

## $\mathfrak{so}(2,2)$ extension of Jackiw-Teitelboim gravity via the Virasoro-Kac-Moody semidirect product

Goffredo Chirco<sup>✉,\*</sup>, Lucio Vacchiano,<sup>†</sup> and Patrizia Vitale<sup>‡</sup>

*Dipartimento di Fisica “Ettore Pancini,” Università di Napoli Federico II and INFN, Sezione di Napoli, Complesso Universitario di Monte Sant’Angelo, Via Cintia, 21, 80126 Napoli, Italy*

 (Received 18 March 2025; accepted 10 June 2025; published 1 July 2025)

We consider a bulk plus boundary extension of Jackiw-Teitelboim gravity coupled with non-Abelian gauge fields. The generalization is performed in the Poisson sigma model formulation and it is derived as a dimensional reduction of the AdS<sub>3</sub> Chern-Simons theory with Wess-Zumino-Witten (WZW) boundary terms. We discuss the role of boundary conditions in relation to the symmetries of the boundary dynamics and we show that the boundary action can be written in terms of coadjoint orbits of an appropriate Virasoro-Kac-Moody group. We obtain a Schwarzian action and interaction terms with additional edge modes that match the effective low-energy action of recent Sachdev–Ye–Kitaev-like tensor models.

DOI: 10.1103/kpkz-rszk

### I. INTRODUCTION

Two-dimensional dilaton gravity has proved to be a powerful tool in describing black hole physics at the classical and quantum levels as an effective description of the near-horizon dynamics obtained from dimensional reduction [1–3]. Among these models, Jackiw-Teitelboim (JT) gravity [4–6] has attracted much attention recently, because of its holographic duality with the one-dimensional Sachdev–Ye–Kitaev model (SYK), which provides a concrete and solvable realization of the (near) AdS<sub>2</sub>/CFT<sub>1</sub> correspondence [7–10]. In the JT/SYK case [11], the duality lies in a shared one-dimensional Schwarzian dynamical sector, which describes the low-energy/strong-coupling regime of the SYK model [12], as well as the boundary dynamics induced by the Gibbons-Hawking-York (GHY) term in the JT theory. In the case of JT gravity, the reduction of the GHY term to the Schwarzian action at the boundary has been computed explicitly (see [13] and references therein) to give

$$S_{\text{JT}}|_{\partial\Sigma} \simeq \int_{\partial\Sigma} \Phi_b \{F(\tau), \tau\}_S d\tau, \quad (1)$$

where  $\tau$  is a coordinate parametrizing the boundary  $\partial\Sigma \simeq S^1$ ,  $F$  is a diffeomorphism of the boundary,  $\Phi_b$  is

the boundary value of the dilaton when adopting Dirichlet boundary conditions, and

$$\{F(\tau), \tau\}_S = \frac{F'''}{F'} - \frac{3}{2} \left( \frac{F''}{F'} \right)^2 \quad (2)$$

is the Schwarzian derivative [14,15]. The latter transforms as a conformal field theory (CFT) stress-energy tensor under reparametrization of the circle, and it is invariant under Moebius transformations.

On the boundary of the near AdS<sub>2</sub> (near-horizon) geometry, the Schwarzian derivative encodes the notion of extrinsic curvature [15,16] while being associated with the breaking of conformal symmetry down to a global  $SL(2, \mathbb{R})$ , both in JT and SYK-like models [8]. In the first case, the conformal symmetry is spontaneously broken by the presence of a boundary and explicitly broken in the passage from pure anti-de Sitter (AdS) to near AdS via the coupling to the dilaton, while, in the second case, the breaking reflects a finite-temperature effect. Besides the duality, the beauty and power of such a framework lies in the generality of the symmetry breaking scheme in low-dimensional gravitational theories with a boundary (see also [17–19]).

The presence of a boundary breaks the asymptotic group of symmetry  $\mathcal{G}$  into some reduced symmetry group  $\mathcal{H}$ , which depends on the choice of boundary conditions. The boundary dynamics is governed by fields belonging to the quotient space  $\mathcal{G}/\mathcal{H}$ , which can be identified with an appropriate coadjoint orbit of  $\mathcal{G}$ . In the case of JT gravity, one has  $\mathcal{G} = \text{Diff}(S^1)$  and  $\mathcal{H} = SL(2, \mathbb{R})$ . The boundary action can then be written as the geometric action on  $S^1$  associated with the coadjoint action of a diffeomorphism  $F$  over an element of the Virasoro dual algebra and, with a

\*Contact author: goffredo.chirco@unina.it

†Contact author: lucio.vacchiano@unina.it

‡Contact author: patrizia.vitale@na.infn.it

*Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Funded by SCOAP<sup>3</sup>.*

suitable choice of this element, the action in (1) can be exactly derived in purely geometric terms. We refer to Sec. II A 1 and Appendix for a detailed review of the derivation.

The generality of the coadjoint orbits method provides a powerful tool for investigating extensions of the boundary JT/SYK dynamics, starting from generalizations of the symmetry group of the dual models. Indeed, ever since the JT/SYK correspondence has been pointed out, much work has aimed at expanding the range of potential holographic relationships in 2D gravity [20–24]. Numerous variants and generalizations of the SYK model have been proposed in recent years [25–31], raising the natural question of whether an extended duality with generalized JT models would exist. In this sense, modifications of JT gravity have been introduced following diverse paths, such as relaxing the boundary conditions [32], considering higher spin extensions of the theory [33–35], or coupling the gravitational sector with additional degrees of freedom, such as non-Abelian gauge fields [35].

Among the various generalizations of the SYK model, on the other side of the correspondence, tensor extensions in particular have been shown to exhibit a broken  $\mathfrak{diff}(S^1) \ltimes \hat{\mathfrak{g}}$  symmetry in the strong coupling limit, with  $\mathfrak{g}$  begin the Lie algebra of some internal symmetry by which the tensor extension is realized [28], and the hat indicating, as in the rest of the paper, the associated loop algebra. One main motivation for tensor generalizations of SYK models lies in the fact that, while preserving solvability in the large  $N$  limit, their effective dynamics turns out to be a generalization of Schwarzian dynamics, without requiring disordered averaging. As argued in [27], the possibility of establishing holographic relations without necessarily averaging over disorder is a remarkable feature of these models because it allows for a more direct correspondence between the tensor models' degrees of freedom and those of gravitational theories in low dimensions. As a further motivation for the introduction of tensor models, we also mention the idea of generalizing to higher dimensions the correspondence between matrix models and two-dimensional geometries [36,37].

The emergent symmetry of tensor extensions of SYK models is generally characterized by the semidirect product of the group of reparametrizations of  $S^1$  with an affine Kac-Moody algebra. Therefore, a natural question is whether a bulk gravitational theory exists whose boundary action is written in terms of coadjoint orbits of such semidirect product.

In this work, we propose a bulk plus boundary theory, which reproduces a broken  $\mathfrak{diff}(S^1) \ltimes \hat{\mathfrak{g}}$  symmetry at the boundary and can be regarded as a gauge extension of the JT gravity model, in the sense indicated in [35]. The way the extension is realized differs from what has been proposed so far in the literature, in many respects. In [35], for instance, the authors consider an  $\mathfrak{sl}(2, \mathbb{R})$ -BF theory for the gravitational sector together with an additional  $\mathfrak{g}$ -BF

theory for the Yang-Mills part, so that the symmetry of the model is given by the direct sum  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{g}$ . However, the proposed bulk duals, to date, do not have a clear gravitational interpretation and the role of the additional degrees of freedom extending the JY/SYK correspondence remains unclear.

We propose a class of extensions of JT where the presence of additional degrees of freedom is fully ascribed to the request that the 2D theory be obtained from a dimensional reduction of a Chern-Simons theory describing a maximally symmetric space in three dimensions, which includes a Banados-Tietelboim-Zanelli (BTZ) black hole solution (see, e.g., [38–40]). This choice drastically reduces the space of possible duality relations, but at the same time allows the additional degrees of freedom to be clearly interpreted as Kaluza-Klein modes emerging from the above dimensional reduction. Notice that, although starting from three-dimensional theories limits the options on the possible symmetry groups for the corresponding 2D theories, the choice of the group does not unambiguously determine a gravitational dual for a given extension of SYK, as much depends on the boundary conditions. Consequently, when a symmetry group is fixed with a given criterion, i.e., compatibility with a pure gravity theory in 3D, as many duality relations can be realized as the allowed choices of boundary conditions.

Our construction can be summarized by the following steps. Building on the previous work, which relates JT gravity, BF theory, and linear Poisson sigma models (see, e.g., [16] and references therein), we first derive the Poisson sigma model with the linear Poisson bracket of  $\mathfrak{so}(2,2)$  type from the dimensional reduction of the  $\mathfrak{so}(2,2)$  Chern-Simons (CS) theory with Wess-Zumino-Witten (WZW) boundary terms. This is a purely gravitational theory with a boundary dynamics, as well as JT gravity is in 2D. The two-dimensional theory obtained in this way is an extension of JT gravity, which also includes additional non-Abelian gauge fields both in the bulk and at the boundary.

The choice of  $\mathfrak{so}(2,2)$  is the simplest nontrivial choice that makes it possible to understand the model as a dimensional reduction of a purely gravitational 3D theory of a kinematical space-time, in the sense of [41], given that  $SO(2,2)$  is the isometry group for  $AdS_3$ .

The WZW boundary term for the 3D theory reduces to a boundary term in the form of a Casimir function for the  $\mathfrak{so}(2,2)$  algebra in the 2D theory. Together with an appropriate choice of boundary conditions, this leads to a boundary dynamics governed by the  $\mathfrak{diff}(S^1) \ltimes \hat{\mathfrak{g}}$  symmetry breaking. The boundary Casimir action is then identified with the action associated with coadjoint orbits of the Virasoro-Kac-Moody semidirect product. We explicitly compute the Kac-Moody terms and investigate their contribution as corrections to the nearly extremal entropy in JT. In particular, we discuss how the obtained results do not depend on the specific parametrization choices for edge

fields, such as the highest weight gauge widely used in the literature [35].

The work is organized as follows. In Sec. II, we recall the formulation of JT gravity as a 2D  $SL(2, \mathbb{R})$ -BF theory and its derivation from a linear  $SL(2, \mathbb{R})$ -Poisson sigma model (PSM). As we shall see, the latter formulation allows for a more natural introduction of boundary terms, differing from the BF action for a boundary contribution. Moreover, as we shall comment in the Conclusions, it lends itself to generalizations. In the PSM framework, we review the construction of dynamical boundary actions in terms of a Casimir function and their relation with coadjoint orbits of infinite-dimensional groups. Finally, we describe the dimensional reduction from 3D CS to our 2D model. The details on the Kirillov method of coadjoint orbits of infinite-dimensional Lie groups are given in the Appendix. Sections III and IV contain the original results of the paper. In Sec. III, we discuss the extension of the JT model via a  $\mathfrak{so}(2, 2)$ -PSM and the need for the Virasoro-Kac-Moody semidirect product. Here, we also provide the explicit computation of the coadjoint orbits, including the Kac-Moody contributions. In Sec. IV, we compute the leading order entropy in the Euclidean theory and compare it with the SYK results. We close in Sec. V with a summary of our results. We also further motivate the choice of working with a PSM with the indication of some future directions of research.

## II. 2D JT GRAVITY AS PSM/BF THEORY

Before discussing the details of the  $\mathfrak{so}(2, 2)$ -PSM, it is worth briefly reviewing the topological gauge theory description of JT gravity in the first-order formulation, the way it reduces to a BF model [42,43] or equivalently to linear PSM [44] (see also [13] for a review). Upon reviewing the formalism, we will consider a two-dimensional manifold  $\Sigma$  with a boundary  $\partial\Sigma$  and focus on the form of the boundary terms in the description of BF/PSM.

### A. JT gravity in the BF/PSM formalism

The Jackiw-Teitelboim two-dimensional gravity theory is defined by the action [45]

$$S_{\text{JT}}[g, \Phi] = \frac{1}{16\pi G_N} \int_{\Sigma} d^2x \sqrt{-g} \Phi (\mathcal{R} + 2) + \int_{\partial\Sigma} du \sqrt{-h} (K - 1) \Phi_b, \quad (3)$$

where  $\Phi$  is the dilaton and  $\mathcal{R}$  is the scalar curvature of the bulk, while  $\Phi_b$ ,  $K$ , and  $du\sqrt{-h}$ , respectively, refer to the restriction of the dilaton field on the boundary, the extrinsic curvature of  $\partial\Sigma$ , and the boundary volume form. The second term in (3), the GHY boundary term [46], is included to make the metric variational problem well defined. The bulk action in (3) can be explicitly derived

from the near-horizon and near-extremality limit of the Reissner-Nordstrom solution in four dimensions, via dimensional reduction [11,47]. The model is topological in the bulk, and the dilaton field just acts as a Lagrange multiplier fixing the value of the curvature. Although there are no propagating degrees of freedom in the bulk, under suitable choices of boundary conditions, the boundary action is dynamical.

Let us first focus on the bulk. The bulk JT action can be formulated, in a first-order formalism, as a 2D topological  $SL(2, \mathbb{R})$  BF theory

$$S_{\text{BF}}[B, A] = \int_{\Sigma} \text{Tr}(\text{BF}), \quad (4)$$

where  $F$  is the curvature of the Lie algebra valued connection 1-form  $A = A_{\mu}^a dx^{\mu} J_a$ ,  $B = B^a J_a$  is a Lie algebra valued scalar field, and  $J_a$  are the  $\mathfrak{sl}(2, \mathbb{R})$  Lie algebra generators. The trace is computed with the Cartan-Killing metric  $\eta = \text{diag}(1, 1, -1)$ . Varying the action with respect to  $B$  leads to the equation of motion  $F = 0$ , which is equivalent to saying that the connection is locally trivial (pure gauge), that is,

$$A = g^{-1} dg, \quad (5)$$

where  $g$  is an element of the gauge group. In components, we get

$$dA^a + \epsilon_{bc}^a A^b \wedge A^c = 0. \quad (6)$$

One then identifies the components of the gauge connection with the zweibein forms on  $\Sigma$  and the Lorentz (or spin) connection (Cartan variables), that is,  $A^{0,1,2} = (e^0, e^1, \omega)$  [42]. Thus, we get

$$de^h + \omega \epsilon^{hk} \wedge e_k = 0, \quad (7)$$

$$d\omega - e^0 \wedge e^1 = 0, \quad (8)$$

with  $h, k = 0, 1$ .

Finally, we identify the torsion  $F^k = T^k$ , for  $k = 0, 1$ , and  $F^2$  as  $R + e^0 \wedge e^1$ , where  $R$  is the curvature 2-form and  $e^0 \wedge e^1$  is the volume form.

The dilaton is encoded in one component of the  $B$  field,  $\Phi = B^2$ , while the other two,  $B^{0,1}$ , define Lagrange multipliers imposing the condition of zero torsion for the spin connection, as in (7), once the BF action is rewritten according to

$$S = \int_{\Sigma} B_h (de^h + \epsilon^{hk} \omega \wedge e_k) - \Phi (R - e^0 \wedge e^1). \quad (9)$$

This is the off-shell JT action, the second term of (9) corresponding to the bulk term of (3).

The on-shell connection is pure gauge and the dynamics of the dilaton corresponds exactly to a gauge transformation that preserves the form of  $A$ , that is,  $B$  on-shell is the stabilizer of  $A$ .

Equivalently, one can write the 2D BF theory as a linear PSM in terms of real fields  $(X, A)$ , with  $X: \Sigma \rightarrow M$  being the usual embedding map and  $A \in \Omega^1(\Sigma, X^*(T^*M))$  a 1-form on  $\Sigma$  taking values in the pull-back of the cotangent bundle over  $M$ . The action of the general PSM is given by ( $i, j = 1, \dots, \dim M$ )

$$S(X, A) = \int_{\Sigma} A_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) A_i \wedge A_j, \quad (10)$$

where  $dX \in \Omega^1(\Sigma, X^*(TM))$ , and the contraction of covariant and contravariant indices is induced by the natural pairing between  $T^*M$  and  $TM$ , yielding a 2-form on  $\Sigma$ . To make contact with the BF formulation of JT gravity described above, one has to consider a linear Poisson tensor of Lie algebra type

$$\Pi^{ij}(X) = f_k^{ij} X^k, \quad (11)$$

with  $f_k^{ij}$  being the structure constants of the  $\mathfrak{sl}(2, \mathbb{R})$  Lie algebra. In particular, integrating by parts the linear PSM action with the Poisson tensor given by (11), we obtain the BF action (4) plus a boundary term

$$S_{\text{PSM}} = S_{\text{BF}} - \int_{\partial\Sigma} X^i A_i. \quad (12)$$

The variation of the action with respect to  $X$  and  $A$  yields

$$\delta S_{\text{PSM}} = \int_{\Sigma} (\text{E.L.}) \delta X^i + (\text{E.L.}) \delta A_i - \int_{\partial\Sigma} \delta X^i A_i, \quad (13)$$

with Euler-Lagrange (E.L) equations

$$D_A A = 0, \quad dX + [X, A] =: \delta_X A = 0, \quad (14)$$

where  $D_A$  denotes the covariant derivative with respect to the gauge connection  $A$ . The first equation implies that  $A$  is pure gauge

$$A = g^{-1} dg, \quad (15)$$

with  $g: \Sigma \rightarrow \text{SL}(2, \mathbb{R})$ , while the second equation states that the on-shell  $X$  field is a stabilizer of  $A$ . As suggested in (14), the dynamics of the dilaton corresponds to an infinitesimal gauge transformation that preserves the form of  $A$  along  $X$  on-shell. In particular, one can check that the boundary term in (12) is such that the gauge invariance restricts to the gauge transformations that satisfy  $\delta_g A|_{\partial\Sigma} = 0$ , which is exactly the equation of motion for the  $X$  field when restricted at the boundary. The reparametrization symmetry

$\text{Diff}(S^1)$  is broken, and this breaking is responsible for the rise of dynamical boundary degrees of freedom (see [40] and references therein).

Now, the presence of the boundary terms in (13) requires fixing the boundary values of the fields to have a well-defined variational principle. However, this choice would prevent any boundary dynamics. Alternatively, adding an extra boundary term allows for more general boundary conditions and preserves the variational principle. The PSM framework suggests a natural choice in this sense, consisting in the Casimir function  $X^2$  [48]. Indeed, one can write

$$S_{(\Sigma+\partial\Sigma)} = S_{\text{PSM}} + \frac{1}{2} \int_{\partial\Sigma} X^i X_i du, \quad (16)$$

with  $du$  being the integration measure over  $\partial\Sigma$ , with the condition that

$$X_i|_{\partial\Sigma} du = A_i|_{\partial\Sigma}. \quad (17)$$

The presence of the extra boundary term in (16) allows for more general boundary conditions and preserves the variational principle. The condition in (17), together with the fact that  $A$  is pure gauge on-shell, and the required continuity of the fields at the boundary, leads to a boundary action that describes the dynamics of a particle-on-a-group manifold, in this case on  $\text{SL}(2, \mathbb{R})$ . Indeed, by continuing Eq. (15) to the boundary, the on-shell boundary action reads [49]

$$S|_{\partial\Sigma} = \int_{\partial\Sigma} \text{Tr}(g^{-1} g')^2 \frac{1}{u} d\tau, \quad (18)$$

that is, a particle-on-a-group action where now  $g$  is an element of the loop group  $\mathcal{L}\text{SL}(2, \mathbb{R})$ .

At this stage, it is not obvious that the boundary quadratic term derived above encodes the same boundary dynamics induced by the GHY term in JT gravity. However, it is possible to show that the boundary dynamics of the particle on the group manifold is related with the Schwarzian action. In the following section, we review the elements of such a correspondence. Thereby, in Sec. III, we propose a generalization of this result for the case of a  $\mathfrak{so}(2, 2)$  PSM.

### 1. Schwarzian action from coadjoint orbits

As we already discussed in the previous section, the presence of a boundary action (18) breaks the reparametrization invariance associated with  $\text{Diff}(S^1)$  to global  $\text{SL}(2, \mathbb{R})$  on the boundary, while the gauge symmetry is automatically mod-out since the equations of motion (14) require the  $A$  connection 1-form to be pure gauge. Therefore, the particle-on-a-group action shows a global  $\text{SL}(2, \mathbb{R})$  symmetry, modulo gauge transformation, which

is trivial on the boundary [50]. Accordingly, the dynamical boundary action is reduced to the coset space  $\text{Diff}(S^1)/SL(2, \mathbb{R})$  defined by the aforementioned symmetry breaking pattern.

Such a reduction can be performed by means of the Kirillov coadjoint orbit method (see Appendix for a detailed review). In brief, coadjoint orbits of the Virasoro group over the dual Virasoro algebra naturally realize homogeneous spaces  $\text{Diff}(S^1)/\text{Stab}(b)$ , where  $\text{Stab}(b)$  indicates the little group associated with the dual element  $b$  over which the coadjoint action is computed. In the present case, we want  $\text{Stab}(b)$  to be the global residual  $SL(2, \mathbb{R})$ . In particular, with  $b = (f, t)$  being an element in the dual of the Virasoro algebra and  $\phi$  a finite diffeomorphism in the Virasoro group, we can compute the coadjoint action [see Eq. (A51)] as follows:

$$\begin{aligned} \tilde{b} &= Ad_{\phi^{-1}}^*(f(\tau), t) = \left( \phi'^2 f(\tau) - \frac{t}{12} \{ \phi(\tau), \tau \}_S, t \right) \\ &\in \mathfrak{diff}^*(S^1). \end{aligned} \quad (19)$$

For  $f(\tau) = -\frac{m^2}{24}$ , one gets that  $\text{Stab}(b) = SL^{(n)}(2, \mathbb{R})$ , and the homogeneous space defined by the coadjoint orbit is  $\text{Diff}(S^1)/SL^{(n)}(2, \mathbb{R})$ , where  $SL^{(n)}(2, \mathbb{R})$  denotes the  $n$ -fold cover of  $SL(2, \mathbb{R})$  (see, e.g., [50]).

Now, a natural action functional on the coset space is given by the pairing of  $\tilde{b}$  with an element  $\xi \in \mathfrak{diff}(S^1)$ , which gives [51]

$$\langle \tilde{b}, \xi \rangle = \int_{S^1} \tilde{b}(\tau) \xi(\tau) d\tau. \quad (20)$$

The latter can be identified with the reduction of the on-shell boundary action

$$S[g]|_{\partial\Sigma} = \int_{\partial\Sigma} \text{Tr}(g^{-1}g')^2 / u' d\tau \quad (21)$$

on the coset space, provided  $Xu'd\tau = g^{-1}g'd\tau$ , where  $g \in \mathcal{L}\mathcal{G}$ , by identifying  $\tilde{b}$  with the Casimir  $X^2$  and setting  $\xi(\tau) = 1/u'$ , for  $du = u'd\tau$  on  $\partial\Sigma \sim S^1$ . The relation between (1) and the boundary action in (16) can then be understood purely from symmetry considerations [52]. The Schwarzian action is nothing but the action associated with the coadjoint orbits of the Virasoro group with global  $SL(2, \mathbb{R})$  symmetry, and it can be identified with the particle-on-a-group-manifold action reduced on the coset space.

A more direct way of deriving the equivalence between the action of a particle-on-a-group-manifold (21) and the natural action on the coadjoint orbits of  $\text{Diff}(S^1)$  is that of [50], where the group element  $g \in SL(2, \mathbb{R})$  is parametrized by means of the Iwasawa decomposition,

$$g(t) = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix}. \quad (22)$$

The action functional is then

$$S[F, a, \theta] = - \int_{S^1} \left( -\frac{1}{4} \theta'^2 + \left( \frac{a'}{a} \right)^2 + \frac{1}{2} a^2 \theta' F' \right) d\tau. \quad (23)$$

Conserved charges of the particle-on-a-group action are given by

$$J_i = \text{Tr}\{T_i g^{-1} g'\}, \quad (24)$$

in particular, one of the conserved charges is the conjugate momentum  $\pi_F$  of  $F$ . The explicit computation leads to

$$J_-(g) = \pi_F = \frac{1}{2} a^2 \theta'. \quad (25)$$

By means of this constraint, the last term of the action is just  $\int F' d\tau$  and, therefore, it vanishes since  $F$  is a periodic function. The action in the only residual degree of freedom  $\theta$  then coincides with the Schwarzian action

$$S[\theta] = \frac{1}{2} \int_{S^1} \left( \frac{1}{2} \theta'^2 + \{ \theta, \tau \} \right) d\tau. \quad (26)$$

See [50] for more details.

In the following, we will seek an extension of the boundary Schwarzian dynamics of JT gravity by extending the derivation to a  $SO(2,2)$ -PSM, which naturally encodes JT gravity plus extra gauge fields. By first writing the boundary action functional in terms of Casimir functions of the larger group and then using the coadjoint orbit method to compute its reduction, we will see that, under a suitable choice of boundary conditions, we get partial gauge symmetry breaking at the boundary, resulting in a semi-direct product structure  $\text{Diff}(S^1) \ltimes \mathcal{L}\mathcal{G}$  for the infinite-dimensional group whose action corresponds to the boundary Casimir action.

The choice of  $SO(2,2)$  is motivated by the possibility of providing a natural gravitational origin to the extra gauge fields emerging in the  $SO(2,2)$ -PSM, once the gravitational  $SL(2, \mathbb{R})$  sector has been singled out, in terms of a dimensional reduction from a 3D CS theory with a boundary [53]. We briefly recall the dimensional reduction from 3D CS theory to the 2D BF/PSM model associated with JT gravity in the following. Thereby, we move to the proposed  $SO(2,2)$  model in Sec. III.

### B. Dimensional reduction from 3D CS + WZW

Let us consider a CS theory in three dimensions, with  $\mathfrak{so}(2,2)$ -valued connection  $\Omega$ . This yields a purely gravitational model describing the  $\text{AdS}_3$  geometry. We shall see

that the dimensional reduction of such a theory is equivalent to a  $\mathfrak{so}(2, 2)$ -PSM in two dimensions. Moreover, we will show that the 2D boundary terms, necessary to derive the 1D Schwarzian dynamics, can be recovered by adding to the CS theory a WZW boundary term that dimensionally reduces to the action of a particle-on-a-group manifold.

To this, let  $\Sigma_3$  be a manifold with the structure  $\Sigma_3 = \Sigma \times I$ , where  $\Sigma$  is a two-dimensional manifold with a boundary and  $I$  a suitably chosen one dimensional submanifold. We write

$$S_{\text{CS}}[\Omega] = \frac{1}{2} \int_{\Sigma_3} \left\langle \Omega \wedge d\Omega + \frac{1}{3} [\Omega, \Omega] \wedge \Omega \right\rangle. \quad (27)$$

It is convenient for the purpose of this section to write  $\mathfrak{so}(2, 2)$  as the direct sum  $\mathfrak{so}(2, 2) = \mathfrak{sl}(2, \mathbb{R})_L \oplus \mathfrak{sl}(2, \mathbb{R})_R$ . Then,  $\Omega = \omega_i^L L_i + \omega_i^R R_i$ , with  $L_i$  and  $R_i$  spanning  $\mathfrak{sl}(2, \mathbb{R})$ . The Killing form is given by

$$\langle L_i, L_j \rangle = \langle R_i, R_j \rangle = \eta_{ij}, \quad \langle L_i, R_j \rangle = 0, \quad (28)$$

with  $\eta = \text{diag}(1, -1, 1)$ . Therefore, the Chern-Simons theory splits into two copies of a single  $\mathfrak{sl}(2, \mathbb{R})$ -CS theory with connection  $\omega = \omega^i \tau_i$ . It is then sufficient to prove that each  $\mathfrak{sl}(2, \mathbb{R})$ -CS sector reduces to a 2D  $\mathfrak{sl}(2, \mathbb{R})$ -BF theory. The explicit form of each  $\mathfrak{sl}(2, \mathbb{R})$  sector of (27) is given by

$$S_{\text{CS}}[\omega] = \frac{1}{2} \int_{\Sigma_3} d^3x \epsilon^{\lambda\mu\nu} \left( \omega_\lambda^h \partial_\mu \omega_\nu^k + \frac{1}{3} \omega_\lambda^h f_{ij}^k \omega_\mu^i \omega_\nu^j \right) \eta_{hk}, \quad (29)$$

where  $f_{ij}^k$  are structure constants of  $\mathfrak{sl}(2, \mathbb{R})$ . Let  $(\phi, \tau, \rho)$  be local coordinates over  $\Sigma_3$ , with  $\phi \in I$ . The dimensional reduction scheme consists in identifying  $\omega_\phi^i \rightarrow \Phi^i$  and  $\omega_{\tau, \rho}^i \rightarrow A_{\tau, \rho}^i$  and discarding any derivative with respect to  $\phi$ ,  $\partial_\phi \rightarrow 0$ . This results in the action

$$S_{\text{CS}}[\omega] = \frac{1}{2} \int_{\Sigma_3} d^3x \left( \omega_\phi^h \epsilon^{\mu\nu} \partial_\mu \omega_\nu^h + \epsilon^{\mu\nu} \omega_\mu^h \partial_\nu \omega_\phi^k + \frac{1}{3} \epsilon^{\lambda\mu\nu} \omega_\lambda^h \epsilon_{ij}^k \omega_\mu^i \omega_\nu^j \right) \eta_{hk}, \quad (30)$$

where the  $\epsilon^{\mu\nu}$  only refers to the  $(\rho, \tau)$  coordinates in  $\Sigma$ . The first two terms in the action are  $\langle \Phi, dA \rangle$  and  $\langle A \wedge d\Phi \rangle$ , where the external derivative and the wedge product are now performed to be those over  $\Sigma$ , i.e., in the  $(\tau, \rho)$  coordinates. The last term corresponds to  $2\langle \Phi, [A, A] \rangle$ . In order to recover a 2D BF theory, we have to recognize  $F = dA + [A, A]$ . From  $d(\Phi A) = d\Phi \wedge A + \Phi dA$ , we deduce that  $\langle \Phi, dA \rangle + \langle A \wedge d\Phi \rangle = 2\langle \Phi, dA \rangle - \langle d(\Phi, A) \rangle$ . One obtains then

$$S_{\text{CS}}[\omega] = \int_{\Sigma_3} d^3x \langle \Phi, F \rangle - \frac{1}{2} \int_{\partial\Sigma_3} d^2x \langle \Phi, A \rangle. \quad (31)$$

In order to make full contact with the 2D dilaton gravity on an  $\text{AdS}_2$  disk  $\Sigma$ , we require that, after integrating out the redundant dimension,

$$\Sigma_3 \xrightarrow{\int_I d\phi} \Sigma, \quad \partial\Sigma_3 \xrightarrow{\int_I d\phi} \partial\Sigma. \quad (32)$$

Therefore, the dimensional reduction of the CS theory not only reproduces a BF theory in the 2D bulk  $\Sigma$ , but also gives a boundary term.

The complete theory whose dimensional reduction leads to the two-dimensional  $\mathfrak{so}(2, 2)$ -PSM model is a CS theory with a WZW boundary term, as in [54]. We refer to [55] for further details. As explained in [54], the 3D theory has a clear interpretation in terms of BTZ black hole geometry, compatible with the near-horizon physics description of JT gravity. The connection between dimensionally reduced  $\text{AdS}_3$  theories and extremal black holes has also been recently explored in [39].

The reduction from 3D gravity to extensions of JT can be induced by the presence of diverse boundary contributions directly in the 3D action (see, e.g., [56–58]). The construction described in this manuscript differs from the results mentioned above. In particular, the absence of a boundary term makes it unnecessary to fix boundary conditions for the metric of  $\text{AdS}_3$  gravity.

A further remark has to do with the role of the gauge invariance of  $\mathfrak{so}(2, 2)$  Chern-Simons theory in the reduction. We have seen that a  $\mathfrak{so}(2, 2)$  Chern-Simons theory can be dimensionally reduced to a BF theory with appropriate boundary terms if the base 3D manifold  $\Sigma_3$  has the topology of  $\Sigma_2 \times I$ , with  $\Sigma_2$  being a closed disc. The boundary of such a manifold is a torus  $\partial\Sigma_3 = S^1 \times S^1$  and the gauge invariance is not guaranteed for topological reasons. Under finite gauge transformation, the Chern-Simons action  $S_{\text{CS}}[A]$  goes to

$$S_{\text{CS}}[A] \rightarrow S_{\text{CS}}[A^g] = S_{\text{CS}}[A] - \frac{k}{3} \int_{\Sigma_3} (g^{-1} dg)^3 + k \int_{\partial\Sigma_3} dg g^{-1} \wedge A. \quad (33)$$

Neglecting boundary terms for the moment, the bulk variation of the action induces a Wess-Zumino (WZ) term, which renders the quantum theory anomalous. This kind of anomaly is not of concern when the group manifold is compact, e.g.,  $\text{SU}(2)$ , and homotopically equivalent to the source manifold  $\Sigma_3$ . In this case, the WZ term is proportional to the volume of the group manifold and the anomaly of the quantum theory can be easily cured by tuning the coupling constant  $k$  in such a way that, up to boundary terms, a finite gauge transformation produces

$$S_{\text{CS}}[A^g] = S_{\text{CS}}[A] + 2\pi i n, \quad (34)$$

therefore,  $k$  must be quantized. This is not the most general situation and, in fact, it does not apply to our case, where the gauge group  $SO(2, 2)$  is not compact and, in particular, the third homotopy group is trivial,  $\pi_3[SO(2, 2)] = 0$ . Moreover, the source manifold is not homotopically equivalent to it. However, one can still cure the bulk anomaly by following the strategy of [59]. The gauge group  $SO(2, 2)$  is locally equivalent to  $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ , and the Chern-Simons theory can be written as the difference of two distinct  $SL(2, \mathbb{R})_{L,R}$  Chern-Simons theories with a suitable choice of the bilinear form on  $\mathfrak{so}(2, 2)$ . Let  $A_L$  and  $A_R$  be the connection one forms with values in  $\mathfrak{sl}(2, \mathbb{R})_{L,R}$ ,

$$S_{\text{CS}}[A_L, A_R] = S_{\text{CS}}[A_L] - S_{\text{CS}}[A_R]. \quad (35)$$

Under finite gauge transformations, one has

$$\begin{aligned} S_{\text{CS}}[A_L, A_R] &\rightarrow S_{\text{CS}}[A_L^{g_L}, A_R^{g_R}] = S_{\text{CS}}[A_L, A_R] \\ &+ k \int_{\Sigma_3} (g_L^{-1} dg_L)^3 - k \int_{\Sigma_3} (g_R^{-1} dg_R)^3 \\ &+ k \int_{\partial\Sigma_3} dg_L g_L^{-1} \wedge A_L - k \int_{\partial\Sigma_3} dg_R g_R^{-1} \wedge A_R. \end{aligned} \quad (36)$$

Therefore, by restricting the gauge transformations to the diagonal subgroup  $SL(2, R)$  given by  $g_L = g_R$ , the bulk anomaly cancels without the need to quantize the coupling constant  $k$ .

Regarding the boundary terms, one can restore gauge invariance by requiring  $A_L = A_R|_{\partial\Sigma_3}$  and even add a dynamical gauge-invariant boundary term in the form of a WZW 2D action for  $A_L$  and  $A_R$ . The same logic can be applied any time the gauge group has the structure  $G \times G$ . This construction shows that a gauge-invariant Chern-Simons theory with nonquantized  $k$  is compatible with the dimensional reduction from AdS<sub>3</sub>-CS to the 2D BF theory with boundaries.

### III. THE $\mathfrak{so}(2, 2)$ POISSON SIGMA MODEL

As anticipated in the Sec. I, our goal is to define a generalized version of JT gravity from a topological gauge theory with a symmetry group containing  $SL(2, \mathbb{R})$ . The dimensional reduction of pure AdS<sub>3</sub>-CS theory suggests the  $\mathfrak{so}(2, 2)$ -Poisson sigma model over a two-dimensional manifold  $\Sigma = \mathbb{R} \times S^1$  as a natural candidate in this sense. In this section, we show how isolating the gravitational  $\mathfrak{sl}(2, \mathbb{R})$  sector in the case of  $\mathfrak{so}(2, 2)$  naturally singles out the dynamics of the residual degrees of freedom, which result in additional non-Abelian gauge fields that also become dynamical at the boundary. In particular, we show how the boundary dynamics is encoded in the Casimir

functions of the full algebra, analogously to the case of the  $\mathfrak{sl}(2, \mathbb{R})$ -PSM, and it is naturally reduced to the  $(\text{Diff}(S^1) \ltimes \mathcal{LG})/SL(2, \mathbb{R})$  coset space.

#### A. The bulk theory

We start by constructing the bulk theory. Let  $\Omega$  be the  $\mathfrak{so}(2, 2)$ -valued connection 1-form over  $\Sigma$ . The way we decompose the connection and the embedding maps in a given basis of the algebra and its dual is convenient for reasons that will be clear soon. The  $\mathfrak{so}(2, 2)$  algebra is isomorphic to two copies of  $\mathfrak{sl}(2, \mathbb{R})$ , so we have a natural chiral basis in  $\mathfrak{so}(2, 2) \simeq \mathfrak{sl}_L(2, \mathbb{R}) \oplus \mathfrak{sl}_R(2, \mathbb{R})$ ,

$$[L_i, L_j] = c_{ij}^k L_k, \quad [R_i, R_j] = c_{ij}^k R_k, \quad [L_i, R_j] = 0, \quad (37)$$

and with  $J_i = L_i + R_i$  we rotate the basis into a “nonchiral” basis with a  $\mathfrak{sl}_L(2, \mathbb{R})$  sector invariant under the action of the  $\mathfrak{sl}_J(2, \mathbb{R})$  subalgebra,

$$[J_i, J_j] = c_{ij}^k J_k, \quad [L_i, L_j] = c_{ij}^k L_k, \quad [J_i, L_j] = c_{ij}^k L_k. \quad (38)$$

We will refer to the  $\mathfrak{sl}(2, \mathbb{R})_J$  subalgebra as the gravitational sector and to the  $\mathfrak{sl}(2, \mathbb{R})_L$  as the “non-Abelian gauge” sector. In the nonchiral basis, we then write the connection  $\Omega$  as

$$\Omega = A^i J_i + B^i L_i. \quad (39)$$

We denote by  $\mathfrak{Z}_i$  the embedding maps

$$\mathfrak{Z}_i: \Sigma \rightarrow \mathfrak{so}(2, 2)^*$$

with the Poisson brackets

$$\{\mathfrak{Z}_i, \mathfrak{Z}_j\} = \Pi_{ij}(\mathfrak{Z}) = f_{ij}^k \mathfrak{Z}_k, \quad (40)$$

where  $f_{ij}^k$  are now the  $\mathfrak{so}(2, 2)$  structure constants. The corresponding Poisson sigma model takes the form

$$S_{P_\sigma} = \int_{\Sigma} d\Omega_i \wedge \mathfrak{Z}^i + \frac{1}{2} \Pi^{ij}(\mathfrak{Z}) \Omega_i \wedge \Omega_j, \quad (41)$$

where we used the invariant bilinear form

$$\langle J_i, J_j \rangle = \langle L_i, L_j \rangle = k_{ij}, \quad \langle J_i, L_j \rangle = \frac{1}{2} k_{ij}, \quad (42)$$

with  $k_{ij}$  being the  $\mathfrak{sl}(2, \mathbb{R})$  Killing form. A decomposition similar to that in (39) can also be performed for the embedding maps. From here on, we denote the dual basis with lowered indices, thus we write

$$\mathfrak{Z} = \mathfrak{X}_i J^i + \mathfrak{Y}_i L^i. \quad (43)$$

It then follows that

$$\begin{aligned}
S_{P_\sigma} = & \int_{\Sigma} dA_i \wedge \mathfrak{X}^i + dB_i \wedge \mathfrak{Y}^i + \frac{1}{2} dA_i \wedge \mathfrak{Y}^i \\
& + \frac{1}{2} dB_i \wedge \mathfrak{X}^i + \frac{1}{2} c_k^{ij} \mathfrak{X}^k A_i \wedge A_j + \frac{1}{2} c_k^{ij} \mathfrak{Y}^k B_i \wedge B_j \\
& + c_k^{ij} \mathfrak{Y}^k A_i \wedge B_j + \frac{1}{2} c_k^{ij} \mathfrak{X}^k A_i \wedge B_j \\
& + \frac{1}{4} c_k^{ij} \mathfrak{X}^k B_i \wedge B_j + \frac{1}{4} c_k^{ij} \mathfrak{Y}^k A_i \wedge A_j, \quad (44)
\end{aligned}$$

where we can recognize the  $\mathfrak{sl}(2, \mathbb{R})_J$  Poisson sigma model appearing as the gravitational subsector of the theory. The equations of motion for the fields  $A_i$  and  $B_i$  are

$$\mathfrak{D}_A A = 0, \quad \mathfrak{D}_\Omega B = c_i^{hk} B_h A_k L^i, \quad (45)$$

while the equations for the embedding maps read

$$\delta_{\mathfrak{X}} A = 0, \quad \delta_{\mathfrak{Y}} \Omega = -c_i^{hk} \mathfrak{X}_h B_k L^i. \quad (46)$$

This means that the on-shell  $A$  field is pure gauge with respect to the  $SL(2, \mathbb{R})_J$  subgroup. On the other hand, the on-shell  $\mathfrak{X}$  field is a stabilizer for  $A$ . Notice that the covariant derivative for the  $B$  field is computed with respect to the entire connection  $\Omega$ , while the same is not true for the  $A$  field, whose equation of motion is equivalent to the case of the  $SL(2, \mathbb{R})$  BF/PSM in (5). This shows that we are allowed to keep interpreting the  $A$  sector of the model as equivalent to ordinary JT gravity. Differently, the covariant derivative acting on the  $B$  field has both contributions from  $A$  and  $B$ . Therefore,  $B$  behaves like a gauge field coupled to gravity.

We now introduce a boundary action term and characterize the boundary dynamics in  $S^1$ . The equation of motion for the gauge connection, that we derive from (41), is

$$\mathfrak{D}_\Omega \Omega = 0 \quad (47)$$

and, without adding any counterterm, the variation on the boundary is just

$$\delta S_{P_\sigma}|_{S^1} = - \int_{S^1} \Omega^i \delta \mathfrak{Z}. \quad (48)$$

As it is the case for the  $\mathfrak{sl}(2, \mathbb{R})$ -PSM, if we insert a boundary Casimir counterterm  $\mathfrak{Z}^2$  and set the boundary condition

$$\Omega|_{S^1} = \mathfrak{Z}|_{S^1} du, \quad (49)$$

we get a particle-on-a-group action [see Eq. (18)]

$$\Omega|_{S^1} = g^{-1} dg \Rightarrow S_{P_\sigma}|_{S^1} = \int_{S^1} \frac{1}{2} \text{Tr}\{(g^{-1} g')^2\} \frac{1}{u'} d\tau, \quad (50)$$

where now  $g$  is an element in the  $SO(2,2)$  gauge group.

We know that the bulk theory has two  $\mathfrak{sl}(2, \mathbb{R})_{J,L}$  sectors in interaction, and we would like to make that manifest also at the boundary. We expect the boundary action to comprise a Schwarzian derivative term corresponding to the gravitational  $\mathfrak{sl}(2, \mathbb{R})_J$ -PSM sector ( $\mathfrak{X}^2$ ) and a particle-on-a-group term for the non-Abelian sector ( $\mathfrak{Y}^2$ ), plus interactions. As we will see, this is indeed the case for a suitable choice of boundary conditions.

## B. Asymptotic symmetries

Let us now focus on the fate of the boundary symmetries and the form of the reduced boundary action. The equation of motion (47) fixes the connection  $\Omega$  to be pure gauge,

$$\Omega = g^{-1} dg, \quad g: \Sigma \rightarrow SO(2,2). \quad (51)$$

This condition, together with (49), reproduces the same symmetry breaking mechanism of the  $SL(2, \mathbb{R})$ -PSM, except for the presence of a global  $SO(2,2)$  symmetry in the action (50). The allowed gauge transformations are, therefore, the  $SO(2,2)$  gauge transformations, which are constant at the boundary. Now, we show that imposing further boundary conditions can dramatically change the symmetry breaking mechanism and the boundary dynamics can be understood in terms of coadjoint orbits of  $\text{Diff}(S^1) \times \mathcal{LG}$ , for some  $\mathcal{G}$  depending on the additional choices made.

Suppose that the embedding fields are chosen at the boundary, such that

$$\mathfrak{X}_i|_{S^1} = -\mathfrak{Y}_i|_{S^1}. \quad (52)$$

Under these boundary conditions, and (49), the  $\mathfrak{Z}$  field at the boundary is no longer  $\mathfrak{so}(2,2)$ -valued but  $\mathfrak{sl}_R(2, \mathbb{R})$ -valued [60]. As a consequence, we are reducing the unfixed fields at the boundary. In fact,

$$\mathfrak{Z} = (\mathfrak{X}_i + \mathfrak{Y}_i) J^i + \mathfrak{Y}_i (L^i - J^i) \xrightarrow{\mathfrak{X}_i = -\mathfrak{Y}_i|_{S^1}} \mathfrak{Z} = -\mathfrak{Y}_i R^i. \quad (53)$$

This additional condition, together with  $\Omega$  being pure gauge, corresponds to a partial gauge fixing in the  $R$  sector. Indeed, also,  $\Omega$  takes the form

$$\Omega|_{S^1} = h_R^{-1} dh_R, \quad h_R \in SL_R(2, \mathbb{R}). \quad (54)$$

Moreover, any  $SL_L(2, \mathbb{R})$  gauge transformation would not preserve the boundary conditions since it would force the boundary fields to get out of the  $R$  sector. Therefore, the  $SL_L(2, \mathbb{R})$  gauge group must now be regarded as an actual

broken gauge symmetry, while the global  $SL_L(2, \mathbb{R})$  symmetry is just trivial since global transformations in this sector are in the little group of the boundary connection. The only nontrivial global symmetry is then the  $SL_R(2, \mathbb{R})$  symmetry. With this in mind, we can conclude that the reduction of the on-shell boundary action involves the coadjoint orbit of  $\text{Diff}(S^1) \ltimes \mathcal{LG}$ , where the given boundary condition identifies  $\mathcal{LG}$  with the  $SL_L(2, \mathbb{R})$  loop group. We can then understand the reduced on-shell particle-on-a-group with the action associated to  $\text{Diff}(S^1) \ltimes \mathcal{L}SL_L(2, \mathbb{R})$  and  $SL_R(2, \mathbb{R})$  global symmetry.

Since symmetries of the boundary theory are fixed by the additional boundary conditions, it is important to ask how many nonequivalent choices of boundary conditions exist. Suppose then that the boundary embedding fields are chosen to be valued on a given subalgebra  $\mathfrak{h} \subseteq \mathfrak{so}(2, 2)$ ,

$$\mathfrak{Z}|_{S^1} = \mathfrak{H} \in \mathfrak{h}. \quad (55)$$

Let then  $\mathfrak{f}$  be the complement of  $\mathfrak{h}$  in  $\mathfrak{so}(2, 2)$ . In order to preserve the boundary condition (55), the following conditions must hold:

$$[\mathfrak{h}, \mathfrak{h}] = 0 \quad \text{or} \quad [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}, \quad (56)$$

$$[\mathfrak{f}, \mathfrak{h}] = 0 \quad \text{or} \quad [\mathfrak{f}, \mathfrak{h}] = \mathfrak{h}. \quad (57)$$

By virtue of these conditions, we can list the allowed scenarios in the case of  $\mathfrak{so}(2, 2)$ . The case  $[\mathfrak{h}, \mathfrak{h}] = 0$  is ruled out since  $\mathfrak{so}(2, 2)$  is semisimple and it has no invariant Abelian subalgebras. The case  $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$  can be realized with  $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{R})_{L,R,J}$  or trivially with  $\mathfrak{h} = \mathfrak{so}(2, 2)$ , which correspond to the absence of any further boundary condition apart from  $\mathfrak{Z}|_{S^1} d\tau = \Omega|_{S^1}$ . Any choice associated with  $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{R})_{R,L,J}$  or  $\mathfrak{h} = \mathfrak{so}(2, 2)$  leads to a well-defined boundary condition, where  $\mathfrak{Z}|_{S^1}$  takes values into a (sub)algebra, which can be either left invariant by its complement or commute with it.

### C. Virasoro-Kac-Moody coadjoint orbits

At the end of the last section, we showed that additional boundary conditions are allowed at the boundary and imply a partial breaking of the gauge symmetry. This means that if we want to make sense of the boundary action, we have to compute the coadjoint orbit of the Virasoro-Kac-Moody semidirect product, as is the case for SYK-like tensor models [28]. Generally speaking, the computation of coadjoint orbits for semidirect products of infinite dimensional groups is not at all an easy task, but the interesting feature of  $\text{Diff}(S^1) \ltimes \mathcal{LG}$  is that the Virasoro algebra and the Kac-Moody algebra are related by the Sugawara construction (see Appendix). In particular, the Virasoro algebra acts naturally in a derivative way over the Kac-Moody sector. The algebra  $\mathfrak{diff}(S^1) \ltimes \hat{g}$  is, in fact, given by

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}; \\ [K_{i,m}, K_{j,m}] &= c^k_{ij}K_{k,m+n} + m\langle K_i, K_j \rangle \delta_{m,-n}; \\ [L_m, K_{i,n}] &= -nK_{i,m+n}. \end{aligned} \quad (58)$$

In our case, we have  $\hat{g} = \widehat{\mathfrak{sl}(2, \mathbb{R})}$ . In order to compute the coadjoint orbit for the semidirect product  $\text{Diff}(S^1) \ltimes \mathcal{LG}$ , we follow [61], where both the Virasoro and the Kac-Moody algebra elements are realized through functions. Let  $(u(\tau), k(\tau), \alpha, \beta)$  be a generic element in the set  $\mathfrak{diff}(S^1) \ltimes \widehat{\mathfrak{sl}(2, \mathbb{R})}$ , with  $u(\tau) \in \mathfrak{diff}(S^1)$ ,  $k(\tau) \in \widehat{\mathfrak{sl}(2, \mathbb{R})}$ , and  $\alpha, \beta$  are the respective central elements. Then, a basis-independent way to write the commutation relation is simply given by

$$\begin{aligned} [(u, k, \alpha, \beta), (v, h, \gamma, \delta)] &= ([u, v]_{\text{Vir}}, [k, h]_{KM} - uh' \\ &\quad + -vk', \Omega_{\text{Vir}}(u''', v), \Omega_{KM}(k, h')), \end{aligned} \quad (59)$$

where the  $\Omega$ 's stand for the respective cocycles (defined in detail in the Appendix). The group multiplication in  $\text{Diff}(S^1) \ltimes \mathcal{LG}$  is defined as

$$(\phi, g)(\psi, h) = (\phi \circ \psi, g \cdot h \circ \phi^{-1}), \quad (60)$$

where  $\phi, \psi$  are finite diffeomorphisms and  $g, h$  are elements of the loop group. In order to compute the coadjoint action of  $\text{Diff}(S^1) \ltimes \mathcal{LG}$ , we must also introduce the pairing between the algebra and its dual space, which is given by the sum of pairings

$$\begin{aligned} \langle (b, \rho, \alpha, \beta), (v, \lambda, \gamma, \delta) \rangle &= \int b(\tau)v(\tau)d\tau + \int \langle \lambda(\tau)\rho(\tau) \rangle_{KM}d\tau + \alpha\gamma + \beta\delta. \end{aligned} \quad (61)$$

Therefore, we have [61]

$$\begin{aligned} Ad^*_{(\phi, g)}(b, \rho, \alpha, \beta) &= \left( (b \circ \phi)\phi'^2 + \alpha\{\phi, \tau\}_S + \langle g^{-1}dg, \rho \rangle \right. \\ &\quad \left. + \frac{1}{2}\beta\|g^{-1}dg\|^2, \phi'(g^{-1}\rho g) \circ \phi \right. \\ &\quad \left. + \beta g^{-1}dg, \alpha, \beta \right). \end{aligned} \quad (62)$$

By pairing the coadjoint orbit with a Lie algebra element, we can construct a natural action, as we already saw in Sec. II A 1 for the JT gravity case, and we can finally write the reduced expression for the on-shell action (50),

$$S|_{S^1} = \int_{S^1} \left\{ (b \circ \phi) \phi'^2 + \alpha \{ \phi, t \}_S + \langle g^{-1} dg, \rho \rangle + \frac{1}{2} \beta \|g^{-1} dg\|^2 \right\} d\tau, \quad (63)$$

which matches with the results of [20]. The element  $b(\tau)$  must be chosen in such a way that the global symmetry is  $\text{SL}(2, \mathbb{R})$ , which is realized through fractional transformations [62]. The geometric action (63), without loss of generality, can be thought of as the pairing of  $Ad_{(\phi, g)}^*(b, \rho, \alpha, \beta)$  with a pure Virasoro element and, given the structure of the algebra in (59), this can always be rotated in a new element with a nonvanishing Kac-Moody component. Notice that this would not be true if we paired the coadjoint action with an element of the Kac-Moody subalgebra, because the latter is an invariant subalgebra. Therefore, this last option could be regarded as a non-equivalent choice that kills the Schwarzian degree of freedom.

#### IV. BLACK HOLE ENTROPY

Once the boundary dynamics for the  $\mathfrak{so}(2, 2)$ -PSM has been established with a given boundary condition, it is an interesting check to compute the leading order entropy. Indeed, it is well recognized that in JT gravity, the latter can be interpreted as the black hole entropy since the theory admits a gravitational interpretation with the boundary playing the role of a near-horizon surface. We shall see that we obtain a consistent result for our model.

Let  $\mathcal{H}[\Omega]$  be the holonomy associated with the gauge connection  $\Omega$ ,

$$\mathcal{H}[\Omega] = \mathcal{P} \exp \left[ - \oint \Omega \right], \quad (64)$$

where the equations of motion fix  $\Omega$  to be pure gauge, i.e.,  $\Omega = g^{-1} dg$ . This implies that, if the integration contour is the  $S^1$  boundary and if we demand for a smooth Euclidean solution, then

$$\mathcal{H} = g(\beta) g^{-1}(0) = \mathbb{1}, \quad (65)$$

with the inverse temperature  $\beta$  being the length of the boundary.

Suppose that no further boundary conditions are imposed apart from  $\mathfrak{Z}|_{S^1} dt = \Omega|_{S^1}$ , so that the boundary dynamics is that of a particle on the entire  $\text{SO}(2, 2)$  group. The global  $\text{SO}(2, 2)$  symmetry for the boundary theory implies the presence of the conserved charges

$$J_i = \langle g^{-1} dg, \tau_i \rangle, \quad (66)$$

where  $\tau_i$  are  $\mathfrak{so}(2, 2)$  generators. This means that, up to constants, the on-shell connection satisfies  $\Omega_i|_{\text{o.s.}} = J_i$ , therefore,

$$\mathcal{H}[\Omega] = \exp[-\beta J] = \mathbb{1}. \quad (67)$$

Let now  $\Lambda$  be the diagonalized  $-\beta J$ , i.e.,  $-\beta J = P \Lambda P^{-1}$ . This implies

$$\mathcal{H}[\Omega] = \exp[P \Lambda P^{-1}] = P \exp[-\beta \Lambda] P^{-1} = \mathbb{1}. \quad (68)$$

Requiring  $\exp[\Lambda] = \mathbb{1}$  implies that the eigenvalues must satisfy  $\lambda_k = -2\pi i n_k / \beta$ , with integers  $n_k$ . In the case of an  $\mathfrak{so}(2, 2)$ -valued boundary connection, the four eigenvalues come in two pairs of eigenvalues with opposite signs,

$$\text{Tr}(J^2) = -\frac{8\pi(n^2 + m^2)}{\beta^2}. \quad (69)$$

In order to evaluate the entropy, we need to compute the leading order free energy  $F$ , which can be identified with the temperature times the on-shell boundary Euclidean action  $I$ ,

$$F = \beta^{-1} I|_{S^1, \text{on-shell}}. \quad (70)$$

The latter can be computed easily. Let  $\mathcal{C}$  be the on-shell boundary Casimir function, then  $\mathcal{C} = \langle J, J \rangle$  and

$$I|_{S^1, \text{on-shell}} = \beta \mathcal{C} = \beta \text{Tr}(J^2) = -\frac{8\pi(n^2 + m^2)}{\beta}. \quad (71)$$

The free energy then comes with the correct sign, thus recovering a positive entropy, which is linear in the temperature  $T = \beta^{-1}$ , that is,

$$S = -\frac{dF}{dT} = 16\pi k(n^2 + m^2)T. \quad (72)$$

This result is consistent with the linear scaling of the entropy in SYK models [20, 63] and it shows that the extra  $\text{SL}(2, \mathbb{R})$  degrees of freedom contribute with a supplementary linear term  $\propto \beta^{-1} m^2$ .

#### V. CONCLUSIONS

We constructed a JT gravity bulk-plus-boundary generalization in terms of a  $\mathfrak{so}(2, 2)$ -Poisson sigma model with a boundary Casimir action. It is a fact that 3D Chern-Simons theories with a WZW term at the boundary, once dimensionally reduced, give 2D BF theories with the particle-on-a-group action at the 1D boundary [55]. The dimensional reduction of  $\text{SO}(2, 2)$  Chern-Simons-WZW theory, which is a 3D theory describing an  $\text{AdS}_3$  geometry, leads to the proposed  $\mathfrak{so}(2, 2)$ -PSM, together with the dynamical particle-on-a-group action [64]. The model provides a gravitational dual for SYK-like tensor models with internal symmetries [28, 30], whose low-energy dynamics is characterized by a  $\mathfrak{diff}(S^1) \ltimes \hat{\mathfrak{g}}$  symmetry. In our case, the  $\mathfrak{diff}(S^1) \ltimes \hat{\mathfrak{g}}$  symmetry, with  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ , arises after a

specific choice of boundary conditions. The additional Kac-Moody sector encodes the dynamics of extra edge modes, whose number is equal to  $\dim(\mathfrak{g})$ . In the near-horizon interpretation, the standard computation of the near-extremal black hole entropy reproduces the JT result with an additional entropy contribution due to the presence of extra gauge fields. The additional contribution is equivalent to the bare JT case because of the particular structure of  $\mathfrak{so}(2, 2)$ . The specific case of  $\mathfrak{so}(2, 2)$ , seen from the CS perspective, allows to interpret the Kac-Moody modes, living at the  $S^1$  boundary and expected to arise in the low-energy regime of SYK tensor models, as Kaluza-Klein modes associated with the dimensional reduction. The proposed  $\mathfrak{so}(2, 2)$ -PSM provides a class of possible JT-Yang-Mills generalizations of the JT/SYK correspondence with a purely gravitational interpretation from a 3D perspective. This interpretation selects a class of SYK-like duals which naturally relates to 3D gravity.

The choice of working with a PSM, which in the present paper is linear and, therefore, equivalent to a BF theory up to boundary terms, lends itself to generalizations that could have interesting gravitational implications. The first generalization is related to the possibility of considering nonlinear PSM, and this has already been noticed in the literature [65], with it being related to nonlinear generalizations of JT gravity [3,66].

The second generalization, entirely novel to our knowledge, would be to consider Jacobi sigma models [67–70], which are models with a deformed Poisson bracket on the target space, known as the Jacobi bracket. The latter is constructed with a quasi-Poisson structure, violating Jacobi identity in a controlled manner. The explicit relation for the quasi-Poisson tensor  $\Pi$  reads  $[\Pi, \Pi] = 2E \wedge \Pi$ , with  $E$  being a vector field on the target space, the so-called Reeb vector field, s.t.  $L_E \Pi = 0$ . This structure may be obtained by a homogeneous Poisson bracket in one dimension higher (technically a fiber bundle over the target space, with fiber  $R - \{0\}$  [70]). Interestingly, the extra dimension, or, better to say, the generator of  $R - \{0\}$ , could be related to a dilaton field, which could play a role in generalized JT gravity models. We plan to analyze in detail this proposal in future investigations.

### ACKNOWLEDGMENTS

The authors acknowledge support from the INFN Iniziativa Specifica GeoSymQFT and from the European COST Action CaLISTA CA21109. P.V. acknowledges support from the Programme STAR Plus, financed by UniNA and Compagnia di San Paolo, and from the PNRR MUR Project No. CN 00000013-ICSC.

### DATA AVAILABILITY

No data were created or analyzed in this study.

## APPENDIX

### 1. The Sugawara Construction

Here, we summarize the basics of the Sugawara construction and give a more detailed computation of the Virasoro coadjoint orbits.

Given a semisimple finite-dimensional Lie algebra  $\mathfrak{g}$ , and the algebra of smooth functions on the circle  $C^\infty(S^1)$ , we denote the corresponding loop algebra as

$$L\mathfrak{g} = \mathfrak{g} \otimes C^\infty(S^1). \quad (\text{A1})$$

Given  $\mathbf{x} \in \mathfrak{g}$ , and  $\{e^{int}\}_{n \in \mathbb{Z}}$  as a basis for  $C^\infty(S^1)$ , a basis for  $L\mathfrak{g}$  is represented by  $\{X_n\}$ , so it is defined as

$$X_n = \mathbf{x}e^{int} \quad (\text{A2})$$

with Lie brackets

$$[X_m, Y_n] = [\mathbf{x}, \mathbf{y}]e^{i(m+n)t} =: X_{m+n}. \quad (\text{A3})$$

Central extensions of  $L\mathfrak{g}$  may be obtained by considering projective representations [71]  $\pi: X_m \mapsto \pi(X_m) =: X(m)$  s.t.  $[\pi(X), \pi(Y)] = \pi([X, Y]) + \alpha(X, Y)\mathbf{e}$ , where  $\alpha(X, Y) \in \mathbb{R}$  is a two-cocycle on  $L\mathfrak{g}$ , namely, a bilinear map from  $L\mathfrak{g} \times L\mathfrak{g}$  to the complex numbers, which is antisymmetric and verifies the cocycle property

$$\begin{aligned} \partial\alpha(X, Y, Z) &:= \alpha(X, [Y, Z]) + \alpha(Z, [X, Y]) \\ &+ \alpha(Y, [Z, X]) = 0, \end{aligned} \quad (\text{A4})$$

while  $\mathbf{e}$  is the generator of a one-dimensional vector space extending the finite Lie algebra  $\mathfrak{g}$ . The representation can be chosen in such a way to fix the value of the cocycle, so as to obtain

$$[X(m), Y(n)] = \pi(X_{n+m}) + m\ell \langle \mathbf{x}, \mathbf{y} \rangle \delta_{n+m,0}, \quad (\text{A5})$$

with  $\langle \cdot, \cdot \rangle$  being the (nondegenerate, invariant) inner product in the semisimple algebra  $\mathfrak{g}$ , and  $\ell$  is a non-negative integer number determined by the dimension of the representation. Hence,  $\mathcal{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{R}\mathbf{e}$  is a central extension of the loop algebra  $L\mathfrak{g}$ , which is called an affine Lie algebra. Note that  $\mathbb{R}\mathbf{e}$  is a central subalgebra of  $\mathcal{L}\mathfrak{g}$ .

By denoting with  $\mathbf{x}_i, i = 1, \dots, \dim \mathfrak{g}$  an orthonormal basis in  $\mathfrak{g}$  and  $X_i(p) = \pi(\mathbf{x}_i e^{ip})$  an orthonormal basis in  $\mathcal{L}\mathfrak{g}$ , the Lie brackets of the affine algebra (A5) acquire the form

$$[X_i(m), X_j(n)] = f_{ij}^k X_k(m+n) + m\ell \delta_{ij} \delta_{n+m,0}. \quad (\text{A6})$$

It is also useful to introduce the current basis [72]

$$\mathcal{X}_i(\tau) = \sum_n e^{in\tau} X_i(n), \quad \tau \in S^1. \quad (\text{A7})$$

It can be checked that the current basis satisfies

$$[\mathcal{X}_i(\tau), \mathcal{X}_j(\tau')] = f_{ij}^k \mathcal{X}_k(\tau) \delta(\tau - \tau') + \delta_{ij} \delta'(\tau - \tau') \ell, \quad (\text{A8})$$

namely,  $\mathcal{X}_i$  are maps from the circle  $S^1$  to the algebra  $\mathfrak{g}$ , which are properly called the generators of the loop algebra associated with  $\mathfrak{g}$  (see [73] for details). The latter shall be indicated with  $\mathcal{L}\mathfrak{g}$  and is often referred to as the current or Kac-Moody algebra of  $\mathfrak{g}$ .

From  $\mathcal{L}\mathfrak{g}$ , a Virasoro algebra is constructed using the Sugawara construction [73,74]. The Virasoro generators are defined according to

$$T_m = -\frac{1}{2(\ell + h)} \sum_i \sum_{p+q=m} :X_i(p)X_i(q):, \quad (\text{A9})$$

where  $h$  is the dual Coxeter number of  $\mathfrak{g}$  [75]. They satisfy

$$[T_m, T_n] = (m - n)T_{m+n} + \frac{C}{12}(m^3 - m)\mathbf{e}\delta_{n+m,0}, \quad (\text{A10})$$

with  $C = \dim \mathfrak{g} \cdot \ell / (\ell + h)$  [see [71] for proof of (A10)]. We shall indicate the Virasoro algebra as  $\mathcal{V} = W \oplus \mathbb{R}\mathbf{e}$ , with  $W$  being the Witt algebra [76].

The Sugawara tensor is then defined in terms of the currents according to

$$T(\tau) = \frac{1}{2(\ell + h)} \sum_i :X_i(\tau)X_i(\tau). \quad (\text{A11})$$

It is a standard result of the Sugawara construction in two-dimensional quantum field theory to verify that it satisfies the Lie brackets

$$[T(\tau), T(\tau')] = (T(\tau) + T(\tau'))\delta'(\tau' - \tau) - \frac{C}{12}(\delta'''(\tau - \tau') + \delta'(\tau - \tau')). \quad (\text{A12})$$

Hence, it can be verified that the generators of the Virasoro algebra (A9) are the Fourier modes of the Sugawara tensor (A11). Finally, notice that the quadratic operator

$$\Delta = \sum_i :X_i(0)X_i(0): \quad (\text{A13})$$

is a Casimir for the affine algebra  $\mathcal{L}\mathfrak{g}$  (A5), or equivalently, the quadratic operator

$$\sum_i :X_i(\tau)X_i(\tau): \quad (\text{A14})$$

is a Casimir for the same algebra, with respect to the bracket (A8).

## 2. Coadjoint orbits

The Kac-Moody and Virasoro algebras introduced in the previous section can be recast in a basis-independent fashion, which allows for a direct computation of the coadjoint orbits of their groups. Let us consider the affine algebra first. Given  $(v, a)$  and  $(u, b)$ , two elements in  $\mathcal{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{R}\mathbf{e}$ , with  $v, u \in L\mathfrak{g}$  and  $a, b \in \mathbb{R}\mathbf{e}$ , the general form of the commutation relations is

$$[(u, a), (v, b)]_{\mathcal{L}\mathfrak{g}} = ([u, v]_{L\mathfrak{g}}, c(u, v)), \quad (\text{A15})$$

where  $c$  is a two-cocycle on the Lie algebra  $L\mathfrak{g}$  [see Eq. (A4)]. We recall that the inequivalent nontrivial central extensions of  $L\mathfrak{g}$  are classified by equivalence classes of two-cocycles (see, for example, [77]). It is easily verified that (A15) reduces to (A6) when computed on the generators  $X_i(m)$ . The cocycle in (A15) can be written in integral form according to

$$c(u, v) = \frac{1}{2\pi} \oint d\tau \langle u'(\tau), v(\tau) \rangle, \quad (\text{A16})$$

with  $\langle \cdot, \cdot \rangle$  being the inner product in  $L\mathfrak{g}$ . It is immediate to check that it coincides with the central term of (A6) when computed on the currents (A7). Analogously, an integral expression can be obtained for the Virasoro cocycle. To this (see, for example, [50,78]), one has to represent the Virasoro algebra  $\mathcal{V} = W \oplus \mathbb{R}\mathbf{e}$  in terms of vector fields on the circle  $\xi = \xi(\tau)\partial_\tau \in W$  plus a central term. Then, for  $(\xi, \alpha), (\eta, \beta) \in \mathcal{V}$ , one has

$$[(\xi, \alpha), (\eta, \beta)]_{\mathcal{V}} = ([\xi, \eta]_W, c_0(\xi, \eta)) \quad (\text{A17})$$

with  $\alpha, \beta \in \mathbb{R}$  and

$$[\xi, \eta]_W = (\xi(\tau)\partial_\tau\eta(\tau) - \eta(\tau)\partial_\tau\xi(\tau))\partial_\tau, \quad (\text{A18})$$

while the two-cocycle is given by

$$c_0(\xi, \eta) = \frac{it}{48\pi} \oint d\tau (\eta'''(\tau)\xi(\tau) - \eta(\tau)\xi'''(\tau)), \quad t \in \mathbb{R}, \quad (\text{A19})$$

known as the Gel'fand-Fuchs cocycle. In order to check that it reduces to the central term in (A10), one has to use the realization of the Witt algebra in terms of the generators  $\xi_n = f_n(\tau)\partial_\tau$  with  $f_n = \exp(in\tau)$ . Then, one has

$$\begin{aligned} c_0(\xi_n, \xi_m) &= \frac{it}{48\pi} \oint d\tau (\xi_n'''(\tau)\xi_m(\tau) - \xi_n(\tau)\xi_m'''(\tau)) \\ &= \frac{t}{12} m^3 \delta_{n+m,0}, \end{aligned} \quad (\text{A20})$$

which is cohomologous to the two-cocycle in (A10) [79]. Indeed, in order to obtain the cocycle in (A10), one has to modify (A19) by a coboundary term as follows:

$$c(\xi, \eta) = \frac{i\tau}{48\pi} \oint d\tau ((\eta''''(\tau) - \eta'(\tau))\xi(\tau) + \eta(\tau)(\xi'''(\tau) - \xi'(\tau))), \quad (\text{A21})$$

it being

$$\oint d\tau ((\eta'(\tau))\xi(\tau) - \eta(\tau)\xi'(\tau)) = \partial b(\xi, \eta) = b([\xi, \eta]), \quad (\text{A22})$$

with  $b$  a one-cochain in the Gel'fand-Fuchs cohomology.

Once the central extensions are defined, in order to compute coadjoint orbits of the corresponding groups, it is necessary to introduce the notion of coadjoint action on the dual algebras. This may be obtained in a standard way by first computing the adjoint action on the algebra and then computing its dual through the natural pairing between algebra and dual algebra. We give here a short review, while referring to [78,80] for more details. The adjoint action of the loop group on its algebra is given by

$$\widetilde{Ad}_g(v(\tau), a) = (Ad_g v(\tau), a + c(g^{-1}g', v(\tau))), \quad (\text{A23})$$

with  $g: S^1 \rightarrow G$  being an element of the affine group, so that  $g^{-1}g' \in L\mathfrak{g}$ . The tilde indicates the action of the affine group on its centrally extended algebra, while  $Ad_g$  is the standard action. Notice that the adjoint action only affects  $v(\tau)$ , with  $a$  being central. However, the adjoint action on  $v(\tau)$  contributes a noncentral and a central term, which adds to  $a$ . On computing (A23) on a Lie algebra generator  $(X(m), a)$ , one can check that its infinitesimal version reproduces the adjoint action of the algebra, Eq. (A15), it being  $g^{-1}g' = \sum c_n X(n)$ . Furthermore, one can verify that (A23) satisfies the ordinary composition property

$$\widetilde{Ad}_{gh} = \widetilde{Ad}_g \circ \widetilde{Ad}_h. \quad (\text{A24})$$

The adjoint action of the Virasoro group on its algebra is more subtle. Analogously to the previous case, the action on the central term is trivial, so we have

$$\widetilde{Ad}_g(v, a) = ([Ad_g v]_{nc}, a + [Ad_g v]_c). \quad (\text{A25})$$

As for the contribution of  $Ad_g v$  to the central term, it is possible to show that it can be formulated in terms of the Schwarzian derivative according to

$$[Ad_g v]_c = \frac{1}{12} \int \frac{d\tau}{2\pi} \frac{v(\phi(\tau))}{\phi'(\tau)} \{\phi, \tau\}_S, \quad (\text{A26})$$

with

$$\{\phi(\tau), \tau\}_S = \frac{\phi'''}{\phi'} - \frac{3}{2} \left( \frac{\phi''}{\phi'} \right)^2. \quad (\text{A27})$$

To this, we shall follow Kirillov [81] and shortly review his theory of coadjoint orbits. Let us first fix the notation. Given  $G = \text{Diff}(S^1)$  and its Lie algebra of smooth vector fields on the circle,  $W$ , the dual algebra  $W^*$  consists of continuous linear functionals on  $W$ . Therefore, it can be identified with  $\Omega(S^1) \otimes \mathcal{D}'(S^1)$ , with  $\mathcal{D}'(S^1)$  being the space of continuous linear functionals on  $\mathcal{F}(S^1)$ . Namely, the elements of  $W^*$ , called moments by Kirillov, are of the form

$$p = f \otimes F \in \Omega^1(S^1) \otimes \mathcal{D}'(S^1), \quad (\text{A28})$$

with  $(f \otimes F)(\xi) = \langle F, f(\xi) \rangle$  and  $\xi \in W$ . Therefore,  $\mathcal{D}'(S^1)$  can be further identified with  $\Omega^1(S^1)$  itself, since it provides the volume form to be used in order to integrate  $f(\xi) \in \mathcal{F}(S^1)$  in (A28). In coordinates form, we have

$$p = f \otimes \omega = f(x)dx \otimes \omega(x)dx, \quad x \in S^1. \quad (\text{A29})$$

Then, given  $\phi \in \text{Diff}(S^1)$ ,  $\phi(x) = y$ , the coadjoint action of the group on  $W^*$  is defined as

$$Ad_{\phi}^* p = f(\phi^{-1}(y)) \frac{\partial \phi^{-1}}{\partial y} dy \otimes \omega(\phi^{-1}(y)) \frac{\partial \phi^{-1}}{\partial y} dy, \quad (\text{A30})$$

while the coadjoint action of the Lie algebra  $W$  on  $W^*$  is given by the Lie derivative

$$ad_{\xi}^* p = L_{\xi} f \otimes \omega + f \otimes L_{\xi} \omega. \quad (\text{A31})$$

The Virasoro group is the centrally extended group of diffeomorphisms of the circle,  $\widetilde{\text{Diff}}(S^1)$ , with product rule

$$(\phi, t) \circ (\psi, s) = (\phi \circ \psi, t + s + B(\phi, \psi)), \quad (\text{A32})$$

with  $B$  being a two-cocycle on  $\text{Diff}(S^1)$ , whose explicit form can be found, for example, in [81]. The Lie algebra of  $\widetilde{\text{Diff}}(S^1)$  is the Virasoro algebra  $\mathcal{V} = W + \mathbb{R}\mathbf{e}$  with Lie bracket (A17). The Gel'fand-Fuchs cocycle (A19) forms a basis in  $H^2(W, \mathbb{R}) = Z^2(W, \mathbb{R})/B^2(W, \mathbb{R})$ , the second cohomology group of the Witt algebra. A representative of the equivalence class,  $c_0 \in Z^2(W, \mathbb{R})$ , can be written in the form

$$c_0(\xi, \eta) = \oint \xi' \eta', \quad (\text{A33})$$

and a generic element in the class has the form

$$c(\xi, \eta) = t c_0(\xi, \eta) + p([\xi, \eta]), \quad (\text{A34})$$

where  $t \in \mathbb{R}$  and  $p \in W^*$ , namely, of the form (A28) ( $c = c_0 + dp$ , with  $dp$  being a coboundary). Prior to computing the coadjoint action of the Virasoro group on

its dual algebra, we state without proof the following theorem [81]:

*Theorem 1.* Given  $W$ ,  $\mathcal{V}$ , and their duals, it is possible to exhibit bijective maps,

$$\begin{aligned}\alpha: W^* &\rightarrow B^2(W), \\ \beta: \mathcal{V}^* &\rightarrow Z^2(W), \\ \gamma: \mathbb{R} &\rightarrow H^2(W),\end{aligned}$$

which are  $G$ -module morphisms [82], with  $G = \text{Diff}S^1$ .

The maps  $\alpha$ ,  $\beta$ , and  $\gamma$  are explicitly chosen to be

$$\alpha(f)(\xi, \eta) = \langle f, [\xi, \eta] \rangle, \quad (\text{A35})$$

$$\beta(f, t)(\xi, \eta) = \langle f, [\xi, \eta] \rangle - tc(\xi, \eta), \quad (\text{A36})$$

$$\gamma(t) = \text{class}tc. \quad (\text{A37})$$

What is interesting for us is the statement for the map  $\beta$ , which implies, in particular, that it is a  $\mathcal{V}$ -module morphism, with  $\mathcal{V}$  being the Virasoro algebra. This amounts to the result

$$\beta(\widetilde{ad^*_{(\xi, \tau)}}(f, t)) = \widetilde{ad^*_{(\xi, \tau)}}(\beta(f, t)). \quad (\text{A38})$$

Let us prove this equality explicitly. We have

$$\begin{aligned}\langle (\widetilde{ad^*_{(\xi, \tau)}}(f, t)), (\eta, \sigma) \rangle &= \langle (f, t), \widetilde{ad_{(\xi, \tau)}}(\eta, \sigma) \rangle \\ &= \langle (f, t), ([\eta, \xi], c(\eta, \xi)) \rangle \\ &= \langle f, ([\eta, \xi]) - tc(\eta, \xi) \rangle.\end{aligned} \quad (\text{A39})$$

Recalling that  $c$  is a two-cocycle on  $W$ , the Lie derivative and the interior product act as follows:

$$\begin{aligned}(L_\xi c)(\eta, \chi) &= ((i_\xi d + di_\xi)c)(\eta, \chi) = c([\eta, \chi], \xi), \\ \langle i_\xi c, \eta \rangle &= c(\eta, \xi),\end{aligned} \quad (\text{A40})$$

where  $df(\xi, \eta) = \langle f, [\xi, \eta] \rangle$  and  $dc = 0$  have been used. Hence, we obtain

$$\widetilde{ad^*_\xi}(f, t) = (ad^*_\xi f - ti_\xi c, 0). \quad (\text{A41})$$

Moreover, for  $f \in \mathfrak{g}^*$  we have

$$ad^*_\xi f = L_\xi f, \quad (\text{A42})$$

then

$$d(ad^*_\xi f - ti_\xi c) = L_\xi(df - tc), \quad (\text{A43})$$

which is, in turn, equal to  $L_\xi(\beta(f, t))$ , which is what we wanted to show. From (A43), it is possible to obtain the full coadjoint action of the Virasoro group [81]. By denoting

with  $L(\phi)$  the 1-parameter group generated by  $L_\xi$  and observing that  $L(\phi)c$  generates a two-cocycle, say  $\tilde{c}$ , which is in the same cohomology class as  $c$ , one has

$$L(\phi)c = c + dh(\phi), \quad (\text{A44})$$

with  $dh(\phi)$  being a coboundary. Thus, upon integrating (A43), we get

$$\widetilde{Ad^*_\phi}(f, t) = (L(\phi)f + th(\phi), t). \quad (\text{A45})$$

We can now explicitly prove that  $h = S(\phi)$  when  $c$  is the Gel'fand-Fuchs cocycle,

$$\begin{aligned}L(\phi)c_0(\xi, \eta) - c_0(\xi, \eta) &= c_0(Ad_\phi^{-1}\xi, Ad_\phi^{-1}\eta) - c_0(\xi, \eta) \\ &= \int \left( \frac{\xi(\phi(\tau))}{\phi'(\tau)} \right)' d \left( \frac{\eta(\phi(\tau))}{\phi'(\tau)} \right)' - \int \xi'(\tau)\eta''(\tau)d\tau.\end{aligned} \quad (\text{A46})$$

After the change of variable  $\tau \rightarrow \phi^{-1}(\tau)$  in the first integral, we get

$$L(\phi)c_0(\xi, \eta) - c_0(\xi, \eta) = \int [\xi\eta' - \xi'\eta] \left( \frac{\{\phi(\tau), \tau\}_S}{\phi'^2} \right) \circ \phi^{-1} d\tau, \quad (\text{A47})$$

where  $\{\phi(\tau), \tau\}_S$  is the Schwarzian derivative (A27). On comparing this result with (A44), we have

$$\begin{aligned}dh(\phi)(\xi, \eta) &= \langle h(\phi), \\ ([\xi, \eta]) \rangle &= \int [\xi\eta' - \xi'\eta] \left( \frac{\{\phi(\tau), \tau\}_S}{\phi'^2} \right) \circ \phi^{-1} d\tau,\end{aligned} \quad (\text{A48})$$

hence the functional  $h$  is explicitly given by

$$h(\phi) = d\tau \otimes \{\phi(\tau), \tau\}_S \circ \phi^{-1} d\tau \in \mathcal{W}^*. \quad (\text{A49})$$

In order to better understand its meaning, we recall that  $W^* \equiv \Omega(S^1) \otimes \Omega(S^1)$  [see Eq. (A28)], where the first differential realizes the pairing with vector fields, whereas the second furnishes the integration measure on the circle. This explains the interpretation of the Schwarzian,

$$S(\phi) := d\tau \otimes \{\phi(\tau), \tau\}_S d\tau, \quad (\text{A50})$$

as a pseudometric on the circle [81]. Finally, replacing the result in (A45), the coadjoint action of the Virasoro group on its dual algebra reads

$$\begin{aligned}\widetilde{Ad^*_\phi}(f, t) &= (Ad^*_\phi f + tS(\phi) \circ \phi^{-1}, t) \\ &= ((f + tS(\phi)) \circ \phi^{-1}, t).\end{aligned} \quad (\text{A51})$$

The homogeneous spaces defined by the coadjoint action are given by the quotient  $\text{Diff}(S^1)/\text{Stab}(f)$ . The stabilizer can be computed in some cases and for some constant values of  $f(\tau)$ . In particular, for  $f(\tau) = -\frac{c\tau^2}{24}$ , we have  $\text{Stab}(b) = SL^{(n)}(2, \mathbb{R})$ , where  $SL^{(n)}(2, \mathbb{R})$  denotes the  $n$ -fold cover of  $SL(2, \mathbb{R})$  (see, e.g., [50]).

- [1] D. Grumiller, W. Kummer, and D. V. Vassilevich, Dilaton gravity in two-dimensions, *Phys. Rep.* **369**, 327 (2002).
- [2] T. Banks and M. O’Loughlin, Two-dimensional quantum gravity in Minkowski space, *Nucl. Phys.* **B362**, 649 (1991).
- [3] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, *Ann. Phys. (N.Y.)* **235**, 435 (1994).
- [4] R. Jackiw, Liouville field theory: A two-dimensional model for gravity?, Report No. MIT-CTP-1049, p. 12 (1982).
- [5] C. Teitelboim, Gravitation and Hamiltonian structure in two space-time dimensions, *Phys. Lett.* **126B**, 41 (1983).
- [6] R. Jackiw, Gauge theories for gravity on a line, *Theor. Math. Phys.* **92**, 979 (1992).
- [7] A. Almheiri and J. Polchinski, Models of AdS<sub>2</sub> backreaction and holography, *J. High Energy Phys.* **11** (2015) 014.
- [8] J. Maldacena, D. Stanford, and Z. Yang, Conformal symmetry and its breaking in two dimensional nearly anti-de Sitter space, *Prog. Theor. Exp. Phys.* **2016**, 12C104 (2016).
- [9] J. Engelsöy, T. G. Mertens, and H. Verlinde, An investigation of AdS<sub>2</sub> backreaction and holography, *J. High Energy Phys.* **07** (2016) 139.
- [10] M. Cvetič and I. Papadimitriou, AdS<sub>2</sub> holographic dictionary, *J. High Energy Phys.* **12** (2016) 008; **01** (2017) 120 (E).
- [11] S. Sachdev, Universal low temperature theory of charged black holes with AdS<sub>2</sub> horizons, *J. Math. Phys. (N.Y.)* **60**, 052303 (2019).
- [12] S. Sachdev, Holographic metals and the fractionalized Fermi liquid, *Phys. Rev. Lett.* **105**, 151602 (2010).
- [13] T. G. Mertens and G. J. Turiaci, Solvable models of quantum black holes: A review on Jackiw–Teitelboim gravity, *Living Rev. Relativity* **26**, 4 (2023).
- [14] H. A. Schwarz, *Gesammelte mathematische abhandlungen* (American Mathematical Society, Providence, 1972), Vol. 260.
- [15] V. Ovsienko and S. Tabachnikov, What is.. the Schwarzian derivative?, *Not. Am. Math. Soc.* **56**, 34 (2009).
- [16] V. L. Iliesiu, On two-dimensional quantum gravity, Ph.D. thesis, Princeton University, 2020.
- [17] A. P. Balachandran, L. Chandar, and A. Momen, Edge states in canonical gravity, [arXiv:gr-qc/9506006](https://arxiv.org/abs/gr-qc/9506006).
- [18] S. Carlip, Statistical mechanics of the (2 + 1)-dimensional black hole, *Phys. Rev. D* **51**, 632 (1995).
- [19] M. Bañados, C. Teitelboim, and J. Zanelli, Black hole in three-dimensional spacetime, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [20] H. A. González, D. Grumiller, and J. Salzer, Towards a bulk description of higher spin SYK, *J. High Energy Phys.* **05** (2018) 083.
- [21] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel, and D. Vassilevich, Menagerie of AdS<sub>2</sub> boundary conditions, *J. High Energy Phys.* **10** (2017) 203.
- [22] T. G. Mertens and G. J. Turiaci, Liouville quantum gravity—holography, JT and matrices, *J. High Energy Phys.* **01** (2021) 073.
- [23] F. Ecker, C. Valcárcel, and D. Vassilevich, 2D holography beyond the Jackiw–Teitelboim model, *J. High Energy Phys.* **09** (2021) 182.
- [24] F. Ecker, D. Grumiller, C. Valcárcel, and D. Vassilevich, Equivalences between 2D dilaton gravities, their asymptotic symmetries, and their holographic duals, *J. High Energy Phys.* **06** (2023) 151.
- [25] D. J. Gross and V. Rosenhaus, A line of CFTs: From generalized free fields to SYK, *J. High Energy Phys.* **07** (2017) 086.
- [26] D. J. Gross and V. Rosenhaus, A generalization of Sachdev–Ye–Kitaev, *J. High Energy Phys.* **02** (2017) 093.
- [27] E. Witten, An SYK-like model without disorder, *J. Phys. A* **52**, 474002 (2019).
- [28] J. Yoon, SYK models and SYK-like tensor models with global symmetry, *J. High Energy Phys.* **10** (2017) 183.
- [29] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Thermoelectric transport in disordered metals without quasiparticles: The Sachdev–Ye–Kitaev models and holography, *Phys. Rev. B* **95**, 155131 (2017).
- [30] P. Narayan and J. Yoon, Supersymmetric SYK model with global symmetry, *J. High Energy Phys.* **08** (2018) 159.
- [31] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, Supersymmetric Sachdev–Ye–Kitaev models, *Phys. Rev. D* **95**, 026009 (2017); [**95**, 069904 (2017)].
- [32] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel, and D. Vassilevich, Menagerie of AdS<sub>2</sub> boundary conditions, *J. High Energy Phys.* **10** (2017) 203.
- [33] K. B. Alkalaev, On higher spin extension of the Jackiw–Teitelboim gravity model, *J. Phys. A* **47**, 365401 (2014).
- [34] D. Grumiller, M. Leston, and D. Vassilevich, Anti-de Sitter holography for gravity and higher spin theories in two dimensions, *Phys. Rev. D* **89**, 044001 (2014).
- [35] H. A. González, D. Grumiller, and J. Salzer, Towards a bulk description of higher spin SYK, *J. High Energy Phys.* **05** (2018) 083.
- [36] R. Gurau, Colored group field theory, *Commun. Math. Phys.* **304**, 69 (2011).
- [37] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, Critical behavior of colored tensor models in the large n limit, *Nucl. Phys.* **B853**, 174 (2011).
- [38] A. Achucarro and M. E. Ortiz, Relating black holes in two-dimensions and three-dimensions, *Phys. Rev. D* **48**, 3600 (1993).
- [39] E. Verheijden and E. Verlinde, From the BTZ black hole to JT gravity: Geometrizing the island, *J. High Energy Phys.* **11** (2021) 092.
- [40] T. G. Mertens, The Schwarzian theory—origins, *J. High Energy Phys.* **05** (2018) 036.
- [41] J. Matulich, S. Prohazka, and J. Salzer, Limits of three-dimensional gravity and metric kinematical Lie algebras in any dimension, *J. High Energy Phys.* **07** (2019) 118.
- [42] T. Fukuyama and K. Kamimura, Gauge theory of two-dimensional gravity, *Phys. Lett. B* **160**, 259 (1985).
- [43] K. Isler and C. A. Trugenberger, A gauge theory of two-dimensional quantum gravity, *Phys. Rev. Lett.* **63**, 834 (1989).
- [44] P. Schaller and T. Strobl, Poisson structure induced (topological) field theories, *Mod. Phys. Lett. A* **09**, 3129 (1994).
- [45] R. Jackiw, Lower dimensional gravity, *Nucl. Phys.* **B252**, 343 (1985).
- [46] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, *Phys. Rev. D* **15**, 2752 (1977).

- [47] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi, and V. Vishal, On the dynamics of near-extremal black holes, *J. High Energy Phys.* **09** (2018) 048.
- [48] Any smooth function of  $X^2$  is a Casimir function for the Poisson algebra  $\mathcal{F}(\Sigma)$  with Poisson bracket (11), it being  $\{X^2, -\} = 0$ .
- [49] One might wonder why not to include the Casimir function directly in the bulk action. The reason, as explained in [44], is that the addition of a Casimir term in the bulk would break the topological nature of the model, resulting in a theory which is equivalent to a Yang-Mills in the bulk, which is not what we want.
- [50] F. Valach and D. R. Youmans, Schwarzian quantum mechanics as a Drinfeld-Sokolov reduction of BF theory, *J. High Energy Phys.* **12** (2020) 189.
- [51] G. Barnich, H. A. González, and P. Salgado-Rebolledo, Geometric actions for three-dimensional gravity, *Classical Quantum Gravity* **35**, 014003 (2017).
- [52] By setting  $n = 1$ , from (19) we get
- $$Ad_{\phi^{-1}}^* \left( -\frac{t}{24}, t \right) = \left( -\frac{t}{12} \left( \frac{1}{2} \phi'^2 + \{ \phi(\tau), \tau \}_S \right), t \right). \quad (\text{A52})$$
- With the change of variables  $F(\tau) \equiv \tan(\frac{1}{2}\phi(\tau))$ , the coadjoint action can be written in terms of a single Schwarzian derivative, that is
- $$Ad_{\phi^{-1}}^* \left( -\frac{t}{24}, t \right) = \left( -\frac{t}{12} \{ F(\tau), \tau \}_S, t \right). \quad (\text{A53})$$
- Therefore, we recognise in (20) the Schwarzian action (1), while disregarding the contribution of the central extension terms in the pairing.
- [53] S. Ebert, C. Ferko, H.-Y. Sun, and Z. Sun,  $T\bar{T}$  in JT gravity and BF gauge theory, *SciPost Phys.* **13**, 096 (2022).
- [54] S. Carlip, The statistical mechanics of the  $(2+1)$ -dimensional black hole, *Phys. Rev. D* **51**, 632 (1995).
- [55] A. Blommaert, T. G. Mertens, and H. Verschelde, The Schwarzian theory—a Wilson line perspective, *J. High Energy Phys.* **12** (2018) 022.
- [56] M. Cvetič and I. Papadimitriou, AdS<sub>2</sub> holographic dictionary, *J. High Energy Phys.* **12** (2016) 008; **01** (2017) 120(E).
- [57] H. Maxfield and G. J. Turiaci, The path integral of 3D gravity near extremality; or, JT gravity with defects as a matrix integral, *J. High Energy Phys.* **01** (2021) 118.
- [58] P. Chaturvedi, I. Papadimitriou, W. Song, and B. Yu, AdS<sub>3</sub> gravity and the complex SYK models, *J. High Energy Phys.* **05** (2021) 142.
- [59] G. Arcioni, M. Blau, and M. O’Loughlin, On the boundary dynamics of Chern-Simons gravity, *J. High Energy Phys.* **01** (2003) 067.
- [60] Notice that the choice in (52), together with the  $\Omega|_{S^1} = \mathfrak{Z}|_{S^1} du$  condition, is similar to the gauge fields parametrisation in the Yang-Mills extension of JT gravity proposed in [35].
- [61] A. Zuevsky, Algebraic proofs for shallow water bi-Hamiltonian systems for three cocycle of the semi-direct product of Kac-Moody and Virasoro Lie algebras, *Open Math.* **16**, 1 (2018).
- [62] A. Alekseev and S. L. Shatashvili, Path integral quantization of the coadjoint orbits of the Virasoro group and 2D gravity, *Nucl. Phys.* **B323**, 719 (1989).
- [63] J. Maldacena and D. Stanford, Remarks on the Sachdev-Ye-Kitaev model, *Phys. Rev. D* **94**, 106002 (2016).
- [64] In three spacetime dimensions,  $\mathfrak{so}(2,2)$  is in fact a kinematic algebra in the sense of [41], and a 3d Chern-Simons theory over this algebra provides a purely gravitational theory of AdS<sub>3</sub> geometry.
- [65] D. Grumiller, R. Ruzzi, and C. Zwickel, Generalized dilaton gravity in 2d, *SciPost Phys.* **12**, 032 (2022).
- [66] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, *Ann. Phys. (N.Y.)* **235**, 435 (1994).
- [67] F. Bascone, F. Pezzella, and P. Vitale, Jacobi sigma models, *J. High Energy Phys.* **03** (2021) 110.
- [68] A. Chatzistavrakidis and G. Šimunić, Gauged sigma-models with nonclosed 3-form and twisted Jacobi structures, *J. High Energy Phys.* **11** (2020) 173.
- [69] F. Bascone, F. Pezzella, and P. Vitale, Topological and dynamical aspects of Jacobi sigma models, *Symmetry* **13**, 1205 (2021).
- [70] F. Di Cosmo, K. Grabowska, and J. Grabowski, Jacobi algebroids and Jacobi sigma models, *Physics* **3**, 2550004 (2025).
- [71] A. Wassermann, Kac-Moody and Virasoro algebras, [arXiv: 1004.1287](https://arxiv.org/abs/1004.1287).
- [72] Notice that  $\tau$  can also be chosen on the real line, with appropriate boundary conditions.
- [73] P. Goddard and D. I. Olive, Kac-Moody and Virasoro algebras: A reprint volume for physicists (1988), <https://api.semanticscholar.org/CorpusID:118308253>.
- [74] H. Sugawara, A field theory of currents, *Phys. Rev.* **170**, 1659 (1968).
- [75]  $h = 1/\kappa^*(\theta_C, \theta_C)$  with  $\kappa^*$  the scalar product in the dual of the Cartan subalgebra of  $\mathfrak{g}$  derived from the Killing metric.
- [76] More precisely, the Witt algebra is the algebra of smooth vector fields on the circle, to which the algebra (A10) with  $C = 0$  is isomorphic. We shall use the same name to simplify the notation.
- [77] M. Schottenloher, Central extensions of lie algebras and Bargmann’s theorem, *Lect. Notes Phys.* **759**, 63 (2008).
- [78] E. Witten, Coadjoint orbits of the Virasoro group, *Commun. Math. Phys.* **114**, 1 (1988).
- [79] J. Grabowski, G. Marmo, A. Perelomov, and A. Simoni, Remarks on Virasoro and Kac-Moody algebras, *Int. J. Mod. Phys. A* **11**, 4969 (1996).
- [80] D. Stanford and E. Witten, Fermionic localization of the Schwarzian theory, *J. High Energy Phys.* **10** (2017) 008.
- [81] A. A. Kirillov, Infinite dimensional lie groups; their orbits, invariants and representations. The geometry of moments, *Lect. Notes Math.* **970**, 101 (1982).
- [82] A  $G$ -module is an Abelian group,  $A$ , on which  $G$  acts respecting the Abelian structure. A  $G$ -module morphism  $\delta$  is a group homomorphism:  $\delta(g(a+b)) = \delta(g(a) + g(b)) = g(\delta(a+b)), g \in G, a, b \in A$ . All of the groups in (A35) are obviously  $G$ -modules for the group of diffeomorphisms.