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Reviewed

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**Unstable modules with only the top $k$ Steenrod operations.** (English summary)

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## Classifications

**55S10 - Steenrod algebra**

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## Review

Let $A$ be the Steenrod algebra over the field $\mathbf{F}_2$ and let $\mathcal{U}$ be the category of unstable left $A$-modules. It is a basic problem in algebraic topology to compute the Ext groups between spheres in $\mathcal{U}$, as they coincide with the $E_2$ page of the unstable Adams spectral sequence. These groups are difficult to compute and the category $\mathcal{U}$ is not of finite homological dimension. The purpose of this paper is to investigate the category $\mathcal{U}_k$ of unstable modules at the prime 2 where only the top $k$ Steenrod operations are allowed. The graded vector spaces in $\mathcal{U}_k$ are no longer modules over the Steenrod algebra. Both $\mathcal{U}$ and $\mathcal{U}_k$ are considered as modules over a ringoid. The category $\mathcal{U}_k$ is easier to work with than $\mathcal{U}$ since it is of finite homological dimension, and the computation of Ext groups in $\mathcal{U}_k$ contributes to the computation of Ext groups in $\mathcal{U}$ because, for a module $N$ in $\mathcal{U}$ that is bounded above, the Ext groups into $N$ in those two categories agree with each other. The author defines functors between the two categories:

(1) the forgetful functor $u{:}\,\mathcal{U} \to \mathcal{U}_k$ (an object of $\mathcal{U}_k$ is not in general an object of $\mathcal{U}$, but an object of $\mathcal{U}$ is an object of $\mathcal{U}_k$ if we forget some Steenrod operations),

(2) the suspension functor $\Sigma{:}\,\mathcal{U}_k \to \mathcal{U}_{k+1}$,

(3) the Frobenius functor $\Phi{:}\,\mathcal{U}_k \to \mathcal{U}_{2k}$, and

(4) the loop functor $\Omega{:}\,\mathcal{U}_k \to \mathcal{U}_{k-1}$ and its first left derived functor $\Omega_1{:}\,\mathcal{U}_k \to \mathcal{U}_{k-1}$.

The author proves that the homological dimension of the category $\mathcal{U}_k$ is at most $k$. The goal is to show that $\mathrm{Ext}^s_k(M,N) = 0$ for all $s > k \geq 0$: first, the author proves it for $N$ a sphere module, by induction on $k$, then for $N$ bounded above, and finally for $N$ a general module. The main result is Theorem 6.1:

Theorem. Let $M$ and $N$ be any two nonzero modules in the category $\mathcal{U}$. Let $s$ be any nonnegative integer. If $N$ is bounded above, then the inverse system

$$\cdots \to \mathrm{Ext}^s_2(uM,uN) \to \mathrm{Ext}^s_1(uM,uN) \to \mathrm{Ext}^s_0(uM,uN)$$

stabilizes and the limit is equal to $\mathrm{Ext}^s_{\mathcal{U}}(M,N)$. If $N^n = 0$ for $n > k+1$, then the maps $\mathrm{Ext}^s_{k+1}(uM,uN) \to \mathrm{Ext}^s_k(uM,uN)$ and $\mathrm{Ext}^s_{\mathcal{U}}(M,N) \to \mathrm{Ext}^s_{\mathcal{U}_k}(uM,uN)$ are isomorphisms.

Further, a contravariant functor $\Lambda_k$ from $\mathcal{U}_k$ to the category of cochain complexes of graded vector spaces over $\mathbf{F}_2$ is introduced. It turns out that the cohomology $H^{s,a}(\Lambda_k(M))$ is naturally isomorphic to $\mathrm{Ext}^s_k(M,S_k(a))$ for all $s$ and $a$, where $S_k(a)$ is the sphere module in $\mathcal{U}_k$ with the degree $a$ part equal to $\mathbf{F}_2$ and the other parts equal to zero. The cochain complex $\Lambda_k(M)$ is relatively small and easy to compute; in particular, $\Lambda^s_k(M) = 0$ for all $s < 0$ and $s > k$, and each module in $\Lambda_k(M)$ is of finite dimension in each degree and bounded above if $M$ has these properties.

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*This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*

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