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REGULAR ARTICLE

Nonlinear dynamics in a public good game

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Abstract

The present work aims to study the problem of individual voluntary anonymous contributions to the financing of public goods in a dynamic setting. To do this, the article departs from a textbook model à la Naimzada and Tramontana (2010) augmented with public goods. The article studies how bounded rationality and dependence on agents' past decisions combine with the problem of voluntary contributions. This favours the emergence of nonlinear dynamics in individual behaviour as well as in the aggregate contribution to the financing of a public good project. The Nash equilibrium can be destabilised through a flip bifurcation when the agent reactivity increases. In addition, some Neimark–Sacker bifurcations can also occur although not around the steady-state equilibrium. A sufficiently high agent reactivity level can also lead to chaotic dynamics with possible multiple attractors. When the chaotic regime prevails, synchronisation phenomena in agent behaviour may occur but are rare. Thus, usually, even if agents are homogeneous, they behave as if they were heterogeneous by making non-synchronised decisions. The work also explicitly deepens the case of a heterogeneous economy in terms of both consumer preferences and income.

Keywords Public goods · Nash equilibrium · Nonlinear dynamics

JEL Classification C62 · D43 · L13 · O31

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1 Introduction

The subject of individual voluntary anonymous contributions (VACs) in a public good game has been extensively studied by the neoclassical literature considering both a *static game* approach (e.g. Sugden 1984; Bergstrom et al. 1986; Varian 1994; Andreoni 2006), in which agents must rationally and strategically choose to contribute to the financing of a public good project, and a *dynamic game* approach (e.g. Marx and Matthews 2000; Duffy et al. 2007), exploring the hypothesis of forward-looking agents in the financing of public goods through private funds over time. More recently, a literature has approached the topic of VACs by using the tools of evolutionary dynamics and has dealt more specifically with whether it is convenient for a subject to contribute or not (binary choice) to the financing of this kind of good (Hauert et al. 2002a, b). In these two strands of literature, several issues have been investigated ranging from the role of the punisher (Lv et al. 2023) to the climate action effects of voluntary contribution (Goeschl et al. 2020), passing through the evolutionary dynamics of cooperation (Liu et al. 2019).

Alongside the well-known results of under-contribution driven by the free-riding phenomenon, as described by neoclassical theory, many empirical papers have shown that situations of over-contribution to the financing of public good projects often emerge in experimental exercises (Keser 1996; McGinty and Milam 2013; Lappalainen 2018). The issue of voluntary contributions has often been related to the phenomenon of donations to non-profit organisations and charitable projects. The Italian experience, for example, shows that data on donations do not follow a trend, but there are fluctuations in the size of contributions and the number of contributors (Osservatorio sul Dono 2023). Likewise, an important experimental work concerning the analysis of agents' behaviour when faced with voluntary contributions to the financing of a public good is that of Feng et al. (2018), who show that the Nash equilibrium does not appear to be significant in explaining this problem. This is because individual contributions exhibit important non-dampening fluctuations thus indicating that the Nash equilibrium may not be a suitable theoretical benchmark. This suggests that an approach guided by the axioms of neoclassical theory may be unable to fully describe the problem of public good games.

The main goal of the present article is to provide a bridge between the neoclassical literature dealing with static and dynamic games, characterised by forward-looking strategies and perfect information, and the evolutionary dynamics literature, where, unlike the neoclassical literature, agents' decisions are guided by instantaneous payoffs. To do this, we start from a static model in which consumers evaluate the consumption of private and public goods in their utility functions and must endogenously decide how much to contribute to the financing of the production of the public good. Then, we move towards the study of a dynamic context, *game dynamics*, in which the decisions made by agents are formulated, given their expectations about the decisions of the others, through heuristics as introduced by Bischi et al. (1998, 1999). A key feature of this approach, enabling us to depart from the dynamic neoclassical approach, is that agents do not necessarily coordinate on a Nash equilibrium, but may over time change their decisions dynamically in response to changes in the behaviour of other agents and the resulting payoffs.

Our line of research is part of a strand initiated by the pioneering contribution of Benhabib and Day (1981), showing how the standard static individual decision-making paradigm—developed in a dynamic context characterised by some forms of bounded rationality—can be compatible with the emergence of nonlinear dynamics and complexity in the definition of consumption patterns (see Rosser 1999, for a deep analysis on complexity and its interplay with bounded rationality). Indeed, the work of Benhabib and Day (1981) has caught the attention of many scholars who have focused on the role of (1) past consumption experiences for the individual consumer and/or consumption decisions of other consumers (Gaertner and Jungeilges 1988; Caravaggio and Sodini 2020; Naimzada and Pireddu 2020; Caravaggio et al. 2022) and (2) the use of heuristic mechanisms in the individual decision-making processes (Naimzada and Tramontana 2010). The present work is in line with and extends this field by considering the existence of a public good entering the individual utility function. This is shown to be responsible for the emergence of nonlinear dynamics in individual behaviour as well as in the contribution to its financing. More specifically, a flip bifurcation can destabilise the Nash equilibrium of the public good game, if the agents' reactivity is relatively high, and also some Neimark–Sacker bifurcations can be observed, but not around the steady-state equilibrium. The agent's reactivity to past behaviour is also responsible for the occurrence of chaotic dynamics, multiple attractors and the loss of synchronisation even with homogeneous agents.

The rest of the article proceeds as follows. Section 2 presents the basic (static) model. Section 3 turns to a dynamic setting and shows the main results of the article. Section 4 concludes.

2 The model

Consider an economy in which there is a positive finite number N of agents (consumers). The problem of the production of the private good and the public good is not directly considered as the paper has a partial equilibrium perspective. Each agent i $(i = 1, ..., N)$ chooses how to allocate her resources (income) between the consumption of a private good c_i and the consumption of a public good G, whose financing is determined by the sum of the individual contributions g_i, that is,

$$G = \sum_{i=1}^{N} g_i. \tag{1}$$

The preferences of agent i, capturing the individual utility from the consumption of both the private good and the public good, are described by a Cobb–Douglas utility function of the form:

$$U_i = c_i^{\alpha_i}(1+G)^{1-\alpha_i}, \tag{2}$$

where $\alpha_i \in (0,1)$ represents the elasticity of the utility to the private good c_i. The Cobb–Douglas utility function in Eq. (2) resembles those used by Benhabib and Day (1981) and Naimzada and Tramontana (2010) in models with two private goods. The reason for using constant 1 is not to exclude corner solutions in which the contribution to the financing of the public good is null for the generic agent i. We note, however, that the results of the consumer problem qualitatively coincide with those obtained using a quasi-linear utility function often used in the public goods literature (see, Mas-Colell et al. 1995, Chapter 11, p. 359; Furusawa and Konishi 2011). In addition, the constant 1 in (2) rules out the implausible case in which the utility is constant and equal to 0 if the contributions to the public good are zero, regardless of the level of c_i.

Normalising the price of the private good to 1, the budget constraint of consumer i reads as follows:

$$c_i + g_i = M_i, \tag{3}$$

where M_i is the income of agent i.

From the consumer optimisation problem, the optimal choice of agent i is:

$$c_i^{\mathrm{Opt}} = \min\left\{ \left(1 + M_i + \sum_{j \neq i} g_j^e\right)\alpha_i, M_i \right\}, \tag{4}$$

and

$$g_i^{\mathrm{Opt}} = \max\left\{ 0, M_i(1-\alpha_i) - \alpha_i(1 + \sum_{j \neq i} g_j^e) \right\}, \tag{5}$$

where the superscript e denotes the expectations of agent i over the contribution decisions of the others. The symbol "min" in Eq. (4) and the symbol "max" in Eq. (5) identify the possibility to have a corner solution in which consumption of the private good of agent i is M_i, that is, all income is used to buy the private good and there is no contribution to the financing of the public good. The superscript "Opt" will be dropped in what follows.

The static game defines several possible equilibria corresponding to which the contribution to the financing of the public good is made by (i) all agents; (ii) no agent; (iii) only a proper subset of agents. These results are highlighted in Fig. 1 for $N = 2$ and $M_i = 1$. The figure shows in the (α_1, α_2) plane the emergence of the equilibrium in which the two players contribute to the public good (red region), equilibria in which one of them does not contribute to the public good (the light grey and the dark grey regions) and the case in which no agent contributes to the public good (white region).

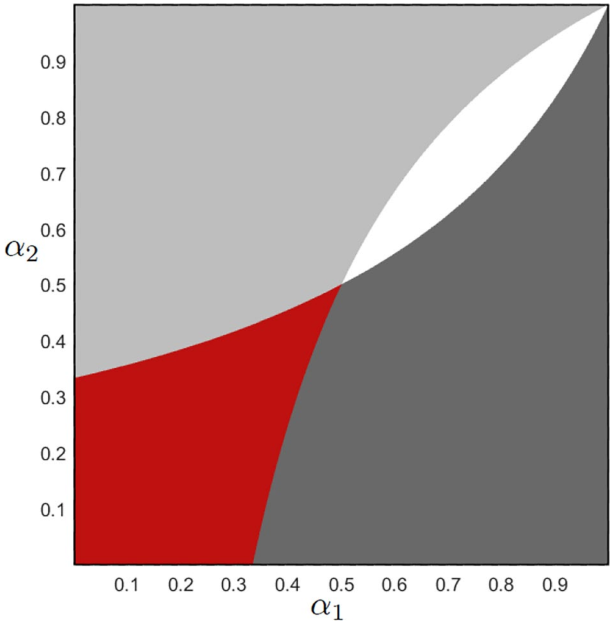

Fig. 1 Plane (α_1, α_2) in which different equilibrium scenarios occur. Red: both agents contribute to the financing of the public good. Light and dark grey: only one agent contributes to the financing of the public good. White: no agent contributes to the financing of the public good (colour figure online)

3 The dynamic setting

Once the static game and its equilibria have been identified, it is relevant to understand whether these equilibria can emerge in a dynamic setting, i.e. whether there exists an adjustment mechanism concerning agents' contributions allowing the system to converge to one of them. Over the years, various dynamic adjustment mechanisms have been proposed in the presence of bounded rationality, especially in the context of dynamic oligopolies. In this regard, the seminal articles by Bischi et al. (1998, 1999) and Bischi et al. (2007), respectively, shed light on the dynamic effects of the gradient adjustment method and the local monopolistic approximation. Although the literature on dynamic oligopolies has been largely developed over the years, less attention has been given to the study of consumer behaviour in partial economic equilibrium and the contribution in this direction focused only on private goods (Gaertner and Jungeilges 1988; Naimzada and Tramontana 2010). To the best of our knowledge, this kind of analysis has never been extended to the case of public goods. To do this, we consider the utility of agent i as a function of her own contribution, g_i, and the contributions of the other $N-1$ agents, $\mathbf{g^e}_{-i}$:

$$V_i(g_i, \mathbf{g^e}_{-i}) = (M_i - g_i)^{\alpha_i}\left(1 + g_i + \mathbf{1}^{\mathbf{T}}\mathbf{g^e}_{-i}\right)^{1-\alpha_i}, \tag{6}$$

where $\mathbf{g^e}_{-i}$ is a column vector reflecting agent i's expectations about the contributions of the other $N-1$ agents, $\mathbf{1}$ denotes a column vector of $N-1$ ones and $\mathbf{T}$ is the transpose operator. Starting from a contribution $g_i(t)$ with $t \in \mathbb{N}$, we assume that at the beginning of time $t+1$ the consumer knows the contributions of the other $N-1$ agents and is able to evaluate the derivative $V_i'(g_i(t)) = \partial V_i(g_i(t), \mathbf{g}_{-i}(t))/\partial g_i(t)$, representing the marginal utility of agent i when g_i varies. Then, agent i uses $V_i'(g_i(t))$ to choose the contribution to the financing of the public good at $t+1$ and thus, by difference, also the consumption of the private good at $t+1$, according to the following behavioural rule (see Naimzada and Tramontana 2010, Eq. 6, p. 50)[1]:

$$g_i(t+1) = g_i(t) + \gamma V_i'(g_i(t)), \tag{7}$$

where the parameter $\gamma \geq 0$, assumed to be equal for all agents, represents the speed of adjustment of the contribution to the public good from one period to the next and measures the intensity of the agent's reaction to this information. When $\gamma = 0$, the decision-making process is static. This implies that the decisions made today are replicated in the future. When $\gamma > 0$, if agent i's utility is increasing at $g_i(t)$ (i.e. $V_i'(g_i(t)) > 0$), this will lead to an increase in its contribution in the subsequent period. Until the beginning of Sect. 3.1, we assume that γ has a magnitude maintaining the dynamics of g_i in the interval $(0, M_i)$. A positive value of γ leads to the emergence of isolated stationary equilibria. More specifically, the interior equilibrium $\mathbf{g}^*$ corresponding to which all agents contribute to the financing of the public good is the solution of the following system:

$$\begin{cases} (1-\alpha_i)g_i + \alpha_i G = M_i(1-\alpha_i) - \alpha_i, \quad i = 1, \ldots, N \\ \sum_{i=1}^{N} g_i = G \end{cases}, \tag{8}$$

and its generic coordinate i is given by

$$g_i^* = M_i - \alpha_i \frac{1 + \sum_{j=1}^{N} M_j}{(1-\alpha_i)\left(1 + \sum_{j=1}^{N} \frac{a_j}{1-a_j}\right)}. \tag{9}$$

A necessary condition for the solution $\mathbf{g}^*$ of (8) to be positive is that the income of each agent is sufficiently high, that is,

$$M_i > \frac{\alpha_i}{1-\alpha_i}, \quad i = 1, \ldots, N, \tag{10}$$

where $\frac{\alpha_i}{1-\alpha_i}$ is the ratio between the elasticity of the utility to the private good c_i and its 1's complement.

[1] We referred to the basic model of Naimzada and Tramontana (2010) described by their Eq. (6), p. 50. This is because their extension given by Eqs. (9)–(11) in Naimzada and Tramontana (2010) qualitatively replicates the results of the basic model, changing only the scale of the value of the reactivity parameter for which the dynamic events occur.

In order to investigate the stability properties of $\mathbf{g}^*$, we consider the utility of agent i expressed as follows:

$$V_i = l_i(g_i)m_i(G), \tag{11}$$

where $l_i(g_i) = \left(1 - g_i\right)^{\alpha_i}$ and $m_i(G) = (1 + G)^{1-\alpha_i}$ so the adjustment mechanism of agent i becomes

$$\begin{aligned} g_i(t+1) &= g_i(t) + \gamma\left(l_i'(g_i(t))m_i(G(t)) + l_i(g_i(t))m_i'(G(t))\right) \\ &= g_i(t) + \gamma V_i \cdot \left(\frac{1-\alpha_i}{1+G} - \frac{\alpha_i}{1-g_i}\right). \end{aligned} \tag{12}$$

Let us consider the Jacobian matrix associated with the system (7) evaluated at $\mathbf{g}^*$, that is,

$$J(\mathbf{g}^*) = \begin{bmatrix} A_1 & B_1 & \dots & B_1 \\ B_2 & A_2 & \dots & B_2 \\ \dots & \dots & \dots & \dots \\ B_N & B_N & \dots & A_N \end{bmatrix}, \tag{13}$$

where

$$A_i = 1 + \gamma(l_i''(g^*)m_i(G^*) + 2l_i'(g^*)m_i'(G^*) + l_i(g^*)m_i''(G^*)), \tag{14}$$

and

$$B_i = \gamma(l_i'(g^*)m_i'(G^*) + l_i(g^*)m_i''(G^*)), \tag{15}$$

with $G^* = \sum_{i=1}^N g_i^*$. The following proposition analyses the stability properties of $\mathbf{g}^*$.

Proposition 1 *Let $\gamma > 0$ hold. The Nash equilibrium $\mathbf{g}^*$ is locally asymptotically stable if for all i*

$$\left(\frac{U_{i,c_i}' U_{i,G}'}{U_i} - U_{i,c_ic_i}''\right)\gamma < 2,$$

and

$$\sum_{i=1}^N \frac{\gamma\left(U_{i,GG}'' - \frac{U_{i,c_i}' U_{i,G}'}{U_i}\right)}{2 + \gamma\left(U_{i,c_ic_i}'' - \frac{U_{i,c_i}' U_{i,G}'}{U_i}\right)} > -1.$$

Proof The characteristic polynomial of (13) is

$$\det(J(\mathbf{g}^*) - \lambda I) = \prod_{i=1}^{N} \left[1 + \gamma(l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*)) - \lambda\right] \cdot \\ \cdot \left[1 + \sum_{i=1}^{N} \frac{\gamma(l_i'(g^*)m_i'(G^*) + l_i(g^*)m_i''(G^*))}{1 + \gamma(l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*)) - \lambda}\right]. \tag{16}$$

Mimicking the procedure in Bischi et al. (2010), the eigenvalues of (13) are given by

$$1 + \gamma(l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*)),$$

and the roots of the equation

$$1 + \sum_{i=1}^{N} \frac{\gamma(l_i'(g^*)m_i'(G^*) + l_i(g^*)m_i''(G^*))}{1 + \gamma(l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*)) - \lambda} = 0.$$

By noticing that $l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*) < 0$, all the eigenvalues are inside the unit circle if and only if for all i

$$\gamma(l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*)) > -2,$$

and

$$\sum_{i=1}^{N} \frac{\gamma(l_i'(g^*)m_i'(G^*) + l_i(g^*)m_i''(G^*))}{2 + \gamma(l_i''(g^*)m_i(G^*) + l_i'(g^*)m_i'(G^*))} > -1.$$

Considering the relationships amongst the derivatives in (16) and the second derivatives of the utility function in (2), we have the result. □

Proposition 1 implies that the interior Nash equilibrium (when it exists) is stable if the speed of adjustment of the players is not too high (this is in line with the literature of agents with bounded rationality). We note that an increase in V_i' (measured in absolute value) increases the reactivity of the adjustment process by favouring the instability of the equilibrium of the map. Further details will be given in the following sections for a two-player economy.

3.1 The two-dimensional case

To investigate the global properties of the model, not confined to the local properties of the steady state, we specify the map in the two-dimensional case $N = 2$. This map is given by:

$$\begin{aligned} g_1' &:= \min\left\{D, \max\left\{g_1 + \gamma\frac{M_1(1-\alpha_1) - (1+g_2)\alpha_1 - g_1}{(M_1 - g_1)^{1-\alpha_1}(1+g_1+g_2)^{\alpha_1}}, 0\right\}\right\}, \\ g_2' &:= \min\left\{D, \max\left\{g_2 + \gamma\frac{M_2(1-\alpha_2) - (1+g_1)\alpha_2 - g_2}{(M_2 - g_2)^{1-\alpha_2}(1+g_1+g_2)^{\alpha_2}}, 0\right\}\right\}, \end{aligned} \tag{17}$$

where the symbol $'$ denotes the unit time advancement of the state variables and D is a parameter smaller than M_i for all i. Parameter D defines the bound to the contribution of the financing to the public good (see Tramontana et al. 2010 for a similar hypothesis in a duopoly). As a contribution equal to M_i (i.e. equal to the available income) would result in a zero utility for each agent i regardless of the contributions of the other agents, we assume that agents, although they may be guided by the local information $V'(g_t)$, do not go as far as that threshold.

3.1.1 The two-dimensional case: the homogeneous economy

Let us start by considering a homogeneous economy as heterogeneous outcomes can already emerge in this context. For doing this, we normalise the exogenous disposable income $M_i = 1$ and set $\alpha_i = \alpha$, for $i = 1, 2$ (i.e. two-agent homogeneous economy). The dynamic system is then given by the following map:

$$T = \begin{cases} g_1' = \min\left\{D, \max\left\{0, g_1 + \gamma\left[\frac{1-g_1-\alpha(2+g_2)}{(1-g_1)^{1-\alpha}(1+g_1+g_2)^{\alpha}}\right]\right\}\right\} \\ g_2' = \min\left\{D, \max\left\{0, g_2 + \gamma\left[\frac{1-g_2-\alpha(2+g_1)}{(1-g_2)^{1-\alpha}(1+g_1+g_2)^{\alpha}}\right]\right\}\right\} \end{cases}. \tag{18}$$

We first note that map T is symmetric. This means that it does not change if g_1 and g_2 are swapped up. It follows that $\Delta = \{(g_1, g_2) : g_1 = g_2\}$ is an invariant manifold, that is, $T(\Delta) \subseteq \Delta$ and this implies that by starting with identical initial conditions $g_1(0) = g_2(0)$, the dynamics evolve on Δ for every t. In this case, the behaviour of the dynamic system is described by the restriction of map T on Δ, and the synchronised trajectories (i.e. $g_1 = g_2, \forall t$) are driven by $T_\Delta : \Delta \to \Delta$, where

$$T_\Delta : g' = f(g) := \min\left\{D, \max\left\{0, g + \gamma\left[\frac{1-2\alpha-(1+\alpha)g}{(1+2g)^{\alpha}(1-g)^{1-\alpha}}\right]\right\}\right\}. \tag{19}$$

Consider first map T_Δ without any bound. The map reads as follows:

$$\widetilde{T}_\Delta : g' = \widetilde{f}(g) := g + \gamma\left[\frac{1-2\alpha-(1+\alpha)g}{(1+2g)^{\alpha}(1-g)^{1-\alpha}}\right]. \tag{20}$$

From (20), we have that $g(0) \gtrless 0$ if $\alpha \lessgtr 1/2$ and $\lim_{g\to 1^-} = \widetilde{f}(g) = -\infty$. By studying the first derivative of $\widetilde{f}$, we conclude that

$$\widetilde{f}'(g) = 1 - \frac{3\alpha\gamma(1-\alpha)(2+g)}{(1-g)^{(2-\alpha)}(1+2g)^{1+\alpha}}, \tag{21}$$

so that the map can be monotonic, uni-modal or bimodal depending on the parameters of the problem. If $\alpha > \frac{1}{2}$, $\widetilde{T}_\Delta$ has no fixed point and the dynamics converge to $g = 0$ for all the initial conditions. If $\alpha < \frac{1}{2}$, $\widetilde{T}_\Delta$ has one fixed point in the interval (0, 1) given by $g = g^* = \frac{1-2\alpha}{1+\alpha}$. This point is stable (resp. unstable) if $\gamma < \gamma_{th}$ (resp. $\gamma > \gamma_{th}$), where $\gamma_{th} = \frac{3\alpha^{1-\alpha}(1-\alpha)^{\alpha}}{(1+\alpha)^2}$ is a threshold that depends non-monotonically on α.

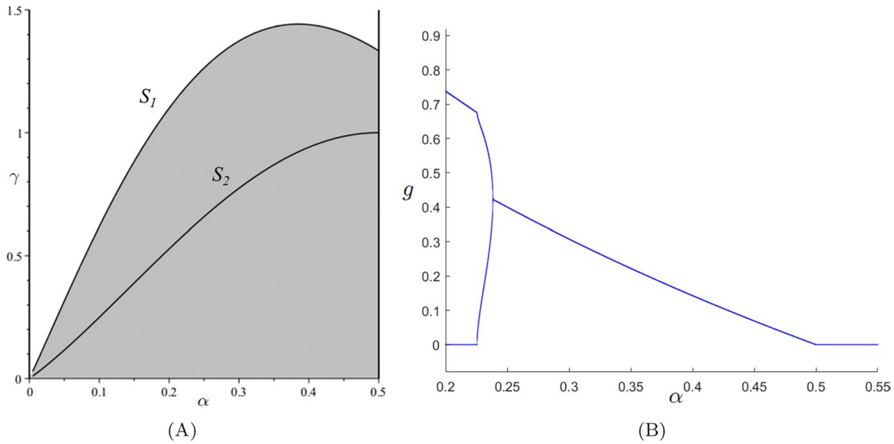

Fig. 2 Panel A. Stability region of map $\widetilde{T}_\Delta$ (the grey area below the curve S_1) in the parameter space (α, γ). The area under the curve S_2 defines the stability region for an economy in which agent i consumes two private goods, a and b, with utility $U_i = a^{\alpha_i}(1+b)^{1-\alpha_i}$ and (normalised) budget constraint $a + b = 1$. Panel B. Bifurcation diagram for α (colour figure online)

In particular, γ_{th} is an inverted U-shaped (uni-modal) function in the interval [0, 1/2] in which the fixed point exists, with $\gamma_{th}\big|_{\alpha=0} = 0$ and $\gamma_{th}\big|_{\alpha=1/2} = 2/3$. This implies that the relationship between α and equilibrium stability is not univocal (see the grey–stability–area below the S_1 curve in Fig. 2, Panel A). This is because a sufficiently large or sufficiently small value of α makes agents very reactive. More specifically, destabilisation can occur either through the emergence of dynamics that do not converge on the equilibrium, but oscillate around it, or through the emergence of attractive corner equilibria for $\widetilde{T}_\Delta$ (see Fig. 2, Panel B). We also wondered whether the presence of a public good made the equilibrium of the economy more or less stable than in the case of an economy in which there are only private goods. To this purpose, Fig. 2, Panel A shows that in an economy with a public good, the stability region of parameters (defined by the area under the S_2 curve) is, ceteris paribus, larger (see the caption of the figure for details) than the area under the S_1 curve. The economic reason for this result is that the utility function of an agent, in the case of a public good, is positively affected by the expectations of the financing of that good by the other agent (positive externality). With the decision mechanism considered in the paper, this configuration reduces the agent's reactivity, in turn, favouring the stability of the Nash equilibrium.[2]

Moving to the study of map T_Δ, that is, by explicitly considering the presence of bounds in the evolution of dynamics, for low values of α ($\alpha < \frac{1-D}{2+D}$) the fixed point g^* does not exist as $D < g^*$. The high evaluation given to the public good makes the equilibrium $g = D$ stable for any value of $\gamma > 0$. For higher values of α

[2] The reactivity of the dynamic system is assessed by both γ and the curvature V'' of the V function.

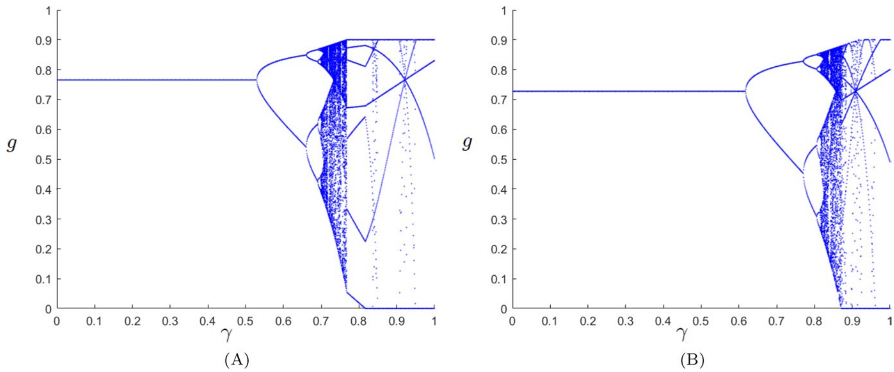

Fig. 3 Dynamics of T_Δ on the diagonal when α varies. Panel A: $\alpha = 0.0849$. Panel B: $\alpha = 0.1$

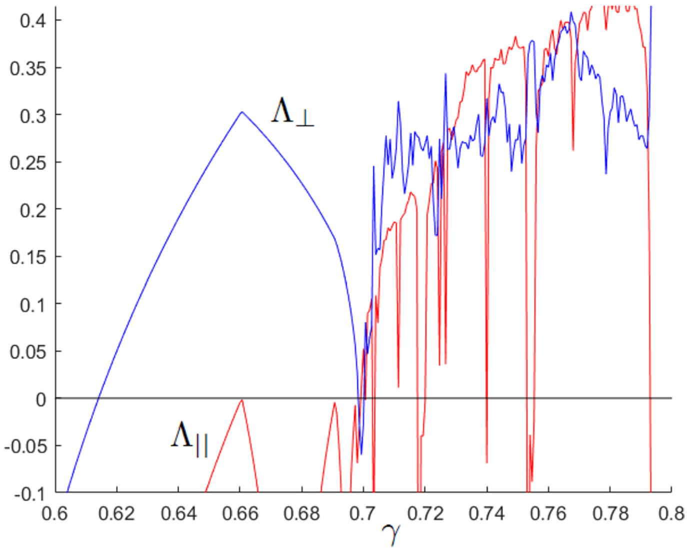

Fig. 4 Lyapunov exponent. Red: $\Lambda_{||}$, non-transverse Lyapunov exponent (see also Fig. 3, Panel A). Blue: $\Lambda_\perp$, transverse Lyapunov exponent (colour figure online)

($\frac{1-D}{2+D} < \alpha < \frac{1}{2}$), the equilibrium g^* exists and—starting from a situation in which γ is sufficiently low so that the equilibrium is locally asymptotically stable—it undergoes a destabilisation by means of a flip bifurcation when γ increases. Thereafter, larger values of γ may be responsible for the classical cascade of period doubling bifurcations until the eventual appearance of a chaotic regime. The evolution of the dynamics can be affected by the existence of the two bounds.[3]

[3] We used $D = 0.9$ in all the simulations (Figs. 2, 3, 4, 5 and 6).

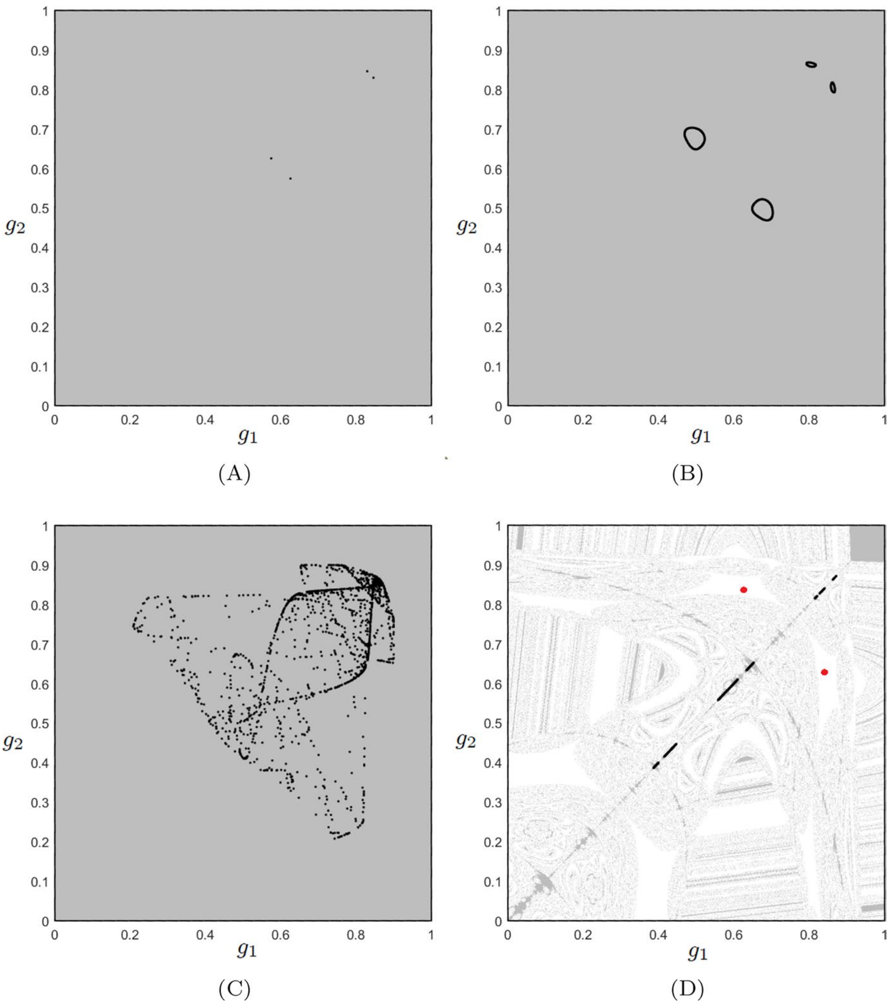

Fig. 5 Phase plane (g_1, g_2) plotted for $\alpha = 0.0849$. Panel A: $\gamma = 0.616$. Panel B: $\gamma = 0.64$. Panel C: $\gamma = 0.68$. Panel D: $\gamma = 0.7$

Let us first consider a fairly low value of α (Fig. 3, Panel A), that is, $\alpha = 0.0849$. In this case, after the onset of the chaotic regime, the broadening of the attractor of T_Δ is halted by the bound D from around $\gamma = 0.7675$. Recall that a very low value of α implies that the public good weighs heavily in the utility function and an increase in γ leads them in some periods to invest almost all $D < M$ resources in its financing. An even higher value of γ ($\gamma \cong 0.8168$) also causes the binding of $g = 0$ in the dynamics. For larger values of γ, the agents, at least in some periods, behave as free riders, relying on the financing of the other agent.

Let us now consider the case $\alpha = 0.1$ (Fig. 3, Panel B). In this case, agents are less interested in the public good and an increase in their speed of adjustment first

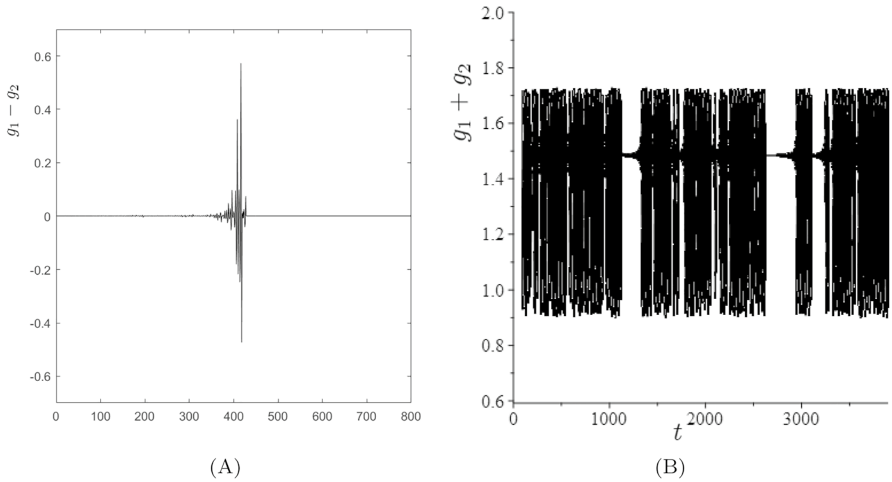

Fig. 6 Panel A: Intermitted behaviour $g_1 - g_2$ when t varies ($\gamma = 0.7001$). Initial conditions: $g_1(0) = 0.861011$, $g_2(0) = 0.861010$. Panel B: G-dynamics ($\alpha = 0.68$)

causes a border collision with bound $g = 0$ (occurring at around $\gamma = 0.8706$) and then another border collision for γ around 0.916 with $g = D$. If the agents' reactivity is larger, even if the agents are less interested in the public good than in the case illustrated in Fig. 3, Panel A, faced with a very low overall contribution for the financing of the public good, the agents can decide for some periods to contribute the maximum possible, D, to its financing. From a mathematical point of view, this means that for higher values of γ there are intervals where the bound $g = D$ is involved or not.

Let us turn now to consider the two-dimensional map. This map has an interior fixed point P if and only if $\alpha < 1/2$, which is given by:

$$P = \left(\frac{1-2\alpha}{1+\alpha}, \frac{1-2\alpha}{1+\alpha}\right). \tag{22}$$

The corresponding Jacobian matrix evaluated at P is

$$J(P) = \begin{pmatrix} 1 - \frac{\gamma\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}(1+\alpha)}{3\alpha} & -\frac{1}{3}\gamma(1+\alpha)(\frac{\alpha}{1-\alpha})^{\alpha} \\ -\frac{1}{3}\gamma(1+\alpha)(\frac{\alpha}{1-\alpha})^{\alpha} & 1 - \frac{\gamma\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}(1+\alpha)}{3\alpha} \end{pmatrix}. \tag{23}$$

The study of the Jury's conditions shows that the stability outcomes of the steady-state equilibrium of the map coincide with those of the map restricted on the diagonal. This implies that the loss of stability of the equilibrium of map (18) when γ increases will occur through a two-period cycle along the diagonal (see Fig. 3).

As we have shown previously, the dynamic system is characterised by the existence of an invariant subset in the phase plane. It is now important to know whether the dynamics of the system are converging to such a set, that is, from an economic perspective, whether the economic dynamics can be summarised by studying the evolution of the decisions of a unique agent (or, analogously, decisions made by agents homogeneously at each decision-making time). A classic tool for analysing the emergence of such a structure in the dynamic system is to characterise the transverse attractiveness of the possible attractors located on Δ. Let us consider the following Jacobian matrix associated with map (18) evaluated at a generic point on the diagonal:

$$J(g,g) = \begin{pmatrix} l(g) & m(g) \\ m(g) & l(g) \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned} l(g) =& 1 + \gamma \left[\frac{\alpha(g-1+(g+2)\alpha)}{(1+2g)^{(1+\alpha)}(1-g)^{1-\alpha}} \right. \\ & \left. + \frac{((g+2)\alpha^2 - 3\alpha - g + 1)(1-g)^{\alpha-2} - (1-g)^{\alpha-1}}{(1+2g)^{\alpha}} \right], \end{aligned}$$

and

$$m(g) = \frac{\alpha\gamma}{(1-g)^{1-\alpha}} \left[\frac{g-1+(g+2)\alpha}{(1+2g)^{1+\alpha}} - \frac{1}{(1+2g)^{\alpha}} \right].$$

The related eigenvalues are the following:

$$\begin{aligned} \lambda_{||} = (l(g)+m(g)) = 1 + \frac{[g-1+(g+2)\alpha]2\alpha\gamma}{(1-g)^{1-\alpha}(1+2g)^{(1+\alpha)}} + \\ - \frac{\gamma}{(1+2g)^{\alpha}} \left(\frac{(1+\alpha)}{(1-g)^{1-\alpha}} + \frac{(1-\alpha)[g-1+(g+2)\alpha]}{(1-g)^{2-\alpha}} \right), \end{aligned} \tag{25}$$

with eigenvector (1, 1) and

$$\lambda_{\perp} = l(g) - m(g) = 1 - \frac{\gamma(1-\alpha)}{(1+2g)^{\alpha}} \left[\frac{g-1+(2+g)\alpha}{(1-g)^{2-\alpha}} + \frac{1}{(1-g)^{(1-\alpha))}} \right], \tag{26}$$

with eigenvector $(1,-1)$. The eigenvalue $\lambda_{||}$ is associated with the eigenvector parallel to the invariant manifold Δ and coincides with the multiplier of the restriction of the map on Δ. The eigenvector associated with the eigenvalue $\lambda_{\perp}$ is always orthogonal to Δ regardless of g. Now, from Eqs. (25) and (26), it is possible to classify the stability of cycles that occur on Δ. Indeed, for a z-cycle $\{(g_1,g_1),(g_2,g_2),...,(g_z,g_z)\}$ of map (18), corresponding to cycle $\{g_1, g_2, ..., g_z\}$ of map (19), the two multipliers are:

$$\lambda_{||}^{(z)} = \sum_{i=1}^{z} (l(g_i) + m(g_i)), \tag{27}$$

with eigenvector (1, 1), and

$$\lambda_{\perp}^{(z)} = \sum_{i=1}^{z} (l(g_i) - m(g_i)), \tag{28}$$

with eigenvector $(1, -1)$.

Regarding transverse stability, when a cycle of finite period exists on the diagonal Δ, results are straightforward as only the eigenvalue $\lambda_{\perp}^{(z)}$ needs to be evaluated. Unlike this, when the dynamics restricted to the invariant sub-manifold shows the emergence of a chaotic set A_s, some definitions and tools are necessary. We now recall the following classical definition of attractiveness:

Definition 2 *A_s is an asymptotically stable attractor (or topological attractor) if it is Lyapunov stable, i.e. for every neighbourhood I of A_s, a neighbourhood W* of *A_s* exists such that $T^n(W) \subset I$ for every $n \geq 0$ and the basin of attraction $B(A_s)$ contains a neighbourhood of A_s.

Following Definition 2, if A_s is an asymptotically stable attractor, then there exists a neighbourhood (with positive Lebesgue measure) Q surrounding A_s, which is included in the basin of attraction of A_s, such that $\lim_{n \to +\infty} T^n(x) = A_s$, for any $x \in Q$. The basin of attraction of A_s is defined as $B(A_s) = \bigcup_{n \geq 0} T^{-n}(Q)$. Then, when a topological attractor exists, the basin has a positive Lebesgue measure. As there can exist a set of initial conditions with positive Lebesgue measure converging to the attractor with no neighbourhood Q of A_s in the basin of attraction of A_s, we introduce the following definition of attractor in Milnor sense (see Maistrenko et al. 1998):

Definition 3 A closed invariant set A_s is said to be a weak attractor in Milnor sense (or Milnor attractor) if its basin of attraction $B(A_s)$ has positive Lebesgue measure.

Given Definitions 2 and 3, a topological attractor is also a Milnor attractor, but the opposite is not generally true.

In order to study the raising of transverse attractiveness in Milnor sense, we can use the transverse Lyapunov exponent, whose definition is the following (Maistrenko et al. 1998).

Definition 4 *Let $\{g_i = f^i(g_0),\ i \geq 0\}$ be a trajectory of* Eq. (19) *embedded in $A_s \subset \Delta$. Then,*

$$\Lambda_{\perp} = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \left| \ln(\lambda_{\perp}(g_i)) \right|, \tag{29}$$

is called a transverse Lyapunov exponent.

To characterise the "average" local behaviour of the trajectories in a neighbourhood of the invariant set $A_s \subset \Delta$, we can consider the *natural transverse*

Lyapunov exponent, $\Lambda_{\perp}^{\mathrm{nat}}$, that is, the transverse Lyapunov exponent calculated for a generic aperiodic trajectory in A_s.

If x_0 belongs to a z-cycle, then $\Lambda_{\perp} = \ln \left| \lambda_{\perp}^{(z)} \right|$ and when $\Lambda_{\perp} < 0$ the z-cycle is transversely stable. If A_s is a chaotic set and $\Lambda_{\perp} < 0$ for all trajectories starting inside A_s, then A_s is asymptotically stable. If we have that for some cycles starting inside A_s the absolute value of the related eigenvalue is larger than 1 but $\Lambda_{\perp}^{\mathrm{nat}} < 0$, then the majority of the trajectories on A_s are transversally attracting, implying that set A_s is a Milnor attractor, if some technical conditions hold.

Let us consider the case $\alpha = 0.0849$ (see Fig. 3, Panel A). From the study of Lyapunov exponents (Fig. 4), the dynamics lose their synchronisation shortly after the onset of the two-period cycle on the diagonal by means of a new flip bifurcation of the two-period cycle, but transversely to the diagonal itself, giving birth to a four-period cycle. This is shown in Fig. 5, Panel A, plotted for $\gamma = 0.616$. A further increase in γ ($\gamma = 0.64$) causes the birth of a Neimark–Sacker bifurcations giving rise to quasi-periodic cyclical dynamics around the four-period cycle, which has become unstable (Fig. 5, Panel B). If γ increases further ($\gamma = 0.68$), there is the birth of a strange attractor as shown in Fig. 5, Panel C. It is worth noting that for this value of γ the dynamics restricted on the diagonal converge to a four-period cycle (transversely unstable). From an economic point of view, this is an interesting scenario as it allows us to pinpoint the importance of the initial conditions. More specifically, the dynamics associated with identical initial conditions for both players (consumers) differs drastically from the dynamics unfolding when initial conditions differ between players. Considering even larger values of γ, we note from Fig. 4 that there is a small interval (just after $\gamma = 0.7$) in which, we note from Fig. 4 that there is a small interval in which the transverse (natural) Lyapunov exponent, $\Lambda_{\perp}$, becomes negative again with a non-transverse Lyapunov exponent, $\Lambda_{||}$, positive, indicating chaotic dynamics along the diagonal, as shown in Fig. 5, Panel D. In this case, the attractor A embedded on the diagonal defines an attractor in the weak Milnor sense for the two-dimensional dynamics. In this case, the on-off intermittency phenomenon can be observed: the behaviour of the two contributors initially almost synchronised loses this property in the interval $t \in (350, 428)$. This behaviour of the dynamics is due to the presence of transversely repelling cycles embedded in A (Fig. 6, Panel A). Small variations in the initial conditions or in γ can cause multiple bursts away from the diagonal characterised by smaller amplitudes than the one shown in the paper. Further details on the dynamic properties of the system can be obtained through the theory of the critical curves (Bischi et al. 2010).

Finally, Fig. 6, Panel B illustrates the dynamics of the supply of the public good ($g_1 + g_2$) for $\alpha = 0.68$. We note that the chaotic nature of the dynamic system is also transferred to the dynamics of this variable. To sum up, even in a context of homogeneity in preferences and income agents behave as if they were heterogeneous by making non-synchronised decisions. These results shown for a specific parametric set are robust when α and M vary.

3.1.2 The two-dimensional case: the heterogeneous economy

Turning to the heterogeneous case, the interior fixed point takes on the following expression

$$(g_1^*, g_2^*) = \left(\frac{M_1 - \alpha_1[M_1 + (1-\alpha_2)(1+M_2)]}{1-\alpha_1\alpha_2}, \frac{M_2 - \alpha_2[M_2 + (1-\alpha_1)(1+M_1)]}{1-\alpha_1\alpha_2} \right), \tag{30}$$

and the corresponding Jacobian matrix evaluated at (g_1^*, g_2^*) becomes

$$J(g_1^*, g_2^*) = \begin{pmatrix} 1 - \frac{\gamma(1-\alpha_1\alpha_2)}{(1+M_1+M_2)(1-\alpha_2)(1-\alpha_1)^{\alpha_1}\alpha_1^{1-\alpha_1}} & -\frac{\gamma(1-\alpha_1\alpha_2)\alpha_1^{\alpha_1}}{(1+M_1+M_2)(1-\alpha_2)(1-\alpha_1)^{\alpha_1}} \\ -\frac{\gamma(1-\alpha_1\alpha_2)\alpha_2^{\alpha_2}}{(1+M_1+M_2)(1-\alpha_1)(1-\alpha_2)^{\alpha_2}} & 1 - \frac{\gamma(1-\alpha_1\alpha_2)}{(1+M_1+M_2)(1-\alpha_1)(1-\alpha_2)^{\alpha_2}\alpha_2^{1-\alpha_2}} \end{pmatrix}, \tag{31}$$

so that the Jury's conditions read as

$$1 - \mathrm{Tr} + \mathrm{Det} = \frac{\gamma^2(1-\alpha_1\alpha_2)^3}{\alpha_1^{1-\alpha_1}\alpha_2^{1-\alpha_2}(1-\alpha_1)^{1+\alpha_1}(1-\alpha_2)^{1+\alpha_2}(1+M_1+M_2)^2} > 0, \tag{32}$$

$$\begin{aligned} 1 - \mathrm{Det} &= \gamma(1-\alpha_1\alpha_2) \\ &\times \frac{(1+M_1+M_2)\left[\alpha_1\alpha_2^{\alpha 2}(1-\alpha_2)(1-\alpha_1)^{\alpha_1} + \alpha_2\alpha_1^{\alpha_1}(1-\alpha_1)(1-\alpha_2)^{\alpha_2}\right] - \gamma\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}(1-\alpha_1\alpha_2)^2}{\alpha_1\alpha_2(1-\alpha_1)^{1+\alpha_1}(1-\alpha_2)^{1+\alpha_2}(1+M_1+M_2)^2}, \end{aligned} \tag{33}$$

and

$$1 + \mathrm{Tr} + \mathrm{Det} = 4 - \frac{2\gamma(1-\alpha_1\alpha_2)[\alpha_1(1-\alpha_2)^{1-\alpha_2}\alpha_2^{\alpha_2} + \alpha_2(1-\alpha_1)^{1-\alpha_1}\alpha_1^{\alpha_1}]}{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)(1+M_1+M_2)} > 0, \tag{34}$$

where Tr and Det are the trace and the determinant of (31).

As the equations involve different parameters, to make the study of the stability of the fixed point more effective and clear, it is convenient to divide the analysis first by focusing on the plane (M_1, M_2) and then move on to the plane (α_1, α_2). The main results are summarised in the two bullets below.

- Analysis in the plane (M_1, M_2). The analysis of the stability conditions shows that given two values of the preference parameters and not too high a level of the speed of adjustment γ compared to M_1 and M_2 an interior Nash equilibrium (g_1^*, g_2^*) is locally attractive if $M_1 + M_2$ is sufficiently high (Fig. 7, Panel A) and the two incomes are not too different. In the figure, the two curves X_1 and X_2 bound the parameter region (M_1, M_2) where the reaction curves meet at an interior points of the first orthant, and the unnamed line, related to the stability of the equilibrium, is defined by the Jury's conditions. A pleasant feature of the analysis in the plane (M_1, M_2) is that the bounds of the regions in

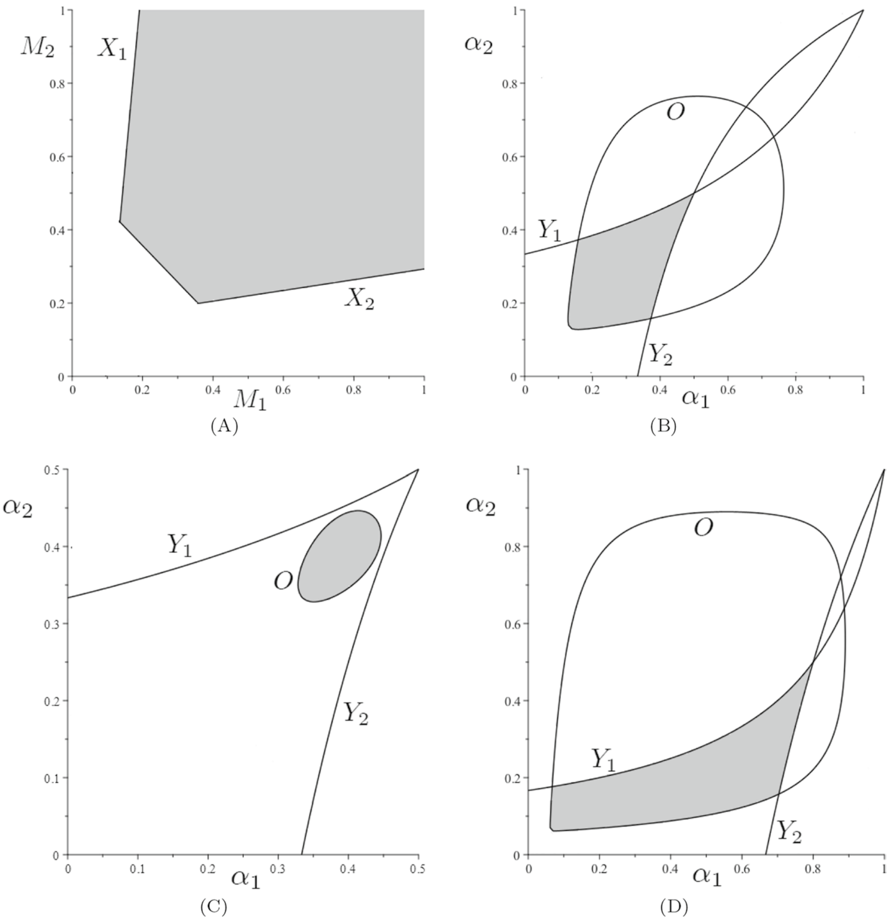

Fig. 7 Panel A. Stability of (g_1^*, g_2^*) in the space (M_1, M_2). Parameter values: $\alpha_1 = 0.1$, $\alpha_2 = 0.14$ and $\gamma = 0.2$. Panel B. Stability of (g_1^*, g_2^*) in the space (α_1, α_2). Parameter values: $M_1 = M_2 = 1$ and $\gamma = 0.8$. The two increasing curves guarantee the positivity of the Nash equilibrium. Grey: stability region. Panel C. Parameter values: $M_1 = 1$, $M_2 = 1$ and $\gamma = 1.418$. Panel D. Parameter values: $M_1 = 4$, $M_2 = 1$ and $\gamma = 0.8$ (colour figure online)

which the interior equilibrium (g_1^*, g_2^*) exists and it is stable can be expressed in explicit form (straight lines).

- Analysis in the plane (α_1, α_2). The stability region in the parameter space (α_1, α_2) appears more intricate due to the nonlinear relationship implied by (33), (34) with respect to these parameters (see Fig. 7, Panel B). The figure shows that given the values of M_1 and M_2 (also possibly equal) and not too high a level of γ, the possibility of an attractive interior equilibrium in which both players contribute to the financing of the public good in the presence of heterogeneous preferences only emerges (1) for intermediate values of α_1 and α_2, and (2) when these parameters are not too different. In the figure, the two curves Y_1 and Y_2 bound the parameter region (α_1, α_2) where the reaction curves meet at interior points of the first orthant, and curve O, related to the stability of the equilibrium, is defined by the Jury's conditions. On the one hand, by increasing the value of γ, for instance, $\gamma = 1.418$ (see Fig. 7, Panel C), due to the increased responsiveness of the agents in contributing to the public good, the region of existence and stability of the interior Nash equilibrium is completely defined by the Jury's conditions. On the other hand, for very low values of γ we have that the interior Nash equilibrium (when it exists) is always locally asymptotically stable (not shown in the figure). Starting from the configuration of Fig. 7, Panel B, we now study how the region of the stability parameters in the space (α_1, α_2) varies as one of the two incomes changes. In particular, as M_1 increases, the stability region undergoes a deformation, as shown in Fig. 7, Panel D. An increase in M_1 from 1 to 4, ceteris paribus, causes enlargement in all directions of the central ovoid region (O) and also a downward deformation of the Y_1 curve ($g_1 = 0$) and a rightward deformation of the Y_2 curve ($g_2 = 0$). This means that the increase in M_1 reduces the maximum levels of α_2 such that the Nash equilibrium is locally asymptotically stable. Conversely, the ranges of α_1 for which the Nash equilibrium is locally asymptotically stable are widened. From an economic point of view, an increase in M_1 makes contributor 1 richer and thus more prone to finance the public good for a greater range of values of α_1. However, an increase in player 1's income has the effect that player 2, who had no change in her income when faced with an increase in the richer player's contribution, might decide to stop contributing to it.

Starting from the case analysed in Fig. 5 (Panel C), the phase plane in Fig. 8 (Panel A-Panel D) shows how heterogeneity acts when the interior equilibrium is unstable and the dynamics are captured by a chaotic attractor. A decrease in α_1 leads to an enlargement of the attractor and a possible involvement of non-negativity constraints (border collision bifurcation) on contributions (Figs. 8, Panel D).

4 Discussion and conclusions

This article considers a public good game under non-fixed voluntary anonymous contributions and bounded rational agents (Bischi et al. 1998, 1999). As a first result, the model is able to generate oscillatory dynamics in consumer contributions. In this regard, the paper shows that the interior Nash equilibrium in which all agents contribute to the financing of a public good project is destabilised when the agents' reactivity to the marginal change in their utilities is high enough. This phenomenon is favoured by a relatively high weight of the public good in the utility function. The paper also pinpoints that *homogeneous* agents can behave as if they were *heterogeneous* by making non-synchronised decisions. The work also explicitly considers the case of a heterogeneous economy in terms of both consumer preferences and income showing how heterogeneity can slightly affect the results of the homogeneous case.

Recent contributions (Hillenbrand and Winter 2018; Kim 2018) study how uncertainty concerning the number of possible contributors to the voluntary financing of

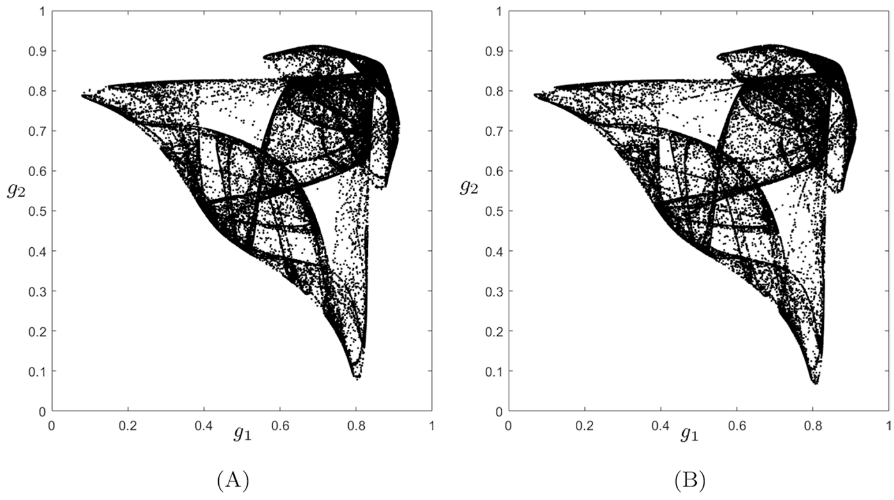

Fig. 8 Phase plane (g_1, g_2) plotted for $\gamma = 0.68$, $M_1 = M_2 = 1$ and $\alpha_2 = 0.0849$. Panel A: $\alpha_1 = 0.0842$. Panel B: $\alpha_1 = 0.084$. Panel C: $\alpha_1 = 0.0832$. Panel D: $\alpha_1 = 0.0828$

a public good is relevant to its production. However, the literature on this topic is, to the best of our knowledge, confined to one-shot games. A similarity between the approach of Hillenbrand and Winter (2018) and Kim (2018) and ours is that the information set of agents is limited compared to what could happen under perfect rationality. One aspect that certainly differentiates the two approaches is that the authors assume the emergence of Nash equilibrium without introducing an adjustment mechanism over time. In their environment, uncertainty concerning the number of participants provides an incentive for a higher single contribution than in the Nash equilibrium of the standard game. In our formulation, however, not knowing the behaviour of the other $N-1$ players may lead each player to continuously change her decisions over time. Thus, in some periods the contribution is higher and in others lower than in the Nash equilibrium solution of the standard game. We also note that the uncertainty on the number of contributors in many contexts, as pointed out by Kim (2018), may not be robust in a dynamic context as this information reveals over time. In this case, by not assuming the spontaneous emergence of the Nash equilibrium, the dynamics of the financing of a public good project should be similar to the dynamics studied in this model.

Being the first contribution in this direction, we have not considered the effects of the intervention of a regulator (in a model à la Lindahl) on the dynamics of the game. Indeed, forms of public intervention can be a further step in the approach used in the paper to the voluntary financing of public goods.

Furthermore, the present paper considered a public good financing game in which the single agent (player) does not know the behaviour of the other $N-1$ agents because the game is simultaneous. We could instead relax this assumption by considering a sequential competition model in which there is an agent that moves first (leader) and takes into account the reaction curves of the followers (see Güth et al. 2007 for an analysis of a voluntary contribution experiment in a Stackelberg setting). Consequently, the information available to the followers is different from that of the leader concerning the problem of voluntarily financing the production of a public good. This approach seems especially promising for understanding individual behaviour in the financing of non-pure public goods.

Finally, we want to stress some limitations of the paper and a possible future research agenda. A potentially promising line of research is to consider the role of conformism, which has been avoided, and the behaviour of an agent compared to a reference population, following for instance Landi and Sodini (2012) and Caravaggio and Sodini (2020), within a dynamic modelling framework like the one developed here. The paper mainly concentrates on the dynamic analysis of two players. An agent-based extension considering a large number of heterogeneous agents with different heuristics can be valuable. In addition, following Benhabib and Day (1981) endogenous values of the elasticity of the utility to the private good can be included in the analysis and/or the study of other decision-making processes, e.g. the mechanism developed by Brock and Hommes (1997) or evolutionary mechanisms such as the replicator dynamics (Clemens and Riechmann 2006; Lamantia et al. 2024) may be promising. Finally, a natural extension of this work is the study of a dynamic model with local public goods.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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