Examples and Counterexamples 5 (2024) 100138

Contents lists available at ScienceDirect

Examples and Counterexamples

journal homepage: www.elsevier.com/locate/exco

Infinite families of trees with equal spectral radius

Francesco Belardo, Maurizio Brunetti *

Department of Mathematics and Applications 'R. Caccioppoli', University of Naples 'Federico II', Naples, Italy

ARTICLE INFO

MSC:
05C50
05C05

Keywords:
Graph
Adjacency matrix
Spectral radius

ABSTRACT

In this note we show that for each positive integer $a \geqslant 2$ there exist infinitely many trees whose spectral radius is equal to $\sqrt{2a}$. Such trees are obtained by replacing the central edge of the double star $S(a, 2a-2)$ with suitable bidegreed caterpillars.

1. Introduction

Let $G = (V_G, E_G)$ be a simple graph with vertex set $V_G = \{v_1, \dots, v_n\}$ and edge set E_G, and let A_G denote the adjacency matrix of G. The *spectrum of* G is by definition the spectrum of A_G and is denoted by $\mathrm{sp}(G)$. The spectral radius $\rho(G)$ is the number $\max\{|\lambda| \mid \lambda \in \mathrm{sp}(G)\}$. As consequence of the Perron–Frobenius Theorem for nonnegative matrices (P–F Theorem for short), $\rho(G)$ is always equal to $\max \mathrm{sp}(A_G)$, i.e. the largest root of the characteristic polynomial $\varphi_G(x) = \det(xI_n - A_G)$. We denote by $\mathcal{C}$ and $\mathcal{T}$ the class of all connected graphs and of all trees respectively. We refer the reader to [1] for basic results on graph spectra and for notation not given here.

For each nonnegative real number t, we consider the sets of graphs

$$\Phi_{\mathcal{C}}(t) = \{G \in \mathcal{C} \mid \rho(G) = t\},$$

$$\Phi_{\mathcal{T}}(t) = \{G \in \mathcal{T} \mid \rho(G) = t\}$$

and

$$\Phi_{\mathcal{C}\setminus\mathcal{T}}(t) = \{G \in \mathcal{C} \setminus \mathcal{T} \mid \rho(G) = t\}.$$

We recall that an algebraic number a is said to be *totally real* if it is a root of a real-rooted monic polynomial with integer coefficients (see, for instance, [2]), whereas it is said an *almost Perron number* if it satisfies $a \geqslant |b|$ for each conjugate b of a (see [3]). We denote the set of algebraic numbers which are totally real (resp. almost Perron) by $\mathbb{TR}$ (resp. $\mathbb{AP}$). Establishing the cardinalities of $\Phi_{\mathcal{C}}(t)$ and $\Phi_{\mathcal{T}}(t)$ is a hard and fascinating problem in spectral graph theory.

The symmetry of A_G and the P-F Theorem yield the following implication:

$$\Phi_{\mathcal{C}}(t) \neq \varnothing \Longrightarrow t \in \mathbb{TR} \cap \mathbb{AP}, \tag{1}$$

and although Estes proved that for every $a \in \mathbb{TR}$, there exists a graph G such that $a \in \mathrm{sp}(G)$ [4], and Salez later showed that every $a \in \mathbb{TR}$ is also a tree eigenvalue [2], it is still dubious whether the implication (1) can be reversed.

In [5] the authors proved that whenever $\Phi_{\mathcal{C}\setminus\mathcal{T}}(t)$ is nonempty, then it contains infinitely many graphs, and posed the following question:

- *Besides $t = 2$, is there another $t \in \mathbb{R}$ such that* $\mathrm{card}(\Phi_{\mathcal{T}}(t)) = \infty$*?*

In this note we answer positively to this question, by showing that

$$\mathrm{card}(\Phi_{\mathcal{T}}(\sqrt{2b})) = \infty \qquad \text{for all integers } b \geqslant 2. \tag{2}$$

The infinite families of graphs we detect in $\Phi_{\mathcal{T}}(\sqrt{2b})$ specialize for $b = 2$ to the graphs known as *double snakes*, which are precisely the trees whose spectral radius is 2 (see [6]).

2. Main results

The tree $S(a, b, c; k)$ depicted in Fig. 1 is defined for every 4-tuple (a, b, c, d) of nonnegative integers. Clearly, the parameter c is significant only if $k > 0$.

Theorem 2.1. *Let b a positive integer larger than 1. For each $k \geqslant 0$, the trees $S(2b-2, b, b-2; k)$ all share the same spectral radius, which is $\sqrt{2b}$.*

Proof. The graph $T = S(2b-2, b, b-2; k)$ contains $n = 3b + k(b-1)$ vertices. We start by labelling the vertex set V_T as in Fig. 2.

Clearly, the uppermost vertices $v_{i,j}$ in Fig. 2 only exist if $k > 0$ and $b > 2$. We could write an adjacency matrix A_T associated to T once we order V_T, for instance, with respect to the lexicographic ordering, i.e.

$$u_i < u_j \quad \text{if } i < j; \quad u_i < v_{j,\ell} \quad \text{for all } i, j \text{ and } \ell$$

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* Corresponding author.

E-mail addresses: fbelardo@unina.it (F. Belardo), mbrunett@unina.it (M. Brunetti).

https://doi.org/10.1016/j.exco.2024.100138
Received 14 September 2023; Received in revised form 8 January 2024; Accepted 22 January 2024
Available online 1 February 2024
2666-657X/

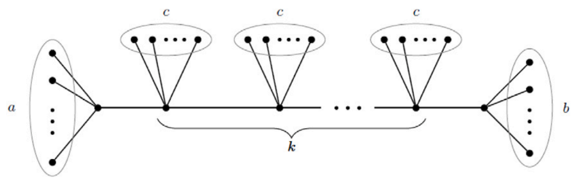

Fig. 1. The tree $S(a,b,c;k)$.

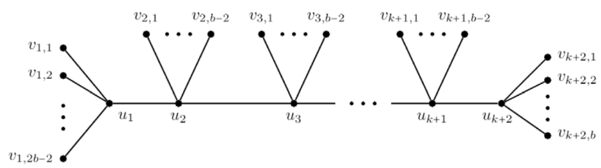

Fig. 2. A vertex labelling for the tree $S(2b-2,b,b-2;k)$.

and

$v_{j_1,\ell_1} < v_{j_2,\ell_2}$ if either $j_1 < j_2$ or $j_1 = j_2$ and $\ell_1 < \ell_2$.

We now denote by z_i (resp. $y_{j,\ell}$) the component of an n-tuple $\mathbf{x}^\top$ correspondent to the vertex u_i (resp. $v_{j,\ell}$), and set

$$\ell(i) = \begin{cases} 2b-2 & \text{if } i = 1, \\ b-2 & \text{if } 1 < i < k+2, \\ b & \text{if } i = k+2. \end{cases}$$

The number $\sqrt{2b}$ actually belongs to the spectrum of T; in fact, the eigenvalue equations

$$\begin{aligned} \lambda z_1 &= z_2 + \sum_{\ell=1}^{2b-2} y_{1,\ell}; \\ \lambda z_i &= z_{i-1} + z_{i+1} + \sum_{\ell=1}^{b-2} y_{1,\ell} \qquad \text{for } 1 < k+2; \\ \lambda z_{k+2} &= z_{k+1} + \sum_{\ell=1}^{b} y_{1,\ell}; \\ \lambda y_{i,\ell} &= z_i \qquad \text{for all } 1 \leqslant i \leqslant \ell(i); \end{aligned}$$

are all satisfied if $\lambda = \sqrt{2b}$,

$$z_i = b\left(\sqrt{\frac{2}{b}}\right)^i, \quad \text{and} \quad y_{i,\ell} = \left(\sqrt{\frac{2}{b}}\right)^{i-1} \tag{3}$$

for $1 \leqslant i \leqslant k+2$ and $1 \leqslant \ell \leqslant \ell(i)$. We have just shown that $\sqrt{2b}$ admits an all-positive eigenvector; namely, the one with components given in (3). By the P–F Theorem, the number $\sqrt{2b}$ is the spectral radius of $A(T)$ (the relevant part of the P-F Theorem is extracted in [7, Theorem 8.3.4]).

As already announced in Section 1, Theorem 2.1 specialized to the case $b = 2$ allows to retrieve the infinite family of graphs originally detected by Smith whose spectral radius is 2 (see [6]). The trees $S(2,2,0;k)$, which can be structurally identified as the bidegreed trees with just two vertices of degree 3, are precisely the double snakes.

The reader may be understandably curious about the way we discovered the families $\mathcal{T}_b := \{S(2b-2,b,b-2;k) \mid k \geqslant 0\}$ for $b \geqslant 2$. Since we were searching for families of trees all having the same spectral radius, and the only existing example we knew was the family of double snakes, we started to focus on double stars, i.e. the trees with diameter 3, and tried to replace their central edge with caterpillars, since this replacement turned out to be fruitful in order to find, for instance, several unicyclic graphs with the same adjacency or signless Laplacian spectral radius (see [8,9]).

The following result has been the intermediate step leading us to Theorem 2.1.

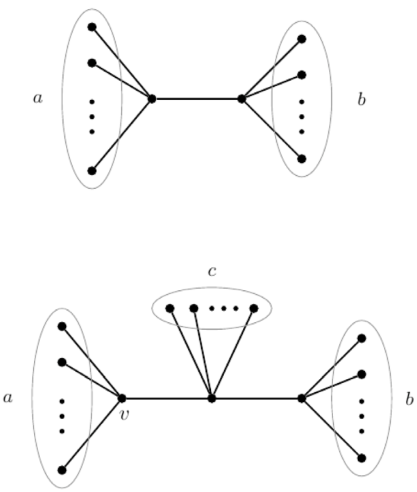

Fig. 3. The double star $S(a,b)$ and the tree $S(a,b,c;1)$.

Theorem 2.2. *For $a \geqslant b \geqslant 2$ and $c \geqslant 0$, let $S(a,b)$ and $S(a,b,c;1)$ be the graphs depicted in Fig. 3. Then $\rho(S(a,b)) = \rho(S(a,b,c;1))$ if and only if $a = 2b-2$ and $c = b-2$.*

Proof. It is not hard to prove that the double star $S(a,b)$ has just four nonzero eigenvalues and

$$\rho(S(a,b)) = \sqrt{\frac{s+\sqrt{s^2-4ab}}{2}} \tag{4}$$

where $s = a+b+1$ (see [10]). By working with the several eigenvalue equations, it is also straightforward to check the following fact: if λ is an eigenvalue of $S(a,b,c;1)$ admits an eigenvector with a nonzero component in correspondence of the vertex v of vertex degree $a+1$ (see Fig. 3), then

$$\lambda^6 - (a+b+c+2)\lambda^4 + (ab+(a+b)(c+1))\lambda^2 - abc = 0. \tag{5}$$

The P-F Theorem ensures that $\rho(S(a,b,c;1))$ is the largest root of (5).

Now, the ‘if’ part of the statement comes from Theorem 2.1; since $S(a,b,c;0) = S(a,b)$. In order to prove the ‘only if’ part, we assume $\rho(S(a,b)) = \rho(S(a,b,c;1))$. In this case, the number (4) is also a root of (5). This happens if and only if

$$(c+1)\left(\sqrt{t}+a+b+1\right) - 2ab = 0, \tag{6}$$

where $t = a^2 - 2a(b-1) + (b+1)^2$.

From the first equality of (6), we immediately deduce that the number t must be a perfect square, whereas the second equality is instead equivalent to

$$a = b-1+\sqrt{t-4b}. \tag{7}$$

An arithmetic argument shows that t and $t-4b$ are both perfect squares if and only if $t = (b+1)^2$. If this is the case, a is equal to $2b-2$ by (7), and $c = b-2$ by (6), ending the proof.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to thank the reviewer for his/her careful reading of the manuscript. The research that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM (Italy).

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