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COMPANION VARIETIES FOR HESSE, HESSE UNION DUAL HESSE ARRANGEMENTS

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Unexpected hypersurface is a name given an element to some particular linear system introduced by Cook, Harbourne, Migliore and Nagel, motivated by work of Di Gennaro, Ilardi and Vallès and of Faenzi and Vallès, and it is a field of great study since then. It attracts many people because of their close ties to various other areas of mathematics including vector bundles, arrangements of hyperplanes, geometry of projective varieties, etc. Harbourne, Migliore, Nagel and Teitler introduced the concept of unexpected hypersurfaces and explained the so-called BMSS duality showing that unexpected curves are in some sense dual to their tangent cones at their singular point. In this paper, we continue the study of BMSS duality. We revisit the configuration of points associated to Hesse arrangement and Hesse union dual Hesse arrangement, and we study the geometry of the associated varieties and their companions.

1. Introduction

In the present note we study companion varieties of unexpected hypersurfaces associated to some famous arrangements in projective spaces.

A central tool in algebraic geometry for studying varieties is to find maps to projective spaces, i.e., to study their linear systems. Even in the simplest cases — for example, the case of $\mathbb{P}^n$ — this study can be very difficult.

It is of particular interest to study linear systems with imposed base loci, i.e., those of the form $\Lambda := H^0(X; L \otimes I(Z))$, where L is a positive (e.g., ample or very ample) line bundle on a smooth variety X and Z is a subscheme of X.

If X is the projective plane, L is the line bundle $\mathcal{O}_{\mathbb{P}^2}(d)$ for $d > 0$ and Z is a zero-dimensional subscheme of $\mathbb{P}^2$ computing the dimension of the vector space Λ is, even in this simple case, an open problem, as it is shown in the following two open conjectures: one due to Nagata (1959) [9] and the other the SHGH-conjecture package due to Segre (1969), Harbourne (1986), Gimigliano (1987) and Hirschowitz (1989). We expect that a single general point, or a fat point scheme concentrated in a single general point impose independent conditions on homogeneous polynomials of any fixed degree in a projective space of arbitrary dimension. It was surprising that a single general fat point might impose less conditions than expected on the linear system of homogeneous polynomials with assigned base loci, as found by Cook, Harbourne, Migliore and Nagel [2], inspired by an example of Di Gennaro, Ilardi and Vallès [3]. In this the so-called *unexpected hypersurfaces* were born; see Section 2 for precise definitions. Szpond [11] began this study for the B_3 root system. The linear system of quadric hypersurfaces vanishing at

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configuration points determines, after passing to the blow up of $\mathbb{P}^2$ in these points, a morphism to $\mathbb{P}^5$ whose image is a surface S such that at every point P of S, there is a hyperplane in $\mathbb{P}^5$ tangent to P to order 2 (in other words: cutting out on S a curve which passes through P with multiplicity at least 3). It has been observed additionally that there is another surface S', which we call a companion surface of S, which also exhibits interesting geometrical properties. This example motivates our present work.

We study the companions of two important arrangements, continuing the work started in [9] and motivated by the introduction of the BMSS duality in [1]. Assume that there is a set of points Z in $\mathbb{P}^N$ which admits a unique unexpected hypersurface $H_{Z,P}$ of degree d and multiplicity m at a general point $P = (a_0 : \cdots : a_N) \in \mathbb{P}^N$. Let

$$F_Z((x_0 : \cdots : x_N), (a_0 : \cdots : a_N)) = 0$$

be a homogeneous polynomial equation of $H_{Z,P}$. Let $g_0, \dots, g_M$ be a basis of the vector space $[I(Z)]_d$ of homogeneous polynomials of degree d vanishing at all points of Z. Under some mild hypothesis the unexpected hypersurface $H_{Z,P}$ comes from a bihomogeneous polynomial $F_Z((x_0 : \cdots : x_N), (a_0 : \cdots : a_N))$ of bidegree (m, d) [6]. Indeed, F_Z can be written in a unique way as a combination

$$F_Z = h_0(a_0 : \cdots : a_N) g_0(x_0 : \cdots : x_N) + \cdots + h_M(a_0 : \cdots : a_N) g_M(x_0 : \cdots : x_N), \tag{1}$$

where $g_0(x_0 : \cdots : x_N), \dots, g_M(x_0 : \cdots : x_N)$ are homogeneous polynomials of degree d and $h_0(a_0 : \cdots : a_N)$, $\dots, h_M(a_0 : \cdots : a_N)$ are homogeneous polynomials of degree m. Therefore, there are two rational maps naturally associated to equation (1):

$$\varphi : \mathbb{P}^N \ni (x_0 : \cdots : x_N) \mapsto (g_0(x_0 : \cdots : x_N) : \cdots : g_M(x_0 : \cdots : x_N)) \in \mathbb{P}^M$$

and

$$\psi : \mathbb{P}^N \ni (a_0 : \cdots : a_N) \mapsto (h_0(a_0 : \cdots : a_N) : \cdots : h_M(a_0 : \cdots : a_N) \in \mathbb{P}^M.$$

The images of these maps are the *companion varieties*. The purpose of this note is to continue to study their properties and relations. We have that H is always the pull-back of the hyperplane bundle under the appropriate blow up.

We study two famous configurations: the Hesse arrangement and the union of Hesse and dual Hesse arrangement, continuing the study of famous arrangements begun in [4], and we prove the following:

Theorem 1.1. *The image S of φ is a smooth arithmetically Cohen–Macaulay* (*aCM for short*) *rational surface in the case of* Hesse *and* Hesse ∪ dHesse. *In particular*:

(1) *In the case of* Hesse, *with S of degree* 13, *it is the plane blown-up in the* 12 *points of* Z(Hesse), *embedded in* $\mathbb{P}^8$ *with the complete linear system of the quintics through* Z(Hesse). *Its ideal* $I(S)$ *is generated by* 15 *quadrics.*

(2) *In the case of* Hesse ∪ dHesse, *with S of degree* 43, *it is the plane blown-up in the* 21 *points of* Z(Hesse ∪ dHesse), *embedded in* $\mathbb{P}^{23}$ *with the complete linear system of the* 8*-tics through* Z(Hesse ∪ dHesse). *Its ideal* $I(S)$ *is generated by* 210 *quadrics.*

We work over the field of complex numbers $\mathbb{C}$.

2. Unexpected hypersurfaces

In [2], Cook, Harbourne, Migliore and Nagel introduced the concept of unexpected curves. This notion was generalized to arbitrary hypersurfaces in the subsequent article [6] by Harbourne, Migliore, Nagel and Teitler.

Definition 2.1. We say that a reduced set of points $Z \subset \mathbb{P}^N$ *admits an unexpected hypersurface* of degree d if there exists a sequence of nonnegative integers $m_1, \dots, m_s$ such that for general points $P_1, \dots, P_s$ the zero-dimensional subscheme $P = m_1 P_1 + \cdots + m_s P_s$ fails to impose independent conditions on forms of degree d vanishing along Z and the set of such forms is nonempty. In other words, we have

$$h^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z) \otimes I(P)) > \max\left\{0, h^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \sum_{i=1}^{s} \binom{N+m_s-1}{N}\right\}.$$

3. Companion varieties for Hesse arrangement

Example 3.1 (Hesse). The most famous example of a free arrangement is the one that corresponds to the *Hesse configuration* of 12 lines through the nine inflection points of a smooth plane cubic. These nine points are defined by the intersection of the smooth Fermat cubic $f = x^3 + y^3 + z^3$ and its hessian curve $H(f) = xyz$. Set $w = e^{2\pi i/3}$; then the projective coordinates of the nine inflection points are

$$\begin{array}{lll} [0:1:-1], & [1:0:-1], & [1:-1:0], \\ [0:1:-w], & [1:0:-w], & [1:-w:0], \\ [0:1:-w^2], & [1:0:-w^2], & [1:-w^2:0]. \end{array}$$

In the pencil $(f, H(f))$ there are four singular cubics, more precisely four triangles $x^3+y^3+z^3-3axyz=0$ with $a = \infty, 1, w, w^2$. The nine points lie on these 12 projective lines, which are the four degenerate cubics corresponding to the parameter value $a = \infty$ and $a^3 = 1$. The equations of the 12 lines are $x = 0$, $y = 0$, $z = 0$, and $x + w^i y + w^j z = 0$, where $i, j = 0, 1, 2$. Then the arrangement of 12 lines has 9 quadruple points ($t_4 = 9$) and 12 double points ($t_2 = 12$) in correspondence with the edges of the triangles. Indeed in any triangle the choice of one side gives an opposite vertex. It is well known that the Hesse arrangement is free with exponents (4, 7); see [5, Example 3.5].

Example 3.2 (dual Hesse). The dual set of the Hesse arrangement, consisting of 9 lines in $\mathbb{P}^{2\vee}$ with 12 triple points ($t_3 = 12$) is also free according to [10, Theorem 6.60], as it is a reflection arrangement corresponding to the irreducible complex reflection group G_{25} by [10, Example 6.30]. This set of 9 points is the well known obstruction to extend the Sylvester problem from $\mathbb{R}$ to $\mathbb{C}$. This problem was proposed by Sylvester in 1893, then by Erdős in 1943 and solved on $\mathbb{R}$ by E. Melchior in 1941 and with a simpler proof by Kelly in 1948. $\mathcal{D}_0(Z)$ is free with exponents (4, 4) [7, Proposition 5.13].

Example 3.3 (union of Hesse and dual Hesse). The union of the two arrangements presented in Examples 3.1 and 3.2 turns out to be an interesting arrangement in its own right. This arrangement, consisting of $21 = 9 + 12$ lines in $\mathbb{P}^2$ with 57 singular points is also free with exponents (7, 13), as we prove in [7, Proposition 4.6]. A computation shows that the singular points split into double, quadruple and quintuple points and the tally of their numbers is $t_2 = 36$, $t_4 = 9$, $t_5 = 12$.

Proposition 3.4. *The union of the Hesse arrangement and the dual Hesse arrangement is a free arrangement.*

Proof. By [10, Example 6.30] the union of the Hesse arrangement and the dual Hesse arrangement is a reflection arrangement corresponding to the irreducible reflection group G_{26}, and thus by [7, Theorem 4.1], this arrangement is free of type (7, 13). □

The Hesse arrangement. The set $Z(\text{Hesse})$ consists of 12 points, whose coordinates can be chosen as

$$\begin{aligned} &P_1=[1:0:0], && P_2=[0:1:0], && P_3=[0:0:1],\\ &P_4=[1:1:1], && P_5=[1:w:w], && P_6=[1:w^2:w^2],\\ &P_7=[1:w:1], && P_8=[1:1:w], && P_9=[1:w^2:1],\\ &P_{10}=[1:1:w^2], && P_{11}=[1:w:w^2], && P_{12}=[1:w^2:w], \end{aligned} \tag{2}$$

where we set, as above, $w := e^{2\pi i/3}$, i.e., w is (up to a renumbering of the points) a primitive cubic root of unity.

The saturated ideal $I(\text{Hesse})$ is generated by

$$x^3z-y^3z, \qquad xy^3-xz^3, \qquad x^3y-yz^3, \tag{3}$$

and we note that these generators are three reducible quartics, each formed by 4 lines:

$$\begin{aligned} x^3z-y^3z &= z(x-y)(x-wy)(x-w^2y),\\ xy^3-xz^3 &= x(y-z)(y-wz)(y-w^2z),\\ x^3y-yz^3 &= y(x-z)(x-wz)(x-w^2z); \end{aligned}$$

in sum, 12 lines, 9 that correspond to the points of the Hesse configuration, i.e.,

$$\begin{aligned} &\ell_1 := Z(x-y), && \ell_2 := Z(x-wy), && \ell_3 := Z(x-w^2y),\\ &\ell_4 := Z(y-z), && \ell_5 := Z(y-wz), && \ell_6 := Z(y-w^2z),\\ &\ell_7 := Z(x-z), && \ell_8 := Z(x-wz), && \ell_9 := Z(x-w^2z), \end{aligned}$$

and each of these lines contains four points of Hesse; and the three coordinate lines

$$X := Z(x), \qquad Y := Z(y), \qquad Z := Z(z),$$

which contain two points each. More precisely,

$$\begin{aligned} &P_1, P_2 \in Z, && P_3, P_4, P_8, P_{10} \in \ell_1, && P_3, P_6, P_9, P_{12} \in \ell_2, && P_3, P_5, P_7, P_{11} \in \ell_3,\\ &P_2, P_3 \in X, && P_1, P_4, P_5, P_6 \in \ell_4, && P_1, P_7, P_{10}, P_{12} \in \ell_5, && P_1, P_8, P_9, P_{11} \in \ell_6,\\ &P_1, P_3 \in Y, && P_2, P_4, P_7, P_9 \in \ell_7, && P_2, P_6, P_{10}, P_{11} \in \ell_8, && P_2, P_5, P_8, P_{12} \in \ell_9. \end{aligned}$$

The unexpected curve then has the following equation:

$$\begin{aligned} F = {} & (2b^3c + c^4)\cdot(x^4y) - (6ab^2c)\cdot(x^3y^2) + (6a^2bc)\cdot(x^2y^3) \\ & - (2a^3c + c^4)\cdot(xy^4) - (b^4 + 2bc^3)\cdot(x^4z) + (2ab^3 - 2ac^3)\cdot(x^3yz) \\ & + (-2a^3b + 2bc^3)\cdot(xy^3z) + (a^4 + 2ac^3)\cdot(y^4z) + (6abc^2)\cdot(x^3z^2) \\ & - (6abc^2)\cdot(y^3z^2) - (6a^2bc)\cdot(x^2z^3) + (2a^3c - 2b^3c)\cdot(xyz^3) \\ & + (6ab^2c)\cdot(y^2z^3) + (2a^3b + b^4)\cdot(xz^4) - (a^4 + 2ab^3)\cdot(yz^4). \end{aligned}$$

I(Hesse) in degree five is generated by the following binomials:

$$\begin{array}{lll} m_0 = xz(x^3 - y^3), & m_1 = yz(x^3 - y^3), & m_2 = z^2(x^3 - y^3), \\ m_3 = x^2(y^3 - z^3), & m_4 = xy(y^3 - z^3), & m_5 = xz(y^3 - z^3), \\ m_6 = xy(x^3 - z^3), & m_7 = y^2(x^3 - z^3), & m_8 = yz(x^3 - z^3). \end{array} \tag{4}$$

The unexpected curve, written down with the aid of the generators (4) then has the following equation:

$$\begin{aligned} F = {} & (-b^4 - 2bc^3)\cdot m_0 - (a^4 + 2ac^3)\cdot m_1 + (6abc^2)\cdot m_2 \\ & + (6a^2bc)\cdot m_3 - (2a^3c + c^4)\cdot m_4 - (2a^3b + b^4)\cdot m_5 \\ & + (2b^3c + c^4)\cdot m_6 - (6ab^2c)\cdot m_7 + (a4 + 2ab^3)\cdot m_8. \end{aligned}$$

Let

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^8, \quad P \mapsto (m_0(P) : \cdots : m_8(P)),$$

be the rational map defined by the generators in (4).

Proposition 3.5. *The image of φ is a smooth rational surface S of degree* 13; *it is the plane blown-up in the* 12 *points of Z*(Hesse) *embedded in $\mathbb{P}^8$ with the complete linear system of the quintics through Z*(Hesse). *Its ideal $I(S)$ is generated by* 15 *quadrics*; *in particular*, *it is arithmetically Cohen–Macaulay* (*aCM*).

Proof. Let $\sigma : X \to \mathbb{P}^2$ be the simultaneous blow up of each of 12 points in Hesse with the exceptional divisor $\mathbb{E}$ (which splits into 12 projective lines E_i, one over each of the points blown-up) and as usual let $H = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$. Since $\{p \in \mathbb{P}^2 \mid m_i(p) = 0 \text{ for } i = 0, \ldots, 8\} = \{P_1, \ldots, P_{12}\}$, the linear system $L = 5H - \mathbb{E}$ is base-point-free and it defines a morphism onto its image $\varphi_L : X \to \mathbb{P}^8$ which lifts the map

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^8, \quad (x : y : z) \mapsto (m_0 : m_1 : \cdots : m_8).$$

Therefore, we have the commutative diagram

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle\sigma} & \searrow^{\varphi_L} & \\ \mathbb{P}^2 & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^8 \end{array}$$

Let us call S the image of φ_L (or the closure of the image of φ).

It is easy to prove that S is smooth and φ is an embedding. Indeed, φ is given by the quintics through the 12 points in Hesse and we know that the ideal of these points is generated by the net of quartics (3);

therefore, even the quartics through Hesse separate the points and the tangent vectors if we consider two of them outside Hesse.

If we take a point $P_i \in$ Hesse, we have that one direction through it — which gives a point $x \in E_i$ — determines a line; therefore if this line is not one of the 12 lines joining the points of Hesse, it separates x and any other point of $X \setminus \{x\}$.

If instead $x \in E_i$ is a point that represents one of the 12 lines, it can be of two types: one of the 9 lines of the Hesse arrangement, or one of the axes. Since all is pretty symmetric, we can prove it for just one of the axes and one of the 9 lines of Hesse.

We start with $P_1, P_2 \in Z$: we want to show that there is a quintic through P_1 and P_2 (and $P_3, \dots, P_{12}$) with tangent Z in P_1 and not in P_2. Since the cubic $x^3 - y^3$ passes through $P_3, \dots, P_{12}$, it is sufficient to find a conic passing through P_1 and P_2 with tangent Z in P_1 and not in P_2: since there are infinitely many conics through P_1 and P_2 with tangent Z in P_1, there exists one conic (indeed, all the infinite conics but one) for which Z is not tangent in P_2.

We then pass to the case, for example, of $P_3, P_4, P_8, P_{10} \in \ell_1$; we want to show that there is a quintic, e.g., through P_3 and P_4 (and $P_1, P_2, P_4, \dots, P_{12}$) with tangent ℓ_1 in P_3 and not in P_4. The conic $x^2 + xy + y^2$ contains the points $P_3, P_5, P_6, P_7, P_9, P_{11}, P_{12}$ so it is sufficient to find a cubic through $P_1, P_2, P_3, P_4, P_8, P_{10}$ with tangent ℓ_1 in P_3 but not in P_4. As above, there are infinitely many cubics passing through $P_1, P_2, P_3, P_4, P_8, P_{10}$ with tangent ℓ_1 in P_3, so there exists one for which ℓ_1 is not tangent to it in P_4.

So, we have proven that the linear system separates the points. Let us prove that it separates the tangent vectors. By symmetry, it is sufficient to show this for, e.g., $u \in E_1$; but this is obvious. In fact, $x^3 - y^3$ passes through $P_3, \dots, P_{12}$, and therefore there is even a net of conics passing through P_1, P_2 which have the tangent line which corresponds to u in P_1. Thus the linear system formed by the strict transforms of the quintics given by the products of $x^3 - y^3$ and these conics have different tangents in u, and so this linear systems separates the tangent vectors in u, and the proof is finished.

The last assertions follow from a standard calculation from the short exact sequence defined by cutting the surface with a (general) hyperplane H,

$$0 \to \mathcal{O}_S(k) \to \mathcal{O}_S(k+1) \to \mathcal{O}_C(k+1) \to 0, \tag{5}$$

where $k \in \mathbb{Z}$ and $C = S \cap H$ is the curve section of S, its long exact sequence in cohomology, Riemann–Roch for C and the fact that S, being rational, is a regular surface. □

Considering the companion surface we take a closer look at the following polynomials, obtaining 9 quartics in the variables $(a : b : c)$:

$$\begin{aligned} q_0 &= -b^4 - 2bc^3, & q_1 &= -a^4 - 2ac^3, & q_2 &= 6abc^2, \\ q_3 &= 6a^2bc, & q_4 &= -2a^3c - c^4, & q_5 &= -2a^3b - b^4, \\ q_6 &= 2b^3c + c^4, & q_7 &= -6ab^2c, & q_8 &= a^4 + 2ab^3. \end{aligned}$$

Let

$$\psi : \mathbb{P}^2 \to \mathbb{P}^8, \quad (a : b : c) \mapsto (q_0 : \dots : q_8),$$

be the map associated to the polynomials $q_0, \dots, q_8$, and let us call X' the image of ψ.

Proposition 3.6. *The map* $\psi : \mathbb{P}^2 \to \mathbb{P}^8$ *is an embedding and its image* X' *is a smooth (rational) surface of degree* 16; *more precisely, it is a projection of the Veronese surface* $V_{2,4}$ — *i.e.,* $\mathbb{P}^2$ *embedded in* $\mathbb{P}^{14}$ *by the complete linear systems of the quartics — from a subspace of dimension* 5 *that does not intersect the secant variety of* $V_{2,4}$. *The ideal of* X' *is generated by* 9 *quadrics,* 5 *cubics,* 15 *quartics. It is not linearly normal and hence it is not aCM.*

Proof. The map ψ is regular since the linear system $(q_0, \dots, q_8)$ is base-point-free. Indeed ψ is an embedding since it is the composition of the 4-tuple Veronese embedding and a projection; more precisely, if

$$v_4 : \mathbb{P}^2 \to \mathbb{P}^{14}, \quad (a : b : c) \mapsto (a^4 : a^3 b : \cdots : c^4)$$

— using the lexicographic order on the monomials — is the 4-tuple Veronese embedding, the projection is

$$\begin{aligned} &\pi : \mathbb{P}^{14} \dashrightarrow \mathbb{P}^8, \\ &(y_0 : \ldots : y_{14}) \mapsto (-y_{10} - 2y_{13} : -y_0 - 2y_6 : 6y_8 : 6y_4 : -2y_2 - y_{14} \\ &\qquad\qquad : -2y_1 - y_{10} : 2y_{11} + y_{14} : -6y_7 : y_0 + 2y_6); \end{aligned}$$

since the vertex $V \cong \mathbb{P}^5$ of the projection, whose equations are

$$I(V) = (-y_{10} - 2y_{13}, -y_0 - 2y_6, 6y_8, 6y_4, -2y_2 - y_{14}, -2y_1 - y_{10}, 2y_{11} + y_{14}, -6y_7, y_0 + 2y_6),$$

does not intersect the secant variety of the 4-tuple Veronese surface, image of v_4, whose ideal is generated by the 3×3-minors of the two *catalecticant matrices* (see [8, Lemma 3.1]),

$$\mathrm{Cat}_F(1,3,3) = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ y_1 & y_3 & y_4 & y_6 & y_7 & y_8 & y_{10} & y_{11} & y_{12} & y_{13} \\ y_2 & y_4 & y_5 & y_7 & y_8 & y_9 & y_{11} & y_{12} & y_{13} & y_{14} \end{pmatrix},$$

$$\mathrm{Cat}_F(2,2,3) = \begin{pmatrix} \frac{y_0}{2} & \frac{y_1}{2} & \frac{y_2}{2} & \frac{y_3}{2} & \frac{y_4}{2} & \frac{y_5}{2} \\ y_1 & y_3 & y_4 & y_6 & y_7 & y_8 \\ y_2 & y_4 & y_5 & y_7 & y_8 & y_9 \\ \frac{y_3}{2} & \frac{y_6}{2} & \frac{y_7}{2} & \frac{y_{10}}{2} & \frac{y_{11}}{2} & \frac{y_{12}}{2} \\ y_4 & y_7 & y_8 & y_{11} & y_{12} & y_{13} \\ \frac{y_5}{2} & \frac{y_8}{2} & \frac{y_9}{2} & \frac{y_{12}}{2} & \frac{y_{13}}{2} & \frac{y_{14}}{2} \end{pmatrix}.$$

In fact — for example — the first minor formed by the first three columns of $\mathrm{Cat}_F(1,3,3)$ always has rank 3 for the points of the vertex.

It follows that the surface image of ψ, X' is a smooth nonlinearly normal surface of degree 16.

With a standard cohomological calculation from the hyperplane exact sequence (5) as we did in the proof of Proposition 3.5, we deduce that the ideal of X' is generated by 9 quadrics, 5 cubics, 15 quartics. It is not linearly normal, and hence it is not aCM. □

Union of Hesse and dual Hesse. We study now the arrangement union of Hesse and dual Hesse.

The set $Z(\text{Hesse} \cup \text{dHesse})$ consists of 21 points, which can be assigned the following coordinates:

$$
\begin{aligned}
&P_1 = [1:0:0], && P_2 = [0:1:0], && P_3 = [0:0:1],\\
&P_4 = [1:1:1], && P_5 = [1:w:w], && P_6 = [1:w^2:w^2],\\
&P_7 = [1:w:1], && P_8 = [1:1:w], && P_9 = [1:w^2:1],\\
&P_{10} = [1:1:w^2], && P_{11} = [1:w:w^2], && P_{12} = [1:w^2:w],\\
&P_{13} = [0:1:-1], && P_{14} = [1:0:-1], && P_{15} = [1:-1:0],\\
&P_{16} = [0:1:-w], && P_{17} = [1:0:-w], && P_{18} = [1:-w:0],\\
&P_{19} = [0:1:-w^2], && P_{20} = [1:0:-w^2], && P_{21} = [1:-w^2:0],
\end{aligned} \tag{6}
$$

where, as above, $w := e^{2\pi i/3}$ is (up to a renumbering of the points) a primitive cubic root of unity.

The saturated ideal $I(\text{Hesse} \cup \text{dHesse})$ is generated by

$$
2x^3yz - y^4z - yz^4, \qquad x^4z - 2xy^3z + xz^4, \qquad x^4y + xy^4 - 2xyz^3. \tag{7}
$$

We note that these three quintics are reducible; each of them is formed by two lines and an irreducible cubic:

$$
\begin{aligned}
2x^3yz - y^4z - yz^4 &= yz(2x^3 - y^3 - z^3),\\
x^4z - 2xy^3z + xz^4 &= xz(x^3 - 2y^3 + z^3),\\
x^4y + xy^4 - 2xyz^3 &= xy(x^3 + y^3 - 2z^3).
\end{aligned}
$$

If we call

$$
C_1 := Z(2x^3 - y^3 - z^3), \qquad C_2 := Z(x^3 - 2y^3 + z^3), \qquad C_3 := Z(x^3 + y^3 - 2z^3),
$$

and, as above, X, Y, Z the coordinate axes, we have

$$
P_2, P_3, P_{13}, P_{16}, P_{19} \in X, \qquad P_1, P_3, P_{14}, P_{17}, P_{20} \in Y, \qquad P_1, P_2, P_{15}, P_{18}, P_{21} \in Z,
$$

and

$$
\begin{aligned}
&P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{16}, P_{19} \in C_1,\\
&P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{14}, P_{17}, P_{20} \in C_2,\\
&P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{15}, P_{18}, P_{21} \in C_3;
\end{aligned}
$$

in other words, C_1, C_2, C_3 are three cubics of the pencil of cubics through the nine points P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, i.e.,

$$
C_i \cap C_j = C_1 \cap C_2 \cap C_3 = \{P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}\} \qquad \text{for all } i \neq j;
$$

moreover,

$$
\begin{aligned}
&X \cap C_1 = \{P_{13}, P_{16}, P_{19}\}, && Y \cap C_2 = \{P_{14}, P_{17}, P_{20}\}, && Z \cap C_3 = \{P_{15}, P_{18}, P_{21}\},\\
&Y \cap Z = \{P_1\}, && X \cap Z = \{P_2\}, && X \cap Y = \{P_3\},
\end{aligned}
$$

and finally,

$$
\begin{aligned}
X \cap C_2 &= \{(0:1:\sqrt[3]{2}),\,(0:1:\sqrt[3]{2}\omega),\,(0:1:\sqrt[3]{2}\omega^2)\},\\
X \cap C_3 &= \{(0:\sqrt[3]{2}:1),\,(0:\sqrt[3]{2}\omega:1),\,(0:\sqrt[3]{2}\omega^2:1)\},\\
Y \cap C_1 &= \{(1:0:\sqrt[3]{2}),\,(1:0:\sqrt[3]{2}\omega),\,(1:0:\sqrt[3]{2}\omega^2)\},\\
Y \cap C_3 &= \{(\sqrt[3]{2}:0:1),\,(\sqrt[3]{2}\omega:0:1),\,(\sqrt[3]{2}\omega^2:0:1)\},\\
Z \cap C_1 &= \{(1:\sqrt[3]{2}:0),\,(1:\sqrt[3]{2}\omega:0),\,(1:\sqrt[3]{2}\omega^2:0)\},\\
Z \cap C_2 &= \{(\sqrt[3]{2}:1:0),\,(\sqrt[3]{2}\omega:1:0),\,(\sqrt[3]{2}\omega^2:1:0)\}.
\end{aligned}
$$

The unexpected curve has then the following equation:

$$
\begin{aligned}
F = {} & (7b^3c^4 - c^7)x^7y - 21ab^2c^4x^6y^2 + 21a^2bc^4x^5y^3 + (-7a^3c^4 + 7b^3c^4)x^4y^4\\
& - 21ab^2c^4x^3y^5 + 21a^2bc^4x^2y^6 + (-7a^3c^4 + c^7)xy^7 + (b^7 - 7b^4c^3)x^7z\\
& + (-7ab^6 + 7ac^6)x^6yz + (21a^2b^5 + 42a^2b^2c^3)x^5y^2z\\
& + (-35a^3b^4 - 56a^3bc^3 - 21b^4c^3 + 7bc^6)x^4y^3z\\
& + (35a^4b^3 + 21a^4c^3 + 56ab^3c^3 - 7ac^6)x^3y^4z + (-21a^5b^2 - 42a^2b^2c^3)x^2y^5z\\
& + (7a^6b - 7bc^6)xy^6z + (-a^7 + 7a^4c^3)y^7z + 21ab^4c^2x^6z^2\\
& + (-42a^2b^3c^2 - 21a^2c^5)x^5yz^2 + (21b^5c^2 - 21b^2c^5)x^4y^2z^2\\
& + (42a^4bc^2 - 42ab^4c^2)x^3y^3z^2 + (-21a^5c^2 + 21a^2c^5)x^2y^4z^2\\
& + (42a^3b^2c^2 + 21b^2c^5)xy^5z^2 - 21a^4bc^2y^6z^2 - 21a^2b^4cx^5z^3\\
& + (56a^3b^3c - 7b^6c + 35a^3c^4 + 21b^3c^4)x^4yz^3 + (-42a^4b^2c + 42ab^2c^4)x^3y^2z^3\\
& + (42a^2b^4c - 42a^2bc^4)x^2y^3z^3 + (7a^6c - 56a^3b^3c - 21a^3c^4 - 35b^3c^4)xy^4z^3\\
& + 21a^4b^2cy^5z^3 + (7a^3b^4 - 7b^4c^3)x^4z^4 + (-21a^4b^3 + 7ab^6 - 35a^4c^3 - 56ab^3c^3)x^3yz^4\\
& + (21a^5b^2 - 21a^2b^5)x^2y^2z^4 + (-7a^6b + 21a^3b^4 + 56a^3bc^3 + 35b^4c^3)xy^3z^4\\
& + (-7a^4b^3 + 7a^4c^3)y^4z^4 + 21ab^4c^2x^3z^5 + (21a^5c^2 + 42a^2b^3c^2)x^2yz^5\\
& + (-42a^3b^2c^2 - 21b^5c^2)xy^2z^5 - 21a^4bc^2y^3z^5 - 21a^2b^4cx^2z^6\\
& + (-7a^6c + 7b^6c)xyz^6 + 21a^4b^2cy^2z^6 + (7a^3b^4 - b^7)xz^7 + (a^7 - 7a^4b^3)yz^7.
\end{aligned} \tag{8}
$$

$I\,(\mathrm{Hesse} \cup \mathrm{dHesse})$ in degree eight is generated by the following 24 generators:

$$
\begin{aligned}
m_0 &= 2x^3y^4z - y^7z - y^4z^4, & m_1 &= 2x^3y^3z^2 - y^6z^2 - y^3z^5,\\
m_2 &= 2x^3y^2z^3 - y^5z^3 - y^2z^6, & m_3 &= 2x^3yz^4 - y^4z^4 - yz^7,\\
m_4 &= x^7z - 2x^4y^3z + x^4z^4, & m_5 &= x^6yz - 2x^3y^4z + x^3yz^4,\\
m_6 &= x^6z^2 - 2x^3y^3z2 + x^3z^5, & m_7 &= x^5y^2z - 2x^2y^5z + x^2y^2z^4,\\
m_8 &= x^5yz^2 - 2x^2y^4z^2 + x^2yz^5, & m_9 &= x^5z^3 - 2x^2y^3z^3 + x^2z^6,
\end{aligned}
$$

$$
\begin{aligned}
m_{10} &= x^4y^3z - 2xy^6z + xy^3z^4, & m_{11} &= x^4y^2z^2 - 2xy^5z^2 + xy^2z^5,\\
m_{12} &= x^4yz^3 - 2xy^4z^3 + xyz^6, & m_{13} &= x^4z^4 - 2xy^3z^4 + xz^7,\\
m_{14} &= x^7y + x^4y^4 - 2x^4yz^3, & m_{15} &= x^6y^2 + x^3y^5 - 2x^3y^2z^3,\\
m_{16} &= x^6yz + x^3y^4z - 2x^3yz^4, & m_{17} &= x^5y^3 + x^2y^6 - 2x^2y^3z^3,\\
m_{18} &= x^5y^2z + x^2y^5z - 2x^2y^2z^4, & m_{19} &= x^5yz^2 + x^2y^4z^2 - 2x^2yz^5,\\
m_{20} &= x^4y^4 + xy^7 - 2xy^4z^3, & m_{21} &= x^4y^3z + xy^6z - 2xy^3z^4,\\
m_{22} &= x^4y^2z^2 + xy^5z^2 - 2xy^2z^5, & m_{23} &= x^4yz^3 + xy^4z^3 - 2xyz^6.
\end{aligned}
$$

The unexpected curve, written down with the aid of above generators has then the following equation:

$$
\begin{aligned}
F = {} & (a^7 - 7a^4c^3)m_0 + (21a^4bc^2)m_1 + (-21a^4b^2c)m_2 + (-a^7 + 7a^4b^3)m_3\\
& + (b^7 - 7b^4c^3)m_4 + \left(\tfrac{2}{3}a^7 - \tfrac{35}{3}a^4b^3 - \tfrac{7}{3}ab^6 - \tfrac{35}{3}a^4c^3 - \tfrac{56}{3}ab^3c^3 + \tfrac{14}{3}ac^6\right)m_5\\
& + (21ab^4c^2)m_6 + (7a^5b^2 + 7a^2b^5 + 28a^2b^2c^3)m_7\\
& + (7a^5c^2 - 14a^2b^3c^2 - 14a^2c^5)m_8 + (-21a^2b^4c)m_9\\
& + \left(-\tfrac{7}{3}a^6b - \tfrac{35}{3}a^3b^4 + \tfrac{2}{3}b^7 - \tfrac{56}{3}a^3bc^3 - \tfrac{35}{3}b^4c^3 + \tfrac{14}{3}bc^6\right)m_{10}\\
& + (-14a^3b^2c^2 + 7b^5c^2 - 14b^2c^5)m_{11}\\
& + \left(-\tfrac{7}{3}a^6c + \tfrac{112}{3}a^3b^3c - \tfrac{7}{3}b^6c + \tfrac{70}{3}a^3c^4 + \tfrac{70}{3}b^3c^4 - \tfrac{4}{3}c^7\right)m_{12}\\
& + (7a^3b^4 - b^7)m_{13} + (7b^3c^4 - c^7)m_{14} + (-21ab^2c^4)m_{15}\\
& + \left(-\tfrac{2}{3}a^7 + \tfrac{35}{3}a^4b^3 - \tfrac{14}{3}ab^6 + \tfrac{35}{3}a^4c^3 + \tfrac{56}{3}ab^3c^3 + \tfrac{7}{3}ac^6\right)m_{16}\\
& + (21a^2bc^4)m_{17} + (-7a^5b^2 + 14a^2b^5 + 14a^2b^2c^3)m_{18}\\
& + (-7a^5c^2 - 28a^2b^3c^2 - 7a^2c^5)m_{19} + (-7a^3c^4 + c^7)m_{20}\\
& + \left(\tfrac{7}{3}a^6b - \tfrac{70}{3}a^3b^4 + \tfrac{4}{3}b^7 - \tfrac{112}{3}a^3bc^3 - \tfrac{70}{3}b^4c^3 + \tfrac{7}{3}bc^6\right)m_{21}\\
& + (14a^3b^2c^2 + 14b^5c^2 - 7b^2c^5)m_{22}\\
& + \left(\tfrac{7}{3}a^6c + \tfrac{56}{3}a^3b^3c - \tfrac{14}{3}b^6c + \tfrac{35}{3}a^3c^4 + \tfrac{35}{3}b^3c^4 - \tfrac{2}{3}c^7\right)m_{23}.
\end{aligned} \tag{9}
$$

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{23}$ be the rational map defined by the generators in (7).

Proposition 3.7. *The image of φ is a smooth surface S of degree* 43*. It is the plane blown-up in the* 21 *points of $Z(\mathrm{Hesse} \cup \mathrm{dHesse})$ embedded in $\mathbb{P}^{23}$ with the complete linear system of the* 8*-tics through $Z(\mathrm{Hesse} \cup \mathrm{dHesse})$. Its ideal $I(S)$ is generated by* 210 *quadrics*; *in particular it is arithmetically Cohen–Macaulay.*

Proof. Let $\sigma : X \to \mathbb{P}^2$ be the simultaneous blow up of each of 21 points in $\mathrm{Hesse} \cup \mathrm{dHesse}$ with the exceptional divisor $\mathbb{E}$ (which splits into 21 projective lines, one over each of the points blown up) and as usual let $H = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$. Since $\{p \in \mathbb{P}^2 \mid m_i(p) = 0 \text{ for } i = 0, \dots, 23\} = \{P_1, \dots, P_{23}\}$, the linear system $L = 8H - \mathbb{E}$ is base-point-free and it defines a morphism onto its image $\varphi_L : X \to \mathbb{P}^{23}$ which lifts the map

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{23}, \quad (x : y : z) \mapsto (m_0 : m_1 : \cdots : m_{23}).$$

Therefore, we have the commutative diagram

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle\sigma} & \searrow^{\varphi_L} & \\ \mathbb{P}^2 & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^{23} \end{array}$$

Let us call S the image of φ_L (or the closure of the image of φ).

It is not difficult to prove that S is smooth and φ is an embedding.

Indeed, φ is given by the 8-tics through the 21 points in Hesse $\cup$ dHesse and we know that the ideal of these points is generated by the net of quintics (7); therefore, even the quintics through Hesse $\cup$ dHesse separate the points and the tangent vectors if we consider two of them outside Hesse $\cup$ dHesse.

Again, since the ideal of the points is generated by a net of quintics, we see that, since we are free to move the 8-tics as a quintic of the net plus any cubic, since the cubics separate the (fat) points up to multiplicity two, we deduce that the linear system separates points and tangent vectors also for Hesse $\cup$ dHesse and the map φ is an embedding.

The last assertions follows as before by sequence (5). □

Considering the companion surface we take a closer look at the following polynomials, obtaining 24 polynomials of degree seven in the variables $(a:b:c)$:

$$q_0 = a^7 - 7a^4c^3, \quad q_1 = 21a^4bc^2, \quad q_2 = -21a^4b^2c, \quad q_3 = -a^7 + 7a^4b^3,$$
$$q_4 = b^7 - 7b^4c^3, \quad q_5 = 2a^7 - 35a^4b^3 - 7ab^6 - 35a^4c^3 - 56ab^3c^3 + 14ac^6, \quad q_6 = 21ab^4c^2,$$
$$q_7 = 7a^5b^2 + 7a^2b^5 + 28a^2b^2c^3, \quad q_8 = 7a^5c^2 - 14a^2b^3c^2 - 14a^2c^5, \quad q_9 = -21a^2b^4c,$$
$$q_{10} = -7a^6b - 35a^3b^4 + 2b^7 - 56a^3bc^3 - 35b^4c^3 + 14bc^6,$$
$$q_{11} = -14a^3b^2c^2 + 7b^5c^2 - 14b^2c^5, \quad q_{12} = -7a^6c + 112a^3b^3c - 7b^6c + 70a^3c^4 + 70b^3c^4 - 4c^7,$$
$$q_{13} = 7a^3b^4 - b^7, \quad q_{14} = 7b^3c^4 - c^7, \quad q_{15} = -21ab^2c^4,$$
$$q_{16} = -2a^7 + 35a^4b^3 - 14ab^6 + 35a^4c^3 + 56ab^3c^3 + 7ac^6, \quad q_{17} = 21a^2bc^4,$$
$$q_{18} = -7a^5b^2 + 14a^2b^5 + 14a^2b^2c^3, \quad q_{19} = -7a^5c^2 - 28a^2b^3c^2 - 7a^2c^5,$$
$$q_{20} = -7a^3c^4 + c^7, \quad q_{21} = 7a^6b - 70a^3b^4 + 4b^7 - 112a^3bc^3 - 70b^4c^3 + 7bc^6,$$
$$q_{22} = 14a^3b^2c^2 + 14b^5c^2 - 7b^2c^5, \quad q_{23} = 7a^6c + 56a^3b^3c - 14b^6c + 35a^3c^4 + 35b^3c^4 - 2c^7.$$

Let

$$\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{23}, \quad (a:b:c) \mapsto (q_0 : \cdots q_{23}),$$

be the rational map defined by the 24 generators $q_0, \dots, q_{23}$, and we call X' the image of φ.

Proposition 3.8. *The map $\psi : \mathbb{P}^2 \to \mathbb{P}^{23}$ is an embedding and its image X' is a smooth* (*rational*) *surface of degree* 49; *more precisely*, *it is a projection of the Veronese surface $V_{2,7}$ — i.e.*, *$\mathbb{P}^2$ embedded in $\mathbb{P}^{35}$ by a complete linear system of the* 7*-tics — from a subspace of dimension* 11 *that does not intersect the secant variety of $V_{2,7}$. The ideal of X' is generated by* 180 *quadrics. It is not linearly normal hence it is not aCM.*

Proof. The map

$$\psi : \mathbb{P}^2 \to \mathbb{P}^{23}, \quad (a:b:c) \mapsto (q_0 : \cdots : q_{23}),$$

is regular since the linear system $(q_0,\dots,q_{23})$ is base-point-free. Indeed ψ is an embedding since it is the composition of the 7-tuple Veronese embedding and a projection; more precisely, if

$$v_7 : \mathbb{P}^2 \to \mathbb{P}^{35}, \quad (a:b:c) \mapsto (a^7 : a^6 b : \cdots : c^7)$$

— using the lexicographic order on the monomials — is the 7-tuple Veronese embedding, the projection is

$$\begin{aligned}
&\pi : \mathbb{P}^{35} \dashrightarrow \mathbb{P}^{23},\\
&(y_0 : \ldots : y_{35}) \mapsto (y_0 - 7y_9 : 21y_8 : -21y_7 : -y_0 + 7y_6 : y_28 - 7y_{31}\\
&\qquad : 2y_0 - 35y_6 - 7y_{21} - 35y_9 - 56y_{24} + 14y_{27} : 21y_{23}\\
&\qquad : 7y_3 + 7y_{15} + 28y_{18} : 7y_5 - 14y_{17} - 14y_{20} : -21y_{16}\\
&\qquad : -7y_1 - 35y_{10} + 2y_{28} - 56y_{13} - 35y_{31} + 14y_{34}\\
&\qquad : -14y_{12} + 7y_{30} - 14y_{33} : -7y_2 + 112y_{11} - 7y_{29} + 70y_{14} + 70y_{32} - 4y_{35}\\
&\qquad : 7y_{10} - y_{28} : 7y_{32} - y_{35} : -21y_{25} : -2y_0 + 35y_6 - 14y_{21} + 35y_9 + 56y_{24} + 7y_{27}\\
&\qquad : 21y_{19} : -7y_3 + 14y_{15} + 14y_{18} : -7y_5 - 28y_{17} - 7y_{20} : -7y_{14} + y_{35}\\
&\qquad : 7y_1 - 70y_{10} + 4y_{28} - 112y_{13} - 70y_{31} + 7y_{34} : 14y_{12} + 14y_{30} - 7y_{33}\\
&\qquad : 7y_2 + 56y_{11} - 14y_{29} + 35y_{14} + 35y_{32} - 2y_{35});
\end{aligned}$$

since the vertex $V \cong \mathbb{P}^{11}$ of the projection, whose equations are

$$\begin{aligned}
I(V) = (&y_0 - 7y_9, 21y_8, -21y_7, -y_0 + 7y_6, y_{28} - 7y_{31},\\
&2y_0 - 35y_6 - 7y_{21} - 35y_9 - 56y_{24} + 14y_{27}, 21y_{23},\\
&7y_3 + 7y_{15} + 28y_{18}, 7y_5 - 14y_{17} - 14y_{20}, -21y_{16},\\
&-7y_1 - 35y_{10} + 2y_{28} - 56y_{13} - 35y_{31} + 14y_{34},\\
&-14y_{12} + 7y_{30} - 14y_{33}, -7y_2 + 112y_{11} - 7y_{29} + 70y_{14} + 70y_{32} - 4y_{35},\\
&7y_{10} - y_{28}, 7y_{32} - y_{35}, -21y_{25}, -2y_0 + 35y_6 - 14y_{21} + 35y_9 + 56y_{24} + 7y_{27},\\
&21y_{19}, -7y_3 + 14y_{15} + 14y_{18}, -7y_5 - 28y_{17} - 7y_{20}, -7y_{14} + y_{35},\\
&7y_1 - 70y_{10} + 4y_{28} - 112y_{13} - 70y_{31} + 7y_{34}, 14y_{12} + 14y_{30} - 7y_{33},\\
&7y_2 + 56y_{11} - 14y_{29} + 35y_{14} + 35y_{32} - 2y_{35})
\end{aligned}$$

does not intersect the secant variety of the 7-tuple Veronese surface, image of v_7, whose ideal is generated by the 3×3-minors of the two *catalecticant matrices* (see [8, Lemma 3.1]); the first one is the 3×28 matrix

$$\mathrm{Cat}_F(1,6,3) = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & \cdots & y_{21} & \cdots & y_{27}\\ y_1 & \hat{y}_2 & \hat{y}_5 & \hat{y}_9 & \hat{y}_{14} & \hat{y}_{20} & \hat{y}_{27} & y_{28} & \cdots & y_{34}\\ y_2 & \hat{y}_3 & \hat{y}_7 & \hat{y}_{10} & \hat{y}_{15} & \hat{y}_{21} & \hat{y}_{28} & y_{29} & \cdots & y_{35}\end{pmatrix},$$

where the variables increase within each row and the hat means that the variable is omitted. So, for example, the first 3×3 matrix formed by the first 3 columns is

$$A := \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_3 & y_4 \\ y_2 & y_4 & y_5 \end{pmatrix};$$

we observe that A always has rank 3 for the points of the vertex.

It follows that the surface image of ψ, X' is a smooth nonlinearly normal surface of degree 16.

With a standard cohomological calculation from the hyperplane exact sequence (5) as we did in the proof of Proposition 3.5, we deduce that the ideal of X' is generated by 180 quadrics. It is not linearly normal, and hence it is not aCM. □

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