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Homogenization in perforated domains with mixed conditions

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Abstract. In this paper we study the asymptotic behaviour of the Laplace equation in a periodically perforated domain of $\mathbf{R}^n$, where we assume that the period is ε and the size of the holes is of the same order of greatness. An homogeneous Dirichlet condition is given on the whole exterior boundary of the domain and on a flat portion of diameter $\varepsilon^{\frac{n}{n-2}}$ if $n > 2$ ($\exp(-\varepsilon^{-2})$, if $n = 2$) of the boundary of every hole, while we take an homogeneous Neumann condition elsewhere.

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1 Introduction

To summarize our homogenization results and related problems we confine ourselves only to very simple geometries.

Let C_r be a cube of side r, B_r a ball of radius r, Ω a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial\Omega$ and ε be a parameter taking values in a decreasing sequence of positive numbers which tends to zero. For every ε, let Ω_ε be the subset of Ω obtained by removing from Ω closed balls or cubes well contained in Ω (the "holes") of size $r(\varepsilon) < \varepsilon$, periodically distribuited with period ε in $\mathbf{R}^n$.

Let us consider the following problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \text{boundary condition} & \text{on } \partial\Omega_\varepsilon \backslash \partial\Omega, \end{cases} \tag{1}$$

with f in $L^2(\Omega)$.

The asymptotic behaviour, as ε tends to zero, of solutions of (1) has been studied by many authors.

If $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$ (Dirichlet condition) in the case both of cubic and spherical holes the problem has been studied by D. Cioranescu and F. Murat in [CM1] and [CM2]. They proved that, if $r(\varepsilon) = \varepsilon^{\frac{n}{n-2}}$, if $n > 2$ ($\exp(-\varepsilon^{-2})$, if $n = 2$) (i.e. holes smaller and smaller in relation to ε), the solution u_ε of problem (1), extended to zero on the holes, converges weakly in $H_0^1(\Omega)$ to the solution u of the problem $-\Delta u + \mu_0 u = f$ in Ω and $u = 0$ on $\partial\Omega$, where the "strange term" μ_0 is the capacity of C_1 (or B_1) in $\mathbf{R}^n$ if $n \geq 3$ (2π if $n = 2$) (see also for general cases the large bibliography contained in [D]).

If $\frac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0$ on $\partial\Omega_\varepsilon \backslash \partial\Omega$ (homogeneous Neumann condition), where $\mathbf{n}$ denotes the exterior unit normal vector to $\partial\Omega_\varepsilon$, the asymptotic behaviour of problem (1) in the case of spherical holes has been studied by D. Cioranescu and J. Saint Jean Paulin in [CSJP1]. They proved that, if $r(\varepsilon)$ is $m\varepsilon$ ($m < 1$) (i.e. holes of size ε), there exists an extension $v_\varepsilon \in H_0^1(\Omega)$ of the solution u_ε of (1) converging weakly in $H_0^1(\Omega)$ to the solution of the problem $-div(\mathcal{A}\nabla u) = \vartheta f$ in Ω and $u = 0$ on $\partial\Omega$, where $\vartheta = \frac{|C_1 \backslash B_{m1}|}{|C_1|}$ and $\mathcal{A}$ is standard homogenized matrix (see (8) for the definition) (see also for more general cases [AM]).

In this paper, we examine the case where $r(\varepsilon) = m\varepsilon$ ($m < 1$) and the condition homogeneous Neumann condition is given on the boundary of cubic holes, but in a flat zone (always the same) of diameter $\varepsilon^{\frac{n}{n-2}}$, if $n \geq 3$ and $\exp(-\varepsilon^{-2})$ for $n = 2$ of every hole, the Neumann condition is replaced by Dirichlet condition (mixed conditions).

We prove that there exists an extension $v_\varepsilon \in H_0^1(\Omega)$ of the solution of (1) converging weakly to the solution of the problem

$$\begin{cases} -div(\mathcal{A}\nabla u) + \frac{1}{2}\mu_0 u = \vartheta f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\, \Omega \end{cases} \tag{2}$$

where μ_0 is exactly the "strange" term of [CM2], ϑ and $\mathcal{A}$ are the constant and the matrix of [CSJP1]. We emphasize the presence of factor $\frac{1}{2}$ before the term μ_0: the simultaneous presence of Dirichlet and Neumann condition on holes at critical size in the homogenization process gives a result where the single effect add but the Dirichlet effect appears halved.

We obtain the same results for $n > 3$ (in this case in a flat zone of $\varepsilon^{\frac{n}{n-2}}$ diameter of the holes is given the Dirichlet condition), but by smoothing the edges and vertices of the holes linked to the known regularity results for the solutions of Neumann problems.

Moreover we examine the case in which holes (obtained by rescaling a reference hole) of size $\varepsilon^{\frac{n}{n-2}}$ if $n \geq 3$ ($\exp(-\varepsilon^{-2})$ if $n = 2$) with Dirichlet conditions, are moving towards Neumann holes of size ε remaining at a distance $\tau\varepsilon^{\frac{n}{n-2}}$ if $n \geq 3$ ($\tau \exp(-\varepsilon^{-2})$ if $n = 2$). In this case the limit problem is

$$\begin{cases} -div(\mathcal{A}\nabla u^\tau) + \frac{1}{2}\mu_\tau u^\tau = \vartheta f & \text{in } \Omega, \\ u^\tau = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_τ is the capacity in $\mathbf{R}^n$ of a set obtained by doubling the Dirichlet reference hole by reflection with respect to a hyperplane at a distance τ.

We recall that in [CDG] the authors examine a related problem. Following [CoD], they consider the condition $\frac{\partial u_\varepsilon}{\partial \mathbf{n}} = g_\varepsilon$ on $\partial\Omega_\varepsilon \backslash \partial\Omega$ (non-homogeneous Neumann condition), where g_ε is obtained by rescaling and periodicizing a L^2-function g defined on ∂B_1 such that $\int_{\partial B_1} g dx \neq 0$ and $r(\varepsilon) = \varepsilon^{\frac{n}{n-1}}$, if $n > 1$; but they replace in a flat zone (always the same) of diameter $\varepsilon^{\frac{n}{n-2}}$, if $n \geq 3$ ($\exp(-\varepsilon^{-2})$ if $n = 2$) of every hole, Neumann condition by Dirichlet condition.

2 Position of the problem and the main result

Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, $n \geq 2$; let $Y = [-1/2, 1/2]^n \subset \mathbf{R}^n$.

Let $l > 0$ such that the cube $R = [0, 2l] \times [-l, l]^{n-1} \subset \overset{o}{Y}$.

For $n > 3$, let us consider a domain Q of $\mathbf{R}^n$ having C^∞ boundary obtained by smoothing the edges and vertices of R and such that

$$\frac{3}{4}R \subseteq Q \subseteq R. \tag{3}$$

For $n = 2$ and $n = 3$, we take $Q = R$.

Let us pose $Y^* = Y \backslash Q$.

Let K a compact subset of $\mathbf{R}^n$ containing the origin; moreover, if $n = 2$, let $K \cap \{x \in \mathbf{R}^n : x_1 = 0\}$ contain a segment.

Let $f \in L^2(\Omega)$.

Let $\varepsilon > 0$ and let $Y_\varepsilon = Y_\varepsilon(\Omega) = \cup\{\varepsilon(Y + \mathbf{k}) : \mathbf{k} \in \mathbf{Z}^n \text{and } \varepsilon(Y + \mathbf{k}) \subset \Omega\}$, $Q_\varepsilon = \varepsilon Q$.

Let us define $T_\varepsilon = T_\varepsilon(\Omega) = (\cup\{Q_\varepsilon + \varepsilon\mathbf{k} : \mathbf{k} \in \mathbf{Z}^n\}) \cap Y_\varepsilon$ and $\Omega_\varepsilon = \Omega \backslash T_\varepsilon$.

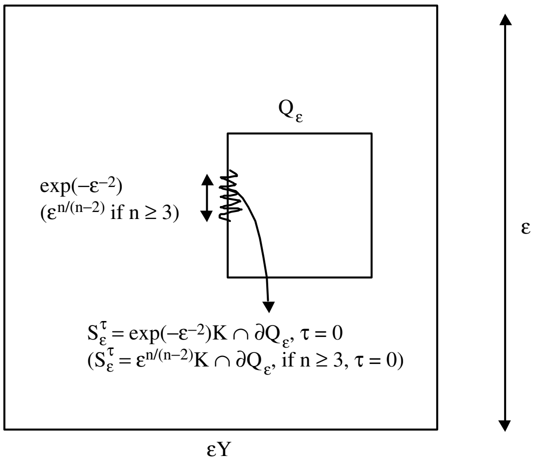
Let $S^\tau_\varepsilon = \begin{cases} \varepsilon^{\frac{n}{n-2}}(K + \tau), & n \geq 3 \\ \exp(-1/\varepsilon^2)(K + \tau), & n = 2 \end{cases}$, where $\boldsymbol{\tau} = (\tau, 0, \dots, 0) \in \mathbf{R}^n$ and $\tau \leq 0$; let us observe that there exists $\varepsilon_\tau > 0$ such that if $0 < \varepsilon < \varepsilon_\tau$ then S^τ_ε is well contained in εY.

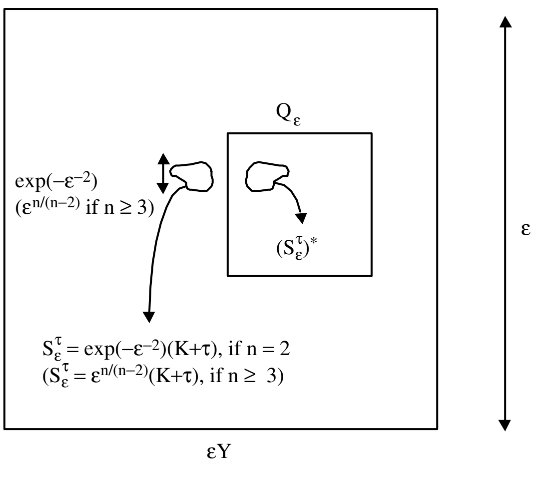
Let $D^\tau_\varepsilon = D^\tau_\varepsilon(\Omega) = (\cup\{S^\tau_\varepsilon + \varepsilon\mathbf{k} : \mathbf{k} \in \mathbf{Z}^n\}) \cap Y_\varepsilon$ if $0 < \varepsilon < \varepsilon_\tau$, $D^\tau_\varepsilon s = \emptyset$ otherwise.

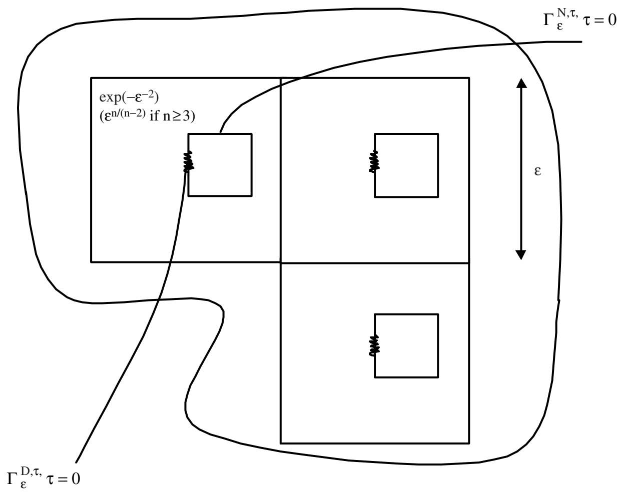
Let $\Omega^\tau_\varepsilon = \Omega \backslash (T_\varepsilon \cup D^\tau_\varepsilon)$ (see fig. 1).

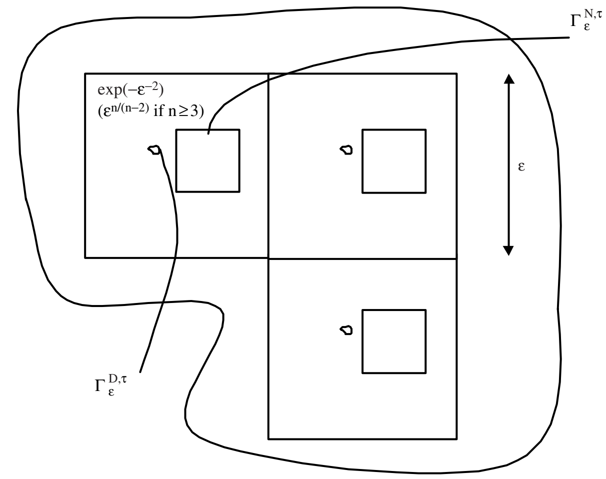
Let $\Gamma^{D,\tau}_\varepsilon = \partial D^\tau_\varepsilon \backslash \overset{o}{T}_\varepsilon$, $\Gamma^{N,\tau}_\varepsilon = \partial T_\varepsilon \backslash D^\tau_\varepsilon$ (see fig. 2).

Let $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_1 \geq 0\}$, $\mathbf{R}^n_- = \{x \in \mathbf{R}^n : x_1 \leq 0\}$.









Let us recall the definition of harmonic capacity (see the definition and remark in Section 4.7.1 of [EG]).

Let K be a compact subset of $\mathbf{R}^n$ and Ω an open set such that $K \subset \Omega$. We define the (harmonic) capacity of K with respect to Ω, and we will denote by *cap* (K, Ω) the following quantity

$$cap\ (K, \Omega) = \inf \left\{ \int_\Omega |\nabla \varphi|^2 dx : \varphi \in C_0^\infty\ (\Omega),\, \varphi \geq \chi_K \right\}, \tag{4}$$

where

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

We will moreover denote by *cap* (K) the quantity *cap* $(K, \mathbf{R}^n)$.

Let us consider, for every $\varepsilon > 0$, the following problem

$$\begin{cases} -\Delta u_\varepsilon^\tau = f & \text{in } \Omega_\varepsilon^\tau, \\ u_\varepsilon^\tau = 0 & \text{on } \partial\Omega \cup \Gamma_\varepsilon^{D,\tau}, \\ \partial u_\varepsilon^\tau / \partial \mathbf{n} = 0 & \text{on } \Gamma_\varepsilon^{N,\tau}. \end{cases} \tag{5}$$

The variational formulation for this problem is the following

$$\begin{cases} \int_{\Omega_\varepsilon^\tau} \langle \nabla u_\varepsilon^\tau, \nabla \varphi \rangle \, dx = \int_{\Omega_\varepsilon^\tau} f\varphi dx, \text{ for every } \varphi \in V_\varepsilon^\tau, \\ u_\varepsilon^\tau \in V_\varepsilon^\tau, \end{cases} \tag{6}$$

where

$$V_\varepsilon^\tau \text{ denote the closure of } C_0^1(\Omega \backslash \Gamma_\varepsilon^{D,\tau}) \text{ in } H^1(\Omega_\varepsilon^\tau).$$

We observe that, by the regularity of $\partial\Omega_\varepsilon$, it can be easily proved that V_ε^τ is also equal to the closure of $\{v \in H^1(\Omega_\varepsilon^\tau),\ v = 0 \text{ on } \partial\Omega \cup \Gamma_\varepsilon^{D,\tau}\}$ in $H^1(\Omega_\varepsilon^\tau)$.

Let us consider now the following ausiliary problem.

For any $\lambda \in \mathbf{R}^n$, let $w_\lambda \in H^1(Y^*)$ be the solution of the following problem

$$\begin{cases} -\Delta w_\lambda = 0 & \text{in } Y^*, \\ w_\lambda(y) - \lambda \cdot y & Y\text{-periodic}, \\ \partial w_\lambda / \partial \mathbf{n} = 0 & \text{on } \partial Q. \end{cases} \tag{7}$$

Since w_λ is linear in λ and the extension operator to zero is linear, we can consider the matrix $\mathcal{A}$ given by (see Theorem 4 of [CSJP])

$$\mathcal{A}\lambda = m_Y(\widetilde{\nabla w_\lambda}) = \frac{1}{|Y|} \int_{Y^*} \nabla w_\lambda dx, \qquad \forall \lambda \in \mathbf{R}^n. \tag{8}$$

where $\widetilde{\nabla w_\lambda}$ denotes the extension to zero of ∇w_λ on the whole Y.

Our main result is the following.

Theorem 1 *Let ε be a parameter taking values in a sequence going to zero and let $\{u_\varepsilon^\tau\}_\varepsilon$ be the sequence of solutions of problems* (6)*. Then*

i) *there exists a bounded sequence $\{v_\varepsilon^\tau\}_\varepsilon \subset H_0^1(\Omega)$ extending $\{u_\varepsilon^\tau\}_\varepsilon$ such that*

$$v_\varepsilon^\tau \to u^\tau \textit{ weakly in } H_0^1(\Omega)$$

where u^τ is the solution of the "homogenized" problem

$$\int_\Omega \langle \mathcal{A}\nabla u^\tau, \nabla\varphi \rangle dx + \frac{1}{2}\mu_\tau \int_\Omega u^\tau \varphi dx$$
$$= \int_\Omega \vartheta f \varphi dx,\ u^\tau \in H_0^1(\Omega), \qquad \textit{for every } \varphi \in \mathcal{D}(\Omega), \tag{9}$$

where $\mathcal{A}$ is the constant matrix given in (8)*, and*

$$\mu_\tau = \begin{cases} cap(((\tau + K) \cap \mathbf{R}_-^n) \cup (-(\tau + K) \cap \mathbf{R}_+^n)), & n \geq 3, \\ 2\pi & \textit{if } n = 2. \end{cases}$$

ii) *if* $\{w_\varepsilon^\tau\}_\varepsilon \subset H_0^1(\Omega)$ *is any bounded sequence extending* $\{u_\varepsilon^\tau\}_\varepsilon$*, then* $\{w_\varepsilon^\tau\}_\varepsilon$ *weakly converges in* $H_0^1(\Omega)$ *to the solution* u^τ *of problem* (9)*;*

iii) *we have the convergence of the energies of problems* (6) *to the one of problem* (9)*, i.e.*

$$\frac{1}{2}\int_{\Omega_\varepsilon^\tau} |\nabla u_\varepsilon^\tau|^2 dx - \int_{\Omega_\varepsilon^\tau} f u_\varepsilon^\tau dx \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}\int_\Omega \langle \mathcal{A}\nabla u^\tau, \nabla u^\tau\rangle dx$$
$$+\frac{1}{4}\mu_\tau \int_\Omega (u^\tau)^2 dx - \int_\Omega \vartheta f u^\tau dx.$$

If K is contained in the set $\{x \in \mathbf{R}^n : x_1 = 0\}$ and we take $\tau = 0$, we obtain the case when the zone Γ_ε^D, where homogeneous Dirichlet conditions is imposed, lies exactly on ∂T_ε.

If $K \cap Q = \{0\}$ and $\tau < 0$ we obtain the case when a hole with Dirichlet condition of size $\varepsilon^{\frac{n}{n-2}}$ if $n \geq 3$ ($\exp(-\varepsilon^{-2})$ if $n = 2$) is moving towards Neumann holes of size ε, remaining at distance $\tau\varepsilon^{\frac{n}{n-2}}$ if $n \geq 3$ ($\tau \exp(-\varepsilon^{-2})$ if $n = 2$). These are the cases explicitely described in the introduction.

3 Preliminary results

We now want to prove the existence of a sequence of extensions of solutions of the problem (6).

Let R a cube in $\mathbf{R}^n$, $C \subset\subset R$ be a compact set with Lipschitz boundary ∂C and $1 \leq p < +\infty$. By Theorem 1 of Section 4.4 in [EG], there exists a linear bounded extension operator $\Phi : W^{1,p}(R\backslash C) \longrightarrow W^{1,p}(R)$, i.e. there exists a constant c such that

$$\|\Phi v\|_{W^{1,p}(R)} \leq c\|v\|_{W^{1,p}(R\backslash C)}, \quad \text{for every } v \in W^{1,p}(R\backslash C). \tag{10}$$

Let $T' = Q\backslash(l \overset{o}{Y})$ and $T'_\varepsilon = \cup\{\varepsilon T' + \varepsilon\mathbf{k} : \mathbf{k} \in \mathbf{Z}^n \text{s.t. } \varepsilon\,(Y+\mathbf{k}) \subset \Omega\}$ and $\Omega'_\varepsilon = \Omega\backslash T'_\varepsilon$.

For every ε and $v \in W^{1,p}(\Omega_\varepsilon)$, we will denote by $R_\varepsilon v$ the extension by reflection of v on Ω'_ε defined by

$$R_\varepsilon v(x_1, x_2, \dots, x_n) = \begin{cases} v(x_1, x_2, \dots, x_n) & \text{on } \Omega_\varepsilon, \\ v(2\varepsilon k_1 - x_1, x_2, \dots, x_n) & \\ \text{for } a.e.\ x \in (\{\varepsilon(lY + \mathbf{k}) : \mathbf{k} \in \mathbf{Z}^n \text{s.t.} & \\ \quad \varepsilon(Y + \mathbf{k}) \subset \Omega\}\backslash\Omega_\varepsilon) \cap \Omega'_\varepsilon. & \end{cases} \tag{11}$$

We have that

$$\|R_\varepsilon v\|_{L^p(\Omega'_\varepsilon)} \leq c_1\|v\|_{L^p(\Omega_\varepsilon)}, \|\nabla(R_\varepsilon v)\|_{L^p(\Omega'_\varepsilon)}$$
$$\leq c_1\|\nabla v\|_{L^p(\Omega_\varepsilon)}, \text{ for every } p \in [1, +\infty]. \tag{12}$$

where the constant c_1 is independent on ε.

Proposition 1 *Let $\Omega \subset \mathbf{R}^n$ a bounded open set, $\varepsilon > 0$ and let Y, l, Q, Y_ε, Q_ε, T_ε, Ω_ε^τ be defined as in problem* (5).

Let $\{u_\varepsilon^\tau\}_\varepsilon \subset H^1(\Omega_\varepsilon^\tau)$.

Then there exists a sequence $\{v_\varepsilon^\tau\}_\varepsilon \subset H^1(\Omega)$ of extension of $\{u_\varepsilon^\tau\}_\varepsilon$ such that

$$v_\varepsilon^\tau(x_1, x_2, \dots, x_n) = v_\varepsilon^\tau(2\varepsilon k_1 - x_1, x_2, \dots, x_n)$$
$$\textit{for a.e. } x \in (\{\varepsilon(lY + \mathbf{k}) : \mathbf{k} \in \mathbf{Z}^n \text{ s.t. } \varepsilon(Y + \mathbf{k}) \subset \Omega\} \backslash \Omega_\varepsilon) \cap \Omega'_\varepsilon, \quad (13)$$

$$\|\nabla v_\varepsilon^\tau\|_{L^2(\Omega)} \leq c_2 \|\nabla u_\varepsilon^\tau\|_{L^2(\Omega_\varepsilon^\tau)} \quad (14)$$

where the constant c_2 is independent on ε and on the particular sequence $\{u_\varepsilon^\tau\}_\varepsilon$.

Proof. Firstly we give a small variation of (10), i.e., if $C = Q \backslash lY$, we prove that there exists a linear bounded extension operator $\Psi : W^{1,p}(Y \backslash C) \to W^{1,p}(Y)$, such that

$$\|\nabla(\Psi v)\|_{L^p(Y)} \leq \overline{c}\|\nabla v\|_{L^p(Y \backslash C)}, \text{for every } v \in W^{1,p}(Y \backslash C), \text{with } 1 \leq p < +\infty. \quad (15)$$

Let $u \in W^{1,p}(Y \backslash C)$, $\overline{u}$ the average of u in $Y \backslash C$ and let us define $\Psi(u) = \overline{u} + \Phi(u - \overline{u})$, where Φ is given in (10); we obviously have $\Psi(u) \in W^{1,p}(Y)$; therefore Ψ is a linear bounded extension operator from $W^{1,p}(Y \backslash C)$ to $W^{1,p}(Y)$. We eventually have

$$\begin{aligned} \|\nabla \Psi(u)\|_{L^p(Y)} &= \|\nabla \Phi(u - \overline{u})\|_{L^p(Y)} \leq \|\Phi(u - \overline{u})\|_{W^{1,p}(Y)} \\ &\leq c\|u - \overline{u}\|_{W^{1,p}(Y \backslash C)} \leq c\|u - \overline{u}\|_{L^p(Y \backslash C)} \\ &\quad + c\|\nabla u\|_{L^p(Y \backslash C)} \\ &\leq c(1 + c')\|\nabla u\|_{L^p(Y \backslash C)} = \overline{c}\|\nabla u\|_{L^p(Y \backslash C)} \end{aligned} \quad (16)$$

where c is given in (10) and c' is the Poincaré-Wirtinger constant of $Y \backslash C$.

Now we prove that there exists a family $\{P_\varepsilon\}_\varepsilon$ of extension operators from $H^1(\Omega_\varepsilon)$ to $H^1(\Omega)$ such that

$$\|\nabla(P_\varepsilon u)\|_{L^2(\Omega)} \leq \sqrt{1 + 2\overline{c}^2}\|\nabla u\|_{L^2(\Omega_\varepsilon)} \quad \text{for every } u \in H^1(\Omega_\varepsilon) \quad (17)$$

where $\overline{c}$ is given in (15), and

$$\begin{aligned} &P_\varepsilon u(\varepsilon(x_1, x_2, \dots, x_n) + \varepsilon\mathbf{k}) = P_\varepsilon u(\varepsilon(-x_1, x_2, \dots, x_n) + \varepsilon\mathbf{k}), \\ &\quad \text{for every } u \in H^1(\Omega_\varepsilon), \text{ for every } (x_1, x_2, \dots, x_n) \in lY \\ &\quad\quad \text{and } k \text{ s.t. } \varepsilon Y + \varepsilon\mathbf{k} \subset \Omega. \end{aligned} \quad (18)$$

In fact, let $u \in H^1(\Omega_\varepsilon)$ and let us consider $\mathbf{k}$ such that the cell $\varepsilon Y + \varepsilon\mathbf{k} \subset \Omega$; we have that $u \in H^1(\varepsilon Y \backslash \varepsilon Q + \varepsilon\mathbf{k})$. Now let us consider $R_\varepsilon u$ given by (11). Let us pose

$$u_{\varepsilon,\mathbf{k}}(y) = R_\varepsilon u(\varepsilon y + \varepsilon\mathbf{k}), \quad \text{for every } y \in Y \backslash C.$$

By (15), the function $v_{\varepsilon,\mathbf{k}}(\varepsilon y+\varepsilon\mathbf{k}) = \Psi(u_{\varepsilon,\mathbf{k}})(y)$ extends $u_{\varepsilon,\mathbf{k}}$ on Y. Let us denote

$$P_\varepsilon u(x) = \begin{cases} v_{\varepsilon,\mathbf{k}}(x) & \text{in } \varepsilon Y + \varepsilon\mathbf{k} \\ u(x) & \text{in } \Omega\backslash Y_\varepsilon \end{cases}. \tag{19}$$

It is straightforward to verify that $P_\varepsilon u \in H^1(\Omega)$ and (18) follows by (11) and (19).

Moreover we have

$$\begin{aligned} \|\nabla P_\varepsilon u\|^2_{L^2(\Omega)} &= \|\nabla u\|^2_{L^2(\Omega\backslash Y_\varepsilon)} + \sum_{\mathbf{k}:\varepsilon Y+\varepsilon\mathbf{k}\subset\Omega} \|\nabla v_{\varepsilon,\mathbf{k}}\|^2_{L^2(\varepsilon Y+\varepsilon\mathbf{k})} \\ &= \|\nabla u\|^2_{L^2(\Omega\backslash Y_\varepsilon)} + \frac{\varepsilon^n}{\varepsilon^2} \sum_{\mathbf{k}:\varepsilon Y+\varepsilon\mathbf{k}\subset\Omega} \|\nabla \Psi(u_{\varepsilon,\mathbf{k}})\|^2_{L^2(Y)} \\ &\le \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + \overline{c}^2\frac{\varepsilon^n}{\varepsilon^2} \sum_{\mathbf{k}:\varepsilon Y+\varepsilon\mathbf{k}\subset\Omega} \|\nabla u_{\varepsilon,\mathbf{k}}\|^2_{L^2(Y\backslash C)} \\ &= \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + \overline{c}^2 \sum_{\mathbf{k}:\varepsilon Y+\varepsilon\mathbf{k}\subset\Omega} \|\nabla (R_\varepsilon u)\|^2_{L^2(\varepsilon(Y\backslash C)+\varepsilon\mathbf{k})} \\ &= \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + 2\overline{c}^2\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} = (1+2\overline{c}^2)\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \end{aligned} \tag{20}$$

where $\overline{c}$ is the constant given in (15), and so we obtain (17).

Now we prove the existence of the sequence $\{v^\tau_\varepsilon\}_\varepsilon \subset H^1(\Omega)$.

Let $u^\tau_\varepsilon \in H^1(\Omega^\tau_\varepsilon)$ and let us define

$$\widetilde{u}^\tau_\varepsilon = \begin{cases} u^\tau_\varepsilon & \text{in } \Omega^\tau_\varepsilon \\ 0 & \text{in } \Omega_\varepsilon\backslash\Omega^\tau_\varepsilon. \end{cases} \tag{21}$$

We observe that $\widetilde{u}^\tau_\varepsilon \in H^1_0(\Omega_\varepsilon)$.

Then the sequence $\{v^\tau_\varepsilon\}_\varepsilon$ given by $v^\tau_\varepsilon = P_\varepsilon\widetilde{u}^\tau_\varepsilon$ meets our requirements by (17) and (18). □

Let us recall now some properties of capacity (see Theorem 2 of Section 4.7.1 of [EG]).

Lemma 1 *Let Ω be an open subset of $\mathbf{R}^n$ and K be a compact subset of Ω. Then*

i) *if $\{\Omega_h\}_h$ is an increasing sequence of open sets such that $\cup_{h\in\mathbf{N}}\Omega_h = \Omega$ then $\lim_h cap(K,\Omega_h) = cap(K,\Omega)$*

ii) *if $t>0$ then $cap(tK,t\Omega) = t^{n-2}cap(K,\Omega)$.*

Let $S'^\tau_\varepsilon = (S^\tau_\varepsilon \cap \mathbf{R}^n_-) \cup (-S^\tau_\varepsilon \cap \mathbf{R}^n_+)$ and $D'^\tau_\varepsilon = (\cup\{S'^\tau_\varepsilon + \varepsilon\mathbf{k} : \mathbf{k}\in\mathbf{Z}^n\}) \cap Y_\varepsilon$ if $0<\varepsilon<\varepsilon_\tau$, $D'^\tau_\varepsilon = \emptyset$ otherwise.

Let $B \subset\subset Y$ an open ball centered at the origin, $1<\nu<\frac{n}{n-2}$, if $n\ge 3$ ($1<\nu<+\infty$, if $n=2$).

Let $A_\varepsilon = (\cup\{\varepsilon^\nu B + \varepsilon\mathbf{k} : \mathbf{k} \in\mathbf{Z}^n\})\cap Y_\varepsilon$ if $0 < \varepsilon < \varepsilon_1$, $A_\varepsilon = \emptyset$ otherwise (let us observe that there exists $\varepsilon_1 > 0$ such that if $0 < \varepsilon < \varepsilon_1$ than $S'^\tau_\varepsilon \subset\subset \varepsilon^\nu B \subset\subset \varepsilon Y$).

Let $C_0^\infty(\Omega, D'^\tau_\varepsilon, 1)$ be the set of functions $v \in C_0^\infty(\Omega)$ such that $v = 1$ in a neighborhood of D'^τ_ε.

Let $H_0^1(\Omega, D'^\tau_\varepsilon, 1)$ be the closure of $C_0^\infty(\Omega, D'^\tau_\varepsilon, 1)$ in $H_0^1(\Omega)$. Let $\widehat{\psi}^\tau_\varepsilon$ be the unique solution of the problem

$$\min\left\{\int_\Omega |\nabla\varphi|^2\, dx : \varphi \in H_0^1(\Omega, D'^\tau_\varepsilon, 1),\ \varphi = 0 \text{ on } \Omega\backslash A_\varepsilon\right\}. \tag{22}$$

Let us pose $\psi^\tau_\varepsilon = 1 - \widehat{\psi}^\tau_\varepsilon$. We can consider ψ^τ_ε as a function of $H_0^1(\Omega)$. Let us observe that

$$\begin{aligned} \psi^\tau_\varepsilon &= 0 \quad \text{on } D'^\tau_\varepsilon \\ \psi^\tau_\varepsilon &= 1 \quad \text{in } \Omega\backslash A_\varepsilon. \end{aligned} \tag{23}$$

Let us now complete some results of [CM2] and [CDG] in the following lemma.

Lemma 2 *Let ψ^τ_ε the unique solution of* (22). *Then there exists a unique distribution $\mu_\tau \in W^{-1,\infty}$, μ_ε and $\gamma_\varepsilon \in H^{-1}(\Omega)$ such that, if ε takes its values in a sequence going to zero,*

$$\begin{aligned} -\Delta\psi^\tau_\varepsilon &= \mu^\tau_\varepsilon - \gamma^\tau_\varepsilon, \\ \mu^\tau_\varepsilon &\to \mu_\tau \quad \text{strongly in } H^{-1}(\Omega) \\ \langle\gamma^\tau_\varepsilon, v_\varepsilon\rangle &= 0, \quad \forall v_\varepsilon \in H_0^1(\Omega\backslash D'^\tau_\varepsilon) \end{aligned} \tag{24}$$

where

$$\mu_\tau = \begin{cases} capD^\tau, & n \geq 3 \\ 2\pi, & n = 2 \end{cases}, \text{ with } D^\tau = ((\tau + K) \cap \mathbf{R}^n_-) \cup (-(\tau + K) \cap \mathbf{R}^n_+). \tag{25}$$

We have also

$$0 \leq \psi^\tau_\varepsilon \leq 1 \tag{26}$$

$$\psi^\tau_\varepsilon \to 1 \textit{weakly in } H^1(\Omega), \tag{27}$$

$$\psi^\tau_\varepsilon \to 1 \textit{ strongly in } L^p(\Omega),\ \forall p \in [1, +\infty[, \tag{28}$$

Moreover, if $1 \leq p < 2$

$$\begin{aligned} \nabla\psi^\tau_\varepsilon \to 0 \textit{ strongly in } L^p(\Omega),\ \textit{for} \quad &\nu \text{ s.t. } 1 < \nu < \tfrac{n}{n-2}, \quad \textit{if } n \geq 3, \\ \textit{for} \quad &\nu \geq \tfrac{2}{2-p}, \qquad\qquad \textit{if } n = 2. \end{aligned} \tag{29}$$

Eventually if we restrict ψ^τ_ε to Ω_ε, we have

$$\begin{aligned} &\psi^\tau_\varepsilon(x_1, x_2, \ldots, x_n) = \psi^\tau_\varepsilon(2\varepsilon k_1 - x_1, x_2, \ldots, x_n) \\ &\quad \textit{for a.e. } x \in (\{\varepsilon(lY + \mathbf{k}) : \mathbf{k} \in \mathbf{Z}^n \text{s.t. } \varepsilon(Y + \mathbf{k}) \subset \Omega\}\backslash\Omega_\varepsilon) \cap \Omega'_\varepsilon. \end{aligned} \tag{30}$$

Proof. By Lemma 3.5 of [CDG] we obtain (24)÷(28).

We first prove (29) in the case $n \geq 3$.

We observe that, by (4), we have

$$\int_{\varepsilon Y+\varepsilon \mathbf{k}} |\nabla \psi_\varepsilon^\tau|^2 dx = cap\ (\varepsilon^{\frac{n}{n-2}} D^\tau,\ \varepsilon^\nu B)$$

and by ii) of Lemma 3 we have

$$cap\ (\varepsilon^{\frac{n}{n-2}} D^\tau, \varepsilon^\nu B) = \varepsilon^n cap\ (D^\tau, \varepsilon^{\nu-\frac{n}{n-2}} B).$$

By Hoelder inequality applied for $\frac{1}{p} = \frac{1}{2} + \frac{1}{s}$, we have

$$\begin{aligned}\int_{\varepsilon Y+\varepsilon \mathbf{k}} |\nabla \psi_\varepsilon^\tau|^p\ dx\ &=\ \int_{\varepsilon^\lambda B} |\nabla \psi_\varepsilon^\tau|^p dx \leq \left(\int_{\varepsilon^\lambda B} |\nabla \psi_\varepsilon^\tau|^2\ dx\right)^{\frac{p}{2}} |\varepsilon^\nu B|^{\frac{p}{s}} \\ &\leq\ \varepsilon^{\frac{np}{2}} cap\ (D^\tau, \varepsilon^{\nu-\frac{n}{n-2}} B)^{\frac{p}{2}}\ \varepsilon^{\frac{\nu np}{s}} |B|^{\frac{p}{s}}.\end{aligned}$$

Then, since the number of cubes Y_ε covering Ω is about $|\Omega|\varepsilon^{-n}$, we have

$$\begin{aligned}\|\nabla \psi_\varepsilon^\tau\|^p_{L^p(\Omega)}\ &\leq\ C\frac{|\Omega|}{\varepsilon^n} \varepsilon^{\frac{np}{2}} cap\ (D^\tau, \varepsilon^{\nu-\frac{n}{n-2}} B)^{\frac{p}{2}} \varepsilon^{\frac{\nu np}{s}} |B|^{\frac{p}{s}} \\ &=\ C|\Omega||B|^{\frac{p}{s}} cap\ (D^\tau, \varepsilon^{\nu-\frac{n}{n-2}} B)^{\frac{p}{2}} \varepsilon^{\frac{n(sp+2\nu p-2s)}{2s}}.\end{aligned}$$

By i) of Lemma 3 $cap(D^\tau, \varepsilon^{\nu-\frac{n}{n-2}} B)$ converges to $cap(D^\tau)$ and since $sp + 2\nu p - 2s > 0$ if $p < 2$, the last term tends to zero for ε that tends to zero.

The case $n = 2$ is similar. We observe that by (4), we have

$$\int_{\varepsilon Y+\varepsilon \mathbf{k}} |\nabla \psi_\varepsilon^\tau|^2 dx = cap(\exp(-\varepsilon^{-2}) D^\tau, \varepsilon^\nu B)$$

and by ii) of Lemma 3, we have

$$cap(\exp(-\varepsilon^{-2}) D^\tau, \varepsilon^\nu B) = cap(D^\tau, \varepsilon^\nu \exp(\varepsilon^{-2}) B)$$

Then

$$\begin{aligned}\|\nabla \psi_\varepsilon^\tau\|^p_{L^p(\Omega)}\ &\leq\ C\frac{|\Omega|}{\varepsilon^2} cap(D^\tau, \varepsilon^\nu \exp(\varepsilon^{-2}) B)^{\frac{p}{2}} \varepsilon^{\frac{2\nu p}{s}} |B|^{\frac{p}{s}} \\ &=\ |\Omega||B|^{\frac{p}{s}} cap(D^\tau, \varepsilon^\nu \exp(\varepsilon^{-2}) B)^{\frac{p}{2}} \varepsilon^{\frac{2(\nu p-s)}{s}}.\end{aligned}$$

By i) of Lemma 3 $cap(D^\tau, \varepsilon^\nu \exp(\varepsilon^{-2}) B)$ converges to $cap(D^\tau)$. If $\nu p - s > 0$ and $p < 2$, the last term tends to zero for ε that tends to zero.

Now let us prove (30).

Let us pose

$$g_\varepsilon^\tau(x_1, \ldots, x_n) = \psi_\varepsilon^\tau(2\varepsilon k_1 - x_1, \ldots, x_n). \tag{31}$$

Since $1 - \psi_\varepsilon^\tau$ satisfies the problem (22), then by the uniqueness of the solution of this problem and by (31), we have that $g_\varepsilon^\tau(x) = \psi_\varepsilon^\tau(x)$ and so the thesis. □

Let C be a compact set such that $C \subset\subset Y \subset \mathbf{R}^n$, $\nu \geq 1$ and let

$$\Omega''_\varepsilon = \Omega \backslash (\cup \{\varepsilon^\nu C + \varepsilon \mathbf{k} : \mathbf{k} \in \mathbf{Z}^n\} \cap Y_\varepsilon). \tag{32}$$

Lemma 3 *Let* $\Omega \subset \mathbf{R}^n$ *a bounded open set,* $\varepsilon > 0$ *and let* Ω''_ε *defined as in* (32).

Let $\{u_\varepsilon\}_\varepsilon$ *and* $\{v_\varepsilon\}_\varepsilon$ *two bounded sequences in* $H^1_0(\Omega)$ *such that* $u_\varepsilon = v_\varepsilon$ *a.e. in* Ω''_ε.

Then

$$u_\varepsilon - v_\varepsilon \longrightarrow 0 \qquad \textit{weakly in } H^1_0(\Omega) \textit{ as } \varepsilon \to 0^+.$$

Proof. See [AtM] and Lemma 3.12 of [CDG]. □

4 Proof of Theorem 2

Let us consider now the problem (7). By Theorem 5.2 of [G3] for $n = 2$ and $n = 3$, $w_\lambda \in H^{1+\alpha}$ for $\alpha \in]\frac{1}{2}, 1]$ and so $\nabla w_\lambda \in H^\alpha$ (see Chapter 7 of [A]). By Rellich-Kondrachov theorem (see for example Theorem 7.57 of [A]) we have that

$$w_\lambda \in L^\infty(Y^*) \cap W^{1,r}(Y^*) \tag{33}$$

for r such that, with $\alpha \in]\frac{1}{2}, 1[$,

$$\begin{aligned} &2 < r \leq \tfrac{4}{2-2\alpha}, \text{ if } n = 2, \\ &3 < r \leq \tfrac{6}{3-2\alpha}, \text{ if } n = 3. \end{aligned} \tag{34}$$

Let us observe that for $\alpha \in]\frac{1}{2}, 1[$, we have that $\frac{4}{2-2\alpha} > 4$ and $\frac{6}{3-2\alpha} > 3$.

Let us consider the function $\overline{w}_\lambda$ defined by

$$\overline{w}_\lambda(x_1, x_2, \ldots, x_n) = \begin{cases} w_\lambda(x_1, x_2, \ldots, x_n) & \text{for } x \in Y^*, \\ w_\lambda(-x_1, x_2, \ldots, x_n) & \text{for a.e. } x \in lY \cap Q. \end{cases} \tag{35}$$

So

$$\overline{w}_\lambda \in L^\infty(Y^* \cup lY) \cap W^{1,r}(Y^* \cup lY), \text{ for rasin (34)}. \tag{36}$$

If we consider the extension operator $\Phi : W^{1,p}(Y \backslash (Q \backslash lY)) \to W^{1,p}(Y)$ given in (10) (where we take $R = Y$ and $C = (Q \backslash lY)$), by (36) we have that

$$\Phi\overline{w}_\lambda \in L^\infty(Y) \cap W^{1,r}(Y), \text{for } r \text{ as in (34)}. \tag{37}$$

If $n > 3$, by regularity of boundary of Q, by Proposition 7.7 of [Ta] $w_\lambda \in H^m(Y^*)$ for every m, and so by Rellich-Kondrachov theorem, if we take $m > \frac{n}{2}$, it results

$$w_\lambda \in W^{1,\infty}(Y^*). \tag{38}$$

So $\overline{w}_\lambda \in W^{1,\infty}(Y^* \cup lY)$. Then

$$\Phi\overline{w}_\lambda \in L^\infty(Y) \cap W^{1,r}(Y), \text{for every } r \text{ s.t. } 1 \leq r < +\infty. \tag{39}$$

Let us pose

$$w_{\varepsilon,\lambda}(x) = \varepsilon(\Phi\overline{w}_\lambda)\left(\frac{x}{\varepsilon}\right) \text{ in } \varepsilon Y, \tag{40}$$

We observe that obviously (by construction)

$$\begin{aligned} &w_{\varepsilon,\lambda}(x_1, x_2, \dots, x_n) = w_{\varepsilon,\lambda}(2\varepsilon k_1 - x_1, x_2, \dots, x_n) \\ &\quad \text{for } a.e.\ x \in (\{\varepsilon(lY + \mathbf{k}) : \mathbf{k} \in \mathbf{Z}^n \text{ s.t. } \varepsilon(Y + \mathbf{k}) \subset \Omega\} \backslash \Omega_\varepsilon) \cap \Omega'_\varepsilon. \end{aligned} \tag{41}$$

We observe that, by (37) and (39)

$$w_{\varepsilon,\lambda} \in L^\infty(\Omega) \cap W^{1,r}(\Omega), \tag{42}$$

for r such that, with $\alpha \in]\frac{1}{2}, 1[$,

$$\begin{aligned} &2 < r \leq \tfrac{4}{2-2\alpha}, \quad \text{if } n = 2, \\ &3 < r \leq \tfrac{6}{3-2\alpha}, \quad \text{if } n = 3, \\ &1 \leq r < +\infty, \ \text{ if } \ n > 3. \end{aligned} \tag{43}$$

We observe also that $\Phi\overline{w}_\lambda(y) - \lambda y$ is Y-periodic. So there exists an Y-periodic function β such that

$$(\Phi\overline{w}_\lambda)(x) = \lambda x + \beta(x). \tag{44}$$

and so

$$w_{\varepsilon,\lambda}(x) = \lambda x + \varepsilon\beta\left(\frac{x}{\varepsilon}\right).$$

Since $\Phi\overline{w}_\lambda$ belongs to $L^\infty(Y)$, by (37) and (39), β is also in $L^\infty(Y)$ and so

$$\|w_{\varepsilon,\lambda}\|_{L^\infty(\Omega)} \leq c_3, \tag{45}$$

with c_3 a constant independent on ε and

$$w_{\varepsilon,\lambda}(x) \to \lambda x \text{ strongly in } L^\infty(\Omega). \tag{46}$$

In the following we use for λx the symbol w^*_λ. By (37) and (39), we have

$$\beta \in L^\infty(Y) \cap W^{1,r}(Y), \text{ for } r \text{ as in (43)} \tag{47}$$

and so, since $\varepsilon\beta(\frac{x}{\varepsilon})$ is equibounded in $W^{1,r}(\Omega)$ for r as in (43), then

$$\|w_{\varepsilon,\lambda}\|_{W^{1,r}(\Omega)} \leq c_4, \text{for as in (43)} \tag{48}$$

with $\overline{c}$ a constant independent on ε.

It easy to see also that

$$w_{\varepsilon,\lambda} \rightharpoonup w_\lambda^* \text{ weakly in } W^{1,p}(\Omega), \tag{49}$$

for p such that, with $\alpha \in]\frac{1}{2}, 1[$,

$$\begin{aligned} &2 < p < \tfrac{4}{2-2\alpha}, && \text{if } n = 2,\\ &3 < p < \tfrac{6}{3-2\alpha}, && \text{if } n = 3,\\ &1 \le p < +\infty, && \text{if } n > 3. \end{aligned}$$

Let $(u_\varepsilon^\tau)_\varepsilon$ be a sequence of the solutions of problem (5).

We observe that $u_\varepsilon^\tau = 0$ on $\partial\Omega \cup \Gamma_\varepsilon^{D,\tau}$. By Proposition 3, there exists a sequence $\{v_\varepsilon^\tau\}_\varepsilon$ of extension of $\{u_\varepsilon^\tau\}_\varepsilon$ satisfying (14) and obviously $v_\varepsilon^\tau = 0$ on $\partial\Omega \cup \Gamma_\varepsilon^{D,\tau}$. So by (13) we have

$$v_\varepsilon^\tau \in H_0^1(\Omega \backslash D_\varepsilon'^\tau). \tag{50}$$

Now we estimate the H^1-norm of v_ε^τ.
We have, by Poincaré inequality, (14) and (6)

$$\begin{aligned} \|v_\varepsilon^\tau\|^2_{H^1(\Omega)} &\le c_\Omega^2 \int_\Omega |\nabla v_\varepsilon^\tau|^2 dx \le c_\Omega^2 c_2^2 \int_{\Omega_\varepsilon^\tau} |\nabla u_\varepsilon^\tau|^2 dx = c_\Omega^2 c_2^2 \int_{\Omega_\varepsilon^\tau} f u_\varepsilon^\tau dx\\ &\le c_\Omega^2 c_2^2 \|f\|_{L^2(\Omega_\varepsilon^\tau)} \|u_\varepsilon^\tau\|_{L^2(\Omega_\varepsilon^\tau)} \le c_\Omega^2 c_2^2 \|f\|_{L^2(\Omega)} \|v_\varepsilon^\tau\|_{H^1(\Omega)}\\ &\le c_5 \|v_\varepsilon^\tau\|_{H^1(\Omega)} \end{aligned}$$

where c_Ω is the Poincaré constant of Ω; then

$$\|v_\varepsilon^\tau\|_{H^1(\Omega)} \le c_5, \tag{51}$$

where c_5 is a constant independent on ε.

So there exists a subsequence, still denoted by ε, such that

$$v_\varepsilon^\tau \rightharpoonup u^\tau \qquad \text{weakly in } H_0^1(\Omega). \tag{52}$$

Obviously, since $v_\varepsilon^\tau = u_\varepsilon^\tau$ on Ω_ε^τ, we deduce by (51) that

$$\|u_\varepsilon^\tau\|_{H^1(\Omega_\varepsilon^\tau)} \le c_6, \tag{53}$$

where c_6 is a constant independent on ε.

By (52) and Rellich theorem we have

$$\begin{aligned} &v_\varepsilon^\tau \in L^q(\Omega), \text{for } q \in [1, +\infty[, \text{ if } n = 2,\\ &v_\varepsilon^\tau \in L^q(\Omega), \text{for } q \in [1, \tfrac{2n}{n-2}], \text{ if } n \ge 3. \end{aligned} \tag{54}$$

and

$$v_\varepsilon^\tau \to u^\tau \qquad \text{strongly in } L^p(\Omega), \text{ for } \quad 2 \le p < \infty, \quad \text{if } n = 2,$$
$$\text{for} \quad 2 \le p < 2\frac{n}{n-2}, \text{if } n \ge 3. \tag{55}$$

Let us pose $\eta_{\varepsilon,\lambda} = \nabla w_\lambda(x\varepsilon)$ in Ω_ε; by (7) we have

$$\begin{cases} -div\tilde{\eta}_{\varepsilon,\lambda} = 0 & \text{in } \Omega_\varepsilon \\ \tilde{\eta}_{\varepsilon,\lambda} \cdot \mathbf{n} = 0 & \text{on } \partial T_\varepsilon. \end{cases}$$

where $\tilde{\eta}_{\varepsilon,\lambda}$ denote the extension to zero of $\eta_{\varepsilon,\lambda}$ on the whole Ω. Its variational formulation is

$$\int_\Omega \langle \tilde{\eta}_{\varepsilon,\lambda}, \nabla\varphi \rangle \, dx = 0, \quad \forall \varphi \in H_0^1(\Omega). \tag{56}$$

We observe that

$$\tilde{\eta}_{\varepsilon,\lambda} \rightharpoonup \mathcal{A}\lambda \text{ weakly in } L^2(\Omega) \tag{57}$$

where $\mathcal{A}\lambda$ is given by (8).

Let us pose $\xi_\varepsilon^\tau = \nabla u_\varepsilon^\tau$. Then by (53), we have (up to a subsequence)

$$\tilde{\xi}_\varepsilon^\tau \rightharpoonup \xi^\tau \qquad \text{weakly in } (L^2(\Omega))^n, \tag{58}$$

where $\tilde{\xi}_\varepsilon^\tau$ denote the extension to zero of ξ_ε^τ on the whole Ω, and

$$\int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla v \rangle dx = \int_\Omega \chi_{\Omega_\varepsilon^\tau} f v dx, \quad \text{for every } v \in V_\varepsilon^\tau. \tag{59}$$

Let us consider, for every $\varphi \in \mathcal{D}(\Omega)$, $\varphi w_{\varepsilon,\lambda}\psi_\varepsilon^\tau$, where ψ_ε^τ is given by (23). By (26), (33) and Proposition 9.4 of [B], we can take $\varphi w_{\varepsilon,\lambda}\psi_\varepsilon^\tau$, as test function in (59); we have

$$\int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\varphi \rangle w_{\varepsilon,\lambda}\psi_\varepsilon^\tau dx + \int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \tilde{\eta}_{\varepsilon,\lambda} \rangle \, \varphi\psi_\varepsilon^\tau dx + \int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau \rangle \, \varphi w_{\varepsilon,\lambda} dx$$
$$= \int_\Omega \chi_{\Omega_\varepsilon^\tau} f \varphi w_{\varepsilon,\lambda}\psi_\varepsilon^\tau dx \tag{60}$$

Moreover if we take in the problem (56) $v_\varepsilon^\tau\varphi$ ($\varphi \in \mathcal{D}(\Omega)$) as test function, we have

$$\int_\Omega \langle \tilde{\eta}_{\varepsilon,\lambda}, \nabla\varphi \rangle v_\varepsilon^\tau dx + \int_\Omega \langle \tilde{\eta}_{\varepsilon,\lambda}, \nabla v_\varepsilon^\tau \rangle \varphi dx = 0 \tag{61}$$

By (60) and (61), we have

$$\int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\varphi \rangle w_{\varepsilon,\lambda}\psi_\varepsilon^\tau dx + \int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \tilde{\eta}_{\varepsilon,\lambda} \rangle \psi_\varepsilon^\tau \varphi dx + \int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau \rangle w_{\varepsilon,\lambda}\varphi dx$$
$$- \int_\Omega \langle \tilde{\eta}_{\varepsilon,\lambda}, \nabla\varphi \rangle v_\varepsilon^\tau dx - \int_\Omega \tilde{\eta}_{\varepsilon,\lambda}, \nabla v_\varepsilon^\tau \varphi dx = \int_\Omega \chi_{\Omega_\varepsilon^\tau} f w_{\varepsilon,\lambda}\psi_\varepsilon^\tau \varphi dx \tag{62}$$

Let us consider the last term in (62). Let us observe that

$$\chi_{\Omega_\varepsilon^\tau} \rightharpoonup \vartheta \text{ weakly}^* \text{ in } L^\infty(\Omega), \tag{63}$$

and, by (28) and (45), that

$$w_{\varepsilon,\lambda}\psi_\varepsilon^\tau \to w_\lambda^* \text{ strongly in } L^2(\Omega). \tag{64}$$

So by (63) and (64)

$$\int_\Omega \chi_{\Omega_\varepsilon^\tau} f\varphi w_{\varepsilon,\lambda}\psi_\varepsilon^\tau dx \to \int_\Omega \vartheta f\varphi w_\lambda^* dx. \tag{65}$$

Let us consider the first term in (62). Then by (58) and (64), it results that, as $\varepsilon \to 0$,

$$\int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\varphi\rangle w_{\varepsilon,\lambda}\psi_\varepsilon^\tau dx \to \int_\Omega \langle \xi^\tau, \nabla\varphi\rangle w_\lambda^* dx. \tag{66}$$

Let us consider the second and the fifth term in (62).

We have by (26) and (53), since the support of $\psi_\varepsilon^\tau - 1$ is contained in A_ε,

$$\begin{aligned}
&\left| \int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \tilde{\eta}_{\varepsilon,\lambda}\rangle \psi_\varepsilon^\tau \varphi dx - \int_\Omega \langle \tilde{\eta}_{\varepsilon,\lambda}, \nabla v_\varepsilon^\tau\rangle \varphi dx \right| \\
&= \left| \int_{\Omega_\varepsilon^\tau} \langle \xi_\varepsilon^\tau, \eta_{\varepsilon,\lambda}\rangle (\psi_\varepsilon^\tau - 1)\varphi dx \right| \\
&\le \|\varphi\|_\infty \|\nabla u_\varepsilon^\tau\|_{[L^2(\Omega_\varepsilon^\tau)]^n} \|\eta_{\varepsilon,\lambda}\|_{[L^2(A_\varepsilon\cap\Omega_\varepsilon)]^n} \\
&\le c_7\varepsilon^n \|\nabla(\Phi w_\lambda)\|_{[L^2(\frac{A_\varepsilon}{\varepsilon})]^n} \\
&\le c_8|\Omega| \|\nabla(\Phi w_\lambda)\|_{[L^2(\frac{A_\varepsilon}{\varepsilon}\cap Y)]^n},
\end{aligned} \tag{67}$$

where c_8 is a constant independent on ε.

We have that $\|\nabla(\Phi w_\lambda)\|_{[L^2(\frac{A_\varepsilon}{\varepsilon}\cap Y)]^n}$ tends to zero, as $\varepsilon \to 0$, by the absolute continuity of the integral because $|\frac{A_\varepsilon}{\varepsilon}| \to 0$.

Let us consider the fourth term in (62). By (55) and (57), we have

$$\int_\Omega \langle \tilde{\eta}_{\varepsilon,\lambda}, \nabla\varphi\rangle v_\varepsilon^\tau dx \to \int_\Omega \langle \mathcal{A}\lambda, \nabla\varphi\rangle u dx. \tag{68}$$

Let us consider the third term in (62).

Let $(\phi_\varepsilon)_\varepsilon$ be a sequence in $\mathcal{D}(\Omega)$ such that

$$\phi_\varepsilon \to \varphi \text{ strongly in } L^\infty(\Omega) \tag{69}$$

and ϕ_ε is constant on $\varepsilon^\nu B + \varepsilon\mathbf{k}$, for every ε, $\mathbf{k} \in \mathbf{Z}^n$ such that $\varepsilon(Q + \mathbf{k}) \subset \Omega$.

Since $\psi_\varepsilon^\tau = 1$ in $\Omega \backslash A_\varepsilon$ and so $\nabla \psi_\varepsilon^\tau = 0$ in $\Omega \backslash A_\varepsilon$, we have

$$\begin{aligned}\int_\Omega \langle \widetilde{\xi_\varepsilon^\tau}, \nabla \psi_\varepsilon^\tau \rangle w_{\varepsilon,\lambda} \varphi dx &= \int_{\Omega_\varepsilon^\tau} \langle \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle w_{\varepsilon,\lambda} \varphi dx \\ &= \int_{\Omega_\varepsilon^\tau} \langle \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle (\varphi - \phi_\varepsilon) w_{\varepsilon,\lambda} dx \\ &\quad + \int_{\Omega_\varepsilon^\tau \cap A_\varepsilon} \langle \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \phi_\varepsilon w_{\varepsilon,\lambda} dx.\end{aligned}$$

If we restrict ξ_ε^τ to Ω_ε and after denote by $R_\varepsilon \xi_\varepsilon^\tau$ the function given by (11), by the properties of simmetry of ψ_ε^τ given by (30) and $w_{\varepsilon,\lambda}$ given by (41) and $R_\varepsilon \xi_\varepsilon^\tau$, since ϕ_ε are constant on $\Omega_\varepsilon^\tau \cap A_\varepsilon$, we have

$$\int_{\Omega_\varepsilon^\tau \cap A_\varepsilon} \langle \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \phi_\varepsilon w_{\varepsilon,\lambda} dx = \frac{1}{2} \int_{A_\varepsilon} \langle R_\varepsilon \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \phi_\varepsilon w_{\varepsilon,\lambda} dx. \tag{70}$$

By the definition of ξ_ε^τ and $R_\varepsilon \xi_\varepsilon^\tau$, and the properties of simmetry of v_ε^τ given by (13), we have

$$\int_{A_\varepsilon} \langle R_\varepsilon \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \phi_\varepsilon w_{\varepsilon,\lambda} dx = \int_{A_\varepsilon} \langle \nabla v_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \phi_\varepsilon w_{\varepsilon,\lambda} dx. \tag{71}$$

So by (70), (71)

$$\begin{aligned}\int_\Omega \langle \widetilde{\xi_\varepsilon^\tau}, \nabla \psi_\varepsilon^\tau \rangle w_{\varepsilon,\lambda} \varphi dx &= \int_{\Omega_\varepsilon^\tau} \langle \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle (\varphi - \phi_\varepsilon) w_{\varepsilon,\lambda} dx \\ &\quad + \frac{1}{2} \int_{A_\varepsilon} \langle \nabla v_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \; \phi_\varepsilon w_{\varepsilon,\lambda dx} \\ &= \int_{\Omega_\varepsilon^\tau} \langle \; \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle (\varphi - \phi_\varepsilon) \; w_{\varepsilon,\lambda} dx \\ &\quad + \frac{1}{2} \int_{A_\varepsilon} \langle \nabla v_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle (\phi_\varepsilon - \varphi) \; w_{\varepsilon,\lambda} dx \\ &\quad + \frac{1}{2} \int_\Omega \langle \nabla v_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \; w_{\varepsilon,\lambda} \varphi dx.\end{aligned} \tag{72}$$

We have that

$$\begin{aligned}&\left| \int_{\Omega_\varepsilon^\tau} \langle \xi_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle (\varphi - \phi_\varepsilon) w_{\varepsilon,\lambda} dx \right| \\ &\quad \le \|\varphi - \phi_\varepsilon\|_{L^\infty(\Omega)} \|w_{\varepsilon,\lambda}\|_{L^\infty(\Omega)} \|\nabla u_\varepsilon^\tau\|_{(L^2(\Omega_\varepsilon^\tau))^n} \|\nabla \psi_\varepsilon^\tau\|_{L^2(\Omega)}\end{aligned} \tag{73}$$

that, by (27), (45), (53) and (69), tends to zero. In a similar way we obtain that, as $\varepsilon \to 0$

$$\frac{1}{2} \int_{A_\varepsilon} \langle \nabla v_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle \, (\phi_\varepsilon - \varphi) w_{\varepsilon,\lambda} dx \to 0. \tag{74}$$

Now we examine the last term in (72). It results

$$w_{\varepsilon,\lambda} v_\varepsilon^\tau \varphi \in H_0^1(\Omega \backslash D_\varepsilon'^\tau). \tag{75}$$

In fact since $v_\varepsilon^\tau \in H_0^1(\Omega \backslash D_\varepsilon'^\tau)$ and $\varphi \in \mathcal{D}(\Omega)$, by (50) we must only prove that $w_{\varepsilon,\lambda} v_\varepsilon^\tau \in H^1(\Omega)$. Obviously $\nabla(w_{\varepsilon,\lambda} v_\varepsilon^\tau) = w_{\varepsilon,\lambda} \nabla v_\varepsilon^\tau + v_\varepsilon^\tau \nabla w_{\varepsilon,\lambda}$. By (45) and since $\nabla v_\varepsilon^\tau \in L^2(\Omega)$ we have that $w_{\varepsilon,\lambda} \nabla v_\varepsilon^\tau \in L^2(\Omega)$. If $n = 2$, by (4) and since $w_{\varepsilon,\lambda} \in L^\infty(\Omega) \cap W^{1,4}(\Omega)$, then $\nabla w_{\varepsilon,\lambda} v_\varepsilon^\tau \in L^2(\Omega)$. If $n \geq 3$, by (4) and since $w_{\varepsilon,\lambda} \in L^\infty(\Omega) \cap W^{1,3}(\Omega)$, then $\nabla w_{\varepsilon,\lambda} v_\varepsilon^\tau \in L^2(\Omega)$. So we have

$$\int_\Omega \langle \nabla v_\varepsilon^\tau, \nabla \psi_\varepsilon^\tau \rangle w_{\varepsilon,\lambda} \varphi dx = \langle -\Delta \psi_\varepsilon^\tau, w_{\varepsilon,\lambda} v_\varepsilon^\tau \varphi \rangle$$
$$- \int_\Omega \langle \nabla \psi_\varepsilon^\tau, \nabla \varphi \rangle w_{\varepsilon,\lambda} v_\varepsilon^\tau dx - \int_\Omega \langle \nabla \psi_\varepsilon^\tau, \nabla w_{\varepsilon,\lambda} \rangle v_\varepsilon^\tau \varphi dx \tag{76}$$

By the equiboundedness of $w_{\varepsilon,\lambda}$ in $L^\infty(\Omega)$, by (55) and since, by (27), $\nabla \psi_\varepsilon^\tau \rightharpoonup 0$ weakly in $L^2(\Omega)$, we obtain that $\int_\Omega \langle \nabla \psi_\varepsilon^\tau, \nabla \varphi \rangle w_{\varepsilon,\lambda} v_\varepsilon^\tau dx$ tends to zero.

We can observe that

$$\|v_\varepsilon^\tau \nabla w_{\varepsilon,\lambda}\|_{L^s(\Omega)} \leq c_9 \,, \text{for}\, s > 2, \tag{77}$$

where c_9 is a constant independent on ε.

In fact if $n = 2$, by (48) it results $\nabla w_{\varepsilon,\lambda}$ is equibounded in $L^r(\Omega)$ with $r > 2$. By (4) v_ε^τ is equibounded in $L^q(\Omega)$ for $1 \leq q < +\infty$ and so by (77) there exists $s > 2$ such that $v_\varepsilon^\tau \nabla w_{\varepsilon,\lambda}$ is equibounded in $L^s(\Omega)$. By (29), $\nabla \psi_\varepsilon^\tau \rightharpoonup 0$ strongly in $L^{s'}(\Omega)$, with $\frac{1}{s'} + \frac{1}{s} = 1$.

If $n = 3$, by (48) there exists r such that $3 < r \leq \frac{6}{3-2\alpha} (\alpha \in]\frac{1}{2}, 1])$ and $\nabla w_{\varepsilon,\lambda}$ is equibounded in $L^r(\Omega)$. By (4) v_ε^τ is equibounded in $L^q(\Omega)$ for $1 \leq q < 6$. So there exists $s > 2$ such that $\nabla w_{\varepsilon,\lambda} v_\varepsilon^\tau$ is equibounded in $L^s(\Omega)$.

If $n > 3$, we observe that, by (48), $\nabla w_{\varepsilon,\lambda}$ is equibounded in $L^\infty(\Omega)$. Then by (29), $\nabla \psi_\varepsilon^\tau \rightharpoonup 0$ strongly in $L^{s'}(\Omega)$, with $\frac{1}{s'} + \frac{1}{s} = 1$.

So we obtain that $\int_\Omega \langle \nabla \psi_\varepsilon^\tau, \nabla w_{\varepsilon,\lambda} \rangle v_\varepsilon^\tau \varphi dx$ tends to zero. We have that $w_{\varepsilon,\lambda} v_\varepsilon^\tau \to w_\lambda^* u^\tau$ weakly in $H_0^1(\Omega)$.

Indeed $w_{\varepsilon,\lambda} \to w_\lambda^*$ strongly in $L^\infty(\Omega)$, by (46) and $v_\varepsilon^\tau \rightharpoonup u^\tau$ weakly in $H_0^1(\Omega)$ by (52).

Moreover $\nabla(w_{\varepsilon,\lambda} v_\varepsilon^\tau) = w_{\varepsilon,\lambda} \nabla v_\varepsilon^\tau + v_\varepsilon^\tau \nabla w_{\varepsilon,\lambda}$ and $v_\varepsilon^\tau \nabla w_{\varepsilon,\lambda}$ is equibounded in $L^s(\Omega)$ for $s > 2$, by (77).

By Lemma 3 and (75), we have

$$\langle -\Delta \psi_\varepsilon^\tau, \varphi w_{\varepsilon,\lambda} v_\varepsilon^\tau \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \to \langle \mu_\tau, \varphi w_\lambda^* u^\tau \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{78}$$

By (62) and (65)÷(78), we have that, as $\varepsilon \to 0$

$$\int_\Omega \langle \xi^\tau, \nabla \varphi \rangle \, w_\lambda^* dx + \frac{1}{2} \langle \mu_\tau, \varphi u^\tau w_\lambda^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$
$$- \int_\Omega \langle \mathcal{A}\lambda, \nabla \varphi \rangle \, u^\tau dx = \int_\Omega \vartheta f \varphi w_\lambda^* dx. \tag{79}$$

Now let us take, for every $\varphi \in \mathcal{D}(\Omega)$, $\varphi\psi_\varepsilon^\tau$ as test function in (59). Then

$$\int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\varphi\rangle \psi_\varepsilon^\tau dx + \int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau\rangle\ \varphi dx = \int_\Omega \chi_{\Omega_\varepsilon^\tau} f\varphi\psi_\varepsilon^\tau dx. \tag{80}$$

As previously proved we have

$$\begin{aligned}\int_\Omega \langle \widetilde{\xi_\varepsilon^\tau}, \nabla\psi_\varepsilon^\tau\rangle\varphi dx \ &= \ \int_{\Omega_\varepsilon^\tau} \langle \xi_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau\rangle(\varphi - \phi_\varepsilon)dx + \frac{1}{2}\int_{A_\varepsilon} \langle \nabla v_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau\rangle\,(\phi_\varepsilon - \varphi)\,dx \\ &\quad + \frac{1}{2}\int_\Omega \langle \nabla v_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau\rangle\varphi dx. \end{aligned} \tag{81}$$

As to obtain (73) and (74), we have that the first and the second term on the right tends to zero. Then

$$\int_\Omega \langle \nabla v_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau\rangle\varphi dx = \int_\Omega \langle \nabla\psi_\varepsilon^\tau, \nabla(v_\varepsilon^\tau\varphi)\rangle dx - \frac{1}{2}\int_\Omega \langle \nabla\psi_\varepsilon^\tau, \nabla\varphi\rangle v_\varepsilon^\tau dx. \tag{82}$$

Now we observe that, since $v_\varepsilon^\tau\varphi \in H_0^1(\Omega\backslash D_\varepsilon'^\tau)$ by (50),

$$\int_\Omega \langle \nabla\psi_\varepsilon^\tau, \nabla(v_\varepsilon^\tau\varphi)\rangle dx = \langle \mu_\varepsilon^\tau - \gamma_\varepsilon^\tau, v_\varepsilon^\tau\varphi\rangle = \langle \mu_\varepsilon^\tau, v_\varepsilon^\tau\varphi\rangle. \tag{83}$$

So by (24) and (52), we have for ε that tends to zero

$$\langle \mu_\varepsilon^\tau, v_\varepsilon^\tau\varphi\rangle \to \langle \mu_\tau, u^\tau\varphi\rangle. \tag{84}$$

By (27) and (52), we have for ε that tends to zero

$$\int_\Omega \langle \nabla\psi_\varepsilon^\tau, \nabla\varphi\rangle v_\varepsilon^\tau dx \to 0. \tag{85}$$

By (81)÷(85), we obtain

$$\int_\Omega \langle \tilde{\xi}_\varepsilon^\tau, \nabla\psi_\varepsilon^\tau\rangle\ \varphi dx \to \langle \mu_\tau, u^\tau\varphi\rangle \tag{86}$$

Then passing to the limit as $\varepsilon \to 0$ in (80), by (28), (58), (63) and (86), we have

$$\int_\Omega \langle \xi^\tau, \nabla\varphi\rangle\ dx + \frac{1}{2}\ \langle \mu_\tau, \varphi u^\tau\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega \vartheta f\varphi dx, \tag{87}$$

for every $\varphi \in \mathcal{D}(\Omega)$.

Now taking φw_λ^* as test function in (87), we obtain

$$\begin{aligned}&\int_\Omega \langle \xi^\tau, \nabla\varphi\rangle w_\lambda^* dx + \int_\Omega \langle \xi^\tau, \lambda\rangle\varphi dx + \frac{1}{2}\ \langle \mu_\tau, \varphi u^\tau w_\lambda^*\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_\Omega \vartheta f\varphi w_\lambda^* dx. \end{aligned} \tag{88}$$

By (79) and (88) we have

$$\int_\Omega \langle \xi^\tau, \lambda\rangle \varphi dx + \int_\Omega \langle \mathcal{A}\lambda, \nabla\varphi\rangle u^\tau dx = 0 \tag{89}$$

But by divergence theorem

$$\int_\Omega \langle \mathcal{A}\lambda, \nabla(\varphi u^\tau)\rangle dx = 0$$

and so

$$\int_\Omega \langle \mathcal{A}\lambda, \nabla\varphi\rangle u^\tau dx = -\int_\Omega \langle \mathcal{A}\lambda, \nabla u^\tau\rangle \, \varphi dx. \tag{90}$$

Then by (89) and (90) we have

$$\int_\Omega \langle \xi^\tau, \lambda\rangle \, \varphi dx - \int_\Omega \langle \mathcal{A}\lambda, \nabla u^\tau\rangle \, \varphi dx = \int_\Omega \langle \xi^\tau - \mathcal{A}\nabla u^\tau, \lambda\rangle \, \varphi dx = 0,$$

and so $\xi^\tau = \mathcal{A}\nabla u^\tau$ a.e. in Ω.

If now $\{w_\varepsilon^\tau\}_\varepsilon$ is any other family of uniform extensions of $\{u_\varepsilon^\tau\}_\varepsilon$ bounded in $H_0^1(\Omega)$ we have that $v_\varepsilon^\tau - w_\varepsilon^\tau = 0$ in Ω_ε^τ and is bounded in $H_0^1(\Omega)$. By Lemma 3 we obtain ii) of Theorem 2.

References

[A] R. ADAMS, *Sobolev spaces*, Academic Press, New York, 1977.

[AM] G. ALLAIRE, F. MURAT, Homogenization of the Neumann problem with non isolated holes, *Asymptotic Analysis* **7** (1993), 81–95.

[At] H. ATTOUCH, Variational Convergence for Functions and Operators, *Applicable Mathematics Series*, Pitman London, 1984.

[AtM] H. ATTOUCH, F. MURAT, Homogenization on fissured elastic materials, *Publications Avamac Université de Perpignan* **85–03** (1985).

[BP] A. BAKHVALOV, G.P. PANASENKO, Homogenization: Averaging Processes in Periodic Media, *Math. Appl.* (Soviet Ser.) **36** Kluwer Academic Publishers, 1989.

[BLP] A. BENSOUSSAN, J.L. LIONS, G. PAPANICOLAOU, Asymptotic Analysis for Periodic Structures, North Holland, Amsterdam, 1978.

[BD] A. BRAIDES, A. DE FRANCESCHI, Homogenization of Multiple Integrals, *Oxford Lecture Ser. Math. Appl.* Oxford University Press **12** (1998).

[B] H. BREZIS, Analisi Funzionale, Liguori Ed., Napoli, 1990.

[CD] D. CIORANESCU, P. DONATO, Homogénéisation du problème de Neumann non homogéne dans des ouverts perforés, *Asymptotic Anal.* **1** (1988), 115–138.

[CM1] D. CIORANESCU, F. MURAT, Un terme étrange venu d'ailleurs, *Nonlinear Partial Differential Equations and their applications. Collège de France Seminar*, Vol. **II**, 58–138, Vol. **III**, 157–178, Research Notes in Mathematics, Pitman, London, 1981.

[CM2] D. CIORANESCU, F. MURAT, A strange term coming from nowhere, in "Topics in the Mathematical Modelling of Composite Materials", A. Cherkaev and R. Kohn editors, Progr. *Nonlinear Differential Equations Appl.* **31** (1997), 45–93.

[CSJP1] D. CIORANESCU, J. SAINT JEAN PAULIN, Homogenization in Open Sets with Holes, *J. Math. Anal. and Appl.* **71** (1978), 590–607.

[CSJP2] D. CIORANESCU, J. SAINT JEAN PAULIN, Homogenization of Reticulated Structures, *Applied Mathematical Sciences*, **136**, Springer.

[CoD] C. CONCA, P. DONATO, Non Homogeneous Neumann Problems in Domains with Small Holes, *Modélisation Mathématique et Analyse Numérique* **22**(4) (1988), 561–608.

[CDG] A. CORBO ESPOSITO, C. D'APICE, A. GAUDIELLO, Homogenization in a perforated domain with both Dirichlet and Neumann boundary conditions on small holes, Asymptotic Analysis (in print).

[D] G. DAL MASO, *An Introduction to* Γ*-convergence*; Birkhauser, Basel, 1993.

[DMM] G. DAL MASO, F. MURAT, Dirichlet problems in perforated domains for homogeneous monotone operators on H_0^1, *Calculus of Variations, Homogenizations and Continuum Mechanics (CIRM-Luminy Marseille, 1993).* World Scientific, Singapore, 1994.

[DM] P. DONATO, G. MOSCARIELLO, On the Homogenization of some Nonlinear Problems in Perforated Domains, *Rend. Sem. Math. Univ. Padova* **84** (1990), 91–108.

[EG] L.C. EVANS, R.F. GARIEPY, Measure Theory and Fine Properties of Functions, *Studies in Advanced Math.*, CRC Press, 1992.

[G1] P. GRISVARD, Elliptic Problems in Non Smooth Domains, *Monographs and Studies in Mathematics* Pitman, London **24** (1985).

[G2] P. GRISVARD, Controlabilité exacte dans les polygones et polyèdres, *C.R. Acad. Sc.* Paris **304** I, n. 13 (1987).

[G3] P. GRISVARD, Controlabilité exacte des solutions de l'équation des ondes en présence de singularités, *J. Math. pures et appl.* **68** (1989), 215–259.

[JKO] V.V. JIKOV, S.M. KOZLOV, O.A. OLEINIK, Homogenization of Differential Operators and Integral Functionals, Springer Verlag, 1994.

[KOS] A. KUFNER, J. OLDRICH, F. SVATOPLUK, Function Spaces, Noordhoff International Publishing, Leyden, 1977.

[MT] F. MURAT, L. TARTAR, H-convergence, in "Topics in the Mathematical Modelling of Composite Materials", A. Cherkaev and R. Kohn editors, Progr. *Nonlinear Differential Equations Appl.* **31** (1997), 21–44.

[SP] E. SANCHEZ PALENCIA, Nonhomogeneous Media and Vibration Theory, *Lecture Notes in Phys.* Springer **127**, (1980).

[T1] E. TARTAR, Problèmes d'homogénéisation dans les équations aux dérivées partielles, Cours Peccot, Collége de France, partially written in [MT].

[T2] L. TARTAR, Quelques remarques sur l'homogénéisation, in Functional Analysis and Numerical Analysis, Proc. Japan-France seminar (1976), ed. H.Fujita, Japanese Society for the promotion of Science.

[Ta] M.E. TAYLOR, Partial Differential Equations, Springer **23** (1996).