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Human behavioral crowds review, critical analysis and research perspectives

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This paper presents a survey and critical analysis of the mathematical literature on modeling and simulation of human crowds taking into account behavioral dynamics. The main focus is on research papers published after the review [N. Bellomo and C. Dogbè, On

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the modeling of traffic and crowds: A survey of models, speculations, and perspectives, *SIAM Rev.* **53** (2011) 409–463], thus providing important research perspectives related to new, emerging trends. The presentation addresses the scaling problem corresponding to microscopic (individual-based), mesoscopic (kinetic), and macroscopic (hydrodynamic) modeling and analysis. A multiscale vision guides the overall content of the paper. The critical analysis of the overall content naturally leads to research perspectives. A selection of them is brought to the attention of the interested reader together with hints on how to deal with them.

Keywords: Complexity, crowd dynamics, living systems, multiscale problems, social dynamics, stress propagation.

AMS Subject Classification: 82D99, 91D10

1. Motivations and Plan of the Paper

This section defines the aim and content of the paper, which begins with a review and critical analysis of the mathematical literature on crowd dynamics and concludes with a look at research perspectives. Our review focuses on key issues that should be considered for new concepts and methods in modeling and simulating human crowds that take into account behavioral state dynamics. The survey essentially covers the period after the publication of the review paper,[25] which provided several possible research perspectives. One of these perspectives is particularly relevant to this paper: the need to develop a modeling approach that accounts for heterogeneous individual behavior, which can significantly affect interactions and the resulting emergent collective dynamics. First, this section introduces some key issues in the modeling approach. Specifically, we provide some motivations, consider some aspects of a multiscale vision, and present some recent trends in crowd modeling. Then, the outline of the paper is presented.

Motivations toward the study of human crowds: Modeling and numerical studies of human crowds have captured the interest of applied mathematicians for decades. One of the main motivations is the impact of this research topic on societal wellbeing. For example, crisis managers could use models and simulations to support decision making in dangerous situations that require safe evacuation, such as those caused by fire, earthquakes and contrast of antagonist groups. A recent additional motivation comes from the SARS-CoV-2 pandemic, specifically from the need to understand how complex crowd interactions can lead to viral contagion and disease spread. We believe that the synergy between artificial intelligence and simulation platforms based on accurate models can result in improved safety. A final motivation that we would like to mention is the variety of challenging analytic and computational/numerical problems generated by the models to study realistic crowd flows. Research activity in this field needs to be developed within the general framework of the study of living (and hence complex) systems. The derivation of mathematical models cannot simply rely on a straightforward application of deterministic causality principles and the methods of classical mechanics.

Modeling scales: An important problem in crowd modeling is the selection of the representation and modeling scale. Indeed, models can be derived at the three usual scales, i.e. microscopic (individual based), macroscopic (hydrodynamic), and mesoscopic (kinetic). The third scale is in between the previous two.[12] However, it is important to search for the links between these scales. A key problem is the derivation of models at all scales based on the same principles and analogous parameters.[30] Then, kinetic models are derived from micro-scale (individual based) models and macro-scale (hydrodynamical) models are developed from asymptotic methods applied to the kinetic theory description.[18, 51, 52] See also specific applications devoted to the micro–macro derivation of other macro-scale models for living organisms.[19, 34, 63, 86, 119] An additional difficulty for the mesoscopic representation with respect to the classical kinetic theory is the search of a pseudo-Maxwellian equilibrium distribution because living systems generally live far from an equilibrium.[11]

New trends in the study of human crowds: Mathematicians have effectively heard the message delivered in Ref. 25 urging them to consider heterogeneous behavioral features in crowds and their influence on people's interactions. Indeed, the recent literature has witnessed an increasing attention to behavioral features in human crowds. This entails interpreting pedestrians as *active* particles, rather than classical particles. Social behaviors, which are modified by vocal and visual interactions, can have a significant influence on the walking strategy that pedestrians use to organize their dynamics. Thus, it is of paramount importance to account for behavioral features in order to reproduce realistic crowd dynamics. The term *social crowds* is used when social features are considered within the general framework of behavioral dynamics.[139]

Plan of the paper: The content of our paper is not limited to a review and critical analysis of the literature. We also provide important research perspectives linked to new and emerging trends. We will refer to a vast literature on crowds and social dynamics. Additional titles can be found in Ref. 107, which is an excellent collection. Although this paper does not include vehicular traffic, for some analogies between crowd dynamics and traffic we refer the interested reader to the pioneering paper by Prigogine and Herman[167] and the modeling of heterogeneous behaviors of vehicle-driver micro-scale systems.[163] Following the reasonings elaborated thus far, the rest of the paper is divided into five sections.

Section 2 provides an introduction to the specific behavioral and mechanical features of human crowds to be considered in the modeling approach when the individual and collective behavioral dynamics depend on interactions and different types of external actions, for instance vocal or visual signals to guide evacuation dynamics. Section 2 describes also a variety of social dynamics that could be considered in the modeling approach.

Section 3 defines the mathematical frameworks that at each scale provide the reference structures to be used in the derivation of the models. These structures include the mechanical variables and behavioral variables related to social dynamics[3, 43] mentioned in Sec. 2. Models are derived by inserting into said structures a phenomenological description of interactions among pedestrians and between pedestrians and external actions. In addition, Sec. 3 states the initial and initial-boundary value problem at each scale.

Section 4 delivers a survey and critical analysis of the literature introduced after Ref. 25. The survey refers to the mathematical structures defined in Sec. 3. Computational problems referred to the selection of the modeling scales are briefly reviewed. Indeed, different types of computational techniques can be used at each scale.

Section 5 provides a look at research perspectives by selecting key topics followed by hints to tackle them. In details, we consider the following topics: analytic and computational problems related to multiscale methods; study of social dynamics in crowds. Then, we show how some tools developed for the study of human crowds can be used to model behavioral swarms, i.e. animal swarms in the presence of heterogeneous social interactions. Finally, we provide some reasonings of a quest toward a mathematical theory of human crowds. All of the perspectives are presented within the general framework of the mathematics of living systems.

2. Complexity Features Towards Multiscale Behavioral Modeling

The study of human crowds requires a deep understanding of the complexity features of this specific living system. From this knowledge, one extracts the key characteristics that should be considered in the derivation of the mathematical models. This way of operating was already proposed in Ref. 30, where the authors have also shown how to apply it to all modeling scales. Our paper takes it as the first step to develop a survey and critical analysis of the state-of-the-art, followed by research perspectives. The general tendency in the field is to propose heuristic modeling approaches, which overlook the complexity of human crowds viewed as a living system. Therefore, research perspectives should be based on our knowledge of such complexity as much as possible.

For pragmatism, we select five key features without claiming that this selection is exhaustive. Indeed, additional features should be considered whenever required by the specific physical situation under consideration. Therefore, it is important to keep the approach flexible.

(1) *Perception ability and behavioral state*: Each individual (walker) has the ability to perceive emotional states by interacting with other individuals. Thus, the behavioral (emotional) state should be considered as a dynamical variable which evolves in time also as a result of vocal and visual stimuli. In general, the behavioral state depends on several factors, e.g. the *mental concentration* to reach a specific target, the *stress* induced by the perception of danger, an *aggressive attitude* to contrast antagonists, and the *awareness to contagion risk* during a pandemic. Possible *irrational behaviors* should also be considered.

(2) *Ability to express a strategy*: Living entities have the ability to develop specific *walking strategies* in relation to their *organization ability.* Therefore, walkers select a trajectory to follow in order to reach the desired target and a speed

to move along that trajectory. The walking strategy depends on the state of the entities in the surrounding environment and on the physical features of the venue where the crowd moves. Moreover, the effect of non-predictable external events on the waking strategy should be considered.

(3) *Role of the environment and the venues*: The *quality of the environment* (i.e. weather conditions for outdoor venues, geometry of the venue, and luminosity) affects the crowd dynamics. Pedestrians receive inputs from their environments and have the ability to learn from past experience. Hence, their rules of interactions evolve in time and space.

(4) *Heterogeneity*: The ability to express a strategy is *heterogeneously distributed* among walkers. This includes, e.g. different walking targets and the presence of leaders, whose aim is to direct all other pedestrians toward their own strategy. All types of heterogeneity induce stochastic features in the interactions. An interesting case study consists in understanding how irrational behaviors of a few entities can generate large deviations from the usual dynamics resulting from rational behaviors.

(5) *Nonlinear interactions*: Pedestrians have a *visibility domain*. Within this domain, interactions are *nonlinearly additive* and *nonlocal* as they involve not only immediate neighbors, but also distant entities, with some weight depending on the distance. The case of interactions with a fixed number of entities, rather than with all the entities in the visibility domain, should also be studied. Finally, it is also an important to consider the *sensitivity domain*, i.e. the domain within which a pedestrian can detect the presence of the other pedestrians. Generally, it is a domain included or equal to visibility domain and depends on the local density and on the level of individual stress.

In our opinion, the above items are the most relevant. One major difficulty to tackle is the lack of a field theory, which exists for the sciences of inert matter. This fact is strongly related to the problem of creating a rigorous mathematical approach to the study of living systems. In addition, we believe that the modeling approach always needs a multiscale vision,[12, 30] as only one observation and representation scale is not sufficient to describe the overall collective dynamics of living systems. Indeed, the dynamics at the microscopic scale define the conceptual basis toward the derivation of models at the higher scales, where observable macroscopic quantities correspond to the collective dynamics. Such dynamics emerge from a collective learning ability,[53–55] i.e., the ability to develop a self-organizing intelligence, which should be taken into account as it progressively modifies the rules of the interactions.

These features should be interpreted as specific flow conditions depending on, for example, local densities and the geometry and quality of the venues through which the crowd moves. In particular, the local density requires an appropriate choice of modeling scale. For example (see Fig. 1), individual-based models may be preferred to approximate the dynamics at low densities (right), while hydrodynamic models are generally valid at high densities (left).

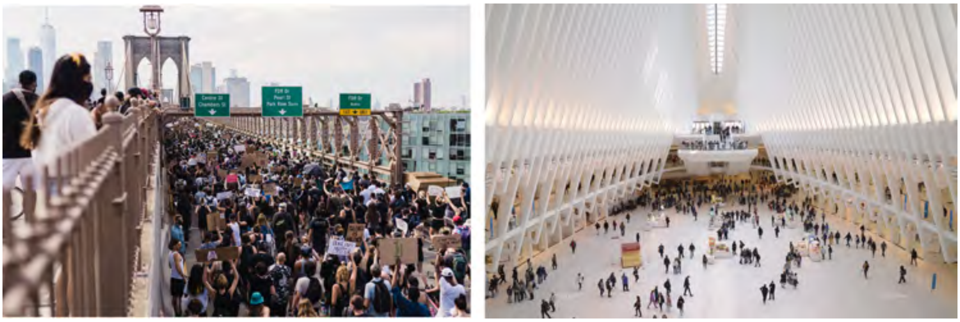

Fig. 1. Left: High density crowds; Right: Low density with obstacles.
Source: https://www.pexels.com/photo/crowd-of-protesters-holding-signs-4614164/.

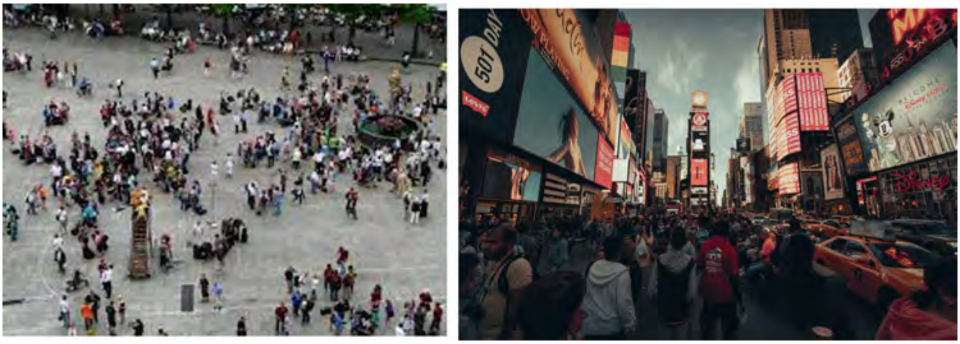

Fig. 2. Left: Mixed (high and low) density crowds; Right: Dynamics in complex venues.
Source: https://pixabay.com/photos/nyc-new-york-times-square-america-5276112.

Venues might present complex features and contain different emotional states. Figure 2 shows, on the left a crowd where dense aggregations occur within a rarefied crowd, while on the right the crowd moves in a complex venue where different types of attraction can modify trajectories.

Figures 1 and 2 contribute to the selection of the most appropriate scale, but also indicate the need to use both scales which, as we will see, is an open problem. In addition, one can have overcrowded states, as shown in Fig. 3, which might denote a high risk safety situation.

The study of crowd dynamics is often related to safety problems, for instance evacuation dynamics in dangerous conditions due to any type of incidents which are often not predictable. In some cases crowd dynamics is related to urban and transportation planning. In these cases, the emotional state is the stress. However, it is very different in each specific case so that the interpretation of the effective emotional state is one of the key action toward modeling, as the emotional state in a crowd can go through very different types of emotional states and can even reach extremes values.

Fig. 3. High risk density.
Source: https://www.pexels.com/photo/bird-s-eye-view-of-group-of-people-1299086.

Additional issues raised by the SARS-CoV-2 pandemic also suggest the development of studies of crowd dynamics in the context of open-air contagion problems, which are more critical in closed environments with limited air circulation. The modeling should take into account additional heterogeneity features such as social and physical distancing and protective devices. All of the figures above show situations of high contagion risk.

3. Scaling and Mathematical Frameworks

The derivation of mathematical models for human crowd dynamics should refer specifically to the scale selected to describe such dynamics by means of differential equations. Thus, in this section, we present the formal mathematical structures that provide the conceptual framework for the derivation of models at each scale. We consider unbounded venues or bounded venues which include walls, inlet-outlet doors and/or internal obstacles. Some notations used in the following are borrowed from Ref. 30.

Before stating the mathematical modeling approach, we introduce some general concepts and parameters. Then, mathematical structures underlying the derivation of models are stated at each specific scale. Finally, a critical analysis looks ahead to the review of models known in the literature.

3.1. *Concept and parameters*

(a) ***The geometry and the quality of the venue.*** In general, models of crowd dynamics should consider the overall geometry and the physical quality of the venue where the crowd moves:

— Σ denotes the overall geometry of the area, which is needed to model the walking trajectories;

— α models the physical quality of the venue, which is used to model the walking speed. As in Ref. 29, we consider a dimensionless parameter $\alpha \in [0, 1]$, with $\alpha = 0$ corresponding to very low quality (i.e. motion is prevented) and $\alpha = 1$ corresponding to very high quality (fast motion is allowed).

(b) ***The behavioral variable.*** An important new trend in the study of human crowds is the introduction of a behavioral variable as an independent variable in social dynamics. Such variable models the emotional state of pedestrians and it is sometimes called *activity*, borrowing the definition used in the kinetic theory of active particles.[22] The activity, denoted by $\boldsymbol{u}$, can be a vector if there is more than one behavioral state.

(c) ***The interaction domain.*** This is a domain within which pedestrians can interact. Nonlocal and nonlinearly additive interactions are considered in this domain. Note that the interaction domain is related to visibility domain or sensory domain in some references. The interaction domain will be specified at each scale.

(d) ***Dimensionless quantities.*** It is convenient to use dimensionless quantities and parameters at each scale. This strategy simplifies the implementation of computational/numerical methods and the interpretation of simulation results coming from the application of the mathematical models. Therefore, the following reference quantities are introduced to make independent and dependent variables dimensionless.

— ℓ: characteristic length of the venue where the crowd moves or the diameter of a specific circle containing the domain Σ in the case of unbounded domains;
— v_M: the highest mean speed walkers can reach in a low density flow within a high quality venue;
— $T = \ell/v_M$: the characteristic time corresponding to the time it takes a fast walker to cover distance ℓ;
— ρ_M: the maximum number of walkers in a square meter, i.e. the maximum people density;
— $u_{i,M}$ and $u_{i,m}$: the maximal and minimum values of the components of the activity $\boldsymbol{u}$, with $u_{i,m} = 0$ to be assumed in general.

Remark 3.1. With the reference quantities introduced as above, the spatial variables are referred to ℓ, the speed to v_M, the time variable to T, and the components of the activity variable to $u_{i,M} - u_{i,m}$. Macroscopic quantities such as local density and mean speed are referred to ρ_M and v_M. In this way, all quantities take values of the order of one.

Remark 3.2. Various papers describe how to model the dependence of the local speed on the quality of the venue α. Then, we denote with v_M^α the highest mean speed that can be reached by pedestrians in a low density flow in a venue with quality α.

Remark 3.3. A crowd can be subdivided into different groups, each characterized by different features. For instance, each group moves to a different direction, the walking strategy or the walking ability differs in each group. Borrowing a terminology used in the kinetic theory of active particles[31] these groups can be called *functional subsystems*, which apply to all scales.

Remark 3.4. The presentation in the next three subsections refers to one functional subsystem only. However, the modeling approach can be extended, by technical calculations, to system of human crowds with several groups or functional subsystems. The mathematical structures underlying the derivation of models are stated at each scale. These are differential systems derived by using conservation and/or balance equations. We refer to crowds in unbounded domain. Some reasonings on how to model the influence of the walls is proposed in the last subsection.

3.2. *Microscopic (individual-based) scale*

(a) ***Dependent variables.*** We consider the dynamics of a human crowd with N pedestrians moving in a domain $\Sigma \subset \mathbb{R}^2$. For $i \in \{1, \ldots, N\}$, the overall state of a pedestrian is specified by

— position variable: $\boldsymbol{x}_i = \boldsymbol{x}_i(t) = (x_i(t), y_i(t))$,
— velocity variable: $\boldsymbol{v}_i = \boldsymbol{v}_i(t) = (v_{ix}(t), v_{iy}(t))$,
— activity variable: $\boldsymbol{u}_i = \boldsymbol{u}_i(t)$.

All the above are evolving functions of the independent time variable t. By rescaling to the characteristic time T, we obtain a dimensionless time variable, which is used in the presentation. Polar coordinates $\boldsymbol{v}_i = \{v_i, \theta_i\}$ are sometimes used to define the velocity of an individual, where v_i is the dimensionless speed and θ_i is the direction of the i-pedestrian.

(b) ***Mathematical structures.*** At the microscopic level, the general mathematical framework is derived by a pseudo-Newtonian mechanics similar to the mathematical theory of behavioral swarms.[33] For this purpose, one introduces a variable $\boldsymbol{z}_i$, denoting the rate of change $d\boldsymbol{u}_i/dt$ of $\boldsymbol{u}_i$. Moreover, let Ω_i be the interaction domain of the i-pedestrian. Then, for $i \in \{1, \ldots, N\}$, a pseudo-Newtonian mechanics for $(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i, \boldsymbol{z}_i)$ is given by:

$$\begin{cases} \dfrac{d\boldsymbol{u}_i}{dt} = \boldsymbol{z}_i, \\[2ex] \dfrac{d\boldsymbol{z}_i}{dt} = \displaystyle\sum_{j \in \Omega_i} \boldsymbol{\psi}_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i, \boldsymbol{x}_j, \boldsymbol{v}_j, \boldsymbol{u}_j; \alpha, \Sigma), \\[2ex] \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \\[2ex] \dfrac{d\boldsymbol{v}_i}{dt} = \displaystyle\sum_{j \in \Omega_i} \boldsymbol{\varphi}_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i, \boldsymbol{x}_j, \boldsymbol{v}_j, \boldsymbol{u}_j; \alpha, \Sigma), \end{cases} \tag{3.1}$$

where the notation $j \in \Omega_i$ indicates that the summation refers to all j-particles in the domain Ω_i. In (3.1), $\boldsymbol{\psi}_i, \boldsymbol{\varphi}_i$ are *pseudo-accelerations* to be specified. Let us clarify a few concepts below.

— The *interaction domain* of the i-pedestrian is the domain where pedestrians can interact with the i-pedestrian. It can be an arc of circular domain symmetric with respect to the pedestrian's velocity direction, denoted by $\Omega_i = \Omega_i(\boldsymbol{x}_i, \theta_i)$ for each i-pedestrian located at $\boldsymbol{x}_i$ with walking direction θ_i. Each pedestrian interacts with all other pedestrians within such a domain by nonlocal and nonlinearly additive interactions.

— The pseudo-accelerations are introduced to incorporate the interaction rules. The action by all pedestrians in Ω_i that produces a pseudo-acceleration to the activity variable of the i-pedestrian is denoted by $\boldsymbol{\psi}_i$, while the action producing a psycho-mechanical acceleration to the velocity variable is denoted by $\boldsymbol{\varphi}_i$. Notice that both $\boldsymbol{\psi}_i$ and $\boldsymbol{\varphi}_i$ depend on the quality of the venue, modeled by α, and on the overall geometry Σ as pedestrians may modify their trajectories to avoid walls or obstacles.

Based on the key features of human behavioral modeling discussed in the previous section, $\boldsymbol{\psi}_i$ and $\boldsymbol{\varphi}_i$ should be specified such that:

— Each pedestrian is able to develop a specific, heterogeneously distributed strategy.

— A decisional hierarchy is applied under the assumption that interactions first modify the activity and subsequently the motion.

(c) ***Related issues.*** The solution of this mathematical model provides the time evolution of the dependent variables corresponding to position, velocity, and activity. Macroscopic quantities can be obtained by a local averaging at each point in the domain where the crowd moves. In practice, in a domain σ surrounding a given point $\boldsymbol{x}$, the local density $\rho(t, \boldsymbol{x})$ and the mean velocity $\boldsymbol{\xi}(t, \boldsymbol{x})$ are given by

$$\rho(t, \boldsymbol{x}) \cong \frac{\sum_{i\in\sigma} 1}{\rho_M |\sigma|}, \quad \boldsymbol{\xi}(t, \boldsymbol{x}) \cong \frac{\sum_{i\in\sigma} \boldsymbol{v}_i}{\rho(t, \boldsymbol{x})\, |\sigma|}, \tag{3.2}$$

where $|\sigma|$ denotes the measure of σ. Note that these are approximations as the limit $\sigma \to 0$ is not allowed. This is due to the fact that the system under consideration is not continuous.

The model (3.1) gives the general mathematical framework for human crowds with activity variables $\boldsymbol{u}_i$. If $\boldsymbol{u}_i \cong \boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is a constant parameter shared by all pedestrians as in traditional crowd dynamics, then this framework simplifies to:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \\[2ex] \dfrac{d\boldsymbol{v}_i}{dt} = \displaystyle\sum_{j\in\Omega_i} \varphi_i(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{x}_j, \boldsymbol{v}_j; \boldsymbol{\beta}, \alpha, \Sigma). \end{cases} \tag{3.3}$$

3.3. *Mesoscopic scale by the kinetic theory for active particles*

(a) ***Dependent variable.*** Let us consider a system of interacting pedestrians who are viewed as *active* particles by the kinetic theory approach. Instead of giving the time evolution of state variables $(\boldsymbol{x}_i, \boldsymbol{v}_i, \boldsymbol{u}_i)$ for each pedestrian, the mesoscopic (kinetic) description of the system is delivered by one particle distribution function at time t over the microscopic states

$$f = f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) = f(t, \boldsymbol{x}, v, \theta, \boldsymbol{u}), \tag{3.4}$$

where $\boldsymbol{x} \in \Sigma$, $\boldsymbol{u} \in D_{\boldsymbol{u}}$ is the activity vector in activity domain $D_{\boldsymbol{u}}$, and $\boldsymbol{v} \in D_{\boldsymbol{v}}$ is the velocity in velocity domain $D_{\boldsymbol{v}}$. We write the velocity in polar coordinates $\boldsymbol{v} = v(\cos\theta, \sin\theta) = v\boldsymbol{\omega}$, where $v \in [0, 1]$ is the *speed*, $\theta \in [0, 2\pi)$ is the velocity *direction* related to an orthogonal plane frame, and $\boldsymbol{\omega}$ is the unit vector denoting the velocity direction.

The distribution function f is linked to the so-called *test particle* (*pedestrian*) assumed to be representative of the whole system. If f is locally integrable, then $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) d\boldsymbol{x}\, d\boldsymbol{v}\, d\boldsymbol{u}$ is the (expected) infinitesimal number of pedestrians whose micro-state, at time t, is comprised within the elementary volume

$$[\boldsymbol{x}, \boldsymbol{x} + d\boldsymbol{x}] \times [\boldsymbol{v}, \boldsymbol{v} + d\boldsymbol{v}] \times [\boldsymbol{u}, \boldsymbol{u} + d\boldsymbol{u}] \tag{3.5}$$

of the space of the micro-states. Note that the function f may be divided by ρ_M, which is the maximal packing density of pedestrians as defined above.

(b) ***Mathematical structures.*** The general mathematical structure for time evolution of the distribution function f can be obtained from a balance of particles in the elementary volume of the space of the micro-states (3.5). This equation is derived by equating the rate-of-change of the number of active particles (a-particles for short) plus the transport due to the velocity variable to the net flux rate within the elementary volume, that is

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f = J[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}), \tag{3.6}$$

where the dot product denotes the standard inner product in $\mathbb{R}^2$, $\nabla_{\boldsymbol{x}}$ denotes the gradient operator with respect to the space variables only, and $J[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u})$ is the net flux rate due to interactions. As in traditional kinetic theory, the interaction term is made of two parts in general:

$$J[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) = \mathcal{G}[f, f] - f\mathcal{L}[f],$$

where $\mathcal{G}$ and $\mathcal{L}$ represent *gain* and *loss* (both nonlinearly acting on f) of pedestrians in the elementary volume of the phase space about the test microscopic state $(\boldsymbol{x}, \boldsymbol{v}\boldsymbol{u})$. The detailed expression of these terms corresponds to different ways of modeling pedestrian interactions at the microscopic scale.

In the kinetic modeling of human crowds, the interaction term is sometimes written as

$$J[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) = \mathscr{J}_G[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) + \mathscr{J}_P[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}), \tag{3.7}$$

where $\mathcal{J}_G$ and $\mathcal{J}_P$ are related to modeling *geometrical effects* and *interactions among pedestrians* within the interaction domain. More precisely, the term $\mathcal{J}_G$ models the desire pedestrians have to reach the exits and their ability to avoid the walls or obstacles, while $\mathcal{J}_P$ represents each individual's tendency for less crowded areas and/or to follow the main stream of motion.

We will mainly focus on the derivation of the term $\mathcal{J}_P[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u})$, which is determined by the interaction rules and walking strategies of each pedestrian. To specify the interaction rules, we need to introduce three types of a-particles (pedestrians):

— *test particles* with distribution function $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u})$: they are representative of the whole system;
— *field particles* with distribution function $f(t, \boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{u}^*)$: by interacting with them test particles may lose their micro-state;
— *candidate particles* with distribution function $f(t, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{u}_*)$: they can acquire, in probability, the micro-state of the test particle after interaction with the field particles.

Interactions among pedestrians lead to a modification of activity, velocity direction, and speed depending on the micro-state and distribution function of the pedestrians in the interaction domain. Basic concepts about the interaction rules include:

— *The local interaction domain* $\Omega = \Omega(t, \boldsymbol{x}, \theta; R, \Theta)$: A circular sector located at the present position $\boldsymbol{x}$, with radius R, symmetric with respect to the velocity direction θ, with "visibility" angles Θ and $-\Theta$. In this domain, pedestrians perceive local density and density gradients in Ω, and interact with the other pedestrians.
— *Perceived density* ρ_θ^p: Pedestrians moving along the direction θ perceive a density ρ_θ^p different from the local density ρ. Models should account for the fact that $\rho_\theta^p > \rho$ when the density increases along θ, while $\rho_\theta^p < \rho$ when the density decreases.
— *Transition probability density* $\mathcal{A}[f](\boldsymbol{v}_* \to \boldsymbol{v}, \boldsymbol{u}_* \to \boldsymbol{u} | \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma)$: The probability that a candidate particle at $\boldsymbol{x}$ with state $\{\boldsymbol{v}_*, \boldsymbol{u}_*\}$ shifts to the state of the test particle $\{\boldsymbol{v}, \boldsymbol{u}\}$ due to the interaction with field particles with state $\{\boldsymbol{v}^*, \boldsymbol{u}^*\}$ in Ω.
— *Interaction rate* $\eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma)$: The frequency with which a candidate (or test) particle at $\boldsymbol{x}$ enters in contact with field particles in Ω.

Based on these concepts, the interaction term $\mathcal{J}_P[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u})$ can be written as follows:

$$\begin{aligned}\mathcal{J}_P[f](t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) = \int_{\Gamma \times D_{\boldsymbol{v}} \times D_{\boldsymbol{u}}} & \eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma) \\ & \times \mathcal{A}[f](\boldsymbol{v}_* \to \boldsymbol{v}, \boldsymbol{u}_* \to \boldsymbol{u} | \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}_*, \boldsymbol{v}^*, \boldsymbol{u}_*, \boldsymbol{u}^*; \alpha, \Sigma)\end{aligned}$$

$$\times f(t, \boldsymbol{x}, \boldsymbol{v}_*, \boldsymbol{u}_*) f(t, \boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{u}^*) d\boldsymbol{x}^*\, d\boldsymbol{v}_*\, d\boldsymbol{v}^*\, d\boldsymbol{u}_*\, d\boldsymbol{u}^*$$
$$- f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) \int_\Gamma \eta[f](\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{v}^*, \boldsymbol{u}, \boldsymbol{u}^*; \alpha, \Sigma)$$
$$\times f(t, \boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}^*, \boldsymbol{u}^*) d\boldsymbol{x}^*\, d\boldsymbol{v}^*\, d\boldsymbol{u}^*, \tag{3.8}$$

where $\Gamma = \Omega \times D_{\boldsymbol{v}} \times D_{\boldsymbol{u}}$. Note that, here and in below, the square brackets denote the functional dependence with respect to its arguments, that is, e.g. dependence on the spatial derivatives of the arguments in brackets.

(c) ***Related issues.*** We remark that the macroscopic observable quantities can be defined, under suitable integrability assumptions, by weighted moments of the distribution function. For instance, the local *density* reads

$$\rho(t, \boldsymbol{x}) = \int_{D_{\boldsymbol{v}}} \int_{D_{\boldsymbol{u}}} f(t, \boldsymbol{x}, v, \theta, \boldsymbol{u}) v\, dv\, d\theta\, d\boldsymbol{u}, \tag{3.9}$$

where $D_{\boldsymbol{v}} = [0, 2\pi) \times [0, 1]$ in polar coordinates, and the *mean velocity* is defined as

$$\boldsymbol{\xi}(t, \boldsymbol{x}) = \frac{1}{\rho(t, \boldsymbol{x})} \int_{D_{\boldsymbol{v}}} \int_{D_{\boldsymbol{u}}} \boldsymbol{v} f(t, \boldsymbol{x}, v, \theta, \boldsymbol{u}) v\, dv\, d\theta\, d\boldsymbol{u}. \tag{3.10}$$

It is worth mentioning that the mesoscopic probability distribution is defined over a multidimensional state space, typically two components for the position and two for the velocity in case of crowds moving in two-dimensional domains, and the activity variables, in addition to its dependence on time and space. Therefore, devising computationally efficient numerical methods for these problems is highly nontrivial. This further motivates the search for mathematical structures at higher scales capable of reducing such computational complexity.

3.4. *Macroscopic hydrodynamic modeling*

(a) ***Dependent variables.*** The macroscopic scale adopts an Eulerian-type hydrodynamic description of a system of human crowds, in which the global crowd dynamics are modeled by macro-scale variables $(\rho, \boldsymbol{\xi}, \boldsymbol{u})$ defining the state of the system:

— $\rho = \rho(t, \boldsymbol{x})$ is the dimensionless local *density* of the crowd at the point $\boldsymbol{x}$ and time t, normalized with respect to the maximum packing density ρ_M;
— $\boldsymbol{\xi} = \boldsymbol{\xi}(t, \boldsymbol{x})$ is the dimensionless mean *velocity* at the point $\boldsymbol{x}$ and time t, normalized with respect to the maximum average speed ξ_M. The mean velocity can also be expressed in polar coordinates as follows: $\boldsymbol{\xi} = \xi(t, \boldsymbol{x}) \boldsymbol{\omega}(t, \boldsymbol{x})$, where ξ is the dimensionless mean speed and $\boldsymbol{\omega}$ is the unit vector giving the direction of the local mean velocity;

— $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ is the dimensionless local mean *activity* representing the specific social-emotional state considered in each case study, with $\boldsymbol{u} \in D_{\boldsymbol{u}}$ for a specific parameter domain.

To describe the local interaction between pedestrians at the macroscopic scale, one also needs to define the interaction domain Ω also used at the lower scales. Recall that the pedestrians at $\boldsymbol{x}$ perceive the action of all pedestrians in $\Omega = \Omega(t, \boldsymbol{x}; \boldsymbol{\omega}(t, \boldsymbol{x}))$, which makes interactions nonlocal.

(b) ***Mathematical structures.*** At the macroscopic scale, the human crowd is described by a second-order differential system for density $\rho(t, \boldsymbol{x})$, velocity $\boldsymbol{\xi}(t, \boldsymbol{x})$ and the activity $\boldsymbol{u}(t, \boldsymbol{x})$:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{\xi}) = 0, \\[2ex] \dfrac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{\xi} = \boldsymbol{A}[\rho, \boldsymbol{\xi}, \boldsymbol{u}], \\[2ex] \dfrac{\partial \boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u} \boldsymbol{\xi}) = \boldsymbol{S}[\rho, \boldsymbol{\xi}, \boldsymbol{u}], \end{cases} \tag{3.11}$$

where $\boldsymbol{A}$ is a pseudo-mechanical acceleration acting on pedestrians in the infinitesimal volume $d\boldsymbol{x}$ and $\boldsymbol{S}$ is a source term that implements locally the emotional state generated by the interaction with the surrounding pedestrians. Both nonlocal and nonlinearly additive interactions are enclosed in these terms. As before, the square brackets denote functional dependence with respect to its arguments.

If one does not wish to consider the social behavioral variable, as in traditional crowd dynamics, the model (3.11) can be simplified by taking the activity $\boldsymbol{u}$ as a uniformly distributed constant in space and time, i.e. $\boldsymbol{u} \cong \boldsymbol{\beta}$ constant, to get

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho\, \boldsymbol{\xi}) = 0, \\[2ex] \dfrac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{\xi} = \boldsymbol{A}[\rho, \boldsymbol{\xi}, \boldsymbol{\beta}]. \end{cases} \tag{3.12}$$

3.5. *Critical analysis*

This subsection proposes some brief remarks on the general framework of the models. Firstly, we distinguish between first- and second-order models, subsequently, some reasonings are proposed on the statement of boundary conditions.

• Let us consider *first- and second-order models.* Models like (3.11) and (3.12) are called second order, as that has been widely used also in the modeling of traffic flow, before the study of crowd dynamics. The essence of second-order models is than both position and velocity are described by using second-order dynamics such as (3.12), a system of conservation laws, for the study of crowd dynamics. This is also the case at the microscopic modeling, see for example in (3.3), a second-order system is used to describe the position-velocity pair of microscopic states for each pedestrian. Note that the perceived distribution of both positions and velocity of the surrounding walkers allows pedestrians to adapt their behavioral strategy in the detailed description of micro-, meso- or macro-scale interactions.

There are another type of models, called first-order models, which are simpler than second-order models. Take $\boldsymbol{\xi} = \boldsymbol{\xi}(\rho)$ be a specified function in (3.12) for example, this second-order model can be replaced by a simpler first-order model

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{\xi}(\rho)) = 0. \tag{3.13}$$

For this simpler version of model, it is assumed that agents adjust instantaneously their velocities according to the density they are experiencing (which implies infinite acceleration) and take into account the slightest change in the density. However, this simpler model might contradict empirical observations in some cases. This can be avoided by using more sophisticated second-order models.[89]

At the microscopic level, a first-order version of the second-order model (3.3) can be obtained, by simply assuming a function $\boldsymbol{v}_i = \boldsymbol{v}_i(\rho)$, where ρ is the local density defined in (3.2). Then the second-order model (3.3) is simply replaced by

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i(\rho). \tag{3.14}$$

The same strategy can also be used in the mesoscopic model (3.6), in which the speed v is suppressed by specifying $v = v(\rho)$ with the concept of perceived density which was introduced in the kinetic theory approach to modeling vehicular traffic.[75] Some precise form of the speed v can be given, for example in Refs. 184, 181 and 182, such that the maximal speed is kept under low density conditions (free flow regime), i.e. up to a certain critical density ρ_c, while the speed decreases to zero, with a polynomial-type dependence on the local density, for values of ρ greater than ρ_c (slowdown zone). See also Refs. 97, 71, 72 and 73 for additional interesting studies on empirical data and their interpretation toward modeling.

• Let us now consider *the role of boundary conditions.* The solution of mathematical problems, typically initial-boundary value problems, needs, at each scale, the statement of initial conditions and boundary conditions, whenever the crowd moves in venues with walls, internal obstacles, and inlet-outlet doors.

Initial conditions, i.e. at $t = 0$, are defined, at the micro-scale by position, velocity and activity for all individuals in the crowd; at the meso-scale by the distribution function over position, velocity and activity for the test particle for all $\boldsymbol{x}$ in the walking domain; and at the macro-scale by the density and velocity for all $\boldsymbol{x}$ in the walking domain.

Boundary conditions require models of a reflection dynamics at walls for all dependent variables.[30] In addition, the presence of walls modifies the walking trajectories by the strategy developed by walkers in order to avoid collision with walls.

This action occurs if the direction of the trajectory meets a wall and increases as the distance from the wall decreases.

• The *perception ability* has an important role in the derivation of models as walkers perceive, at each scale, the distribution of both positions and velocity of the surrounding walkers, allows pedestrians to adapt their behavioral strategy in the detailed description of micro-, meso- or macro-scale interactions.

4. Review and Critical Analysis of the Existing Literature

This section provides a survey and critical analysis of the literature on mathematical approaches on modeling, simulations, and numerical analysis of dynamics of human crowds. We consider some pioneering papers and mainly those which appeared after review.[25] The report specifically refers to the framework reported in Sec. 3. It is presented in the next three subsections accounting for a critical analysis, which pervades the whole section, focused on how far the various model and applications consider the complexity features reported in Sec. 2. Section 4.4 proposes some reasonings, valid at each scale, on the conceptual difficulties of the modeling approach. These reasonings lead to the research perspectives which are treated extensively in the next section.

4.1. *Microscopic (individual based)*

Here, we review some key works on microscopic models for crowd dynamics. These range from molecular (individual based) and Brownian dynamics to cellular automata and agent-based models.

4.1.1. *Molecular/Brownian dynamics-like based models: The social force model and its variants*

The majority of the models in this category are based on the celebrated social force model (SFM) for pedestrian dynamics[115] proposed back in the mid '90s. Its derivation is based on the Langevin-type equation that contains a frictional term and a random force term reading:

$$m_i \frac{d\boldsymbol{x}_i}{dt} = -\frac{1}{\tau_i}(v_{0i}\boldsymbol{e}_i - \boldsymbol{v}_i) + \sum_{j \in \Omega_i} \boldsymbol{\phi}_i(\cdot) + \sigma_i \boldsymbol{\xi}_i(t), \tag{4.1}$$

where v_{0i} is the maximum walking speed for an individual i, $\boldsymbol{e}_i$ is the desired direction, τ_i is a characteristic relaxation time, $\boldsymbol{\phi}_i(\cdot)$ is the force exerted to the ith pedestrian, and $\boldsymbol{\xi}_i(t)$ is a random force and σ_i its amplitude. The random term, usually represents fluctuations that occur around the preferred paths[73] and it is considered to be a Gaussian process. Thus, the expected value of random force vanishes.

The idea behind the SFM is that the motion of individuals is governed by "virtual forces". These forces include the acceleration toward the desired destination, a resultant repellent force "exerted" by obstacles/borders and by the pairwise interaction between nearby individuals, that keeps an individual at a distance from another. Furthermore, one may also consider the effect of a resultant attractive force due to mimetic behavior that may observed for example in situations such as those of emergency evacuation or the movement toward objects of common interest.[115]

For the numerical integration of the equations of motion of the SFM and its variants, one should take into account that the resulting system is not Hamiltonian due to the friction term. Thus, established methods that have been developed for Molecular dynamics should be used with caution.[138]

The SFM has been widely used to simulate crowd dynamics with applications ranging from guided crowd dynamics[204] and the behavior of pedestrian at signalized intersections[209, 212] to emergency evacuation[112, 114, 169, 210] and the mitigation of the spread of infectious diseases,[82] to name just a few.

On the other hand, various works have been focused on the adaptation/extension of the SFM in various directions. For example, in Ref. 207, the authors have combined the pairwise Mutual Information[90] and the SMF to evaluate the disorder of an escaping crowd, thus adjusting the parameters of the SFM in a dynamic way in order to achieve an optimal evacuation. In another work, the SFM has been adapted to include stochasticity in order to classify and quantify changes due to the collision-avoidance movement.[74] Moreover, the SFM has been the basis for formulating the collision-avoidance problem between two individuals, treated like a Nash-equilibrium problem, Ref. 176 in order to derive a Fokker–Planck (convection-diffusion PDE) equation in the thermodynamic limit.[60]

4.1.2. *Cellular automata models*

The concept of cellular automaton (CA) has been created by Stanislav Ulam and John von Neumann in 1950s and finalized in its current form by John von Neumann in 1960s.[197] Since then, CA have been rapidly developed and widely used to model the dynamics of complex systems including crowd dynamics.[38, 158] In a nutshell, CA models are discrete dynamical systems defined by three main components, namely the discrete lattice of cells, the states, i.e. a set of discrete variables, and the transition rules that govern the way cells change their state in discrete time steps.

Therefore, in a general way, the evolution of crowd dynamics can be represented by a discrete-time/discrete space model of the form:

$$\boldsymbol{s}_i(t+1) = \boldsymbol{R}(\boldsymbol{s}_i(t), \boldsymbol{s}_{i-l_1}(t), \ldots \boldsymbol{s}_{i+l_2}(t), \boldsymbol{u}(t)). \tag{4.2}$$

$\boldsymbol{s}_i(t) \in \mathbb{R}^d$ is a d-dimensional vector containing the states at the cell i that can be discrete or continuous variables, including for example information about if the cell i is empty or occupied, if it corresponds to a border/object, the speed and direction of the individual that occupies the cell at time t, his/her desired direction, etc.;

the integers l_1, l_2 define the radius of the pairwise interactions,[42] $\boldsymbol{R} : \mathbb{R}^N \to \mathbb{R}^d$ denotes the evolution operator that can be deterministic or probabilistic (usually a Markovian process), and N denotes the number of cells at time t that influence the state of the cell at time $t + 1$. Finally the vector $\boldsymbol{u} \in \mathbb{R}^l$ reflects the influence of external stimuli, such as alarms, announcements, external notifications, and weather conditions.

A two-dimensional lattice is usually formed by square cells of approximately $40\,\text{cm} \times 40\,\text{cm}$,[137] which is the average space around an individual on a layout plan view; taking into account that the average walking speed at normal situations (i.e. when people are not in a hurry or under panic) is $1.3\,\text{m/s}$ (see Ref. 115) one can derive the unit time step of the simulation (the time of transition from a cell to a neighbor cell) which is $0.3\,\text{s}$.[137]

In the simplest case of a two-dimensional square lattice, the state of each cell is a binary variable (empty of occupied), while simple rules of motion can be described in a probabilistic way for example as follows (see also Ref. 56). In each time step, individuals stay still or move simultaneously according to a transition probability p_{ij} in their Moore neighborhood defined by the eight cells surrounding the cell occupied by the individual. This transition probability depends on the desired direction and the speed of the individual and the "free cells" in its neighborhood. If more than one individual decides to move to the same not-occupied cell, then only one is allowed to go there; this choice is made according to the corresponding relative transition probabilities. If the cell in which the individual has "decided" to move is already occupied, then he/she does not move.

Cellular Automata models have been used to approximate the emergent flows for a bidirectional pedestrian walkway.[74] The idea of chemotactic responses to attractant and repellent sources, has been used in the so-called *floor field* model[56, 137] to describe long-range interactions, in order to approximate collective dynamics in counter flows, such as lane formation, and to study the effect of the cell size and the maximum walking speed. Cellular Automata have been also combined with game-theoretical approaches to model rationality, herding, and conflict during an emergency evacuation.[211] Other studies have coupled CA with fuzzy logic to simulate crowd evacuation processes, with the usage of a Mamdani-type fuzzy inference system for defining the transition rules.[102] For a review of CA modeling approaches for crowd dynamics, see Ref. 145.

4.1.3. *Agent-based models*

Agent-based (AB) simulators are the state-of-the-art in individual-based, say micro-scale, modeling of crowd dynamics. AB models introduce heterogeneity in the behavior of individuals and can approximate real-world environments/urban architectures, thus integrating concepts and techniques from both the SFM and CA modeling approaches but also from other disciplines, ranging from game theory and complex networks to epidemiology, neuroscience and sociology.[110, 201]

Agent-based models have been used to model many complex crowd dynamics under realistic situations and scenarios. One of the first attempts in this particular field was that of the simulation of evacuation from a hospital.[64] In this particular model, agents possess six attributes that influence their mobility, namely walking speed, physical size, ability to traverse, perception, psychological profile, and assistance needs including motorized and non-motorized wheel-chair users. Another application of AB models is for the simulation of pedestrian flow in a continuous space which is represented as a network.[148]

Agent-based models have been also developed to simulate evacuation under structural damages of buildings subjected to ground motions due to earthquakes.[147] In this model, agents were categorized according to their gender, age, body size, walking speed, stride length and step frequency based on experimental data. Recently, AB models have been used to simulate evacuation from the emergency department at the Johns Hopkins Hospital during the COVID-19 pandemic due to a fire emergency.[109] The agents were categorized in two main types, namely patients and staff that have the mutual interest of reaching the safe zone. Patients were further categorized into visually impaired, hearing impaired, mobility impaired, mentally impaired, and non-disabled. Mobility impaired patients were further classified into wheelchair users, motorized wheelchair users, stamina impaired, high-acuity bed-bound, and low-acuity bed-bound patients, while non-disabled patients were categorized as elderly, children, and adults.

4.1.4. *Critical discussion on microscopic simulations*

Models derived at the microscopic models and, in particular, highly detailed agent-based models are the state-of-the-art in the simulation of realistic crowd dynamics. However, all models are just approximations of the real-world dynamics. Because of the inherent complexity, strong heterogeneity and nonlinearities in the interactions between individuals, AB-models are built with uncertainty on the various parameters, variables and evolution rules ranging from physical to sociological, cognitive and emotional ones.

What is usually done with such detailed simulators in order to relax the inherent uncertainty and to study the emergent dynamics is to run "brute force" temporal simulations. One usually sets up many initial (macroscopic) conditions, for each one of them create a large enough number of ensemble (microscopic) realizations, probably changes some of the rules and then runs the detailed dynamics for a long time. The aim is to investigate how parameters such as initial conditions, walking speeds, "social-forces", architectural and urban interventions, the use of alerts and signs, may influence the collective behavior, the time evacuation, the capacity of a shared space or the rate of spread of an epidemic in a structure.

However, such "experimental" simulations suffer from the "curse of dimensionality" and are therefore insufficient for the systematic numerical analysis, optimization and design of controllers for shaping the emergent dynamics.

An alternative is the statistical-mechanics-based approach presented above aiming at extracting macroscopic evolution laws and then analyze and control their dynamics with the best available continuum-level techniques. The gap between the high-dimensional agent-based space and the low-dimensional emergent/macroscopic/hydrodynamical scale is bridged through closures, relating higher-order, moments to a few, low-order moments of the underlying detailed distributions. However, such closures are based on assumptions introducing certain biases in the modeling and numerical analysis. Infinite size of population, homogeneous agents, homogeneous interaction networks are some of the assumptions that may bias the analysis at the density/crowd level.

Thus, bridging systematically the individual to the emergent crowd dynamics constitutes an important, open challenge in the contemporary crowd dynamics modeling, systematic qualitative and numerical analysis, and control. In Sec. 4.4, we briefly touch on this problem.

4.2. *Mesoscopic (kinetic)*

This subsection reports about different methods and applications concerning modeling approaches developed by methods of generalized kinetic theories. In detail, we consider models with discrete velocities, models with behavioral-social dynamics, and contagion problems in crowds. These topics are selected among a variety of possible case studies.

4.2.1. *Kinetic models with discrete velocities*

Let us consider *discrete velocity models*, in which the velocity modulus v (speed) and the velocity direction θ take discrete values in

$$I_v = \{v_1 = 0, \dots, v_i, \dots v_m = 1\}$$

and

$$I_\theta = \left\{\theta_1 = 0, \dots, \theta_j, \dots \theta_n = \frac{n-1}{n} 2\pi\right\}.$$

Therefore, the overall state of the system is described by the set of probabilities:

$$f = f(t, \boldsymbol{x}) = \{f_{ij}(t, \boldsymbol{x})\}, \tag{4.3}$$

where $f_{ij}(t, \boldsymbol{x})$ denotes the probability, at time t and position $\boldsymbol{x}$ that a pedestrian moves with speed v_i and direction θ_j.

Accordingly, the mathematical structure writes as a system of $n \times m$ equations

$$(\partial_t + \boldsymbol{v}_{ij} \cdot \nabla_{\boldsymbol{x}}) f_{ij}(t, \boldsymbol{x}) = J_{ij}[f](t, \boldsymbol{x}), \tag{4.4}$$

where $J_{ij}[f]$ is the operator modeling interactions.

Macroscopic quantities are obtained as in Eqs. (3.9) and (3.10), where finite sums replace integrals

$$\rho(t, \boldsymbol{x}) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(t, \boldsymbol{x}), \tag{4.5}$$

while analogous calculations lead to mean velocity

$$\boldsymbol{\xi}(t, \boldsymbol{x}) = \frac{1}{\rho(t, \boldsymbol{x})} \sum_{i=1}^{n} \sum_{j=1}^{m} v_i (\cos\theta_j \boldsymbol{i} + \sin\theta_j \boldsymbol{j}) f_{ij}(t, \boldsymbol{x}), \tag{4.6}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors of an orthogonal frame.

Remark 4.1. In general, models can introduce discrete or continuous activities, as well as different groups of pedestrians, i.e. functional subsystems (FSs). Then, a superscript h, with $h = 1, \dots, r$, denotes each FS.

Remark 4.2. It is useful to normalize the terms f_{ij} with respect to the maximal packing density ρ_M, defined in Sec. 3, and similarly the speed is divided by v_M, see Remark 3.1.

The model proposed in Ref. 20 considers a dynamics of different FSs in unbounded domains, including antagonist groups, where each FS uniformly shares different activity variables. The dynamics refer to:

(a) ***Selection of preferred velocity direction.*** The search for the preferred direction is obtained by combining, weighted by the local density, the following trends:

— **T1**: Trend to the exit or a preferred direction;
— **T2**: Attraction by the mean stream induced by the other pedestrians;
— **T3**: Search of less crowded direction with minimal density gradient.

In particular, increasing local density increases **T2** and **T3** and weakens **T1**.

(b) ***Modeling the speed.*** The speed is modeled by a simple phenomenological model accounting for the local density and the quality of the venue. More precisely, within the framework of first-order model, the speed is given by $v = v[\rho]$, i.e. it is supposed to depend on the local density in a functional way.

Remark 4.3. The dependence of v on ρ can be based on empirical data in uniform flow conditions, see Refs. 180–182, 184. Some authors, see Refs. 24 and 29, include density gradients to account for the so-called, perceived density, which is greater (lower) than the real one for positive (negative) slopes.

This approach corresponds to a *hybrid model*, as both kinetic and macroscopic structures are used. In detail:

— The velocity *direction* is modified in a kinetic way, by modeling collective behaviors from microscopic interactions with tools of kinetic game theory;

— The *speed* is heuristically supposed to be induced by macro-scale quantities, i.e. it depends on the local density.

Detailed descriptions of the model are reported in Sec. 2 of the pioneering article,[20] where, in addition to modeling, existence of solutions is proved for arbitrarily large times, while simulations are obtained through computational schemes based on splitting methods, where the transport equations are treated by finite difference methods for hyperbolic equations. Some preliminary reasonings toward the modeling of stress conditions, related to awareness of danger, are also proposed. Subsequently, the derivation of macro-scale equations from the underlying mesoscopic description was developed by an asymptotic limit of the kinetic models.[18]

The contents of Ref. 20 have motivated various developments generally focused on specific applications, for instance, the study of evacuation dynamics during stress conditions, see Refs. 2, 133 and 146. Further developments correspond to crowd dynamics in domains with boundaries and obstacles.[133, 146] In this case, an additional trend must be considered in the modeling approach, i.e.

— **T4**: Trend to avoid collisions with walls or obstacles.

Technical calculations to model how the selection of trajectories accounts for all trends **T1**–**T4** are reported in Ref. 12. Applications mainly refer to the study of evacuation dynamics focusing both on pattern formation and evacuation time. Patterns of the flow are related to overcrowding phenomena that indicate high risk against safety. In detail, the following studies have been developed:

— Numerical simulations of evacuation time depending on the size of the exit zone, on the initial distribution of the crowd, and on a parameter which weighs the unconscious attraction of the stream and the search for less crowded walking directions.[2]
— Lane formation in bidirectional flow in corridors, evacuation from a room with one or more exits with variable size, with and without obstacles.[27, 133]
— Crowd dynamics of several groups characterized by different *motility* and walking strategy. Some pattern formations are shown by numerical simulations, for instance, lane formation and clustering of a crowd with several groups having different motility.[146]

4.2.2. *Kinetic models with behavioral-social dynamics*

The models reviewed in the preceding subsection indicate that the overall dynamics depend on specific parameters modeling the quality of the venue, the walking ability, and the emotional state of the pedestrians.

The role of the activity variable at all scales, from microscopic to macroscopic, has been studied in Ref. 12. In principle, this emotional state is heterogeneously distributed in the crowd. Such state significantly affects the overall crowd dynamics in extreme real-life situations such as a peaceful demonstration that turns violent[88] and the spreading of panic in emergency evacuations.[108] Studies devoted to safety problems clearly indicate that crisis management can take advantage of models that account for human behaviors such as spreading of emotional states.[172, 173, 175, 203]

On the other hand, fully consistent behavioral models should describe the dynamics of interactions and propagation of specific emotional state. The foundations of the mathematical approach on this topic were proposed in Ref. 27 by inserting in the microstate of the a-particles an additional social variable and accounting for the dynamics of this variable also in the transition probability density $\mathcal{A}[f]$. Numerical simulations show clear emergence behaviors such as pedestrian segregation into two groups walking in opposite directions in a crowded street.[27]

The key problem is then modeling the dynamics of the activity variable and consequently the dynamics of the mechanical variables. Recent studies have been focused on the aforementioned concepts. For instance, a kinetic modeling of human crowds with behavioral-social dynamics is considered in Ref. 29, in which interactions are supposed to trigger a decision process which comprises the following sequential steps:

(1) Exchange of the stress state;
(2) Selection of the walking direction;
(3) Selection of the walking speed.

The crowd evacuation from a metro station is simulated in Ref. 29 to enlighten the role of the emotional state in the overall dynamics. The numerical results show that stress propagation significantly affects crowd density patterns and overall crowd dynamics.

An interesting model for crowd dynamics with emotional contagion is considered in Ref. 41, in which interacting pedestrians modify their psychological status and, in turn, the walking strategy. The model involves pedestrian movement with a speed proportional to a "fear" variable that undergoes a temporal consensus averaging based on the distance to other agents. The problem is approached by an agent-based model as well as in a continuum limit. However, in the case that when pedestrian paths cross, the continuum PDE model does not capture the dynamics of the particle system accurately due to crossing of the characteristics of the PDE, then a kinetic equation is introduced to provide a continuous description of the particle model. More precisely, therein, the emotional state, fear, is propagated by a Bhatnagar–Gross–Krook (BGK) type kinetic model

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial x}(uf) = \frac{\partial}{\partial u}\gamma((u - u^*)f), \tag{4.7}$$

where $f(t, x, u)$ is the probability density depending on time t, position x and local fear level u. The quantity $u^*(t, x)$ is the "average" fear level for location x at time t computed as the mean fear level of individuals weighted by their distance from x:

$$u^*(t,x) = \frac{\iint \kappa(|x-y|) f(t,y,u) u \, du \, dy}{\iint \kappa(|x-y|) f(t,y,u) du \, dy}, \tag{4.8}$$

with a interaction kernel $\kappa(r)$ that decays with r and integrates to one.

The results are limited to one space dimension. The numerical examples with the kinetic description can recover the behavior of emotional contagion propagation observed at the microscopic level. This model is further studied in Ref. 200 by appropriate numerical methods, which further illustrates that kinetic description provides better resolution than the macroscopic model whose viscosity solution becomes incorrect when the characteristics at the microscopic level cross. Note that Ref. 41 has provided a multiscale approach, from microscopic to macroscopic, for crowd dynamics with emotional contagion. Recent studies have further developed this research line.[206]

Motivated by the studies in above references, the following hybrid model of crowd dynamics with emotional contagion has been developed[131]:

$$\frac{\partial}{\partial t} f_i + u(\cos\theta_i, \sin\theta_i) \cdot \nabla_{\boldsymbol{x}} f_i = J[f] + \gamma((u-u^*)f_i)_u, \tag{4.9}$$

where $f_i(t, \boldsymbol{x}, u)$ is the distribution function at time t, position $\boldsymbol{x}$, with moving direction θ_i and emotion level u. Several evacuation scenarios involving two groups of interacting pedestrians are investigated in Ref. 131 to assess the impact of domain geometry on the fear propagation and evacuation dynamics.[131, 135]

Remark 4.4. Note that the interaction term $J[f]$ in (4.9) takes into account **T1**–**T4** as the standard model for crowd dynamics.[12] However, **T2**–**T3** are weighted by the emotional state u, such that, a higher level of emotional state (more specifically, fear level) results in a higher tendency to follow the stream unconsciously (**T2**) and lower ability to search for the less congested direction (**T3**).

4.2.3. *Kinetic models with epidemic spread*

The onset of the SARS-CoV-2 pandemic and subsequent space diffusion has motivated studies somehow related to crowd dynamics. In general, contagion occurs by contact, or simply clustering, of people in a crowd. Diffusion is a subsequent dynamics related on the movement of the crowd in a territory also by using public or individual transportation.

It is worthwhile making precise the role of the social variable in the dynamics thus, distinguishing emotional and awareness to the risk of epidemic contagion, despite of their apparent similarity. The key social state in the former is the level of stress, while in the latter is the level of awareness, which lead to completely different consequences, as stress promotes aggregation of pedestrians and leads to the herd behavior under stress conditions,[140] while the level of awareness pushes pedestrians to follow social distancing guidelines to prevent epidemic spread.

Kinetic models of crowd dynamics are also applied, after the onset of the SARS-CoV-2 pandemic, to the study of epidemic spread in human crowds. In details, a hybrid approach, coupling a kinetic model of crowd dynamics[27] and epidemic contagion model in one dimensional crowd motion, is introduced in Refs. 134 and 135:

$$\begin{cases} \dfrac{\partial}{\partial t} f_i + \dfrac{\partial}{\partial x}(v_i[\rho] f_i) = J_i[f], \\ \dfrac{\partial h_i}{\partial t} + \dfrac{\partial}{\partial x}(v_i h_i) = \gamma((u - u_i^*) g_i)_u, \end{cases} \tag{4.10}$$

where $f_i(t,x)$ is the pedestrian distribution function at time t, position x, with moving direction θ_i, $h_i(t,x,u)$ is the probability of finding people with contagion level u at time t, position x, and with walking direction θ_i, for two directions $i = 1,2$ in one-dimensional case. The quantity $u_i^*(t,x)$ is the local average contagion level of pedestrians with moving direction θ_i weighted by their distance from x as similarly defined in (4.8). This model is one way coupled, in which the first equation is closed but the second one depends on the solution of the first one, meaning that the contagion spreading depends on but does not influence the crowd motion.

Another kinetic model of crowd dynamics with epidemic spread is the one considered in Ref. 179, in which the pedestrians are described by different groups (i.e. susceptible, exposed, and infected) and the pedestrian dynamics are modeled by a kinetic equation for multi-group pedestrian flow accounting for a nonlocal Susceptible–Exposed–Infected–Susceptible (SEIS) contagion model for disease spread:

$$\frac{\partial f^k}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f^k + R[f^k] = T^k, \tag{4.11}$$

where $f^k = f^k(t, \boldsymbol{x}, \boldsymbol{v})$, $k = S, E, I$, are the distribution functions of susceptible (S), exposed (E) and infected (I) pedestrians. The operator R is used to represent interactions between pedestrians. A social force model is utilized in Ref. 179. However, it can be replaced by the usual interaction term $J[f]$ as in the kinetic theory of active particles. Finally, the operator T^k in (4.11) is defined by using an SEIS-type kinetic disease spread model

$$\begin{cases} T^S = \nu f^I - \beta_I f^S, \\ T^E = \beta_I f^S - \theta f^E, \\ T^I = \theta f^E - \nu f^I, \end{cases} \tag{4.12}$$

with constants ν, θ and some well-defined nonlocal infection rate β_I depending on the rate of infected pedestrians. The numerical simulations show qualitative behaviors of each group, which may also shed some light on contagion problems in crowds.

Finally, we also mention a novel multiscale mathematical framework informed by multidisciplinary data modeling pandemic dynamics proposed in Ref. 21, in which the interaction of different spatial scales, from the small scale of the virus itself and cells to the large scale of individuals and further up to the collective behavior of populations, are taken into account, such that both the propagation of virus and the spatial patterns of the crowd are studied by kinetic and lattice models. It sheds light on further development of new mathematical theories, ideas and techniques.

4.3. *Macroscopic (hydrodynamical) models*

Macroscopic (hydrodynamical) modeling of human crowd describes dynamics in a coarse-grained way (via densities, mean velocities, and mean flows). Firstly, this subsection presents a review of models and problems at the macroscopic scale and, subsequently, some reasonings on computational problems within a multiscale vision.

4.3.1. *On macro-scale models and problems*

The reference structure for the derivation of models is defined in Sec. 3.4 by Eq. (3.11). This structure involves the local density $\rho = \rho(t, \boldsymbol{x})$, mean velocity $\boldsymbol{\xi} = \boldsymbol{\xi}(t, \boldsymbol{x})$ and activity $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$. It can be simplified by assuming that the activity is uniformly shared by all pedestrian (*second-order models*) or even by *first-order models*, if $\boldsymbol{\xi}$ is approximated by heuristic models linking the mean speed to local density and gradients of the density along the trajectories.

In general, we will call *third-order models* those derived by using of all equations in (3.11). In principle, one may invent first-order models which include also the dynamics of the activity variable.

It is worth mentioning three key objectives/problems of the modeling approach:

(1) Velocity direction accounting for the need of reaching a target or a meeting point and of avoiding obstacles walls.
(2) Adaptation of the speed to local density conditions.
(3) Social dynamics in the crowd.

Examples of first- and second-order models can be found in Refs. 76 and in Ref. 26, respectively. Both models are derived according to an oversimplified modeling of the selection of the velocity directions. See also the critical analysis in Ref. 121 and further developments in Ref. 126.

The main reference for macro-scale models is the book,[77] where the derivation of models is referred to a multiscale vision. A theoretical approach to the three key modeling problems has been proposed in Ref. 30 which has been already cited in reference to a multiscale vision of the modeling approach. Indeed, this paper opens at new perspectives toward the modeling of macro-scale models.

Bearing all the above in mind, let us now present a concise description to show how the modeling approach evolved in time. The first modeling attempts can be traced back in the early 1970s,[116, 117] where the crowd motion was modeled as a molecular fluid based on gas-kinetic and fluid-dynamic models. In particular, the crowd was modeled as a Hamiltonian system at the thermodynamic limit, thus assuming conservation of momentum and energy.

Following studies[123, 124] based on the theory of hydrodynamics[143] and theories of continuum mechanics addressed phenomenological continuum models in the form of PDEs for large scale crowds starting from the modeling of a single type of pedestrian and extended to a multiple type flow. The derivation of the models was based on three main hypotheses, namely that the speed of pedestrians of a single type in a multiple type flow is determined by a function of the total density, pedestrians seek to minimize their motion in time and while avoiding places of high densities, and, that their motion is governed by a potential determined by the destination they want to reach.

In another study,[68] the authors used the conservation law of mass analogous to the ones used for traffic dynamics in order to capture patterns that are typical in crowds but not in traffic dynamics (such as evacuation under panic). That was achieved by modifying both the speed law, i.e. the fundamental diagram, and the definition of the solution per-se, thus introducing a non-entropic Riemann solver.

Other studies, have used continuum mechanics theories, based on balance equations of mass and momentum in the form of PDEs which are "closed" by phenomenological algebraic equations linking the local velocity to density and density gradients.[76] The closures take into account the presence of obstacles and pedestrian strategies, such as the avoidance of high density places that are associated with and increased life-threat.

Based on the concept of contact dynamics for deformable solids[95] and by applying the principle of virtual work on the crowd, in Ref. 127 a macroscopic model was proposed to approximate the dynamics, pressure, and contact forces in a moving dense crowd. In Ref. 69, the authors proposed a model based on nonlocal conservation laws in two space dimensions, thus incorporating explicitly the influence of walls, obstacles and exits into the crowd dynamics in order to predict observed phenomena such as clogging and the spontaneous formation of queues at the exits. Recently, in Ref. 179, the authors derived a hydrodynamic approximation of a social force model coupled with an Eikonal equation and a non-local SEIS model for disease spread to study the effect of crowd motion in the spread of an infection disease among pedestrians.

An additional interesting application is proposed in Ref. 150, where the authors coupled a macroscopic model for crowd evacuation (inspired to a 2D vehicular traffic flow model) with a hydrodynamic model of flood inundation described by the Shallow Water Equation. The model is in charge of describing the collective behavior of the crowd during an evacuation scenario due to a flood.

The qualitative analysis of mathematical problems has received much attention from mathematicians, leading to valuable contributions. As examples, the reader can refer to the Hughes models,[7, 62] see also Ref. 165 for the proof of measure continuity. On the other hand, the research activity has not yet succeeded to provide a fully convincing reply to the criticism that the number of pedestrians, in most cases, is not high enough to justify the assumption of continuity of the matter. In addition, the modeling approach should also consider that the flow of pedestrians often mixes almost continuous and rarefied conditions. This is an important topic also in the case of flows through different venues, for instance from small size venues (with high pedestrian density) to large size venues (with low pedestrian density), or vice versa. This feature of the physics of crowds motivates attention to multiscale computational problems such as those treated in the second part of this subsection.

4.3.2. *Numerical-analysis-based bridging of microscopic and macroscopic scales: The Equation-free approach*

In this section, we briefly present the concept of the Equation-free multiscale numerical analysis-based framework that allows the bridging of micro- and macro-modeling scales.

Let us assume a microscopic (SFM, CA, AB) simulator with N individuals that can be in general presented as

$$\boldsymbol{U}(t+T_U) = \boldsymbol{\Phi}_{T_U}(\boldsymbol{U}(t), \boldsymbol{p}). \tag{4.13}$$

$\boldsymbol{U}(t) \in \mathbb{R}^{MN}$ denotes the vector of the states containing M attributes for each individual (in the simplest case the positions and velocities for each individual). $\boldsymbol{\Phi}_{T_U} : \mathbb{R}^{MN} \times \mathbb{R}^p \to \mathbb{R}^{NM}$ is the microscopic evolution operator, which given the values of the states at time t will report the values of the evolved microscopic distribution after a time horizon T_U (considered as the sampling time of the observations); and $\boldsymbol{p} \in \mathbb{R}^p$ is the vector of the model parameters.

A key hypothesis for the existence of macroscopic models in the form of PDEs is that after sometime $t \gg T_U$ the emergent crowd collective dynamics can be in principle described by a few macroscopic quantities, say, $\boldsymbol{u} \in \mathbb{R}^n, n \ll MN$. Usually these "few" quantities are the first moments (e.g. densities $\rho(t, \boldsymbol{x})$ and the mean velocity field $\boldsymbol{v}(t, \boldsymbol{x})$ distributed in space as in the hydrodynamical models) of the underlying microscopic distribution.

This implies that there is a slow, low-dimensional manifold that can be parameterized by $\boldsymbol{u}$. The hypothesis of the existence of such a slow low-dimensional manifold dictates that the higher-order moments $\boldsymbol{y} \in \mathbb{R}^{MN-n}$ of the microscopic distribution $\boldsymbol{U}$ become very fast (compared to the macroscopic time of observation, say $T \gg T_U$) functions of the n lower-order moments.

At the moments-space, this dependence can be written as a singularly perturbed system of the form:

$$\begin{cases} \boldsymbol{u}(t+T) = \boldsymbol{F}_T(\boldsymbol{u}(t), \epsilon\boldsymbol{y}(t), \boldsymbol{p}), \\ \boldsymbol{y}(t+T) = \boldsymbol{W}_T(\boldsymbol{u}(t), \epsilon\boldsymbol{y}(t), \boldsymbol{p}), \end{cases} \tag{4.14}$$

where $\epsilon > 0$ is a sufficiently small number.

Under the above description and assumptions, Fenichels' theorem[92] on the existence of an invariant low-dimensional "slow" manifold as specified above can be extended to discrete systems of the form of (4.14) (see also Refs. 45 and 191):

$$\boldsymbol{u}(t+T) = \boldsymbol{F}_T(\boldsymbol{u}(t), \boldsymbol{\chi}(\boldsymbol{u}(t), \boldsymbol{p}, \epsilon), \boldsymbol{p}). \tag{4.15}$$

Such a smooth manifold is defined as

$$M_\epsilon = \{(\boldsymbol{u}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{MN-n} : \boldsymbol{y}(t) = \boldsymbol{\chi}(\boldsymbol{u}(t), \boldsymbol{p}, \epsilon)\}. \tag{4.16}$$

The manifold M_ϵ is diffeomorphic, $O(\epsilon)$ close to the M_0 manifold defined for $\epsilon = 0$, and locally invariant under the dynamics given by Eq. (4.14).

Thus, *on* the slow manifold, one can in principle define the (discrete in time) macroscopic model:

$$\boldsymbol{u}(t+T) = \boldsymbol{F}_T(\boldsymbol{u}(t), \boldsymbol{p}), \tag{4.17}$$

where $\boldsymbol{F}_T : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a smooth multi-variable, vector-valued, function having $\boldsymbol{u}(t)$ as initial condition. The above macroscopic map (called coarse-timestepper) describes the emergent collective dynamics and can be obtained by finding the closure expressed by $\boldsymbol{\chi}$ relating the higher-order moments of the microscopic distribution $\boldsymbol{U}(t)$ to the macroscopic variables $\boldsymbol{u}(t)$.

The Equation-free approach,[130, 152] via the above idea, bypasses the need to extract such a closure analytically, thus providing it on demand in a strict numerical way as follows:

Given the set of the macroscopic variables at time t_0:

(a) Set the coarse-grained initial conditions $\boldsymbol{u}(t_0) \equiv \boldsymbol{u}_0$.
(b) Transform the coarse-grained initial conditions to consistent microscopic distributions $\boldsymbol{U}_0 = \boldsymbol{\mu}\boldsymbol{u}_0$, where $\boldsymbol{\mu}$ is a lifting operator.
(c) Run the microscopic (SFM, CA, AB) simulator for a short macroscopic interval T to get the resulting microscopic distributions $\boldsymbol{U}(t+T)$. The choice of T is associated with the (estimated) gap of the eigenspectrum of the Jacobian of the coarse-timestepper around the stationary state.
(d) Obtain the values of the coarse-grained variables using a restriction operator $\boldsymbol{M}$: $\boldsymbol{u}(t+T) = \boldsymbol{M}\boldsymbol{U}(t+T)$.

Around the above coarse-timestepper one can "wrap" iterative linear algebra numerical methods (and for large-scale system, matrix-free iterative methods in the Krylov subspace) to perform numerical bifurcation analysis[187, 189, 190] and design both linear and nonlinear control methods.[188, 190]

Within this framework, in Refs. 70, 151 and 159 the authors studied the dynamics of two crowds moving along a corridor from opposite directions with a door-like bottleneck in the middle of the corridor using the social-force model as microscopic simulator. Using the Equation-free approach, they constructed the bifurcation diagram for the collective dynamics with respect to the width of the door, thus identifying the existence of a coarse-grained Hopf bifurcation point, marking the onset of oscillations in the collective flow in both directions.

More recently, in Ref. 162, the Equation-free approach has been coupled with machine learning for the numerical bifurcation analysis and data-driven control of the collective dynamics of complex/multiscale systems modeled via microscopic/agent-based simulators. The proposed approach has been applied to agent-based simulations of traffic dynamics, however, it can be also used in the case of crowd dynamics.

4.4. *Additional remarks, further applications and perspectives*

The review of the existing literature on modeling by active particle methods has revealed an intense research activity. We expect this to be further developed in the next years, which will be devoted to the challenging objective of developing mathematical tools to couple mechanics and human behaviors. This objective can be pursued not only for crowd dynamics, but also for a broad variety of studies involving living systems in general.

Indeed, further investigations are very attractive due to many complex aspects of human psychology in different scenarios and the challenges in describing with mathematical equations how heterogeneous behaviors affect the collective dynamics of crowds. Therefore, suitable studies on the psychology of the crowd are useful, even necessary, toward a fully consistent modeling approach.[40, 59, 193] Different emotional behaviors have been studied to understand their ability to modify the crowd walking strategy.[87, 142, 186]

The impact of social dynamics on individual interactions and their influence at higher scales has been studied in various papers, for instance, Refs. 80 and 81. In addition, as noted in Refs. 4 and 30, crowd dynamics should be described at all the three possible modeling scales (i.e. microscopic, mesoscopic, macroscopic) by a consistent approach for many practical applications. Namely, models must be derived at each scale using the same principles and similar parameters, i.e. within a multiscale vision. This is a key to derive macroscopic models from the underlying description at the micro-scale, see Refs. 51 and 52. This approach is also useful to the application of computational methods as we shall see in Sec. 5.

Therefore, we suggest to look ahead to perspectives in two sequential steps. First, in this subsection, we propose some general reasonings on recent applications of mathematics to model crowd dynamics. Then, in Sec. 5, we select a number of key problems which we believe will capture the attention of applied mathematicians in the next years. Hints to tackle these problems will be given as well.

Let us now mention some noteworthy research articles dealing with topics that are presently at an initial stage, but arguably will be further developed. Besides their theoretical interest, the following topics are of practical importance as they may significantly contribute to crowd management in emergency situations where overcrowding may cause fatal accidents.[17, 54]

(1) The concept of crowd is broadening in order to consider different types of interactions and modeling scopes. An example of such scopes is to account for

contagion problems in a population.[164] Further studies toward the search of new mathematical structures are motivated by this broader vision. In addition, possible alternatives can be studied, like, e.g. methods derived from the Fokker–Plank framework.[39]

(2) Control problems have been studied focusing on the role of leaders, trained personnel to guide pedestrians to egress from a complex environment whose connectivity may not even be known or may modified by incidents.[5] Modeling of crowds with passive and active agents can contribute to the formulation of these control problems,[65] see also Ref. 6. Models and empirical data of assisted evacuation dynamics[178, 198] can provide further support. An important topic appears to be the detection of stress conditions.[149]

(3) Crowd dynamics should be studied also in complex frameworks. An example is the coupling of crowd dynamics with vehicular traffic to organize the motion of people in combination with different types of vehicles.[44, 96]

(4) The study of the crowd psychology should go beyond the concept of *panic*, which has received criticisms. Well-defined feelings, such as *stress*, or even positive behaviors, such as *awareness of danger situations*,[15] should be investigated. However, independently of the use or misuse of the term "stress", simulations of high stress conditions are useful.[199, 213] The pioneering papers by Helbing and coworkers offer an important framework to start from Refs. 100, 111–115 and 156.

(5) The motion of crowds depends on the quality of the environment. For example, so far a parameter $\alpha \in [0, 1]$ has been introduced to account for such quality. The parameter α can be used at all scales as shown in Ref. 30. However, the literature in engineering journals should consider a multiplicity of specific features of venues where the crowd moves, such as presence of smoke[174] or footbridges.[46, 196] In general, the modeling approach should include the awareness of the specific features of the environment where the crowd moves.[67]

(6) A key problem for all different frameworks is the modeling of interactions, which can benefit from the contribution of theoretical tools of game theory.[48, 58, 168] Further theoretical studies can contribute to the derivation of social models of micro-scale interactions, for instance the study of fast and slow thinking[128] somehow related to decision making.[128] Indeed, understanding interactions leads to understanding collective dynamics at all scales.[120, 205] Thus, the search of universal interaction models is well motivated.[129] This research objective might require mathematical structures different from the classical ones reviewed in this paper. An interesting results of this type of quest is given in Ref. 194.

(7) Safety in evacuation is often mentioned as an application crowd dynamics model. In addition to previously cited articles, see also Refs. 125 and 195. For related congestion problems, see Ref. 144. In addition, contagion problems have received much attention recently, as stressed in Sec. 4.2. Another important application is related to engineering problems. For example, the study of crowd

evacuation can lead to improve the design of buildings with enhanced safety.[171] Analogous reasonings apply to urban planning. Models of crowd dynamics on lively structures, such as footbridges, can contribute to their design again with the aim of improving safety.[46]

5. On a Forward Look to Research Perspectives

The review and critical analysis presented in the preceding sections, mainly in Sec. 4.4, has already identified some pros and cons of the current state-of-the-art and a few research perspectives, which have been brought to the attention of the interested reader. Beyond these indications, a further selection of research targets is proposed in this conclusive section. We consider five key topics worth to be developed within well-focused research programs because, according to the authors opinion (it might be even bias), these topics will be pervasive future research activity in the field.

These topics are treated in the next five subsections which include also some hints on how to tackle each of them. This is a selection, mostly based on the authors' research expertise, not an exhaustive list of relevant future directions. Certainly, additional future directions can be identified, and developed according to the background knowledge of each researcher active in the field.

We will start with analytical problems. Then, we will look at computational/numerical problems. Next, we will focus on the modeling of complex social dynamics to understand how different types of social interactions modify the collective motion of the crowd. Then, we will consider how the knowledge acquired to model human crowds can be exported to describe the dynamics of animal swarms.

Finally, in the last subsection, we will go back to the main problem at the origin of our survey, i.e. we propose some speculations toward the development of a mathematical theory of human crowd dynamics.

5.1. *Analytic problems*

The derivation of models, and their application to the simulation of real dynamics, generates challenging analytic and computational problems through the multiscale vision mentioned various times in this paper. Some perspective ideas are treated in Ref. 30, where the following methodological indications were proposed:

— Models should be derived, at all scales, by the same physical principles as reported in Sec. 4.
— The activity is an additional micro-scale variable whose dynamics is described, at all scales, by an additional dynamical equation.

Therefore, we believe all previous results obtained with models that treat the activity variable as constant parameter should be re-visited in favor of results obtained with models which treat the dynamics of the activity variable.

A classical problem is the qualitative analysis of the initial value problem. The analysis in Ref. 20 for models with discrete velocity directions might be extended to more general models with dynamical activity variable. The study should be focused on the existence and regularity of solutions and may consider also initial-boundary value problems. Papers linking the qualitative analysis to computing problems and identification of pattern formation[9] are of special interest for this task.

An additional problem worth to be considered is the derivation of the macro-scale models from the underlying micro-scale description in the spirit of the sixth Hilbert problem.[118] This problem was treated in Ref. 49 by a mean field approach, while methods known in the kinetic theory of classical particles have been further developed to study the case of crowd dynamics in Ref. 18. Recent studies[50–52] have developed a general approach to deal with the Hilbert problem, which is worth to be extended to specific case studies, such as the micro–macro derivation.

The qualitative analysis of macro-scale mathematical models has attracted the attention of applied mathematicians who have studied various interesting problems related to the study of real flow conditions, see Refs. 8 and 9. The open problems refer to the qualitative analysis of problems, if possible at all scales, related to the broad variety of crowd dynamics including those reviewed in this paper.

5.2. *Computational problems*

The application of computational tools refers to the specific scale selected for the representation and modeling of the system. Indeed, micro-scale models require integration codes for ODEs, where the main difficulty consists in dealing with large number of nonlinear equations and stiffness;[47, 185] while macro-scale models need computational methods for hyperbolic type PDEs. Specific tools have been applied in the case of kinetic type models where the dependent variables is the distribution function $f = f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u})$ and where, the right-hand side of (3.6) is a nonlinear and nonlocal integration term. This intrinsic complexity brings difficulties not only to the analysis but also to numerical computations.

Note that both macro- and meso-scale models involve both transport and interaction terms. The natural way to consider these terms is given by operator splitting methods,[122] where the overall evolution step is decomposed into two sub-steps — a transport step and an interaction step. Specifically, in the first step, one has to solve a system of hyperbolic conservation laws which can be done by standard methods such as finite-difference, finite-volume, finite-element, or spectral methods.[141] The second step can be technically challenging, mainly in the case of kinetic models, as high-dimensional integral defining the interaction operator needs to be computed. The reader interested to numerical computation tools of kinetic equations is referred to Refs. 11, 85 and 161 and references therein.

The following discussion mentions two basic computational tools: the stochastic Monte Carlo method and the deterministic discrete-velocity models. See, for example, Refs. 27–29, where the kinetic model has been solved in by a stochastic particle scheme which closely resembles the Direct Simulation Monte Carlo (DSMC) method originally proposed by Bird.[37] The distribution function is represented by a collection of computational particles whose positions and velocities evolve in time by a sequence of time steps, each consisting of a free transport and a local interaction sub-steps. The former corresponds to the transport operator in the kinetic equation. In contrast, the latter is performed according to stochastic rules, consistent with the structure of the interaction term in the kinetic equation and the corresponding transition probability densities. The space domain to be simulated is divided by a mesh of cells. These cells are used to collect together particles that may interact and to sample macroscopic properties such as density and mean velocity.

After the above introductory reasonings, let us now consider some research perspectives on computational methods that are induced by the multiscale vision, just like in the case of analytic problems. In particular, we consider the following perspectives:

- **The activity viewed as a dynamical variable:** The numerical solution of problems where the activity is a dynamical variable are definitely worth to be studied. Specifically, we refer the development of Monte Carlo methods, as well as to the dynamics at the micro- and macro-scale. Some preliminary results can be found in Ref. 29, where simulations have been developed in the specific case study of counterflow in corridors. However, a systematic study for more general methods still needs to be exhaustively tackled.
- **Linking equations at different scales:** An additional tricky problem is the computational interfacing of equations at different scales, for instance kinetic models versus continuum models. This study is useful to simulations of crowd flows through venues, where local density undergoes sharp variations. Then, the selection of the specific model-scale might be related to the local density. For instance, low density flows in large areas demand micro-scale models, while high density flows, which can occur, in small areas, demand macro-scale models. The transition from micro- to macro-scale poses technical problems, as variables at the micro-scale provide, by local averaging, the required information about macro-scale variables. On the other hand, the inverse transition poses problems as the required information about micro-scale variables cannot be obtained from the information at the macro-scale in a straightforward way.
- **Discrete velocity models:** Discrete-velocity models are popular for approximating kinetic type equations in velocity space, which originated as simplified models of the Boltzmann equation for a qualitative study of rarefied gases.[57, 101, 160] Discrete velocity models of kinetic type equations can be obtained assuming that particles are allowed to move with a finite number of velocities. In semi-discrete-velocity models instead, the assumption is that the velocity directions attain a finite number of values while the speed is continuous along each direction.

In the setting of modeling crowd dynamics, discrete-velocity models are widely accepted as already mentioned in Sec. 4.2 in reference to Boltzmann-like methods. The distribution function $f(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u})$ is represented by a set of unknowns $\{f_{ij}(t, \boldsymbol{x})\}$ in lower dimensions as (4.3). Then, the original differential-integral equation in higher dimensions is replaced by a system of coupled differential equations of the form (4.4) in lower dimensions, which can be split into a transport part and a system of ordinary differential equations for the interaction. At this point, standard full discretization methods can be applied, see Refs. 2, 20, 133 and 146.

- **Discrete space dynamics:** Discrete velocities lead to treat the interaction operator by a finite sum of interaction terms corresponding to the discrete velocity. Further, discretization of space would lead to systems of ODEs. This approach has been developed in the case of vehicular traffic by taking advantage of the one-dimensionality of the flow.[93] In this case, the transport term is approximated by the inlet and outlet fluxes at the boundary of the finite elements. Space discretization can be achieved by, e.g. lattice methods (see Ref. 83) where the technical difficulty consists in selecting a lattice appropriate with the geometry of the venue where the crowd moves. In addition, it is worth mentioning that the use of discrete velocities can help with the criticism that the number of pedestrian is never large enough to justify the continuity assumption of the distribution function over the micro-state variable. In fact, this approach considers homogeneous groups of individual entities within a finite domain of the velocity space.

5.3. *Social dynamics in crowds*

In Sec. 4, we have shown how emotional states, heterogeneously distributed among people in a crowd, can have an important influence on the walking strategy. The main reference for this is Ref. 30. Emotional states can be modeled by the activity variable viewed as a dynamical variable whose space propagation depends on interactions, external actions, as well as on the geometry of venues.

Starting from these reasonings, we can argue that modeling of crowds goes beyond rational behaviors and that irrational behaviors should also be taken into account. In addition, different types of social dynamics should be modeled based on social and psychological studies.[87, 100, 140, 142, 193] These emotional states might affect density sensitive interactions and natural distancing.[36, 91, 153] Without claiming to be exhaustive, we report on a selections of possible case studies and define, for each of them, the specific interactions to be considered followed by some hints toward their modeling.

The common feature in the modeling approach to the following case studies is that we model the walking strategy as described in Sec. 4. Specifically, each walker interacting with other walkers and with the external environment, first modifies one's emotional state, i.e. the activity, and subsequently, the mechanical variables corresponding to the selection of the direction and to the adjustment of the speed.

- *Safety problems in evacuation dynamics.* We consider a crowd in venues consisting in a sequence of connected domains. Modeling and simulations can discover the trajectories which reduce the risk of overcrowding. Vocal and visual signaling can act to induce walkers to "learn" about these trajectories. In detail, signaling should promote a collective learning toward a consensus to the optimal values of the activity variable deemed to reduce evacuation time and avoid overcrowded areas. Indeed, it is a problem of collective consensus by learning[53–55] that can be promoted also by leaders trained to select optimal trajectories. These leaders promote consensus toward their own walking strategy.
- *Dynamics of antagonistic groups.* Models of counter-flows have already been studied in various papers by models describing patterns formation with finger-like shapes in crowds moving to opposite directions. Further studies have been developed in Refs. 27 and 29, which show how social dynamics modify these patterns. These papers suggest to develop the modeling of antagonistic groups in complex venues. The study of this topic might contribute to describe the complex dynamics happening in safety problems, e.g. the contrast between security forces and rioters. Detection of violent behaviors and classification of images can contribute to the modeling approach.[35] The literature in this field is not limited to the kinetic theory approach. For instance, macroscopic models have been developed to describe motion in corridors and counter-flows[94] and to account for asymmetric interactions[99] and self-organization.[78, 103, 170, 177, 192]
- *Virus contagion dynamics in crowds.* Mathematical models refer to crowds that include individuals carrying a virus. Pioneering studies have been developed in Refs. 131, 134, 135 and 136, where the authors compute the trajectories of individuals moving in complex venues and relate the probability of contagion to the local density and speed, see also Ref. 1. In fact, the aforementioned probability increases with the density and decreases with the speed, as high speeds reduce the contact time. Therefore, it is important to account for the awareness of contagion risk since this would modify trajectories with the aim to reduce the quantitative effects of contacts.

  Obviously, the modeling approach should advance jointly with the progress in the study of virus epidemics, which has stressed the multiscale feature of the contagion dynamics and immune response,[21, 23, 157, 183, 208] since internal state includes both the awareness to the contagion risk and the viral charge. An objective of modeling and simulations is showing how contagion patterns propagate in time and space depending on the probability distribution of the aforementioned internal variables. The dynamics of pattern formation should consider the structure of local networks as shown in Ref. 104.

5.4. *From crowds to swarms*

The mathematical theory of swarms has been promoted by the celebrated paper by Cucker and Smale[79] which has been followed by a broad variety of research articles devoted to analytic and computational topics and specific applications to simulate real-world collective dynamics. A detailed survey of the literature on modeling of swarms and related problems goes beyond the aims of our paper. Therefore, we simply address the interested reader to the survey[4] and references therein, as well as the special issue[61] devoted to this specific topic.

An important development in the modeling approach has been the introduction of internal mechanical variables, see Refs. 105 and 106, corresponding to the so-called thermodynamic swarm, or to activity variables[33] somehow analogous to those in crowd dynamics. This development is important whenever the study is focused on applications beyond the classical consensus dynamics of animal swarms. For instance, it can be referred to exotic applications, such as financial markets,[13, 14] and to a broad variety of systems in the natural sciences, see Ref. 10 and references therein. Applications can be pursued from both individual-based models and kinetic theory approaches.[32]

Bearing all the above reasonings in mind, it appears worthy to understand how far the modeling techniques developed for human crowds can be considered toward the derivation of models of animal swarms. The key problem consists in the modeling of interactions by taking advantage of the studies developed in the case of crowds and transferring this knowledge to the modeling of swarms. Moreover, the modeling of swarms should include not only the dynamics of position and velocity, but also social dynamics generated, e.g. by the interaction between predators and prey.[84]

5.5. *Toward a mathematical theory of crowds*

Let us now return to the stone guest behind the overall content of our paper. We wish to understand how far we have gone in the quest of a mathematical theory of human crowds.

Right from the beginning, we have stated that crowds must be treated as a collective living system. Therefore, we cannot hope in the identification of causality principles based on background physical theories valid for the inert matter, as clearly explained by Robert May.[154] Further, one should consider also evolutive features which are visible in all biological systems and, more in general, all living systems.[155] A strategy to pursue this objective consists in the following approach,[22] which is here specialized to the case of crowd dynamics:

(1) Deriving, at each scale, a mathematical structure capable to capture the key features of human, hence living, crowds.
(2) Modeling of the interactions which first modify the activity variable and then the mechanical variables, i.e. velocity direction and speed.
(3) Deriving crowd models by simply inserting these interaction models into the structure mentioned in Item (1).

The literature on the first item has been reviewed in our paper showing (in agreement with Ref. 30) that it is possible to derive, at each scale, mathematical structures that couple the dynamics of the activity variable with the dynamics of the mechanical variables. Therefore, the behavioral features of human crowd can be taken into account. Further, it has been observed that the kinetic theory approach captures, quite naturally, some specific features of crowds, such as the different types of heterogeneity handled by functional subsystems. Transferring the achievements of kinetic theory models to micro- and macro-scale approaches is a possible perspective. See hints proposed in Ref. 30.

On the other hand, a key open problem is the modeling interactions. Therefore, it is interesting to observe that some active teams (see for instance Refs. 71 and 72) are moving to sharp investigations of empirical data on different aspects of interaction dynamics to understand, e.g. avoidance,[73] orientation,[202] distancing[166] and fluctuations.[98] Some challenging problems are still open, one of these being if interactions are visual based or sensitivity based, where sensitivity refers to a fixed number of individual entities as conjectured in the theory of swarms.[16, 32]

Finally, integrating empirical research with the mathematical structures reviewed in this paper would provide an important (decisive) contribution toward the complex quest of a mathematical theory of human crowds.

Therefore, we can state that the key toward the derivation of a mathematical theory of human crowds is the modeling of interactions accounting for the heterogeneous emotional and behavioral features of people in the crowd, see Refs. 66 and 194. This philosophy should be followed at each scale by mathematical tools suitable to move from one scale to the other.

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Publisher's Note

The last author, Constantinos Siettost's name was misspelled in the initial published version and has been corrected as of 21st June 2023.

References

1. J. P. Agnelli, B. Buffa, D. Knopoff and G. Torres, A spatial kinetic model of crowd evacuation dynamics with infectious disease contagio, *Bull. Math. Biol.* **85** (2023) Article 23.

2. J. P. Agnelli, F. Colasuonno and D. Knopoff, A kinetic theory approach to the dynamics of crowd evacuation from bounded domains, *Math. Models Methods Appl. Sci.* **25** (2015) 109–129.
3. G. Ajmone Marsan, N. Bellomo and L. Gibelli, Stochastic evolutionary differential games toward a systems theory of behavioral social dynamics, *Math. Models Methods Appl. Sci.* **26** (2016) 1051–1093.
4. G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato and J. Soler, Traffic, crowds, and swarms. From kinetic theory and multiscale methods to applications and research perspectives, *Math. Models Methods Appl. Sci.* **29** (2019) 1901–2005.
5. G. Albi, E. Cristiani, L. Pareschi and D. Peri, Mathematical models and methods for crowd dynamics control, in *Crowd Dynamics*, Vol. 2, Series: Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2020), pp. 159–197.
6. G. Albi, F. Ferrarese and C. Segala, Optimized leaders strategies for crowd evacuation in unknown environments with multiple exits, in *Crowd Dynamics,* Vol. 3, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2022), pp. 97–132.
7. D. Amadori, P. Goatin and M. D. Rosini, Existence results for Hughes' model for pedestrian flows, *J. Math. Anal. Appl.* **420** (2014) 387–406.
8. B. Andreianov, C. Donadello, U. Razafison and M. D. Rosini, Qualitative behaviorand numerical approximation of solutions to conservation laws with non-local point constraints on the flux and modeling of crowd dynamics at the bottlenecks, *ESAIM: Math. Models Numer. Anal.* **50** (2016) 1269–1287.
9. B. Andreianov and A. Sylla, Existence analysis and numerical approximation for a second-order model of traffic with orderliness marker, *Math. Models Methods Appl. Sci.* **32** (2022) 1295–1348.
10. G. Ariel, A. Ayali, A. Be'er and D. Knebel, Variability and Heterogeneity in natural swarms: Experiments and modeling, in *Active Particles,* Vol. 3, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2022), pp. 1–33.
11. V. V. Aristov, *Direct Methods for Solving the Boltzmann Equation and Study of Nonequilibrium Flows* (Springer-Verlag, 2001).
12. B. Aylaj, N. Bellomo, L. Gibelli and A. Reali, On a unified multiscale vision of behavioral crowds, *Math. Models Methods Appl. Sci.* **30** (2020) 1–22.
13. H.-O. Bae, S.-Y. Cho, S.-K. Lee and S.-B. Yun, A particle model for herding phenomena induced by dynamic market signals, *J. Statist. Phys.* **177** (2019) 365–398.
14. H.-O. Bae, S.-Y. Cho, J. Kim and S.-B. Yun, A kinetic description for the herding behavior in financial market, *J. Statist. Phys.* **176** (2019) 398–424.
15. R. Bailo, J. A. Carrillo and P. Degond, Pedestrian models based on rational behaviors, in *Crowd Dynamics,* Vol. 1, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2018), pp. 259–292.
16. M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study, *Proc. Natl. Acad. Sci. USA* **105** (2008) 1232–1237.
17. M. K. Banda, M. Herty and T. Trimborn, Recent developments in controlled crowd dynamics, in *Crowd Dynamics*, Vol. 2, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2020), pp. 133–157.

18. N. Bellomo and A. Bellouquid, On multiscale models of pedestrian crowds from mesoscopic to macroscopic, *Commun. Math. Sci.* **13** (2015) 1649–1664.
19. N. Bellomo, A. Bellouquid and N. Chouhad, From a multiscale derivation of nonlinear cross-diffusion models to Keller–Segel models in a Navier–Stokes fluid, *Math. Models Methods Appl. Sci.* **26** (2016) 2041–2069.
20. N. Bellomo, A. Bellouquid and D. Knopoff, From the micro-scale to collective crowd dynamics, *Multiscale Model. Simul.* **11** (2013) 943–963.
21. N. Bellomo, R. Bingham, M. A. J. Chaplain, G. Dosi, G. Forni, D. A. Knopoff, J. Lowengrub, R. Twarock and M. E. Virgillito, A multi-scale model of virus pandemic: Heterogeneous interactive entities in a globally connected world, *Math. Models Methods Appl. Sci.* **30** (2020) 1591–1651.
22. N. Bellomo, D. Burini, G. Dosi, L. Gibelli, D. Knopoff, N. Outada, P. Terna and M.-E. Virgillito, What is life? A perspective of the mathematical kinetic theory of active particles, *Math. Models Methods Appl. Sci.* **31** (2021) 1821–1866.
23. N. Bellomo, D. Burini and N. Outada, Multiscale models of COVID-19 with mutations and variants, *Netw. Heterog. Media* **17** (2022) 293–310.
24. N. Bellomo, D. Clarke, L. Gibelli, P. Townsend and B. J. Vreugdenhil, Human behaviours in evacuation crowd dynamics: From modeling to "big data" toward crisis management, *Phys. Life Rev.* **18** (2016) 1–21.
25. N. Bellomo and C. Dogbè, On the modeling of traffic and crowds: A survey of models, speculations, and perspectives, *SIAM Rev.* **53** (2011) 409–463.
26. N. Bellomo and C. Dogbè, On the modelling crowd dynamics from scaling to hyperbolic macroscopic models, *Math. Models Methods Appl. Sci.* **18** (2008) 1317–1345.
27. N. Bellomo and L. Gibelli, Toward a mathematical theory of behavioral-social dynamics for pedestrian crowds, *Math. Models Methods Appl. Sci.* **25** (2015) 2417–2437.
28. N. Bellomo and L. Gibelli, Behavioral crowds: Modeling and Monte Carlo simulations toward validation, *Comput. Fluids* **141** (2016) 13–21.
29. N. Bellomo, L. Gibelli and N. Outada, On the interplay between behavioral dynamics and social interactions in human crowds, *Kinet. Related Models* **12** (2019) 397–409.
30. N. Bellomo, L. Gibelli, A. Quaini and A. Reali, Towards a mathematical theory of behavioral human crowds, *Math. Models Methods Appl. Sci.* **32** (2022) 321–358, https://www.worldscientific.com/doi/epdf/10.1142/S0218202522500087.
31. N. Bellomo, A. Bellouquid, L. Gibelli and N. Outada, *A Quest Towards a Mathematical Theory of Living Systems* (Birkhäuser, 2017).
32. N. Bellomo and S.-Y. Ha, A quest toward a mathematical theory of the dynamics of swarms, *Math. Models Methods Appl. Sci.* **27** (2017) 745–770.
33. N. Bellomo, S.-Y. Ha and N. Outada, Towards a mathematical theory of behavioral swarms, *ESAIM: Control Theory Var. Calculus* **26** (2020) 125.
34. A. Bellouquid and N. Chouhad, Kinetic models of chemotaxis towards the diffusive limit: Asymptotic analysis, *Math. Methods Appl. Sci.* **39** (2016) 3136–3151.
35. Y. Bi, D. Li and Y. Luo, Combining keyframes and image classification for violent behavior recognition, *Appl. Sci.* **12** (2022) 8014, https://doi.org/10.3390/app1216801.
36. D. H. Biedermann, J. Clever and A. Borrmann, A generic and density-sensitive method for multi-scale pedestrian dynamics, *Autom. Constr.* **122** (2021) 103489.
37. G. A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows* (Oxford Univ. Press, 1994).
38. V. Blue and J. L. Adler, Cellular automata microsimulation of bidirectional pedestrian flows, *J. Transp. Res. Board* **1678** (1999) 135–141.

39. A. Borzi, The Fokker–Plank framework in the modeling of pedestrians' motion, in *Crowd Dynamics,* Vol. 2, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2020), pp. 111–131.
40. T. Bosse, M. Hoogendoorn, M. C. Klein, J. Treur, C. N. Van Der Wal and A. Van Wissen, Modelling collective decision making in groups and crowds: Integrating social contagion and interacting emotions, beliefs and intentions, *Auton. Agents Multi-Agent Syst.* **27** (2013) 52–84.
41. A. L. Bertozzi, J. Rosado, M. B. Short and L. Wang, Contagion shocks in one dimension, *J. Statist. Phys.* **158** (2015) 647–664.
42. N. Boccara, Phase transitions in cellular automata, in *Cellular Automata,* ed. A. Adamatzky, Encyclopedia of Complexity and Systems Science Series (Springer, 2009).
43. P. Bonacich and P. Lu, *Introduction to Mathematical Sociology* (Princeton Univ. Press, 2012).
44. R. Borsche, A. Klar, S. Köhn and A. Meurer, Coupling traffic flow networks to pedestrian motion, *Math. Models Methods Appl. Sci.* **24** (2014) 359–380.
45. R. Bouyekhf and A. El Moudni, On analysis of discrete singularly perturbed nonlinear systems: Application to the study of stability properties, *J. Franklin Inst.* **334** (1997) 199–212.
46. L. Bruno and F. Venuti, Crowd-structure interaction in footbridges: Modelling, application to a real case-study and sensitivity analyses, *J. Sound Vib.* **323** (2009) 475–493.
47. G. Byrne and A. C. Hindmarsh, Stiff ODE solvers: A review of current and coming attractions, *J. Comput. Phys.* **70** (1987) 1–62.
48. M. Burger, M. Di Francesco, P. A. Markowich and M. T. Wolfram, Mean field games with nonlinear mobilities in pedestrian dynamics, *Discr. Conti. Dynam. Syst. B* **19** (2014) 1311–1333.
49. M. Burger, P. Markowich and J. F. Pietschmann, Continuous limit of a crowd motion and herding model: Analysis and numerical simulations, *Kinet. Related Models* **4** (2011) 1025–1047.
50. D. Burini and N. Chouhad, Hilbert method toward a multiscale analysis from kinetic to macroscopic models for active particles, *Math. Models Methods Appl. Sci.* **27** (2017) 1327–1353.
51. D. Burini and N. Chouhad, A multiscale view of nonlinear diffusion in biology: From cells to tissues, *Math. Models Methods Appl. Sci.* **29** (2019) 791–823.
52. D. Burini and N. Chouhad, Virus models in complex frameworks towards modeling space patterns of SARS-CoV-2 epidemics, *Math. Models Methods Appl. Sci.* **32** (2022) 2917–2036.
53. D. Burini and S. De Lillo, On the complex interaction between collective learning and social dynamics, *Symmetry* **11** (2019) 967, doi:10.3390/sym11080967.
54. D. Burini, S. De Lill and L. Gibelli, Stochastic differential "nonlinear" games modeling collective learning dynamics, *Phys. Life Rev.* **16** (2016) 123–139.
55. D. Burini, S. De Lillo and L. Gibelli, Learning dynamics towards modeling living systems: Reply to comments on "Collective learning dynamics modeling based on the kinetic theory of active particles", *Phys. Life Rev.* **16** (2016) 158–162.
56. C. Burstedde, K. Klauck, A. Schadschneider and J. Zittartz, Simulation of pedestrian dynamics using a two-dimensional cellular automaton, *Physica A*: *Statist. Mech. Appl.* **295** (2001) 507–525.
57. H. Cabannes, R. Gatignol and L. Luo, The discrete Boltzmann equation: Theory and applications, Henri Cabannes, Paris (2003); Revised from the lecture notes given at the University of California, Berkeley, CA (1980).

58. K. Cao and Y. Chen, A fractional micro-macro model for crowds of pedestrians based on fractional mean field games, *IEEE/CAA J. Autom. Sin.* **3** (2016) 261.
59. G. Carbone and I. Giannoccaro, Model of human collective decision-making in complex environments, *Eur. Phys. J. B* **88** (2015) 339.
60. E. Carlini and F. J. Silva, On the discretization of some nonlinear Fokker–Planck–Kolmogorov equations and applications, *SIAM J. Numer. Anal.* **56** (2018) 2148–2177.
61. J. A. Carrillo, S.-Y. Ha, L. Pareschi and B. Piccoli, Special issue on mathematical models for collective dynamics, *Netw. Heterog. Media* **15** (2020) 307–553.
62. J. A. Carrillo, S. Martin and M.-T. Wolfram, An improved version of the Hughes model for pedestrian flow, *Math. Models Methods Appl. Sci.* **26** (2016) 671–697.
63. F. Chalub, P. Markowich, B. Perthame, C. Schmeiser, Kinetic models for chemotaxis and their drift-diffusion limits, *Monatsh. Math.* **142** (2004) 123–141.
64. K. Christensen and Y. Sasaki, Agent-based emergency evacuation simulation with individuals with disabilities in the population, *J. Artifi. Soc. Soc. Simul.* **11** (2008) 1–13.
65. M. Colangeli, A. Muntean, O. Richardson and T. K. T. Thieu, Modeling interactions between active and passive agents moving through heterogeneous environments, in *Crowd Dynamics,* Vol. 1, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2018), pp. 211–102.
66. A. Colombi and M. Scianna, Modelling human perception processes in pedestrian dynamics: A hybrid approach, *Roy. Soc. Open Access* **4** (2017) 160561.
67. A. Colombi, M. Scianna and A. Alaia, A discrete mathematical model for the dynamics of a crowd of gazing pedestrians with and without an evolving environmental awareness, *Comput. Appl. Math.* **36** (2017) 1113–1141.
68. R. M. Colombo and M. D. Rosini, Pedestrian flows and non-classical shocks, *Math. Methods Appl. Sci.* **28** (2005) 1553–1567.
69. R. M. Colombo and E. Rossi, Modelling crowd movements in domains with boundaries, *IMA J. Appl. Math.* **84** (2019) 833–853.
70. O. Corradi, P. G. Hjorth and J. Starke, Equation-free detection and continuation of a Hopf bifurcation point in a particle model of pedestrian flow, *SIAM J. Appl. Dynam. Syst.* **11** (2012) 1007–1032.
71. A. Corbetta, L. Bruno, A. Mountean and F. Toschi, High statistics measurements of pedestrian dynamics, models via probabilistic method, *Transp. Res. Proc.* **2** (2014) 96–104.
72. A. Corbetta, A. Mountean and K. Vafayi, Parameter estimation of social forces in pedestrian dynamics models via probabilistic method, *Math. Biosci. Eng.* **12** (2015) 337–356.
73. A. Corbetta, L. Schilders and F. Toschi, High-statistics modeling of complex pedestrian avoidance, in *Crowd Dynamics*, Vol. 2, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2020), pp. 33–53.
74. A. Corbetta, J. Meeusen, C. Lee, R. Benzi and F. Toschi, Physics-based modeling and data representation of pairwise interactions among pedestrians, *Phys. Rev. E* **98** (2018) 062310.
75. V. Coscia, M. Delitala and P. Frasca, On the mathematical theory of vehicular traffic flow models II. Discrete velocity kinetic models, *Int. J. Nonlinear Mech.* **42** (2007) 411–421.
76. V. Coscia and C. Canavesio, First-order macroscopic modelling of human crowd dynamics, *Math. Models Methods Appl. Sci.* **18** (2008) 1217–1247.

77. E. Cristiani, B. Piccoli and A. Tosin, *Multiscale Modeling of Pedestrian Dynamics* (Springer, 2014).
78. E. Cristiani, F. S. Priuli and A. Tosin, Modeling rationality to control self-organization of crowds: An environmental approach, *SIAM J. Appl. Math.* **75** (2015) 605–629.
79. F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Autom. Control* **52** (2007) 853–862.
80. P. Degond, C. Appert-Rolland, M. Moussaid, J. Pettré and G. Theraulaz, A hierarchy of heuristic-based models of crowd dynamics, *J. Statist. Phys.* **152** (2013) 1033–1068.
81. P. Degond, C. Appert-Rolland, J. Pettré and G. Theraulaz, Vision based macroscopic pedestrian models, *Kinet. Related Models* **6** (2013) 809–839.
82. P. Derjany, S. Namilae, D. Liu and A. Srinivasan, Multiscale model for the optimal design of pedestrian queues to mitigate infectious disease spread, *PloS One* **15** (2020) e0235891.
83. A. Deutschand and S. Dormann, *Cellular Automaton Modeling of Biological Pattern Formation*, Modeling and Simulation in Science, Engineering and Technology (Birkhäuser, 2017).
84. M. Di Francesco and S. Fagioli, A nonlocal swarm model for predators–prey interactions, *Math. Models Methods Appl. Sci.* **26** (2016) 319–355.
85. G. Dimarco and L. Pareschi, Numerical methods for kinetic equations, *Acta Numer.* **23** (2014) 369–520.
86. Y. Dolak and C. Schmeiser, Kinetic models for chemotaxis: Hydrodynamic limits and spatio-temporal mechanisms, *J. Math. Biol.* **51** (2005) 595–615.
87. F. Durupinar, U. Guüdükbay, A. Aman and N. I. Badler, Psychological parameters for crowd simulation: From audiences to mobs, *IEEE Trans. Visual. Comput. Graph.* **22** (2015) 2145–2159.
88. J.-M. Epstein, Modeling civil violence: An agent based computational approach, *Proc. Natl. Acad. Sci. USA* **99** (2002) 7243–7250.
89. S. Fan and B. Seibold, Data-fitted first-order traffic models and their second-order generalizations: Comparison by trajectory and sensor data, *Transp. Res. Record* **2391** (2013) 32–43.
90. R. M. Fano, Transmission of information: A statistical theory of communications, *Amer. J. Phys.* **29** (1961) 793–794.
91. S. Faure and B. Maury, Crowd motion from the granular standpoint, *Math. Models Methods Appl. Sci.* **25** (2015) 463–493.
92. N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, *J. Differential Equations* **31** (1979) 53–98.
93. L. Fermo and A. Tosin, A fully-discrete-state kinetic theory approach to modeling vehicular traffic, *SIAM J. Appl. Math.* **73** (2013) 1533–1556.
94. M. Fischer, G. Jankowiak and M.-T. Wolfram, Micro- and macroscopic modeling of crowding and pushing in corridors, *Netw. Heterog. Media* **15** (2020) 405–426.
95. M. Fremond, Rigid bodies collisions, *Phys. Lett. A* **204** (1995) 33–41.
96. H. Fujii, H. Uchida, T. Yamada and S. Yoshimura, Mixed traffic simulation of cars and pedestrians for transportation policy assessment, in *Crowd Dynamics*, Vol. 2, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2020), pp. 199–222.
97. Z. Fu, L. Luo, Y. Yang, Y. Zhuang, P. Zhang, L. Yang, H. Yang, J. Ma, K. Zhu and Y. Li, Effect of speed matching on fundamental diagram of pedestrian flow, *Physica A* **458** (2016) 31–42.

98. A. Gabbana, F. Toschi, P. Ross, A. Haans and A. Corbetta, Fluctuations in pedestrian dynamics routing choices, preprint (2022), arXiv:2202.02108.
99. Y. B. Gaididei, C. Marschler, M. P. Soerensen, P. L. Christiansen, J. J. Rasmussen and J. Starke, Pattern formation in flows of asymmetrically interacting particles: Peristaltic pedestrian dynamics as a case study, *Evolut. Equations Control Theory* **8** (2019) 73–100.
100. P. Gantt and R. Gantt, Disaster psychology: Dispelling the myths of panic, *Profess. Saf.* **57** (2012) 42–49.
101. R. Gatignol, Theorie Cinétique des Gaz a Repartition Discréte de Vitesses, *Lecture Notes in Physics*, Vol. 36 (Springer, 1975).
102. I. Gerakakis, P. Gavriilidis, N. Dourvas, I. G. Georgoudas, G. A. Trunfio and G. C. Sirakoulis, Accelerating fuzzy cellular automata for modeling crowd dynamics, *J. Comput. Sci.* **32** (2019) 125–140.
103. G. H. Goldstein, Self-organization when pedestrian move in opposite directions. Multi-lane circular track model, *Appl. Sci.* **10** (2020) 563.
104. M. Grave, A. Viguerie, G. F. Barros, A. Reali, R. F. S. Andrade and A. L. G. A. Coutinho, Modeling nonlocal behavior in epidemics via a reaction–diffusion system incorporating population movement along a network, *Comput. Methods Appl. Mech. Eng.* **401** (2022) 115541.
105. S.-Y. Ha and T. Ruggeri, Emergent dynamics of a thermodynamically consistent particle model, *Arch. Ration. Mech. Anal.* **223** (2017) 1397–1425.
106. S.-Y. Ha, J. Kim and T. Ruggeri, Emergent behaviors of thermodynamic Cucker–Smale particles, *SIAM J. Math. Anal.* **50** (2018) 3092–3121.
107. M. Haghani, The knowledge domain of crowd dynamics: Anatomy of the field, pioneering studies, temporal trends, influential entities and outside–domain impact, *Physica A* **580** (2021) 126145.
108. M. Haghani and M. Sarvi, Social dynamics in emergency evacuations: Disentangling crowds attraction and repulsion effects, *Physica A* **475** (2017) 24–34.
109. F. Haghpanah, K. Ghobadi and B. W. Schafer, Multi-hazard hospital evacuation planning during disease outbreaks using agent-based modeling, *Int. J. Disaster Risk Reduct.* **66** (2021) 102632.
110. D. Helbing, *Social Self-Organization: Agent-Based Simulations and Experiments to Study Emergent Social Behavior* (Springer, 2012).
111. D. Helbing, Traffic and related self-driven many-particle systems, *Rev. Mod. Phys.* **73** (2001) 1067–1141.
112. D. Helbing, I. Farkas and T. Vicsek, Simulating dynamical features of escape panic, *Nature* **407** (2000) 487–490.
113. D. Helbing and A. Johansson, Pedestrian crowd and evacuation dynamics, *Encyclopedia Complex. Syst. Sci.* R. Meyers, (eds) (Springer, New York, NY, 2009), 6476–6495, https://doi.org/10.1007/978-0-387-30440-3_382.
114. D. Helbing, A. Johansson and H.-Z. Al-Abideen, Dynamics of crowd disasters: An empirical study, *Phys. Rev. E* **75** (2007) 046109.
115. D. Helbing and P. Molnár, Social force model for pedestrian dynamics, *Phys. Rev. E* **51** (1995) 4282–4286.
116. L. F. Henderson and D. M. Jenkins, Response of pedestrians to traffic challenge *Transp. Res.* **8** (1974) 71–74.
117. L. F. Henderson, On the fluids mechanics of human crowd motion, *Transp. Res.* **8** (1974) 509–515.
118. D. Hilbert, Mathematical problems, *Bull. Amer. Math. Soc.* **8** (1902) 437–479.

119. T. Hillen and H. G. Othmer, The diffusion limit of transport equations derived from velocity jump processes, *SIAM J. Appl. Math.* **61** (2000) 751–775.
120. S. P. Hoogendoorn, F. van Wageningen-Kessels, W. Daamen and D. C. Duives, Continuum modelling of pedestrian flows: From microscopic principles to self-organised macroscopic phenomena, *Physica A* **416** (2014) 684–694.
121. S. Hoogendoorn and P. H. L. Bovy, Gas-kinetic modeling and simulation of pedestrian flows, *Transp. Res. Record* **1710** (2000) 28–36.
122. H. Holden, K. Karlsen, K. Lie and N. Risebro, *Splitting Methods for Partial Differential Equations with Rough Solutions*: *Analysis and Matlab Programs*, EMS Series of Lectures in Mathematics (European Mathematical Society, 2010).
123. R. L. Hughes, The flow of human crowds, *Annu. Rev. Fluid Mech.* **35** (2003) 169–182.
124. R. L. Hughes, A continuum theory for the flow of pedestrians, *Transp. Res. B* **36** (2002) 507–536.
125. A. M. Ibrahima, I. Venkata and P. De Wildec, Uncertainty in a spatial evacuation model, *Physica A* **479** (2017) 485–497.
126. Y.-Q. Jiang, R.-Y. Guo, F.-B. Tian and S.-G. Zhou, Macroscopic modeling of pedestrian flow based on a second-order predictive dynamic model, *Appl. Math. Model.* **40** (2016) 9806–9820.
127. A. Jebrane, P. Argoul, A. Hakim and M. El Rhabi, Estimating contact forces and pressure in a dense crowd: Microscopic and macroscopic models, *Appl. Math. Model.* **74** (2019) 409–421.
128. D. Kahneman, *Thinking Fast and Slow* (Penguin Books, 2012).
129. I. Karamouzas, B. Skinner and S. J. Guy, Universal power law governing pedestrian interactions, *Phys. Rev. Lett.* **113** (2014) 238701.
130. I. G. Kevrekidis, C. W. Gear, J. M. Hyman, P. G. Kevrekidis, O. Runborg and C. Theodoropoulos, Equation-free, Coarse-grained multiscale computation: Enabling microscopic simulators to perform system-level analysis, *Commun. Math. Sci.* **1** (2003) 715–762.
131. D. Kim, K. O'Connell, W. Ott and A. Quaini, A kinetic theory approach for 2D crowd dynamics with emotional contagion, *Math. Models Methods Appl. Sci.* **31** (2021) 1137–1162.
132. J. Kim, D. Poyato and J. Soler, Hydrodynamic limit of a coupled Cucker–Smale system with strong and weak internal variable relaxation, *Math. Models Methods Appl. Sci.* **31** (2021) 1163–1235.
133. D. Kim and A. Quaini, A kinetic theory approach to model pedestrian dynamics in bounded domains with obstacles, *Kinet. Related Models* **12** (2019) 1273–1296.
134. D. Kim and A. Quaini, Coupling kinetic theory approaches for pedestrian dynamics and disease contagion in a confined environment, *Math. Models Methods Appl. Sci.* **30** (2020) 1893–1915.
135. D. Kim and A. Quaini, Kinetic theory approach to model crowd dynamics with disease contagion, in *Crowd Dynamics*, Vol. 3, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2022), pp. 157–184.
136. N. L. Kontorovsky, C. G. Ferrari, J. P. Pinasco and N. Saintier, Kinetic modeling of coupled epidemic and behavior dynamics: The social impact of public policies, *Math. Models Methods Appl. Sci.* **32** (2022) 2037–2076.
137. A. Kirchner, H. Klüpfel and K. Nishinari, A. Schadschneider and M. Schreckenberg, Discretization effects and the influence of walking speed in cellular automata models for pedestrian dynamics, *J. Statist. Mech.*: *Theory Exper.* **2004** (2004) P10011.

138. G. Köster, F. Treml and M. Gödel, Avoiding numerical pitfalls in social force models, *Phys. Rev. E* **87** (2013) 063305.
139. H. R. Kwon and E. A. Silva, Mapping the landscape of behavioral theories: Systematic literature review, *J. Plan. Literat.* **35** (2020) 161–179, DOI: https://doi.org/10.1177/0885412219881135.
140. G. Le Bon, *Psychology of Crowds* (*Annotated*) (Sparkling Books, 2009).
141. R. LeVeque, *Finite Volume Methods for Hyperbolic Problems* (Cambridge Univ. Press, 2002).
142. X. Li, F. Guo, H. Kuang and H. Zhou, Effect of psychological tension on pedestrian counter flow via an extended cost potential field cellular automaton model, *Physica A* **487** (2017) 47–57.
143. M. Lamb, *Hydrodynamics*, 6th edn. (Cambridge Univ. Press, 1932).
144. L. Li, H. Liu and Y. Han, An approach to congestion analysis in crowd dynamics models, *Math. Models Methods Appl. Sci.* **20** (2020) 867–890.
145. Y. Li, M. Chen, Z. Dou, X. Zheng, Y. Cheng and A. Mebarki, A review of cellular automata models for crowd evacuation, *Physica A*: *Statist. Mech. Appl.* **526** (2019) 120752.
146. J. Liao and L. Zhou, A kinetic modeling of crowd evacuation with several groups in complex venues, *Math. Models Methods Appl. Sci.* **32** (2022) 1785–1805.
147. Z. Liu, C. C. Jacques, S. Szyniszewski, J. K. Guest, B. W. Schafer, T. Igusa and J. Mitrani-Reiser, Agent-based simulation of building evacuation after an earthquake: Coupling human behavior with structural response, *Natural Hazards Rev.* **17** (2016) 04015019.
148. S. Liu, S. Lo, J. Ma and W. Wang, An agent-based microscopic pedestrian flow simulation model for pedestrian traffic problems, *IEEE Trans. Intelli. Transp. Syst.* **15** (2014) 992–1001.
149. J. Ma, M. Wang and L. Li, Research on crowd dynamic risk management based on the psychological stress perception function, *J. Statist. Mech.* **2022** (2022) 123405.
150. S. Maity and S. Sundar, A coupled model for macroscopic behavior of crowd in flood induced evacuation, *Physica A*: *Statist. Mech. Appl.* **607** (2022) 128161.
151. C. Marschler, J. Sieber, R. Berkemer, A. Kawamoto and J. Starke, Implicit methods for equation-free analysis: Convergence results and analysis of emergent waves in microscopic traffic models, *SIAM J. Appl. Dynam. Syst.* **13** (2014) 1202–1238.
152. C. Marschler, J. Starke, P. Liu and G. Kevrekidis, Coarse grained particle model for pedestrian flow using diffusion maps, *Phys. Rev. E* **89** (2014) 013304.
153. B. Maury and J. Venel, A discrete contact model for crowd motion, *ESAIM*: *Math. Models Numer. Anal.* **45** (2011) 145–168.
154. R. M. May, Uses and abuses of mathematics in biology, *Science* **303** (2004) 338–342.
155. E. Mayr, The philosophical foundation of darwinism, *Proc. Amer. Philos. Soc.* **145** (2001) 488–495.
156. M. Moussaïd, D. Helbing, S. Garnier, A. Johansson, M. Combe and G. Theraulaz, Experimental study of the behavioural mechanisms underlying self-organization in human crowds, *Proc. Roy. Soc. A* **276** (2009) 2755–2762.
157. P. Musiani and G. Forni, *Basic Immunology* (Piccin, 2018).
158. K. Nishidate, M. Baba and R. J. Gaylord, Cellular automaton model for random walkers, *Phys. Rev. Lett.* **77** (1996) 1675.
159. I. Panagiotopoulos, J. Starke, J. Sieber and W. Just, Continuation with non-invasive control schemes: Revealing unstable states in a pedestrian evacuation scenario, *SIAM J. Appl. Dynam. Syst.* (2022), https://arxiv.org/abs/2203.02484.

160. V. Panferov and A. Heintz, A new consistent discrete-velocity model for the Boltzmann equation, *Math. Methods Appl. Sci.* **25** (2002) 571–593.
161. L. Pareschi and G. Toscani, *Interacting Multiagent Systems*: *Kinetic Equations and Monte Carlo Methods* (Oxford Univ. Press, 2013).
162. D. G. Patsatzis, L. Russo, I. G. Kevrekidis and C. Siettos, Data-driven control of agent-based models: An Equation/Variable-free machine learning approach, *Journal of Computational Physics* **478** (2023) 111953, https://doi.org/10.1016/j.jcp.2023.111953.
163. S. Paveri Fontana, On Boltzmann like treatments for traffic flow, *Transp. Res.* **9** (1975) 225–235.
164. G. Pescarmona, P. Terna, A. Acquadro, P. Pescarmona, G. Russo, E. Sulics and S. Terna, An agent-based model of COVID-19 diffusion to plan and evaluate intervention policies, in *Crowd Dynamics*, Vol. 3, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2022), pp. 203–257.
165. B. Piccoli and A. Tosin, Time-evolving measures and macroscopic modeling of pedestrian flow, *Arch. Ration. Mech. Anal.* **199** (2011) 707–738.
166. C. A. S. Pouw, F. Toschi, F. van Schadewijk and A. Corbetta, Monitoring physical distancing for crowd management: Real-time trajectory and group analysis, *PloS One* **15** (2020) e024096360.
167. I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, 1971).
168. Y. Rahmati and A. T. Zachry, Learning-based game theoretical framework for modeling pedestrian motion, *Phys. Rev. E* **98** (2018) 032312.
169. S. T. Rassia and C. I. Siettos, Escape dynamics in office buildings: Using molecular dynamics to quantify the impact of certain aspects of human behavior during emergency evacuation, *Environ. Model. Assess.* **15** (2010) 411–418.
170. S. M. Reia and J. F. Fontanari, Effect of group organization on the performance of cooperative processes, *Ecol. Complex.* **30** (2017) 47–56.
171. E. Ronchi, Disaster management: Design buildings for rapid evacuation, *Nature* **528** (2015) 333.
172. E. Ronchi, E. D. Kuligowski, D. Nilsson, R. D. Peacock and P. A. Reneke, Assessing the verification and validation of building fire evacuation models, *Fire Technol.* **52** (2016) 197–219.
173. F. Ronchi, F. Nieto Uriz, X. Criel and P. Reilly, Modelling large-scale evacuation of music festival. *Fire Saf.* **5** (2016) 11–19.
174. E. Ronchi and D. Nilsson, Pedestrian movement in smoke: Theory, data and modelling approaches, in *Crowd Dynamics*, Vol. 1, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2018), pp. 37–62.
175. E. Ronchi, P. A. Reneke and R. D. Peacock, A conceptual fatigue-motivation model to represent pedestrian movement during stair evacuation, *Appl. Math. Model.* **40** (2016) 4380–4396.
176. S. Roy, A. Borzì and A. Habbal, Pedestrian motion modelled by Fokker–Planck Nash games, *Roy. Soc. Open Sci.* **4** (2017) 2017.
177. M. Saberi, K. Aghabayk and A. Sobhani, Spatial fluctuations of pedestrian velocities in bidirectional streams: Exploring the effects of self-organization, *Physica A* **434** (2015) 120–128.
178. Z. Sabeur and B. A. Zavar, Crowd behaviour understanding using computer vision, in *Crowd Dynamics*, Modelling Simulations Science Engineering Technology, Vol. 3 (Birkhäuser, Cham, 2022), pp. 49–72.
179. P. S. A. Salam, W. Bock, A. Klar and S. Tiwari, Disease contagion models coupled to crowd motion and mesh-free simulation, *Math. Models Methods Appl. Sci.* (2021), doi:10.1142/S0218202521400066.

180. A. Schadschneider, M. Chraibi, A. Seyfried, A. Tordeux and J. Zhang, Pedestrian dynamics: From empirical results to modeling, in *Crowd Dynamics*, Vol. 1, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2018), pp. 63–102.
181. A. Schadschneider, W. Klingsch, H. Kläpfel, T. Kretz, C. Rogsch and A. Seyfried, Evacuation dynamics: Empirical results, modeling and applications, *Encycl. Complex. Syst. Sci.* in R. Meyers, (eds) (Springer, New York, NY, 2009), 3142–3176, https://doi.org/10.1007/978-0-387-30440-3_187.
182. A. Schadschneider and A. Seyfried, Empirical results for pedestrian dynamics and their implications for modeling, *Netw. Heterog. Media* **6** (2011) 545–560.
183. G. Seminara, B. Carli, G. Forn, S. Fuzzi, A. Mazzino and A. Rinaldo, Biological fluid dynamics of airborne COVID.19 infection, *Rendiconti Lincei. Scienze Fisiche e Naturali* **31** (2020) 505–537.
184. A. Seyfried, B. Steffen, W. Klingsch and M. Boltes, The fundamental diagram of pedestrian movement revisited, *J. Statist. Mech.*: *Theory Exper.* **360** (2006) 232–238.
185. L. F. Shampine and C. W. Gear, A users view of solving stiff ordinary differential equations, *SIAM Rev.* **21** (1979) 1–17.
186. A. Shipman and A. Majumdar, Fear in humans: A glimpse into the crowd-modeling perspective, *Transp. Res. Record* **2672** (2018) 183–197.
187. C. I. Siettos, D. Maroudas and I. G. Kevrekidis, Coarse bifurcation diagrams via microscopic simulators: A state-feedback control-based approach, *Int. J. Bifurc. Chaos* **14** (2004) 207–220.
188. C. I. Siettos, N. Kazantzis and I. G. Kevrekidis, An equation-free approach to nonlinear control: Coarse feedback linearization with pole-placement, *Int. J. Bifurc. Chaos* **16** (2006) 22029–2041.
189. C. I. Siettos, Equation-Free multiscale computational analysis of individual-based epidemic dynamics on networks, *Appl. Math. Comput.* **218** (2011) 324–336.
190. C. I. Siettos, C. W. Gear and I. G. Kevrekidis, An equation-free approach to agent-based computation: Bifurcation analysis and control of stationary states, *Europhys. Lett.* **99** (2012) 48007.
191. C. Siettos and L. Russo, A numerical method for the approximation of stable and unstable manifolds of microscopic simulators, *Numer. Algor.* **89** (2022) 1335–1368.
192. H. Sun, L. Hu, W. Shou and J. Wang, Self-organized crowd dynamics: Research on earthquake emergency response patterns of drill-trained individuals based on GIS and Multi-agent systems methodology, *Sensors* **21** (2021) 135, https://doi.org/10.3390/s21041353.
193. A. Templeton and F. Neville, Modeling collective behavior: Insights and applications from crowd psychology, in *Crowd Dynamics*, Vol. 2, Modelling Simulations Science Engineering Technology (Birkhäuser, Cham, 2020), pp. 55–81.
194. A. Tordeux and C. Totzeck, Multi-scale description of pedestrian collective dynamics with Port-Hamiltonian Systems, preprint (2022), arXiv:2211.06503v1 [physics.soc-ph].
195. M. Twarogowska, P. Goatin and R. Duvigneau, Macroscopic modeling and simulations of room evacuation, *Appl. Math. Model.* **38** (2014) 5781–5795.
196. F. Venuti and L. Bruno, Crowd-structure interaction in lively footbridges under synchronous lateral excitation: A literature review, *Phys. Life Rev.* **6** (2009) 176–206.
197. J. von Neumann, *Theory of Self-reproducing Automata*, ed. Arthur W. Burks (Univ. of Illinois Press, 1966).
198. A. U. K. Wagoum, M. Chraibi, J. Mehlich, A. Seyfried and A. Scadschneider, Efficient and validated simulation of crowds for an evacuation assistant, *Comput. Anim. Virtual Worlds* **23** (2012) 3–15.

199. J. Wang, L. Zhang, Q. Shi, P. Yang and X. Hu, Modeling and simulating for congestion pedestrian evacuation with panic, *Physica A: Statist. Mech. Appl.* **428** (2015) 396–409.
200. L. Wang, M. B. Short and A. L. Bertozzi, Efficient numerical methods for multiscale crowd dynamics with emotional contagion, *Math. Models Methods Appl. Sci.* **27** (2017) 205–230.
201. J. Was and R. Lubaś, Towards realistic and effective agent-based models of crowd dynamics, *Neurocomputing* **146** (2014) 199–209.
202. J. Willems, A. Corbetta, V. Menkovski and F. Toschi, Pedestrian orientation dynamics from high-fidelity measurements, *Sci. Rep.* **10** (2020) 11653.
203. N. Wijermans, C. Conrado, M. van Steen, C. Martella and J.-L. Li, A landscape of crowd management support: An integrative approach, *Saf. Sci.* **86** (2016) 142–164.
204. X. Yang, H. Dong, Q. Wang, Y. Chen and X. Hu, Guided crowd dynamics via modified social force model, *Physica A: Statist. Mech. Appl.* **411** (2014) 63–73.
205. Y. Yuan, B. Goñi-Ros, H. H. Bui, W. Daamen, H. L. Vu and S. P. Hoogendoorn, Macroscopic pedestrian flow simulation using Smoothed Particle Hydrodynamics (SPH), *Transp. Res. C* **111** (2020) 334–351.
206. M. Xu, X. Xie, P. Lv, J. Niu, H. Wang, C. Li and B. Zhou, Crowd behavior simulation with emotional contagion in unexpected multi-hazard situations, *IEEE Trans. Syst. Man Cybern. Syst.* **51** (2021) 1567–1581.
207. M. Xu, Y. Wu, P. Lv, H. Jiang, M. Luo and Y. Ye, miSFM: On combination of mutual information and social force model towards simulating crowd evacuation, *Neurocomputing* **168** (2015) 529–537.
208. M. Zanella, C. Bardelli, G. Dimarco, S. Deandrea, P. Perotti, M. Azzi, S. Figini and G. Toscani, A data-driven epidemic model with social structure for understanding the COVID-19 infection on heavily affected italian province, *Math. Models Methods Appl. Sci.* **31** (2021) 2533–2570.
209. W. Zeng, P. Chen, H. Nakamura and M. Iryo-Asano, Application of social force model to pedestrian behavior analysis at signalized crosswalk, *Transp. Res. C: Emerg. Technol.* **40** (2014) 143–159.
210. H. Zhang, H. Liu, X. Qin and B. Liu, Modified two-layer social force model for emergency earthquake evacuation, *Physica A: Statist. Mech. Appl.* **492** (2018) 1107–1119.
211. X. Zheng and Y. Cheng, Conflict game in evacuation process: A study combining Cellular Automata model, *Physica A: Statist. Mech. Appl.* **390** (2011) 1042–1050.
212. Z. Zhou, Y. Zhou, Z. Pu and Y. Xu, Simulation of pedestrian behavior during the flashing green signal using a modified social force model, *Transp. A: Transp. Sci.* **15** (2019) 1019–1040.
213. Y. Zou, J. Xie and B. Wang, Evacuation of pedestrians with two motion modes for panic system, *PloS One* **11** (2016) e0153388.