than the classic progress measure algorithms. This motivates the conjecture that the integration significantly accelerates the convergence to a progress measure. The experiments show that the proposed approach scales better than any known algorithm on games with a complex structure, such as clustered random games and the worst-case families. In particular, the speed-up can be of several orders of magnitude when compared to other algorithms based on progress measures. We believe that this integration approach may also lead to practically efficient quasi-polynomial algorithms based on succinct progress measures.

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^dThe instances were generated by issuing the following PGSolver command: `clusteredrandgame n n/10 2 2 5 3 7 3 7`.

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