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Robust tracking design for uncertain MIMO systems using proportional–integral controller of order ν

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Abstract
This paper provides a systematic method to design robust tracking controllers of reference signals with bounded derivatives of order ν for uncertain multi-input multi-output (MIMO) systems with bounded parametric uncertainties, in particular, of rational multi-affine type, and/or in presence of disturbances with bounded derivatives of order ν. The proposed controllers have state-feedback structures combined with proportional–integral regulators of order ν (PI_ν). Theoretical tools and systematic methodologies are provided to effectively design robust controllers for the considered systems, also in case of additional bounded nonlinearities and/or not directly measurable states. Applicability and efficiency of the proposed methods are validated through three examples: the first one is theoretical and useful to validate the proposed methodology, the second case study presents a metal-cutting problem for an industrial robot, and the third example deals with a composite robot, such as a milling machine.

KEYWORDS
Cartesian robots, PI_ν controller, processes with unmeasurable states, robust tracking method, systems with additional bounded nonlinearities, uncertain linear time-invariant (LTI) MIMO systems

1 | INTRODUCTION

Numerous industrial systems operate subject to parametric uncertainties and non-standard references and disturbances, which need to be efficiently controlled. For such systems evolving in an increasingly dynamic and global society, it is indispensable to design robust controllers able to track rapidly varying reference signals (i.e., with larger derivatives) quickly and precisely, despite any external disturbances. A number of publications on this topic are available (e.g. [1–7]), including recent ones [8–20]. However, the following practical limitations still remain in effect: (a) the considered classes of systems are often of particular structure; (b) the considered reference signals and disturbances almost always have standard waveforms (polynomial and/or sinusoidal ones); (c) the controllers are not robust enough and/or do not applicable to systems satisfying more than one specification; (d) the control inputs are excessive and/or unfeasible in some cases [1–11, 21–28].

A basic control problem is to force an uncertain process or a plant to track with an acceptable precision sufficiently smooth but generic references, e.g., with bounded ν - order derivatives and/or in the presence of generic but sufficiently smooth disturbances. This paper provides a systematic method to design robust tracking controllers using the majorant systems approach for LTI uncertain multi-input multi-output (MIMO) systems with bounded parametric uncertainties, in particular, of rational multi-affine type, and reference signals and disturbances with bounded derivatives of order ν. The proposed controller possesses a state-feedback structure combined with a PI_ν regulator. Systems with additional bounded nonlinearities and/or not directly measurable states are also considered.

This development presents a broad generalization of [29], where the case $\nu = 1$ has been considered with more restrictive uncertainty type and a controller without proportional action. The obtained results also generalize the results of [30] concerning uncertain MIMO systems, considering bounded uncertainties of rational multi-affine type, with an integral controller and observer. The applied majorant systems approach has already successfully used for particular classes of linear and nonlinear systems (see [29–31]). Another approach to design robust tracking controllers for relevant classes of nonlinear systems can be found in [32]. The provided results are particularly useful for mechatronic systems (e.g., rigid Cartesian robots, rolling mills, AGVs, conveyor belts, active suspension systems, printing machines), whose reference signals and disturbances are represented by non-standard waveforms (see, e.g., [30, 33, 34]).

Some distinctive features and advantages of the proposed study are: (a) The designed controllers avoid the derivative action causing realization problems, especially in the presence of measurement noises; (b) the presence of realistic uncertainties, which are bounded and, in particular, of rational multi-affine type; (c) the possibility to obtain acceptable tracking errors for reference signals and disturbances represented by generic sufficiently regular non-standard waveforms.

The paper is organized as follows. In Section 2, the considered MIMO uncertain systems are introduced, and the synthesis problems are stated. Section 3 provides the theoretical background. Section 4 presents the main theorems used to design robust controllers for the considered systems. Section 5 extends some results obtained in Section 4. Section 6 specifies the control design algorithms based on the presented theoretical background. In Section 7, three examples are provided: the first one is given to illustrate the proposed methodology, the second example presents a metal-cutting problem for an industrial robot, and the third case study considers a composite robot such as a milling machine, to show practical advantages and effectiveness of the designed controller. Finally, Section 8 presents conclusions and future developments to this study.

2 | PRELIMINARIES AND PROBLEM STATEMENT

The notation is introduced as follows:

$$\|x\|_P = \sqrt{x^T P x}, \|x\| = \|x\|_{I_n} = \sqrt{x^T x}, S_{P,\rho} = \{x : \|x\|_P \le \rho\},\ \rho \ge 0,$$
$$C_{P,\,\rho} = \{x : \|x\|_P = \rho\}, \hat{C}_{P,\,\rho} \supseteq C_{P,\,\rho}, \tag{1}$$

where $P \in R^{n\times n}$ is a symmetric and positive definite (*p.d.*) matrix, I_n denotes the identity matrix of order n, x^T is the transpose of $x \in R^n$, and $\hat{C}_{P,\,\rho}$ is a compact set.

Given a *p.d.* matrix $P \in R^{n\times n}$, $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) denotes the maximum (minimum) eigenvalue of P.

Given a real matrix $A \in R^{n\times n}$, $\lambda(A)$ is the set of eigenvalues (spectrum) of A; $\hat{\alpha}$ is the maximum real part of the eigenvalues of A, i.e., $\hat{\alpha} = \max(\mathrm{real}(\lambda(A)))$; $\hat{\tau} = -1/\hat{\alpha}$ denotes the maximum time constant of A; $\alpha \ge \hat{\alpha}$ is an upper estimate of $\hat{\alpha}$; $\tau \ge \hat{\tau}$ is an upper estimate of $\hat{\tau}$.

Let $A = \{a_{ij}\}$ be a real $n \times m$ matrix. $|A|$ is its absolute values matrix, i.e., $|A| = \{|a_{ij}|\}$..

Given a function $f(t)$, $f^{(\nu)}(t)$ denotes its derivative of ν - th order.

Now, consider an uncertain LTI MIMO plant given by

$$\begin{aligned} \dot{x}(t) &= A(p)x(t) + B(p)u(t) + E(p)d(t), \\ y(t) &= C(p)x(t) + D(p)d(t), \end{aligned} \tag{2}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^r$ is the control input, $d(t) \in R^l$ is a disturbance, $y(t) \in R^m$ is the output, $p \in \wp \subset R^\nu$ is the vector of uncertain parameters, and $A(p)$, $B(p)$, $E(p)$, $C(p)$, $D(p)$ are matrices of appropriate dimensions.

Suppose that $\wp$ can be covered by a finite number N of hyper-rectangles $\wp_j = \left[p_j^-, p_j^+\right]$, the conditions

$$\begin{aligned} &rank\left[B(p)\ A(p)B(p)\ \dots\ A^{n-1}(p)B(p)\right] = n, \\ &rank\begin{bmatrix} A(p) & B(p) \\ C(p) & 0 \end{bmatrix} = n + m, \\ &rank\left[C^T(p)\ \ A^T(p)C^T(p)\ \ \dots\ \ \left(A^T(p)\right)^{n-1}C^T(p)\right] = n \end{aligned} \tag{3}$$

are satisfied for each $p \in \wp$, and the matrices $A(p)$, $B(p)$, $E(p)$, $C(p)$, $D(p)$ are defined as ratios of multi-affine matrix functions to non-zero multi-affine polynomials in $\wp$ as

$$A(p)=\frac{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} A_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} a_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}\in R^{nxn},$$

$$B(p)=\frac{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} B_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} b_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}\in R^{nxr},$$

$$E(p)=\frac{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} E_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} e_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}\in R^{nxl}, \tag{4}$$

$$C(p)=\frac{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} C_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} c_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}\in R^{mxn},$$

$$D(p)=\frac{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} D_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}{\sum\limits_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} d_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}\in R^{mxl}.$$

In the following, for simplicity of notation, the explicit dependency of $A(p)$, $B(p)$, $E(p)$, $C(p)$, $D(p)$ on p will be omitted.

Remark 1. The considered class of systems is relevant from the engineering point of view, since many mechatronic and transportation processes are described by models referable to them (see Figure 1).

Remark 2. Note that many mechanical, electrical, mechatronic, thermal, and fluid dynamic systems can be modeled by systems in the form $L\dot{x}+Mx+Nu=0\Rightarrow\dot{x}=-L^{-1}Mx-L^{-1}Nu=Ax+Bu$, where matrices L,M,N are linear with respect to the physical system parameters, which are often uncertain. Hence, the dependency on uncertain parameters of the matrices A and B is of rational multi-affine type (or transformable to this by using suitable changes of the parameters, as demonstrated in Corollaries 1 and 2).

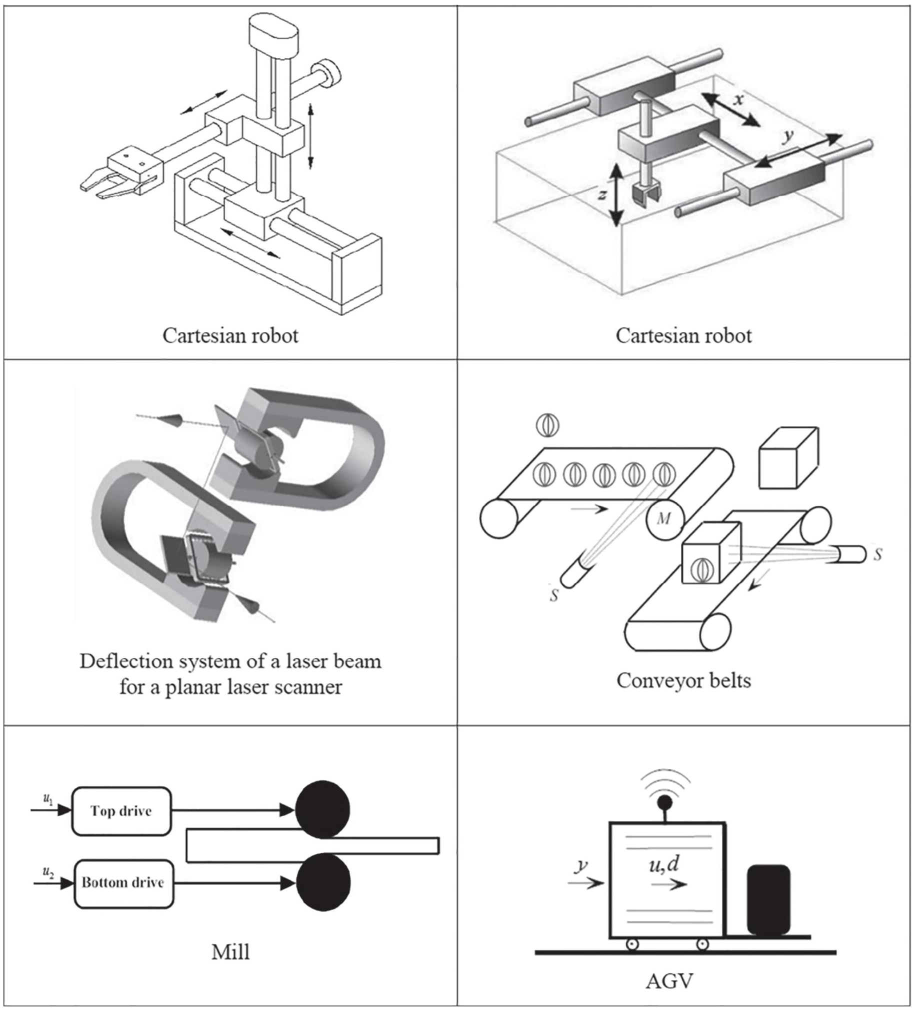


FIGURE 1 Some mechatronic and transportation processes described by 2

Remark 3. Conditions 3 imply that $rank\ C = m \le n$ and $rank\ B \ge m$, i.e., the m outputs of the plant are independent and the number of the independent control inputs has to be greater or at least equal to the number of the outputs to be controlled.

Suppose that the system 2 is controlled using a PI_ν controller with a state-feedback structure as shown in Figure 2.

From the control scheme in Figure 2, it follows that

$$\begin{aligned}\dot{x} &= Ax + Bu + Ed = \left(A + B\left(K_s - K_pC\right)\right)x + BK_{i1}z_1 \\ &\quad + BK_{i2}z_2 + \cdots + BK_{i\nu}z_\nu + BK_pr + \left(E - BK_pD\right)d, \\ \dot{z}_1 &= -Cx + r - Dd, \dot{z}_2 = z_1, \vdots \dot{z}_\nu = z_{\nu-1}, \\ e &= -Cx + r - Dd.\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned}\dot{\xi} &= A_c\xi + B_cr + E_cd, \\ e &= C_c\xi + r - Dd\ ,\end{aligned} \tag{6}$$

where

$$A_c = \begin{bmatrix} A + BK_e & BK_{i1} & \cdots & BK_{i\nu-1} & BK_{i\nu} \\ -C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$B_c = \begin{bmatrix} BK_p \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ E_c = \begin{bmatrix} E - BK_pD \\ -D \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \xi = \begin{bmatrix} x \\ z_1 \\ z_2 \\ \vdots \\ z_\nu \end{bmatrix}, \tag{7}$$

$$K_e = K_s - K_pC,\quad C_c = \left[-C\ \ 0\ \ 0\ \ \cdots\ \ 0\right].$$

The considered problem is to estimate for each $p \in \wp$ the maximum time constant $\hat{\tau}_c$ of the closed-loop control system and the maximum norm $\hat{e}_{\max}$ of the tracking error $e(t)$ of a generic reference signal r with an almost everywhere bounded derivative of order ν, in the presence of a generic disturbance d with an almost everywhere bounded derivative of ν - th order, and use $\hat{\tau}_c$ and $\hat{e}_{\max}$ as parameters for the subsequent controller design. In other words, the objective is to find:

i. a constant $\tau_c > 0$ such that $\hat{\tau}_c = \max_{p\in\wp}\tau_{\max}(A_c(p)) \le \tau_c$, where $A_c(p)$ is the dynamics matrix of the closed-loop control system,
ii. non-negative constants H_r and H_d, called "generalized gain constants", such that, after a possible transient phase, the tracking error satisfies the conditions

$$\begin{aligned}&\|e(t)\| = \|r(t) - y(t)\| \le \hat{e}_{\max} \le H_rR_\nu + H_dD_\nu, \\ &\forall p \in \wp, \forall r(t),\ d(t) : \left\|r^{(\nu)}(t)\right\| \le R_\nu, \left\|d^{(\nu)}(t)\right\| \le D_\nu.\end{aligned} \tag{8}$$

Remark 4. Nowadays, a generic reference signal for a manufacturing or transportation system is a non-standard (non-polynomial or non-sinusoidal) signal, whose derivative represents "working velocity" or "transportation velocity", while the second derivative represents acceleration.

In regard to the above systems, it can also be useful to compute the maximum norms $\hat{\dot{e}}_{\max}$, $\hat{\ddot{e}}_{\max}$ of the tracking errors $\dot{e}(t) = \dot{r}(t) - \dot{y}(t)$, $\ddot{e}(t) = \ddot{r}(t) - \ddot{y}(t)$ of $\dot{r}(t), \ddot{r}(t)$, since high values of $\dot{e}(t)$ can produce high-frequency oscillations and/or vibrations, while high values of $\ddot{e}(t)$ can produce skids (e.g., for some transportation systems), and/or require high values of the control action.

Remark 5. A reference signal $r(t)$ with a bounded ν - th derivative can be obtained by interpolating a set of given points $(t_k,\ r_k)$, $k = 0, 1, \ldots, n_r$, with appropriate splines or by filtering any piecewise constant or piecewise linear signal $\tilde{r}(t)$ with the following MIMO filter:

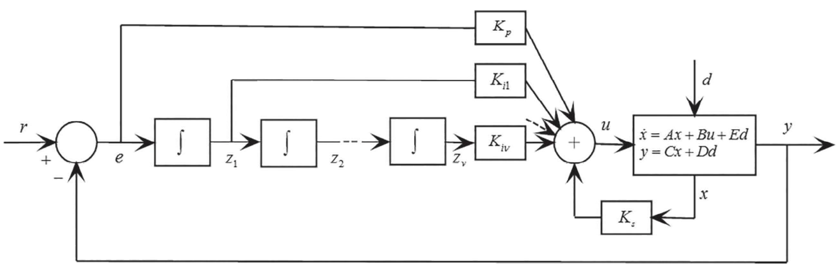


FIGURE 2 Proposed state-feedback control scheme with PI_ν controller

$$\dot{\zeta} = \begin{bmatrix} 0 & I & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & I \\ -f_1 I & -f_2 I & \cdots & -f_\nu & -f_{\nu+1} I \end{bmatrix} \zeta \tag{9}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ If_1 \end{bmatrix} \tilde{r}, \quad \begin{bmatrix} r \\ \dot{r} \\ \cdots \\ r^{(\nu-1)} \\ r^{\nu} \end{bmatrix} = \zeta.$$

In this case, it is also possible to reduce the control action, in particular, during the transient phase by suitably choosing initial conditions of the filter and its cutoff frequency.

Note that if the filter is a Bessel one with cutoff angular frequency ω_b, larger or equal to the angular frequency of $\tilde{r}(t)$, then $r(t) \cong \tilde{r}(t-t_r), t_r = (\nu+1)\pi/(4\omega_b)$.

To solve the stated problem, the following preliminaries are given.

3 | THEORETICAL BACKGROUND

Definition 1. Given the system

$$\dot{x} = f(t,x,u,p), y = Cx, t \in \mathcal{T} = [t_0, t_f] \subseteq R, x \in R^n, u \in R^r, \\ p \in \wp \subset R^\mu, y \in R^m, C \in R^{m\times n}, \tag{10}$$

a constant $U \geq 0$, and a *p.d.* symmetric matrix $P \in R^{n\times n}$. A positive first-order system

$$\dot{\rho} = \varphi(\rho,\ U),\ \rho_0 = \|x_0\|_P, Y = c\rho, \tag{11}$$

where $\rho(t) = \|x(t)\|_P$, such that $\|y(t)\| \leq Y(t)$, for each $t \in \mathcal{T}, u : \|u\| \leq U$, and $p \in \wp$, is said to be a majorant system of the system 10.

Consider a matrix function $H(p) \in R^{n\times l}$, with $p = [p_1\ p_2 ... p_\mu]^T \in \wp \subset R^\mu$, defined as a ratio of a multi-affine matrix function to a multi-affine polynomial

$$H(p) = \frac{\sum_{i_1,i_2,\cdots,i_\mu \in \{0,1\}} H_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2} \cdots p_\mu^{i_\mu}}{\sum_{i_1,i_2,\cdots,i_\mu \in \{0,1\}} h_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2} \cdots p_\mu^{i_\mu}}, \tag{12}$$

where $H_{i_1,i_2,\cdots,i_\mu} \in R^{n\times l}$ and $h_{i_1,i_2,\cdots,i_\mu} \in R$.

Suppose that the compact set $\wp$ is a hyper-rectangle given by

$$\wp = [p_1^-, p_1^+] \times [p_2^-, p_2^+] \times \cdots \times [p_\mu^-, p_\mu^+] = [p^-, p^+], \tag{13}$$

and the denominator $d(p)$ of $H(p)$ is different from zero $\forall p \in \wp$, i.e.,

$$d(p) = \sum_{i_1,i_2,\cdots,i_\mu \in \{0,1\}} h_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2} \cdots p_\mu^{i_\mu} \neq 0, \forall p \in \wp. \tag{14}$$

The following lemmas and corollaries hold (see [29, 30] for proofs).

Lemma 1. *Consider a matrix H(p) (12) and a symmetric p.d. matrix $P \in R^{n\times n}$. Then, i $l = n$, $\lambda_{\max_{p\in\wp}}(Q(p)P^{-1})$, where $Q(p) = H^T(p)P + PH(p)$, is achieved in one of the 2^μ vertices V_p of $\wp$. Similarly, $\lambda_{\max_{p\in\wp}}\left(H(p)^T PH(p)\right) = \lambda_{\max_{p\in V_p}}\left(H(p)^T PH(p)\right)$.*

Corollary 1. *Let*

$$H(p) = \frac{\sum_{i_1,i_2,\cdots,i_\mu \in \{0,1,2\}} H_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2} \cdots p_\mu^{i_\mu}}{\sum_{i_1,i_2,\cdots,i_\mu \in \{0,1,2\}} h_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2} \cdots p_\mu^{i_\mu}}, \tag{15}$$

$$H_{i_1,i_2,\cdots,i_\mu} \in R^{n\times l},\quad h_{i_1,i_2,\cdots,i_\mu} \in R,$$

be a nonsingular matrix function, defined as a ratio of a quadratic matrix function to a quadratic polynomial with respect to the parameters $[p_1 p_2 ... p_\mu]^T = p \in \wp = [p^-,\ p^+]$, and $P \in R^{n\times n}$ be a symmetric p.d. matrix. Then, if $l = n$, *an upper estimate of $\lambda_{\max_{p\in\wp}}(Q(p)P^{-1})$, where $Q(p) = H^T(p)P + PH(p)$, is given by $\lambda_{\max_{p\in V_{ap}}}(Q_a(p_a)P^{-1})$, where V_{ap} is the set of $2^{2\mu}$ vertices of $\wp_a = \wp \times \wp, p_a = \left[p_1 \cdots p_\mu p_{\mu+1} \cdots p_{2\mu}\right]^T$, $Q_a(p_a) = H_a^T(p_a)P + PH_a(p_a)$ and $H_a(p_a)$ is obtained from matrix H(p) by replacing p_i^2 with the product $p_i p_{\mu+i}$ $i = 1,2,...,\mu$. Similarly, an upper estimate of $\lambda_{\max_{p\in\wp}}\left(H^T(p)PH(p)\right)$ is given by $\lambda_{\max_{V_{ap}}}\left(H_a^T(p_a)PH_a(p_a)\right)$.*

Corollary 2. *Let*

$$H(g(\pi)) = \frac{\sum_{i_1,i_2,\cdots,i_\mu \in \{0,1\}} H_{i_1,i_2,\cdots,i_\mu} g_1(\pi)^{i_1} g_2(\pi)^{i_2} \cdots g_\mu(\pi)^{i_\mu}}{\sum_{i_1,i_2,\cdots,i_\mu \in \{0,1\}} h_{i_1,i_2,\cdots,i_\mu} g_1(\pi)^{i_1} g_2(\pi)^{i_2} \cdots g_\mu(\pi)^{i_\mu}}, \tag{16}$$

$$\pi \in \Pi \subset R^\nu, g = [g_1\ g_2 ... g_\mu]^T, H_{i_1,i_2,\cdots,i_\mu} \in R^{n\times l},$$

$$h_{i_1,i_2,\cdots,i_\mu} \in R,$$

be a nonsingular matrix function, where Π is a compact set and each function g_i, $i = 1,2,..,\mu$, is continuous with respect *to π, and $P \in R^{n\times n}$ be a p.d. symmetric matrix. Then, if $l = n$, an upper estimate of $\lambda_{\max\limits_{\pi\in\Pi}}(Q(g(\pi))P^{-1})$, $Q(g(\pi)) = H^T(g(\pi))P+PH(g(\pi))$, is given by $\lambda_{\max\limits_{p\in V_p}}(Q(p)P^{-1})$, where V_p is the set of vertices of*

$$\wp = \left\{ p \in R^{\mu} : \left[\min g_1 \cdots \min g_{\mu}\right]^T \le p \le \left[\max g_1 \cdots \max g_{\mu}\right]^T \right\}. \quad (17)$$

Similarly, an upper estimate of $\lambda_{\max\limits_{\pi\in\Pi}}\left(H^T(g(\pi))PH(g(\pi))\right)$ is given by $\lambda_{\max\limits_{p\in V_p}}\left(H^T(p)PH(p)\right)$..

Lemma 2. *Let $A \in R^{n\times n}$ be a matrix with ν real distinct eigenvalues $\lambda_i, i = 1,\ldots,\nu$, and $\mu = \frac{n-\nu}{2}$ distinct pairs of complex conjugate eigenvalues $\lambda_{h\pm} = \alpha_h \pm j\omega_h$, $h = 1,\ldots,\mu$, and let $u_i, i = 1,\ldots,\nu$, and $u_{h\pm} = u_{ah} \pm ju_{bh}$, $h = 1,\ldots,\mu$, be the associated eigenvectors. Then, denoting as Z^* the conjugate transpose of the matrix of the eigenvectors $Z = [u_1 \ldots u_{\nu}\ u_{a1}+ju_{b1}\ u_{a1} - ju_{b1} \ldots u_{a\mu}+ju_{b\mu}u_{a\mu} - ju_{b\mu}]$, the matrix*

$$\hat{P} = (ZZ^*)^{-1} = \left[\sum_{i=1}^{\nu} u_i u_i^T + 2\sum_{h=1}^{\mu}\left(u_{ah}u_{ah}^T + u_{bh}u_{bh}^T\right)\right]^{-1} \quad (18)$$

is always p.d. Furthermore,

$$\lambda_{\max}\left(Q\hat{P}^{-1}\right)/2 = \hat{\alpha} = \max(\mathrm{real}(\lambda(A))), \quad (19)$$

where $Q = A^T\hat{P} + \hat{P}A$.

Taking Remark 2 into account, it is important to analyze uncertain LTI MIMO systems, whose matrices are ratios of multi-affine functions with respect to the uncertain parameters. Concerning this, the following lemmas (see [29, 30] for proofs) are useful.

Lemma 3. *Given the system*

$$\begin{aligned} &\dot{x}(t) = A(p)x(t) + B(p)u(t) + E(p)w(t),\\ &y(t) = C(p)x(t),\\ &p \in [p^-,\ p^+,] = \wp \subset R^{\mu}, \|u\| \le U, \|w\| \le W, \end{aligned} \quad (20)$$

where matrices $A \in R^{n\times n}$, $B \in R^{n\times r}$, $E \in R^{n\times l}$, $C \in R^{m\times n}$ are rational multi-affine with respect to the parameters $p \in \wp$, and a symmetric p.d. matrix $P \in R^{n\times n}$. Then, a majorant of 20 *is given by*

$$\dot{\rho} = \alpha\rho + bU + eW, Y = c\rho, \rho = \|x\|_P, \quad (21)$$

where

$$\begin{aligned} &\alpha = \max_{p\in V_p}\left(\lambda_{\max}\left(Q(p)P^{-1}\right)\right)/2 \ge \hat{\alpha}, Q(p) = A^T(p)P + PA(p),\\ &b = \max_{p\in V_p}\sqrt{\lambda_{\max}\left(B^T(p)PB(p)\right)}, e = \max_{p\in V_p}\sqrt{\lambda_{\max}\left(E^T(p)PE(p)\right)},\\ &c = \max_{p\in V_p}\sqrt{\lambda_{\max}\left(C(p)P^{-1}C^T(p)\right)}, \end{aligned} \quad (22)$$

and V_p is the set of vertices of the hyper-rectangle $\wp$.

Lemma 4. *Given the system*

$$\begin{aligned} &\dot{x}(t) = A(p)x(t) + G(p)\gamma(t),\\ &y(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^T = C(p)x(t),\\ &p \in [p^-,\ p^+] = \wp \subset R^{\mu},\ \gamma(t) \in [-\hat{\gamma},\ \hat{\gamma}] = \Gamma \subset R^{\eta},\ \hat{\gamma} \ge 0, \end{aligned} \quad (23)$$

where matrices $A \in R^{n\times n}$, $G \in R^{n\times\eta}$, $C \in R^{m\times n}$ are rational multi-affine with respect to the parameters $p \in \wp$, and a symmetric p.d. matrix $P \in R^{n\times n}$. Then, some majorants of 23 *are given by*

$$\dot{\rho} = \alpha\rho + \beta, \begin{cases} Y_j = c_j\rho, \rho = x_P, j = 1, 2, \cdots, m,\\ Y = c\rho, \rho = x_P, \end{cases} \quad (24)$$

where

$$\begin{aligned} &\alpha = \max_{p\in V_p}\left(\lambda_{\max}\left(Q(p)P^{-1}\right)\right)/2, Q(p) = A^T(p)P + PA(p),\\ &\beta = \max_{p\in V_p}\sqrt{\hat{\gamma}^T G^T(p)PG(p)\hat{\gamma}},\\ &c_j = \max_{p\in V_p}\sqrt{C_j(p)P^{-1}C_j^T(p)},\\ &c = \max_{p\in V_p}\sqrt{\lambda_{\max}\left(C(p)P^{-1}C^T(p)\right)}, \end{aligned} \quad (25)$$

$C_j(p)$ is the j-th row of $C(p)$, and V_p is the set of vertices of the hyper-rectangle $\wp$.

Remark 6. Note that the majorant 21 yields an upper estimate for evolution of the system 20 for each $t \in R$, $x_0 \in R^n$, $p \in [p^-,\ p^+]$, $u(t) : \|u(t)\| \le U$, $w(t) : \|w(t)\| \le W$.

For example, if $\tau = -1/\alpha > 0$, after a time equal to 4.6τ, an upper estimate of $\|y(t)\|$ is given by

$$\|y(t)\| \le Y \cong \tau c(bU + eW). \quad (26)$$

Similarly, the majorants 24 yield an upper estimate for evolution of the absolute value of each output

y_i (or $\|y(t)\|$) of system 23 for each $t \in R, x_0 \in R^n$, $p \in [p^-, p^+], \gamma(t) \in [-\hat{\gamma}, \hat{\gamma}]$.

Evidently, it depends on P how close this estimate would be to the real value. An inappropriate choice of P may result in a positive value of α, even if $A(p)$ has all eigenvalues with negative real parts $\forall p \in \wp$.

If for an assigned $\hat{p} \in \wp$ the matrix $A(\hat{p})$ has distinct eigenvalues (which is an easily verifiable condition), the corresponding matrix $\hat{P}$ given by 18 is always *p.d.*, and the equality 19, $\lambda_{\max}\left(Q(\hat{p})\hat{P}^{-1}\right)/2 = \max(\text{real}(\lambda(A(\hat{p}))))$, always holds, even if $A(\hat{p})$ has some eigenvalues with non-negative real parts.

A better estimate of the maximum absolute output values of the system 23, but more difficult to compute, can be obtained using the following theorem. This estimate is also valid in case of uncertain matrices of non-multi-affine type and in the presence of an additional bounded nonlinearity γ.

Theorem 1. *Given the system*

$$\begin{aligned} \dot{x}(t) &= A(p)x(t) + B\gamma(t, x(t), \upsilon(t), p), \quad x(0) = 0 \\ y(t) &= C(p)x(t), \end{aligned} \tag{27}$$

where $x \in R^n$, $y \in R^m$, $\gamma(t) \in [-\hat{\gamma}, \hat{\gamma}] = \Gamma \subset R^\eta$, $\hat{\gamma} \geq 0$, $\upsilon \in R^\nu$ is an external signal, and A, B, C are constant matrices of appropriate dimensions. Then,

$$|y_j(t)| \leq \int_0^t |C_j(p)e^{A(p)\sigma}B(p)|\hat{\gamma}d\sigma, \tag{28}$$

where C_j is the j-th row of C.

Proof. Note that the solution to the system 27 satisfies the equation

$$\begin{aligned} x(t) &= \int_0^t e^{A(p)(t-\tau)}B(p)\gamma(\tau, x(\tau), \upsilon(\tau), p)d\tau \\ &= \int_0^t e^{A(p)\tau}B\gamma(t-\tau, x(t-\tau), \upsilon(t-\tau), p)d\tau. \end{aligned} \tag{29}$$

Indeed,

$$\begin{aligned} &\frac{d}{dt}\left(e^{At}\int_0^t e^{-A(p)\tau}B(p)\gamma(\tau, x(\tau), \upsilon(\tau), p)d\tau\right) \\ &= Ae^{At}\int_0^t e^{-A(p)\tau}B(p)\gamma(\tau, x(\tau), \upsilon(\tau), p)d\tau \\ &\quad + e^{A(p)t}e^{-A(p)t}B(p)\gamma(t, x(t), \upsilon(t), p) \\ &= Ax(t) + B\gamma(t, x(t), \upsilon(t), p) = \dot{x}(t). \end{aligned} \tag{30}$$

The equation 29 implies that

$$\begin{aligned} |y_j(t)| &= \left| C_j \sum_{i=1}^{\mu} \int_0^t e^{A(p)\tau}B_i\gamma_i(t-\tau, x(t-\tau), \upsilon(t-\tau), p)d\tau \right| \\ &\leq \sum_{i=1}^{\mu} \int_0^t \left|C_j e^{A(p)\tau}B_i\right| d\tau \hat{\gamma}_i, \end{aligned} \tag{31}$$

where B_j is the j-th column of B, and γ_j, $\hat{\gamma}_j$ are the j-th rows of γ, $\hat{\gamma}$, respectively. The inequality 28 follows from 31 and the second last introduced notation, i.e., $|\{f_{ij}\}| = \{|f_{ij}|\}$.

4 | CONTROLLER DESIGN

This section presents the main results allowing one to design robust controllers to track generic reference signals with an acceptable precision, despite the presence of disturbances with bounded ν-th order derivatives.

Note that

$$A_c = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ -C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{32}$$

$$[-K_e - K_{i1} - K_{i2} \cdots - K_{i\nu}] = A_0 - B_0K,$$

where

$$A_0 = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ -C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \in R^{(n+\nu m)\times(n+\nu m)}, \tag{33}$$

$$B_0 = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^{(n+\nu m)\times r}.$$

Hence, if (A_0, B_0) is reachable, it is well-known that the eigenvalues of A_c can be assigned arbitrarily for a pre-fixed p by making a suitable choice of K, taking into account that they are placed symmetrically with respect to the real axis of the complex plane.

Concerning reachability of the pair (A_0, B_0), the following result holds.

Theorem 2. *Let $A \in R^{n\times n}$,$B \in R^{n\times r}$,$C \in R^{m\times n}$ be a triple of matrices. If*

$$rank\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = rank\begin{bmatrix} -A & B \\ -C & 0 \end{bmatrix} = rank\begin{bmatrix} -A & B \\ C & 0 \end{bmatrix} = n+m \quad (34)$$

and (A,B) *is reachable, then* (A_0, B_0) *is also reachable, where* A_0 and B_0 *are defined in* 33.

Proof. Given a pair of matrices ($F \in R^{\nu\times\nu}$,$G \in R^{\nu\times\rho}$). If

$$rank([\lambda I - F \quad G]) = \nu, \forall \lambda \in \mathbb{C}, \quad (35)$$

where $\mathbb{C}$ is the set of complex numbers, the pair (F, G) is reachable. Hence, if

$$rank[\lambda I_{n+m\nu} - A_0 \quad B_0] = n + m\nu, \forall \lambda \in \mathbb{C}, \quad (36)$$

(A_0, B_0) is reachable.

Since (by swapping some columns)

$$rank[\lambda I_{n+m\nu} - A_0 \quad B_0]$$

$$= rank\left[\begin{array}{cc|cccccc} \lambda I_n - A & B & 0 & 0 & \cdots & 0 & 0 \\ C & 0 & \lambda I_m & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & -I_m & \lambda I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda I_m & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_m & \lambda I_m \end{array}\right] \quad (37)$$

$$= rank\left[\begin{array}{cc|cccccc} \lambda I_n - A & B & 0 & 0 & \cdots & 0 & 0 \\ \hline C & 0 & \lambda I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & -I_m & \lambda I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda I_m & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_m & \lambda I_m \end{array}\right],$$

the relation 36 follows from the first equality of 37 and 34, if $\lambda = 0$. Otherwise, if $\lambda \neq 0$, the relation 36 follows from the second equality of 37, taking into account that (A,B) is reachable.

The following theorems are useful to compute majorants of the control system.

Theorem 3. *The control system* 6 *can also be represented as*

$$\begin{aligned} &\dot{\zeta} = A_c\zeta + B_c r^{(\nu)} + E_c d^{(\nu)}, \\ &e = H\zeta, \ H = \left[0_{m\times(n+m(\nu-1))} \quad I_m\right]. \end{aligned} \quad (38)$$

Proof. Upon making the change of variables

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = A_c\xi^{(\nu-1)} + B_c r^{(\nu-1)} + E_c d^{(\nu-1)}, \ \zeta_1 \in R^{n+m(\nu-1)}, \zeta_2 \in R^m, \quad (39)$$

one obtains

$$\dot{\zeta} = A_c\xi^{(\nu)} + B_c r^{(\nu)} + E_c d^{(\nu)}. \quad (40)$$

From 6, it follows that

$$\xi^{(\nu)} = A_c\xi^{(\nu-1)} + B_c r^{(\nu-1)} + E_c d^{(\nu-1)} = \zeta. \quad (41)$$

Now, the first equation of 38 follows from 39, 40, and 41.

The first equation in 6 and 39 implies that

$$\begin{aligned} \zeta = &A_c^{\nu}\xi + A_c^{\nu-1}B_c r + A_c^{\nu-2}B_c\dot{r} + \cdots + B_c r^{(\nu-1)} \\ &+ A_c^{\nu-1}E_c d + A_c^{\nu-2}E_c\dot{d} + \cdots + E_c d^{(\nu-1)}, \end{aligned} \quad (42)$$

which yields, after some manipulations, $\zeta_2 = e$, and, therefore, the second equation of 38.

Corollary 3. *Consider the control system* 6*. If* $\nu = 2$ *then*

$$\begin{aligned} &\dot{\zeta} = A_c\zeta + B_c r^{(\nu)} + E_c d^{(\nu)}, \\ &\dot{r} - \dot{y} = \dot{e} = H_1\zeta, \ H_1 = [0_{m\times n} \quad I_m \quad 0_{m\times m}]; \end{aligned} \quad (43)$$

if $\nu = 3$ *then*

$$\begin{aligned} &\dot{\zeta} = A_c\zeta + B_c r^{(\nu)} + E_c d^{(\nu)}, \\ &\dot{e} = \dot{r} - \dot{y} = H_1\zeta, \ H_1 = \left[0_{m\times(n+m)} \quad I_m \quad 0_{m\times m}\right], \\ &\ddot{e} = \ddot{r} - \ddot{y} = H_2\zeta, \ H_2 = \left[0_{m\times n} \quad I_m \quad 0_{m\times(2m)}\right]. \end{aligned} \quad (44)$$

Proof. The proof follows from 40, 7, and the relations

$$\begin{aligned} &\dot{e} = H\dot{\zeta} = H\left(A_c\zeta + B_c r^{(\nu)} + E_c d^{(\nu)}\right), \\ &\ddot{e} = H_1\dot{\zeta} = H_1\left(A_c\zeta + B_c r^{(\nu)} + E_c d^{(\nu)}\right). \end{aligned} \quad (45)$$

Corollary 4. *Consider the control system (6). If* $r(t)$ *and* $d(t)$ *have bounded derivatives of order* $\nu+\mu$, $\mu \geq 1$, *and are equal to zero for* $t = 0$, *then*

$$\dot{\zeta} = A_c\zeta + B_c r^{(\nu+\mu)} + E_c d^{(\nu+\mu)},$$
$$e^{(\mu)} = H\zeta\ . \tag{46}$$

Proof. Denoting as $F(s) = L(f(t))$ the Laplace transform of a generic function $f(t)$, one obtains $E(s) = H(sI - A_c)^{-1}s^{\nu}(B_cR(s)+E_cD(s))$.

Hence, $L(e^{(\mu)}(t)) = s^{\mu}E(s) = H(sI - A_c)^{-1}s^{\nu+\mu}(B_cR(s)+E_cD(s))$, and the proof follows.

The control system 38 is observable, considering the tracking error $e(t)$ as the output, since the following theorem states.

Theorem 4. *If the pair of matrices* $(A \in R^{n\times n}, C \in R^{m\times n})$ *is observable, then the pair of matrices* (A_c, H) *is also observable.*

Proof. The proof readily follows by noting that

$$rank\left[\lambda I_{n+\nu m} - A_c^T\ H^T\right] = n+\nu m, \forall\lambda\in\mathbb{C}. \tag{47}$$

Consider now the control system in Figure 2, where the plant is given by 2. In view of Theorem 3, this control system can be represented as

$$\dot{\zeta} = A_c\zeta + B_c r^{(\nu)} + E_c d^{(\nu)},$$
$$e = H\zeta, \tag{48}$$

where

$$A_c = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ -C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} K = A_0 - B_0K,$$
$$B_c = \begin{bmatrix} BK_p \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ E_c = \begin{bmatrix} E - BK_pD \\ -D \\ 0 \\ \vdots \\ 0 \end{bmatrix}, H = \left[0_{m\times(n+m(\nu-1))}\ \ I\right], \tag{49}$$

$$K = \left[-K_e\ \ -K_{i1}\ \ -K_{i2}\ \cdots\ K_{i\nu}\right],\ K_e = K_s - K_pC.$$

The following main theorem holds.

Theorem 5. *Suppose that the plant 2 satisfies the condition 3 and* $\wp = [p^-, p^+]$. *If* $\hat{p}$ *is the nominal value of* p *then there exists a matrix* K *such that the eigenvalues of the matrix* $\hat{A}_c = A_c(\hat{p})$ *are equal to* $n+\nu m$ *pre-fixed symmetric values* $\hat{\lambda}$ *of* $\mathbb{C}$. *In addition, if the eigenvalues* $\hat{\lambda}$ *are distinct, then,* $\forall r(t), d(t) : \|r^{(\nu)}(t)\| \le R_\nu, \|d^{(\nu)}(t)\| \le D_\nu$, *a majorant of the system 48 is given by*

$$\dot{\rho} = a_c\rho + b_cR_\nu + e_cD_\nu,\quad E = h_c\rho, \tag{50}$$

with

$$a_c = \max_{p\in V_p}\frac{\lambda_{\max}\left(Q(p)\hat{P}^{-1}\right)}{2}, Q = A_c(p)^T\hat{P} + \hat{P}A_c(p), \tag{51}$$

$$b_c = \max_{p\in V_p}\sqrt{\lambda_{\max}\left(B_c{}^T\hat{P}B_c\right)},\quad e_c = \max_{p\in V_p}\sqrt{\lambda_{\max}\left(E_c{}^T\hat{P}E_c\right)}, \tag{52}$$

$$h_c = \sqrt{\lambda_{\max}\left(H\hat{P}^{-1}H^T\right)},$$

where $\rho = \|\zeta\|_{\hat{P}}$, $\hat{P} = \left(\hat{Z}\hat{Z}^*\right)^{-1}$, $\hat{Z}$ *is the matrix of the eigenvectors of* $\hat{A}_c$, *and* V_p *being the set of vertices of the hyperrectangle* $\wp$.

Proof. The proof follows from Lemma 1 and Theorem 3.

Remark 7. Note that if $\wp = \hat{p}$ and all the eigenvalues of A_c have negative real parts, the time constant $\tau_c = -1/a_c$ of the majorant system is positive and coincides, in view of Lemma 2, with the maximum time constant $\hat{\tau}_c$ of the control system.

If $\tau_c > 0$, after a transient phase whose practical duration depends on τ_c, the tracking error $e(t)$ satisfies the relation

$$\|e(t)\| \le Y = H_r\max\left\|r^{(\nu)}(t)\right\| + H_d\max\left\|d^{(\nu)}(t)\right\|$$
$$= H_rR_\nu + H_dD_\nu,\quad H_r = \tau_ch_cb_c,\ H_d = \tau_ch_ce_c. \tag{53}$$

In view of Corollary 4, the i-th derivative of the tracking error $e^{(i)}(t)$ is given by

$$\left\|e^{(i)}(t)\right\| \le Y = H_r\max\left\|r^{(\nu+i)}(t)\right\| + H_d\max\left\|d^{(\nu+i)}(t)\right\|, \forall i\ge 1. \tag{54}$$

Alternatively, in view of Corollary 3,

$$\left\|e^{(i)}(t)\right\|\leq H_{ri}\max\left\|r^{(\nu)}(t)\right\|+H_{di}\max\left\|d^{(\nu)}(t)\right\|,$$
$$i=1\text{ if }\nu=2,\ i=1,2\text{ if }\nu=3, H_{ri}=\tau_c h_{ci} b_c,\ H_{di}=\tau_c h_{ci} e_c,\ h_{ci}=\sqrt{\lambda_{\max}\left(H_i \hat{P}^{-1} {H_i}^T\right)}. \tag{55}$$

Similar estimates of $\left|e_j^{(i)}(t)\right|$, $j = 1,\ldots,m$, $i = 0,1,2$, can be obtained using Lemma 4, Theorems 1, 3, and Corollaries 3, 4.

Remark 8. Given a reference signal $r(t)$, the change of variable $t = \tau/\rho$ yields $dr/d\tau = \frac{dr/dt}{\rho}$, $d^\nu r/d\tau^\nu = \frac{d^\nu r/dt^\nu}{\rho^\nu}$. Hence, halving "velocity" (i.e., assuming $\rho = 2$) makes the second derivative ("acceleration") four times less and reduces the maximum tracking error accordingly. Dividing the "velocity" by three ($\rho = 3$) makes the second derivative ("acceleration") nine times less, etc. Similarly, if "velocity" is halved ($\rho = 2$), the third derivative becomes eight times less, and the maximum tracking error is reduced accordingly.

Remark 9. It is well-known that the proportional action makes the control system faster and generally leads to a reduction of the error $e(t)$. On the other hand, the control signal may increase due to sudden variations of $r(t)$ and/or $d(t)$. For example, if $\zeta_0 = 0$, then $u(0) = K_p(r(0) - Dd(0))$. Hence, it is appropriate to make the matrix K_p bounded, $\left|K_p\right| \leq \hat{K}_p$. Note that once the matrix K (and, therefore, the matrix K_e) is computed, for a fixed matrix K_p the relation $K_e = K_s - K_pC$ implies $K_s = K_pC+K_e$.

If the initial state of the control system is not very large (which can be obtained using a suitably filtered reference or using an appropriate joint reference signal before the desired reference), the matrix K_p can be chosen such that for $p=\hat{p}$ the tracking error norm $\|e\|$ depends on $\|r^{(\nu+1)}(t)\|$ and not on $\|r^{(\nu)}(t)\|$. Indeed, the following theorem holds.

Theorem 6. For $p=\hat{p}$ it is possible to choose a matrix K_p such that

$$\|e(t)\|=\|r(t)-y(t)\|\leq\hat{e}_{\max}\leq H_r R_{\nu+1}+H_d D_\nu,\ \forall r(t),\ d(t):\left\|r^{(\nu+1)}(t)\right\|\leq R_{\nu+1}, \left\|d^{(\nu)}(t)\right\|\leq D_\nu. \tag{56}$$

Proof. For prefixed eigenvalues of $\hat{A}_c=\hat{A}_0-K\hat{B}_0$ and, hence, of K, one obtains

$$E(s)=H_r(s)(s^\nu R(s))+H_d(s)(s^\nu D(s)),$$
$$H_r(s)=H\left(sI-\hat{A}_c\right)^{-1}\hat{B}_c=H\left(sI-\hat{A}_c\right)^{-1}\begin{bmatrix}\hat{B}K_p\\ I\end{bmatrix}=\frac{N(s)}{d(s)}=\frac{N_1(s)s+N_2}{d(s)}, N_2=N_{21}K_p+N_{22}. \tag{57}$$

Therefore, it is possible to choose K_p such that $N_2 = 0$ and then, from the relation $K=\left[K_p\hat{C}-K_s\quad -K_i\right]$, to compute K_s and K_i. Making such a choice of K_p it is $E(s) = H_r(s)(s^\nu R(s)) = (N_1(s)/d(s))(s^{\nu+1}R(s))$, from which 56 follows.

5 | EXTENSIONS

The methodology provided in the previous sections can be used to establish other new results. In the following, for brevity, only two are stated in the case $\nu = 1$. The first result deals with the case of systems with unmeasurable states, and the second one treats a class of nonlinear system with additional bounded nonlinearities.

5.1 | Observer-based controller

If the state is unmeasurable, an "observer" can be introduced as in Figure 3, where

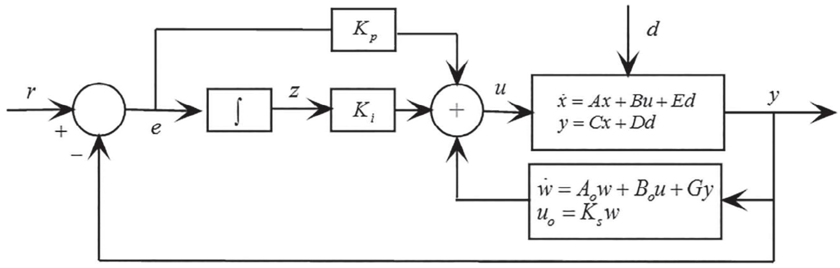
$$A_o=\hat{A}-G\hat{C}, \hat{A}=A(\hat{p}), \hat{B}=B(\hat{p}), \hat{C}=C(\hat{p}), \tag{58}$$

$\hat{p}$ is the nominal value of p and G is a design matrix. The control scheme in Figure 3 yields

$$\dot{\xi}=\begin{bmatrix}A-BK_pC & BK_i & BK_s\\ -C & 0 & 0\\ (G-\hat{B}K_p)C & \hat{B}K_i & \hat{A}-G\hat{C}+\hat{B}K_s\end{bmatrix}\xi+\begin{bmatrix}BK_p\\ I\\ \hat{B}K_p\end{bmatrix}r+\begin{bmatrix}E-BK_pD\\ -D\\ (G-\hat{B}K_p)D\end{bmatrix}d=A_c\xi+B_cd+E_cd,$$
$$\xi=\begin{bmatrix}x\\ e\\ z\end{bmatrix},$$
$$e=[-C\ \ 0\ \ 0]\xi+r-Dd\ . \tag{59}$$

Upon making the change of variables

FIGURE 3 Control scheme with observer



$$\zeta = \begin{bmatrix} \zeta_1 \\ e \\ \zeta_2 \end{bmatrix} = A_c\xi + B_c r + E_c d, \quad \zeta_1, \zeta_2 \in R^n \tag{60}$$

the system 59 can be rewritten as follows:

$$\begin{aligned} &\dot{\zeta} = A_c\zeta + B_c\dot{r} + E_c\dot{d}, \\ &e = H\zeta, \ \ H = [0_{m\times n} \ \ I_m \ \ 0_{m\times n}]. \end{aligned} \tag{61}$$

Note that for $p = \hat{p}$

$$\begin{aligned} &eig(\hat{A}_c) \\ &= eig\left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & -I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A} - \hat{B}K_p\hat{C} & \hat{B}K_i & \hat{B}K_s \\ -\hat{C} & 0 & 0 \\ (G - \hat{B}K_p)\hat{C} & \hat{B}K_i & \hat{A} - G\hat{C} + \hat{B}K_s \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & -I \end{bmatrix} \right) \\ &= eig\left(\begin{bmatrix} \hat{A} - \hat{B}(K_p\hat{C} - K_s) & \hat{B}K_i & -\hat{B}K_s \\ -\hat{C} & 0 & 0 \\ 0 & 0 & \hat{A} - G\hat{C} \end{bmatrix} \right) = eig\left(\begin{bmatrix} \hat{A} - \hat{B}(K_p\hat{C} - K_s) & \hat{B}K_i \\ -\hat{C} & 0 \end{bmatrix} \right) \cup eig(\hat{A} - G\hat{C}). \end{aligned} \tag{62}$$

Therefore, if $p = \hat{p}$, the separation principle for eigenvalues is valid. Specifically, note that if $d \neq 0$ and/or $p \neq \hat{p}$, the observer does not provide the state of the system and, hence, has to be considered as a dynamic compensator.

From the above considerations, the controller can be designed by computing matrices K, G to assign prefixed negative real-part eigenvalues to the matrices

$$\hat{A}_1 = \begin{bmatrix} \hat{A} & 0 \\ -C & 0 \end{bmatrix} - \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} K, \ \ K = [K_p\hat{C} - K_s \ \ -K_i], \ \hat{A}_2 = \hat{A} - G\hat{C} \tag{63}$$

and, afterwards, optimize some performance indices of the control system with respect to K_p and the eigenvalues of $\hat{A}_1, \hat{A}_2 \ \ \forall p \in \wp$.

5.2 | Systems with additional bounded nonlinearities

Consider the system

$$\begin{aligned} &\dot{x}(t) = A(p)x(t) + B(p)u(t) + Eg(x, u, d, p), \\ &y(t) = C(p)x(t), \end{aligned} \tag{64}$$

where $g(x,u,d,p) \in R^l$ is a bounded nonlinearity, i.e., there exists a $\delta > 0$ such that $\|g\| \leq \delta$.

If the system 64 is controlled with the control law

$$\dot{z} = r - y = e, \ \ u = K_s x + K_p e + K_i z, \tag{65}$$

the control system turns out to be

$$\dot{\xi} = \begin{bmatrix} A + BK_e & BK_i \\ -C & 0 \end{bmatrix}\xi + \begin{bmatrix} BK_p \\ I \end{bmatrix} r + \begin{bmatrix} E \\ 0 \end{bmatrix} g = A_c\xi + B_c r + E_c g, \tag{66}$$

$$K_e = K_s - K_p C, e = [-C \ \ 0]\xi + r = C_c\xi + r.$$

By making the change of variables

$$\zeta = \begin{bmatrix} \zeta_1 \\ e \end{bmatrix} = A_c\xi + B_c r, \tag{67}$$

the system 66 can be rewritten as

$$\begin{aligned} \dot{\zeta} &= A_c\zeta + B_c\dot{r} + G_c g, \quad G_c = A_c E_c, \\ e &= H\zeta, \quad H = [0 \;\; I]. \end{aligned} \tag{68}$$

Since the nonlinearity g is bounded, if $\dot{r}$ is also bounded, the controller can be designed via the majorant system approach using Lemmas 3, 4 or via input–output (*i-o*) increase, applying Theorem 1 to the control system 68.

Remark 10. Clearly, the tracking error can be acceptable using the proposed controller also if the nonlinearity g is not bounded, but bounded in a sufficiently large neighborhood, $r, \dot{r}, d, p-\hat{p}$ and the initial state of the control system are sufficiently bounded.

An estimate of the asymptotic practical stability region can be determined via majorant system (see 33 in [30]) or via randomized simulations [35].

6 | DESIGN ALGORITHMS

In this section, some algorithmic procedures to design robust controllers are provided.

Remark 11. Once K and K_p are fixed, some less conservative (smaller) values of τ_c, H_r and H_d can be obtained using the following algorithm.

Algorithm. Step 1 *Divide the set* $\wp$ *in N hyper-rectangles* $\wp_i = [p_i^-, \, p_i^+]$..

Step 2 *Determine the majorant systems corresponding to each hyper-rectangle* $\wp_i$, *with* $\hat{p}_i = \frac{p_i^- + p_i^+}{2}$, *or equal to a close value, if cond*(P_i) $\gg$ 1. *Then, compute the corresponding values of* a_{ci}, H_{ri} and H_{di} *by (51)-(53).*

Step 3 *Compute* α_c, τ_c, H_r and H_d *by using the relations*

$$\alpha_c = \max(\alpha_{ci}), \tau_c = -1/\alpha_c, H_r = \max\{H_{ri}\}, H_d = \max\{H_{di}\}. \tag{69}$$

Similar algorithms can be developed, if the majorants of the control systems are computed using Lemma 4 and Theorem 1 for each $\wp_i = [p_i^-, \, p_i^+]$.

Remark 12. A majorant system of the *i-o* type can be obtained as an envelope of N majorant systems.

Remark 13. Since the pair (A_0, B_0) is reachable in view of Theorem 2, the eigenvalues of A_c can be assigned arbitrarily for a fixed p. Furthermore, in view of Theorem 4, the control system is observable, considering the tracking error $e(t)$ as output. Hence, suitably choosing the matrices K and K_p, it is possible to stabilize the control system and optimize a quality index related to the tracking error.

Let $\hat{p}$ be the nominal value of p. The eigenvalues of $\hat{A}_c = A_c(\hat{p})$ can be chosen equal to $g\,\Lambda$, where g is a positive real optimization parameter belonging to an interval or set Υ, and Λ is a set of n+mv complex numbers with negative real parts, symmetric with respect to the real axis. A good choice of Λ is to consider the set of poles Λ_B of a low-pass n+mv-th order Butterworth (Bessel) filter with cutoff frequency $\omega_n = 1$, or the set of poles with real parts equal to -1 and imaginary parts such that the pole arguments are equal to the ones of Λ_B, or a set of n+mv poles belonging to the trapezoidal region $T = \{\lambda = \alpha + j\omega \in \mathbb{C} : \alpha \in [\breve{\alpha}, -1], |\omega| \le |\alpha| \sin\zeta, \zeta \in (0,1)\}$.

Now, it is possible to design the proposed controllers by considering certain specifications.

I. For instance, for a desired maximum error $\hat{e}_d$, setting $e = H_r R_\nu + H_d D_\nu$, the design algorithm consists in solving the min-max conditioned problem

$$\min_{g \in \Upsilon} \; \min_{K_p : |K_p| \le \hat{K}_p} \; \max_{p \in \wp} (\hat{e}_d - e)^2, \tag{70}$$

computing K for each fixed g with $p = p_n$.

This problem can be solved by using the MATLAB commands *place* and *fmincon* (see e.g. [27]).

If $\hat{e}_d = 0$ then 70 provides the controller minimizing e. If the obtained tracking error is not satisfactory, then it is possible to reduce the velocity of $r(t)$ by a factor ρ (see Remark 8).

II. It is also possible to design a controller solving the min-max conditioned problem

$$\min_{g \in \Upsilon} \; \min_{K_p : |K_p| \le \hat{K}_p} \; \max_{p \in \wp} (\hat{\tau}_{cd} - \tau_c)^2, \tag{71}$$

where $\hat{\tau}_{cd}$ is the desired maximum time constant. Then, compute H_r and H_d to obtain an estimate of the maximum value of e.

III. Finally, it is possible to design a controller minimizing the performance index

$$\min_{g\in\Upsilon}\ \min_{K_p:|K_p|\le\hat{K}_p}\ \max_{p\in\wp}\left(p_e(\hat{e}_d-e)^2+p_\tau(\hat{\tau}_{cd}-\tau_c)^2\right),\tag{72}$$

where p_e, p_τ are suitable positive weights.

7 | EXAMPLES

Three examples are provided: the first one is theoretical to illustrate the proposed methodology; the second and third ones are industrial, whose objective is to demonstrate utility and feasibility of the stated results and effectiveness of the designed controller.

Example 1. Consider the plant

$$\begin{aligned}\dot{x}&=Ax+Bu+Ed,\\ y&=Cx,\end{aligned}\tag{73}$$

where

$$A=\begin{bmatrix}0&1&0\\1&0&0\\1&1&(p+1)/(3p-1)\end{bmatrix},B=\begin{bmatrix}1&0\\0&1\\1&p\end{bmatrix},E=\begin{bmatrix}1\\0\\1\end{bmatrix},$$
$$C=\begin{bmatrix}2&0&0\\0&1&0\end{bmatrix},p\in[0.9,1.1]\ .\tag{74}$$

The objective is to design controllers to force the plant 73 to track sufficiently smooth reference signals with acceptable errors. In the following, the cases $\nu=1$ and $\nu=2$ are considered.

If

$$\begin{aligned}&\nu=1,\\ &\begin{vmatrix}\dot{r}_1(t)\\ \dot{r}_2(t)\\ \dot{d}(t)\end{vmatrix}\le\begin{bmatrix}1\\1.5\\1.25\end{bmatrix}=\hat{\gamma},\\ &\Lambda=\{-1.0000\pm3.0777i,-1.0000,\ -1.0000\pm0.7265i\},\\ &\Upsilon=\{0.5,0.75,1\},|K_p|\le\begin{bmatrix}1&1\\1&1\end{bmatrix}=\hat{K}_p,\end{aligned}\tag{75}$$

then an optimal controller (i.e., minimizing $\|e(t)\|$), using the provided Algorithm with Lemma 4, $p_n=1$, and $N=20$, is given by

$$\dot{z}_1=e,\ u=K_pe+K_{i1}z_1+K_sx,\tag{76}$$

where

$$K_p=\begin{bmatrix}-0.8236&-0.9624\\0.5130&-0.3026\end{bmatrix},K_{i1}=\begin{bmatrix}-0.2313&-0.6441\\-0.3211&0.1867\end{bmatrix},\tag{77}$$
$$K_s=\begin{bmatrix}0.2267&0.3973&-4.0040\\1.5732&-0.6015&-1.0710\end{bmatrix}x.$$

For the reference signals and disturbance shown in Figure 4, the tracking errors obtained for $p\in[0.9,1.1]$ with increments by 0.025 are displayed in Figure 5.

Figure 5 shows that $\|e(t)\|\le5.9125$, whereas using the provided Algorithm with Lemma 4 and $N=20$ yields $\|e(t)\|\le8.4116$.

The maximum time constant $\hat{\tau}_c$ is numerically calculated as $\hat{\tau}_c=2.7620$, whereas the estimate obtained by using the provided Algorithm with Lemma 4 and $N=20$ is equal to $\tau_c=2.8975$. In addition, Theorem 1 implies that

$$\begin{bmatrix}|e_1(t)|\\|e_2(t)|\end{bmatrix}\le\begin{bmatrix}3.4658&1.5173&1.7533\\1.3903&2.8695&1.4872\end{bmatrix}\hat{\gamma}.\tag{78}$$

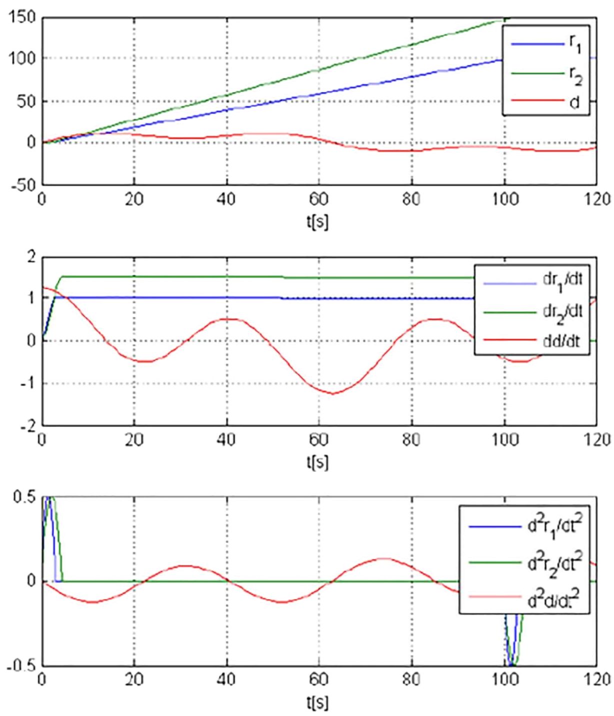


FIGURE 4 Time histories of $r_1,r_2,d,\ \dot{r}_1,\dot{r}_2,\dot{d},\ \ddot{r}_1,\ddot{r}_2,\ddot{d}$. [Color figure can be viewed at wileyonlinelibrary.com]

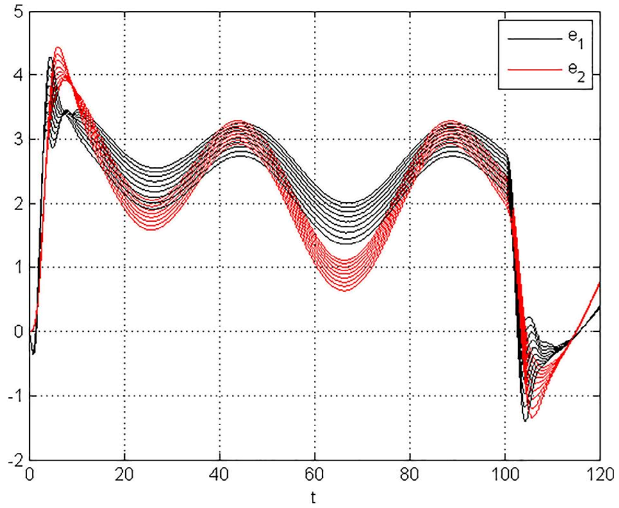


FIGURE 5 Time histories of e_1,e_2 using the controller 77 [Color figure can be viewed at wileyonlinelibrary.com]

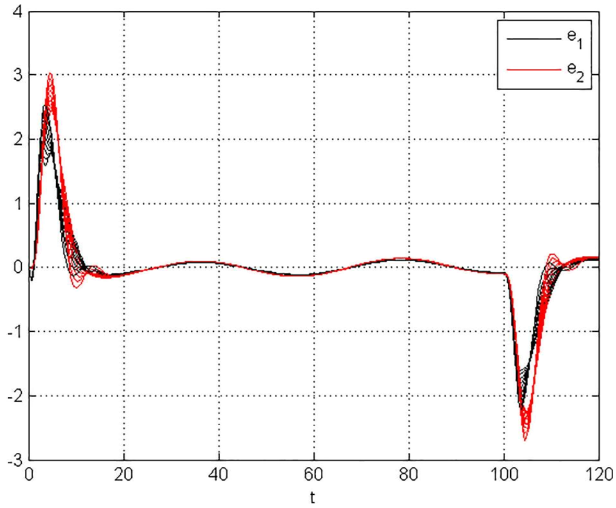


FIGURE 6 Time histories of e_1,e_2 using the controller 80 [Color figure can be viewed at wileyonlinelibrary.com]

If

$$\nu=2,\quad \begin{vmatrix} \ddot{r}_1(t) \\ \ddot{r}_2(t) \\ \ddot{d}(t) \end{vmatrix} \le \begin{bmatrix} 0.5 \\ 0.5 \\ 0.125 \end{bmatrix} = \hat{\gamma},$$
$$\Lambda=\{-1\pm4.3813i, -1\pm1.2540i, -1\pm0.4816i, -1\},$$
$$\Upsilon=0.5, |K_p| \le \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \hat{K}_p, \tag{79}$$

then an optimal controller (i.e., minimizing $\|e(t)\|$), using the provided Algorithm with Lemma 4, $p_n = 1$, and $N = 20$, is given by

$$\dot{z}_1=e, \dot{z}_2=z_1, u=K_pe+K_{i1}z_1+K_{i2}z_2+K_sx, \tag{80}$$

where

$$K_p=\begin{bmatrix} -0.9612 & -0.9946 \\ -0.0277 & -0.0296 \end{bmatrix}, \tag{81}$$
$$K_{i1}=\begin{bmatrix} -1.3033 & -3.1828 \\ -0.9201 & 0.7040 \end{bmatrix},$$
$$K_{i2}=\begin{bmatrix} -0.2459 & -0.6891 \\ -0.2815 & 0.2279 \end{bmatrix},$$
$$K_s=\begin{bmatrix} 3.9798 & 4.5477 & -8.7330 \\ 1.0923 & -0.6401 & -1.0587 \end{bmatrix}.$$

For the reference signals and disturbance shown in Figure 4, the tracking errors obtained for $p \in [0.9,1.1]$ with increments by 0.025 are displayed in Figure 6.

Figure 6 shows that $|e_1(t)| \le 2.5518, |e_2(t)| \le 3.0402$, whereas using Theorem 1 yields

$$\begin{bmatrix} |e_1(t)| \\ |e_2(t)| \end{bmatrix} \le \begin{bmatrix} 4.1149 & 3.9251 & 1.3682 \\ 1.3028 & 7.9299 & 1.3798 \end{bmatrix} \begin{bmatrix} \max|\ddot{r}_1| \\ \max|\ddot{r}_2| \\ \max|\ddot{d}| \end{bmatrix} = \begin{bmatrix} 4.1914 \\ 4.7891 \end{bmatrix}, \forall p \in [0.9, 1.1]. \tag{82}$$

The maximum time constant $\hat{\tau}_c$ is numerically calculated as $\hat{\tau}_c = 3.3675$, whereas the estimate obtained by using the provided Algorithm with Lemma 4 and $N = 20$ is equal to $\tau_c = 3.5917$.

Note that during the phase where $\ddot{r}_1(t)=\ddot{r}_2(t)=0$ the tracking error is smaller. Obviously, if also $\ddot{d}(t)=0$, then after a transient phase the tracking error is null.

Example 2. Consider a Cartesian planar robot. Assuming that each activation system is an electric DC motor, the model of each axis is of the form 2 with

$$A=\begin{bmatrix} -R/L & -K/L & 0 \\ K/M & -K_a/M & 0 \\ 0 & 1 & 0 \end{bmatrix}, B=\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}, \tag{83}$$
$$E=\begin{bmatrix} 0 \\ -1/M \\ 0 \end{bmatrix}, C=[0\ \ 0\ \ 1], D=0,$$

and $x^T=[x_1\ \ x_2\ \ x_3]=[i\ \ \upsilon=\dot{y}\ \ y]$; i is measured in A, υ in cm/s, and y in cm.

Suppose that $R = 1$, $L = 0.010$, $K = 5$, $M = 0.50$, $K_a = p \in [0.40, 0.60]$.

If

$$\begin{aligned} &\nu = 1, \\ &\begin{vmatrix} \dot{r}_i(t) \\ \dot{d}(t) \end{vmatrix} \le \begin{bmatrix} 1.2 \\ 0 \end{bmatrix} = \hat{\gamma}, \\ &\Lambda = \{-1 \pm 2.4142i, -1 \pm 0.4142i\}, \\ &\Upsilon = 10, |K_p| \le 4 = \hat{K}_p, \end{aligned} \tag{84}$$

then an optimal controller (i.e., minimizing $\|e(t)\|$), using the provided Algorithm with Lemma 4, $p_n = 0.5$, and $N = 4$, is given by

$$\begin{aligned} \dot{z}_1 = e,\ u = K_p e + K_{i1} z_1 + K_s x = 2.40e \\ + 80.00 z_1 + [0.610 \quad 3.839 \quad -13.60] x. \end{aligned} \tag{85}$$

Suppose that the robot's task is to cut the metal ring in Figure 7 from a metal sheet by using a laser beam with a constant cutting velocity in 120*s*.

Figure 8 displays the smoothed reference signals $r_1 = r_x$ and $r_2 = r_y$ reported with the corresponding velocities and accelerations, which are obtained by using two third-order single-input single-output (SISO) Bessel filters with $\omega_b = 5 rad/s$ (initial conditions $[r_i(0)\ 0\ 0]^T$, $i = 1,2$) from the initial piecewise-linear references obtained by linearly interpolating a finite number of points.

Assuming that the reference signals are the ones in Figure 8 and the disturbance is absent, the tracking errors are displayed in Figure 9, while the control voltages are shown in Figure 10. Figure 11 displays the obtained metal ring.

The maximum time constant $\hat{\tau}_c$ is numerically calculated as $\hat{\tau}_c = 0.1019$, whereas the estimate provided by using the provided Algorithm with Lemma 4 and $N = 4$ is equal to $\tau_c = 0.1026$.

Using Theorem 1 yields

$$|e_i(t)| \le 0.1706 \max|\dot{r}_i| + 9.7675\text{e}-4 \max|\dot{d}|, \forall p \in [0.4, 0.6]. \tag{86}$$

If

$$\begin{aligned} &\nu = 2, \\ &\begin{vmatrix} \ddot{r}_i(t) \\ \ddot{d}(t) \end{vmatrix} \le \begin{bmatrix} 3.4 \\ 0 \end{bmatrix} = \hat{\gamma}, \\ &\Lambda = \{-1 \pm 3.0777i, -1, -1 \pm 0.7265\}, \\ &\Upsilon = 10, |K_p| \le 10 = \hat{K}_p, \end{aligned} \tag{87}$$

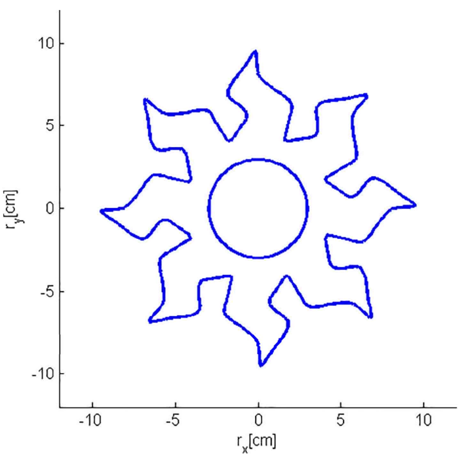


FIGURE 7 Desired metal ring [Color figure can be viewed at wileyonlinelibrary.com]

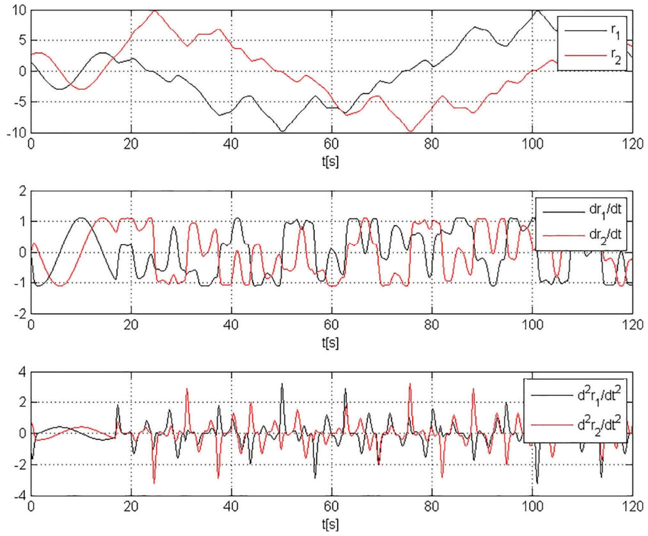


FIGURE 8 Time histories of $r_1, r_2,\ \dot{r}_1, \dot{r}_2,\ \ddot{r}_1, \ddot{r}_2$. [Color figure can be viewed at wileyonlinelibrary.com]

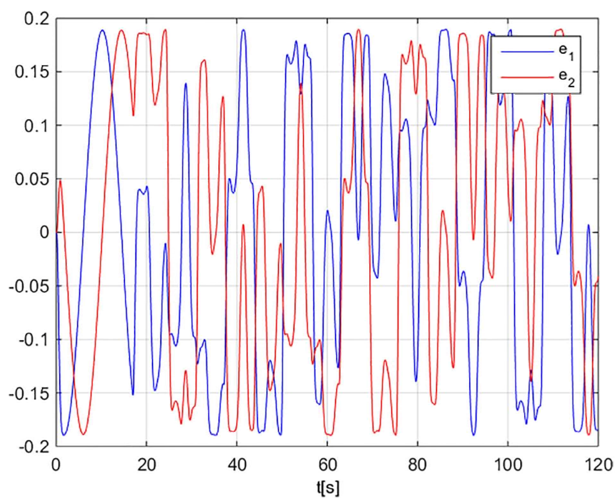


FIGURE 9 Time histories of e_1, e_2 obtained with the controller 85 [Color figure can be viewed at wileyonlinelibrary.com]

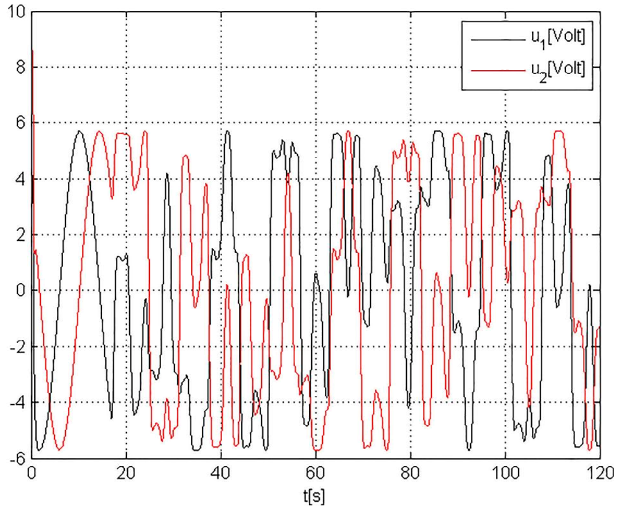


FIGURE 10 Time histories of control voltages [Color figure can be viewed at wileyonlinelibrary.com]

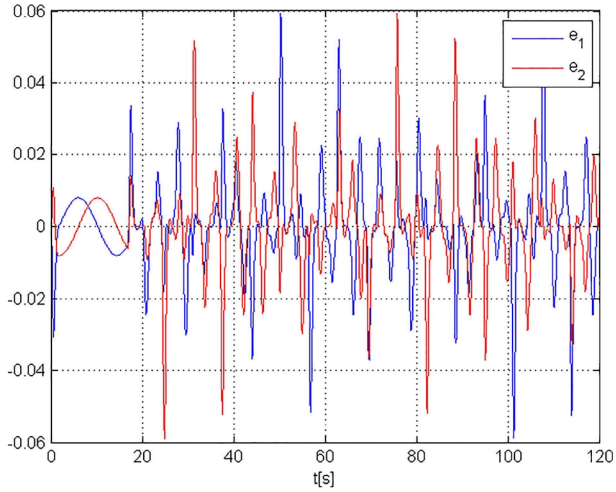


FIGURE 12 Time histories of e_1,e_2 using the controller 88 ($\nu = 2$). [Color figure can be viewed at wileyonlinelibrary.com]

then an optimal controller (i.e., minimizing $\|e(t)\|$) using the provided Algorithm with Lemma 4, $p_n = 0.5$, and $N = 4$, is given by

$$\dot{z}_1 = e,\ \dot{z}_2 = z_1,\ u = K_p e + K_{i1} z_1 + K_{i2} z_2 + K_s x$$
$$= 8.90e + 400.00z_1 + 1600.00z_2 + [0.510 \quad 3.049 \quad -31.100]x. \tag{88}$$

Assuming that the references are shown in Figure 8 and the disturbance is absent, the tracking errors are displayed in Figure 12.

The maximum time constant $\hat{\tau}_c$ is numerically calculated as $\hat{\tau}_c = 0.1020$, whereas the estimate provided by using the provided Algorithm with Lemma 4 and $N = 4$ is equal to $\tau_c = 0.1031$.

Using Theorem 1 yields

$$|e_i(t)| \le 0.0194 \max|\ddot{r}_i| + 6.1263e-5 \max|\ddot{d}|, \forall p \in [0.4, 0.6]. \tag{89}$$

If the cutting velocity is halved (Figure 13), then the tracking errors for the controller 85 ($\nu = 1$) and the

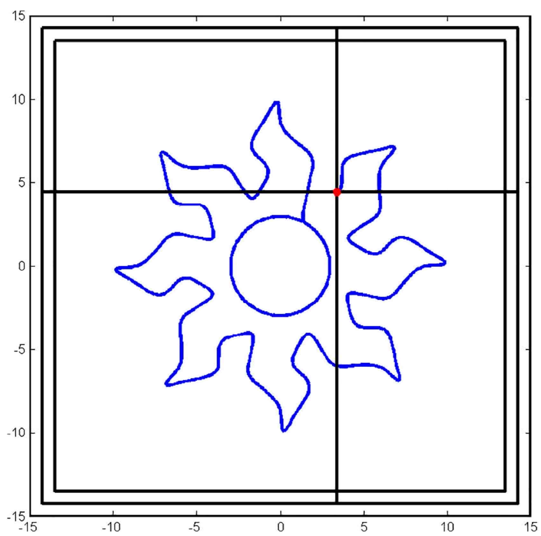


FIGURE 11 Obtained metal ring [Color figure can be viewed at wileyonlinelibrary.com]

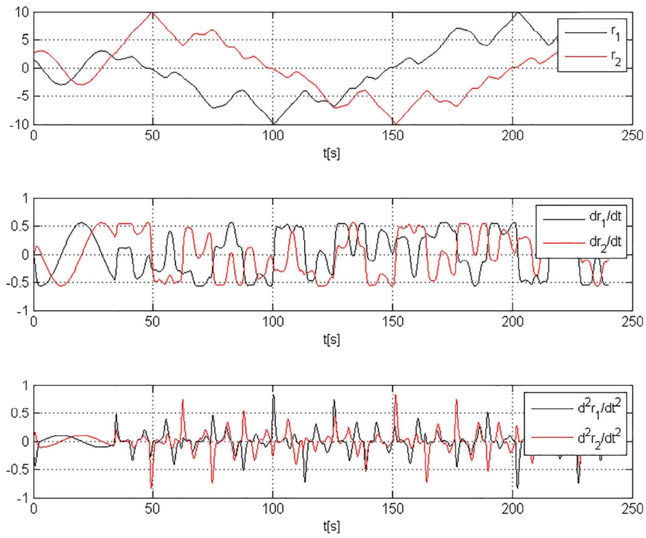


FIGURE 13 Time histories of r_1,r_2, $\dot{r}_1$,$\dot{r}_2$, $\ddot{r}_1$,$\ddot{r}_2$ with a halved cutting velocity [Color figure can be viewed at wileyonlinelibrary.com]

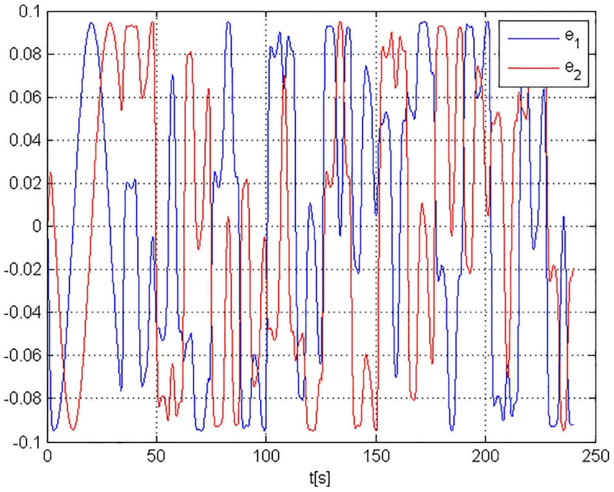


FIGURE 14 Time histories of e_1,e_2 for controller 85 with a halved cutting velocity (ν = 1). [Color figure can be viewed at wileyonlinelibrary.com]

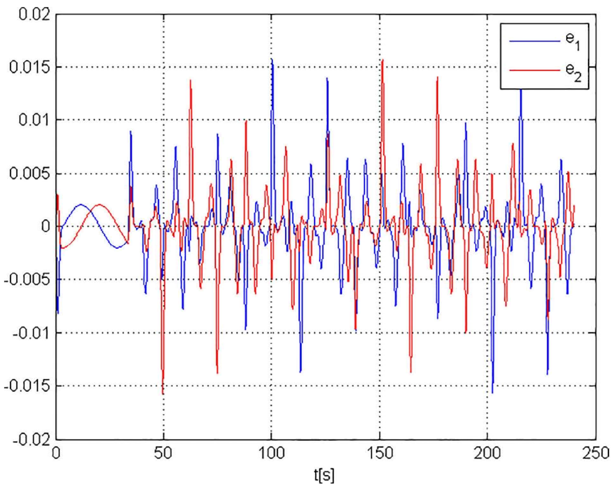


FIGURE 15 Time histories of e_1,e_2 for controller 88 with a halved cutting velocity [Color figure can be viewed at wileyonlinelibrary.com]

ones for the controller 88 (ν = 2) are reported in Figures 14 and 15, respectively. Note that the obtained errors are respectively the half and one-fourth of those in the previous case, in accordance with Remark 8. Clearly, upon reducing the velocity and acceleration, the control signal also decreases, as well as the motor power.

To reduce the tracking errors or increase the cutting velocity without reducing the errors, it is possible to increase the size of the optimization parameter set Υ. However, this approach may result in higher control signals.

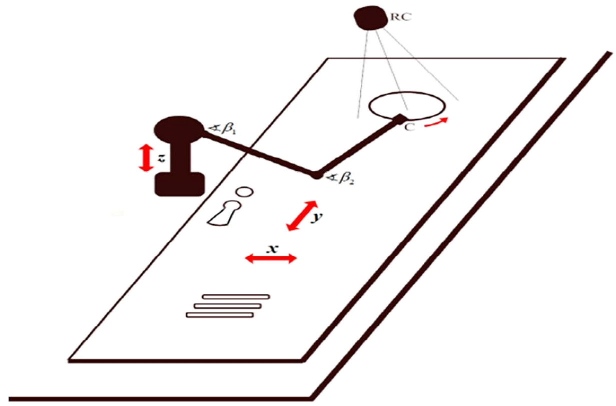


FIGURE 16 Milling machine [Color figure can be viewed at wileyonlinelibrary.com]

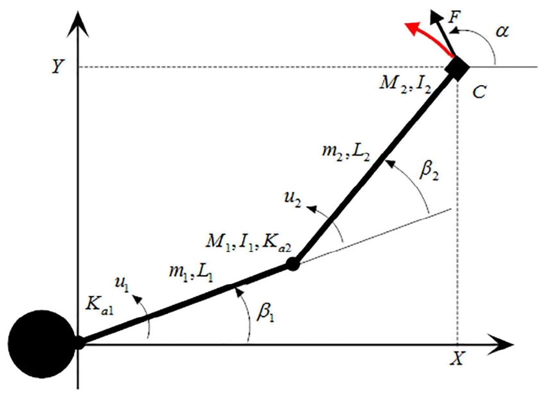


FIGURE 17 Handling robot of the milling cutter [Color figure can be viewed at wileyonlinelibrary.com]

Example 3. Consider a milling machine composed of a Cartesian conveyor to move the working table along the directions X – Y and a planar robot with rotoidal joints to move the milling cutter C (see Figures 16 and 17).

The robot model is given by

$$M(\beta_2)\ddot{\beta} + K_a\dot{\beta} + C(\beta_2, \dot{\beta}) = u + d, \tag{90}$$

where $\beta = [\beta_1 \quad \beta_2]^T, u = [u_1 \quad u_2]^T$, and $d = [d_1 \quad d_2]^T$ is the vector of disturbance torques due to interaction of the milling cutter with a workpiece. If the working area is sufficiently bounded, the controller can be designed using the linearized model of the robot.

Under the hypothesis that

$$L_1 = 0.6m, m_1 = 6Kg/m, M_1 = 0.5Kg, I_1 = 0.01Kgm^2,$$
$$L_2 = 0.6m, m_2 = 5Kg/m, M_2 = 0.5Kg, I_2 = 0.02Kgm^2, \quad (91)$$
$$K_a = diag([0.2\,0.1]),$$

the linearized model around the operating point $\beta_e = [-\pi/4 \ \ \pi/2]^T$, $u_e = 0$, and $d_e = 0$ is given by

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1063 & 0.1063 \\ 0 & 0 & 0.1063 & -0.4634 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.5313 & -0.5313 \\ -0.5313 & 2.3171 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.4243 & 0.4243 \\ 0 & 0.8485 \end{bmatrix} d, \quad (92)$$
$$y = \begin{bmatrix} 0 & -0.4243 & 0 & 0 \\ 0.8485 & 0.4243 & 0 & 0 \end{bmatrix} x,$$

where

$$x = \beta - \beta_e, u = [u_1 \ u_2]^T, d = F[\cos\alpha \ \sin\alpha]^T, \quad (93)$$
$$y = [X \ Y]^T - y_e, \ y_e = \left[0.6\sqrt{2} \ 0\right]^T.$$

$$\left[|e_{x1}| \ |e_{y1}|\right] \le [0.5156 \ \ 0.4827]mm, \left[|e_{x2}| \ |e_{y2}|\right] \le [0.2904 \ \ 0.2742]mm,$$
$$\left[|e_{x3}| \ |e_{y3}|\right] \le [0.2112 \ \ 0.2043]mm, \left[|e_{x4}| \ |e_{y4}|\right] \le [9.1255 \ \ 36.5622]mm. \quad (98)$$

If

$$\Lambda = \{-3 \pm 11.1962i, -3.0000 \pm 3.0000i, \ -3.0000 \pm 0.8038i\}, \quad (94)$$

the controller designed using Theorem 5 takes the form

$$u = K_p e + K_i \int e d\tau + K_s \begin{bmatrix} \beta - \beta_e \\ \dot{\beta} \end{bmatrix} \quad (95)$$
$$= K_p e + K_i \int e d\tau + K_{\dot{\beta}} \dot{\beta}, e = r - y,$$

where

$$K_p = \begin{bmatrix} -289.8144 & 122.9067 \\ -146.5444 & 3.2447 \end{bmatrix}, \ K_i = \begin{bmatrix} -576.0291 & 251.4648 \\ -290.2232 & 8.1366 \end{bmatrix},$$
$$K_s = [K_\beta \ \ K_{\dot{\beta}}] = \begin{bmatrix} 0 & 0 & -16.3490 & -28.9928 \\ 0 & 0 & -0.2071 & -10.4696 \end{bmatrix}. \quad (96)$$

Remark 14. Note that the designed controller is easy to realize, since it does not require the measurement of β (being $K_\beta = 0$), and the measurement of e can be obtained with a robot camera if the reference r is depicted on the workpiece. Moreover, this controller is also robust against the measurement noises, due to the integral action included in the control law 95, and the fact that the reaction of the joints' velocity is inside of the control cycle which uses the tracking error.

Now, suppose that the goal is to cut the elliptic windows (see Figure 18).

$$r_{xi} = a_i \cos(\vartheta(it_r)), \ r_{yi} = b_i \sin(\vartheta(it_r)), \quad (97)$$
$$a_i = 7.5icm, \ b_i = 5icm, \ i = 1, 2, 3, 4,$$

with the time history of $\vartheta(t_r)$ shown in Figure 19.

Assuming that the cutting force is equal to $F = -\eta\dot{y}, \eta = 15Ns/m$, the tracking error bounds are given by

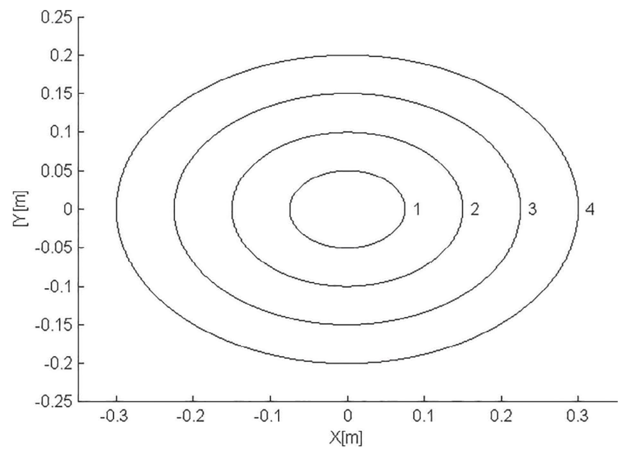


FIGURE 18 Elliptic windows

Using the designed controller with $\Lambda = \{-4 \pm i14.9282, -4 \pm i4, -4 \pm i\,1.0718\}$, the tracking error bounds are reduced to

$$\begin{aligned} \left[|e_{x1}|\ |e_{y1}|\right] &\le [0.2545\ \ 0.1889]mm, \left[|e_{x2}|\ |e_{y2}|\right] \le [0.1412\ \ 0.1062]mm, \\ \left[|e_{x3}|\ |e_{y3}|\right] &\le [0.1018\ \ 0.0788]mm, \left[|e_{x4}|\ |e_{y4}|\right] \le [0.0796\ \ 0.0617]mm. \end{aligned} \tag{99}$$

The time histories of r_x,r_y, tracking errors e_x,e_y, and control torques u_1,u_2 are reported for $i = 2$ and controller 95, 96 in Figure 20.

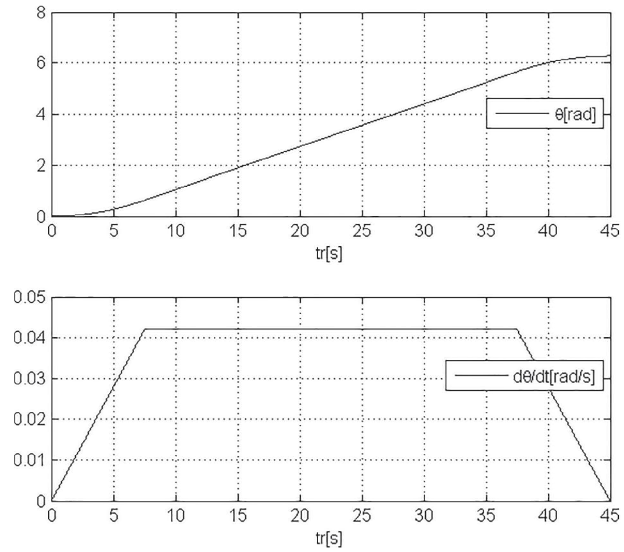


FIGURE 19 Time histories of ϑ and $\dot{\vartheta}$

Finally, for $i = 2$ and the controller 95, 96, if $\dot{\beta}$ is affected by a measurement noise uniformly distributed in the interval $\left[n^-_{\dot{\beta}_i}, n^+_{\dot{\beta}_i}\right] = [-2e^{-3}, 2e^{-3}]rad/s$ (6.7% of $\max\{\max|\dot{\beta}_1|, \max|\dot{\beta}_2\}$) and y is affected by a measurement noise uniformly distributed in the interval $\left[n^-_{y_i}, n^+_{y_i}\right] = [-0.2, 0.2]mm$, the tracking errors e_x,e_y and control torques u_1,u_2 are shown in Figure 21.

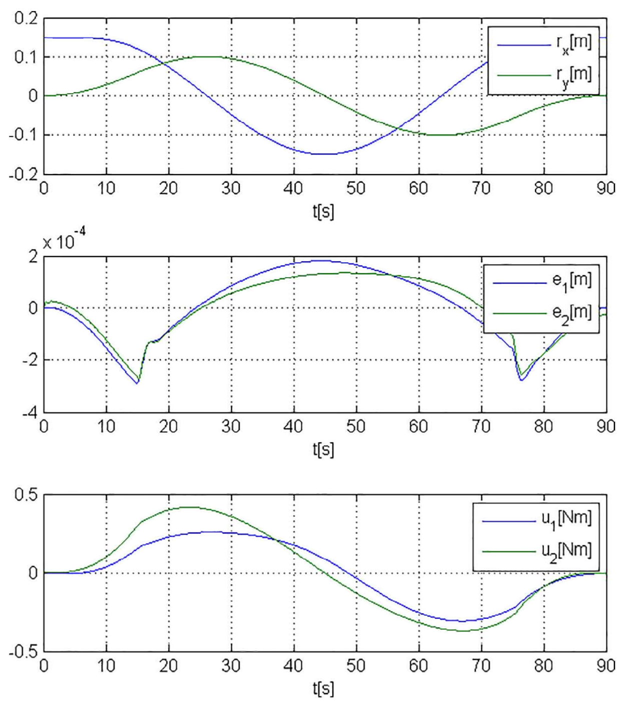


FIGURE 20 Time histories of r,e,u. [Color figure can be viewed at wileyonlinelibrary.com]

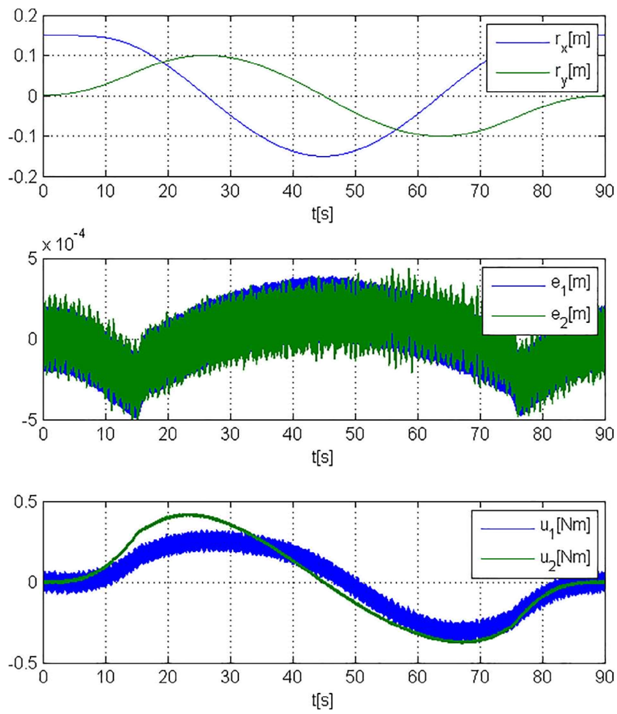


FIGURE 21 Time histories of r,e,u with $\dot{\beta}$ and y affected by noises [Color figure can be viewed at wileyonlinelibrary.com]

8 | CONCLUSIONS

In this paper, a broad class of LTI uncertain MIMO systems with bounded parametric uncertainties and/or affected by disturbances with bounded derivatives of order ν has been considered. The tracking problem has been systematically presented for generic sufficiently smooth reference signals, for instance, with bounded ν-th order derivatives. The employed majorant systems approach results in designing the state-feedback proportional-integral controllers of order ν. The case of systems with additional bounded nonlinearities and/or not directly measurable states has been studied as well.

The main advantages of the obtained results can be summarized as follows:

- uncertain LTI MIMO plants are considered generic;
- reference signals and/or disturbances are considered generic, with bounded ν - th order derivatives
- the obtained results allow one to systematically design robust controllers for tracking generic reference signals with an acceptable precision, despite the presence of measurement noises;
- the proposed controllers avoid the derivative action causing realization problems, especially in the presence of measurement noises.

Finally, the obtained results have been validated by three examples: the first one is given to illustrate the proposed methodology, whereas the second and third examples deal with a metal-cutting problem for an industrial robot and a composite robot, such as a milling machine, to show practical advantages and effectiveness of the designed controller.

The ongoing research is conducted on extending the above results to some classes of uncertain LTI MIMO processes and/or other types of controllers, to design robust control laws guaranteeing pre-fixed maximum tracking errors, considering also the physical constraints of the control signals.

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Laura Celentano: Conceptualization; Formal analysis; Funding acquisition; Methodology; Software; Supervision; Validation; Writing - original draft; Writing - review and editing. **Michael Basin:** Formal analysis; Funding acquisition; Supervision; Writing - original draft; Writing - review and editing. **Mohammed Chadli:** Supervision; Writing - original draft; Writing - review and editing.

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