

Area law from loop quantum gravity

Alioscia Hamma,¹ Ling-Yan Hung,² Antonino Marcianò,² and Mingyi Zhang^{2,3}

¹*Department of Physics, University of Massachusetts Boston, Massachusetts 02125-3393, USA*

²*Department of Physics, Fudan University, 220 Handan Road, 200433 Shanghai, China*

³*Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut),
Am Mühlenberg 1, 14476 Golm Germany, Europe*



(Received 4 July 2015; published 29 March 2018)

We explore the constraints following from requiring the area law in the entanglement entropy in the context of loop quantum gravity. We find a unique solution to the single-link wave function in the large j limit, believed to be appropriate in the semiclassical limit. We then generalize our considerations to multilink coherent states, and find that the area law is preserved very generically using our single-link wave function as a building block. Finally, we develop the framework that generates families of multilink states that preserve the area law while avoiding macroscopic entanglement, the space-time analogue of “Schrödinger’s cat.” We note that these states, defined on a given set of graphs, are the ground states of some local Hamiltonian that can be constructed explicitly. This can potentially shed light on the construction of the appropriate Hamiltonian constraints in the LQG framework.

DOI: [10.1103/PhysRevD.97.064040](https://doi.org/10.1103/PhysRevD.97.064040)

I. INTRODUCTION

In recent years, the study of entanglement has shed new lights in the fields of quantum field theory, quantum statistical mechanics and condensed matter theory. Whenever quantum fluctuations are important, entanglement plays an important role [1]. In the context of condensed matter physics, it has played a key role in classifying phases of matter, notably those with mass gaps, leading to the notion of topological orders [2–5] characterized by long-range entanglement [6–9]. A crucial quantifier of quantum entanglement is the entanglement entropy. For a wave-function that is the ground state of a local Hamiltonian, it is believed that the entanglement entropy should obey the so called area law [10]: in the limit of a large subregion A , its entanglement entropy is given by $S_A = -\text{Tr} \rho_A \ln \rho_A$, where ρ_A is the reduced density matrix of subsystem A , and S_A scales to leading order as the area of the boundary of A .

From lessons that have been drawn from black hole entropy, which is proposed to be understood as an entanglement entropy between degrees of freedom within and outside the horizon [11–13], and more recently evidence from the computation of entanglement entropy via the Ryu-Takayanagi formula [14] in the AdS/CFT correspondence, it appears that one very important feature of quantum gravity at least in the semiclassical limit, in which some classical background geometry can be defined, is that the leading contribution to the entanglement entropy associated to a sub-region A should also satisfy the area law [15].

Loop quantum gravity (LQG) [16–18] provides a perfect arena to test the requirement of area law, in which explicit calculations can be done. A long standing and central

question in the LQG is to understand the semi-classical limit. Thus far, semi-classical states have been proposed [19–22] in LQG, and studied in many contexts [23–40]. However, it is still by far an open question whether a semiclassical geometry has been undisputedly recovered. The current paper is partly inspired by recent work in LQG that recovers the Bekenstein’s black hole entropy [41–51] and several attempts in computing entanglement entropy in terms of spin network states [52–57], which is intimately related to the area law of the entanglement entropy [15,57].

In this letter, we inspect the semi-classical limit through the lens of entanglement entropy of a bounded subregion in LQG. We find that imposing the area law across a single triangular surface, the simplest possible scenario, already leads to a very tight constraint on the possible wave function, which admits an almost unique solution, one that acquires an interesting correction from what is envisaged in [15].

Ultimately the main goal of this paper is to show that entanglement imposes very stringent requirement to a theory of quantum gravity. We show how to use this guide in a constructive way. In particular, we impose that, together with the area-law, entanglement has to obey (i) SU(2) gauge invariance, and (ii) it must be *microscopic*, implying that fluctuations in quantum space-times should be confined to microscopic scales. These are very natural requirements if Entanglement is physically measurable. Gauge invariance means that entanglement does result in stronger than classical correlations in actual physical observables [58], and microscopic entanglement means that we do not allow arbitrarily large Schrödinger cat-spacetimes [59]. These considerations, together with the

single-link result, allows us to construct the many-body or multi-facet state of a general quantum geometry. As a bonus, we obtain a constructive method to build local Hamiltonians for given set of graphs, potentially important towards understanding the Hamiltonian constraints of LQG.

II. ONE-LINK STATES

Let us begin by setting the stage and notations. Our starting point is the $SU(2)$ spin-network states in the graph-fixed kinematical Hilbert space \mathcal{H}_Γ of LQG. An $SU(2)$ spin-network state is a triplet

$$|S\rangle = |\Gamma, j_l, i_n\rangle.$$

Γ is the given *proper* graph with L oriented links and N nodes. $j_l \in \mathbb{Z}^+/2$ is an assignment of an $SU(2)$ unitary irreducible representation to each link l , and i_n is an assignment of an $SU(2)$ intertwiner to each node n . The spin- j_l representation is isomorphic to a $SL(2, \mathbb{C})$ representation $(\gamma j_l, j_l)$ selected by the linear version of the simplicity constraints [60], namely

$$\vec{K} = \gamma \vec{L},$$

in which \vec{K} and \vec{L} are boost and rotation generators, respectively. The simplicity constraints are used in the spin foam formalism [61–64] for recovering gravity amplitudes from BF(topological) amplitudes [65,66] and for regaining the real connection [64,67]. Spin-network states correspond to the so called *twisted geometries* [68,69], which describe the geometries of three dimensional *fuzzy* discrete manifolds. Each N -valent node corresponds to a N -facet polyhedron, while each link is dual to the face of the polyhedron [70]. The areas of the faces are realized as the expectation value of the $SU(2)$ Casimir operator acting on the spin-network states, i.e., the area of the face dual to link l is

$$\langle \hat{\mathcal{A}}_l \rangle = 8\pi\gamma\ell_p^2 \sqrt{j_l(j_l + 1)}, \quad (1)$$

where $\gamma \in \mathbb{R}^+$ in the rest of the paper, and ℓ_p is the Planck length.

Given a state $|\Psi\rangle$ in \mathcal{H}_Γ and cut the graph Γ into two regions A and \bar{A} , the entanglement entropy S_A between these two regions is defined as the von Neumann entropy of the reduced density matrix

$$\rho_A \equiv \text{Tr}_{\bar{A}} |\Psi\rangle\langle\Psi|.$$

$\text{Tr}_{\bar{A}}$ is the partial trace over the states (i.e., the intertwiners' and spins' degrees of freedom) inside A . To recover locally a Minkowskian vacuum that respects the Bisognano-Wichmann theorem, it is proposed that the reduced density matrix ρ_A should be proportional to the exponentiation of

the boost operator K , leading to the following proposal for a single-link state [15,71]

$$|\Psi\rangle = \sum_j \alpha_j \exp(-\gamma K)_{mn} |j, m, n\rangle.$$

A careful check however suggests that it *does not* satisfy the area law in the large j limit, which is another crucial ingredient of the semiclassical limit, opposing naive expectation. Indeed, one can readily check that the normalization of the reduced density matrix always cancel the area term in this limit. Surprisingly, in the large j regime, there is a natural and arguably unique solution that recovers the area law across each link.

By construction, we will implement invariance under gauge and space-diffeomorphisms of the states $|\Psi\rangle$ that we seek to fulfill the area law requirement. More in general, any reduced density matrix that is related by unitary transformation to the density matrix associated to the $|\Psi\rangle$ state will have the entanglement entropy scaling as area, because of the expressions of the entanglement entropy involved. This ensures that our procedure is effective in order to determine a generic state $|\Psi\rangle$ with area law scaling in the large j limit, which is unique up to unitary transformations. At this purpose, we first recover in this section the single-link state that achieves this goal, then generalize in the next sections our proposal to multilink coherent states and to multinodes states, so to fully encode gauge and space-diffeomorphism invariance in the construction.

For a single-link, see e.g. Fig. 1, we can conveniently choose a state in the diagonalized basis with fixed spin j . This is expressed by

$$|\Psi\rangle = \frac{1}{\mathcal{N}} \sum_{m,n=-j}^j F_{mn}^j |j, m, n\rangle = \frac{1}{\mathcal{N}} \sum_{m=-j}^j F_{mm}^j |j, m, m\rangle,$$

having used the form of the coefficients in the diagonalized basis, i.e. $F_{mn}^j = F_{mm}^j \delta_{mn}$ and $|j, m\rangle \equiv |j, m, m\rangle$. The reduced density matrix casts

$$\rho_A \equiv \frac{1}{\mathcal{N}} \sum_{n=-j}^j \bar{F}_{nm} F_{nm'} |j, m, n\rangle \langle j, m', n|,$$

and because of the diagonalized basis involved, satisfies

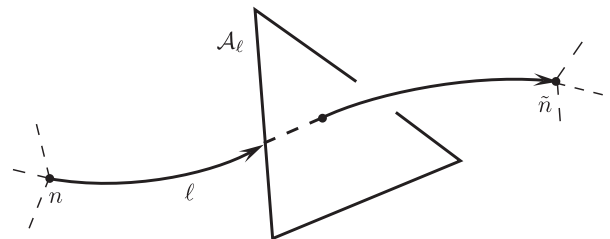


FIG. 1. A face A_l dual to link l .

$$\rho_A = \sum_{m=-j}^j p(m) |j, m\rangle \langle j, m|,$$

$$p(m) \equiv \frac{f(m)}{\sum_{m=-j}^j f(m)}, \quad (2)$$

with $f(m) = \bar{F}_{mm} F_{mm}$. Note that the function $f(m)$ is assumed not to carry explicit j dependence.

In what follows, by determining the form of $f(m)$, we show that

$$F_{mn}^j = D_{mn}^j \left(e^{-\pi\gamma L_z - \frac{\exp(1-2\pi\gamma L_z)}{4\pi\gamma}} \right),$$

where $D_{mn}^j(\dots)$ is defined as $\langle j, m | \dots | j, n \rangle$ and L_z represents the $\mathfrak{su}(2)$ generator in the z axis. In the next sections, we will use this single-link state as the building block to construct generalized multilink and multinode states.

We now focus on (2). Since we are interested in the large spin regime, the sum can be replaced by an integral

$$\sum_{m=-j}^j \rightarrow \int_{-j}^j dj.$$

Requiring that the entropy admits an area law and taking into account (1), we impose

$$S_A = cj + \dots,$$

the derivative of which with respect to j is thus a constant $c > 0$. This leads to a constraint on $f(m)$ ¹:

$$\frac{c}{p(j)} + cj - 1 + \ln p(j) = 0. \quad (3)$$

Equation (3) cannot be solved exactly in complete generality. However, anticipating that we are considering the large spin limit, and motivated by the original proposal, we consider the limit in which

$$\frac{c}{p(j)} \ll |cj - 1|.$$

In this limit, we have $p(j) = \exp(1 - cj)$, which finally gives

$$f(m) = \exp \left[-cm - \frac{\exp(1 - cm)}{c} \right]. \quad (4)$$

¹We have made the simplifying assumption that the probability density for $m < 0$ is negligible compared to $m > 0$. Relaxing this assumption does not lead to any material change.

One can readily check that $c/p(j) \ll |cj - 1|$ is satisfied in the large j limit, namely given $J \in \mathbb{R}^+$ in the limit $J \ll j$, if c satisfies

$$0 < J^{-1} \ll c \ll J^{-\frac{1}{2}} \ll 1. \quad (5)$$

In this limit the leading term in the entanglement entropy is indeed linear in cj , recovering the long sought area law. The expression for c is then straightforwardly determined to be $c = 2\pi\gamma$.

We remark that the area law is indeed an upper bound for the entanglement reachable by these states. In particular, there are non-logarithmic corrections scaling like the volume of the system. One consequence is that these states cannot be seen even locally as finite temperature equilibrium states. In quantum cosmology, if one sought to find reduced density matrices that represent thermal states, these should be obtained by tracing out high energy states. The ground state—or the low lying states—will only feature the area law. On the other hand, these states can be topologically ordered. Topological order does indeed consist in a global constraint on the allowed configurations on the boundary, resulting in a negative correction to the entanglement entropy [7–9, 72].

The function $p(m)$ also naturally suppresses contributions at small values of m in this limit, justifying the approximation that replaces the sum by the integral. It is very interesting that the state that we have found is very similar to the original proposal, up to an exponentially suppressed factor, which is negligible where $f(m)$ actually contributes, but suppresses regions which would otherwise have contributed in the original proposal. The constant c also emerges naturally and plays precisely the same role as the Barbero-Immirzi parameter γ , and (5) is exactly the semiclassical large j regime in the covariant formalism of LQG [31]. Recall that the boost operator K_z was related to L_z via the simplicity constraint

$$\vec{K} = \gamma \vec{L},$$

which followed from an action quadratic in the tetrad. One might be tempted to interpret the extra exponential correction we find here as a nonlinear correction arising from quantum effects.

We can extend our construction to the more complicated case of one node, with many links protruding from it. For simplicity, the graph Γ used in the following calculation is the graph with L out-pointing links attaching to only one node.

III. MULTILINK STATES

Combining our previous one-link proposal with the well known construction of multiple link coherent spin-network state [19–22], we are led to consider the state $|\Psi\rangle$ supported on \mathcal{H}_Γ , which reads

$$|\Psi\rangle = \frac{1}{\mathcal{N}} \sum_{\{j_l\}} \int_{\text{SU}(2)} dh \left(\prod_l^L \Delta_{(t_l, J_l)}^{j_l} \right) \times \sum_{\{m_l\}\{k_l\}} \sum_{\{n_l\}} \prod_l^L F_{m_l n_l}^{j_l} D_{n_l k_l}^{j_l}(h) |j_l, m_l, k_l^\dagger\rangle. \quad (6)$$

In (6) we defined: the normalization factor \mathcal{N} , which is recovered in Sec. B of the Appendix;

$$\Delta_{(t_l, J_l)}^{j_l} \equiv d_{j_l} e^{-(j_l - J_l)^2 t_l}$$

with $d_{j_l} = 2j_l + 1$; J_l , a given spin for the link l , and t_l , similar to a heat kernel time; the Wigner matrix of the group elements that encodes the correction found in (4), i.e.

$$F^{j_l} \equiv D^{j_l}(g_l e^{-\pi\gamma L_z + i\phi_l L_z} \frac{\exp(1-2\pi\gamma L_z)}{4\pi\gamma} \tilde{g}_l^\dagger), \quad (7)$$

in which $D_{mn}^j(\dots)$ is defined as $\langle j, m | \dots | j, n \rangle$; ϕ_l is a phase; g_l and \tilde{g}_l are SU(2) elements; L_z is the $\mathfrak{su}(2)$ generator in the z axis; v_i is the normalized intertwiner with

$$\bar{v}_i \cdot v_{i'} = \delta_{ii'}.$$

Note that this is a state that is factorizable into wavefunctions of individual links up to gauge constraints.

We cut the graph into two regions as in Fig. 2. Region \bar{A} contains the node. The reduced density matrix ρ_A is recovered by tracing out the link source states (or the intertwiner degrees of freedom) in $|\Psi\rangle$. Furthermore, ρ_A can be rewritten in the block-diagonal form

$$\rho_A = \sum_{\{j_l\}} \rho_{\{j_l\}} = \sum_{\{j_l\}} P_{\{j_l\}} \hat{\rho}_{\{j_l\}},$$

where each block is characterized by the configuration of the $\{j_l\}$ eigenvalues of the boundary links. Crucially, such a block factorization follows from the SU(2) invariance

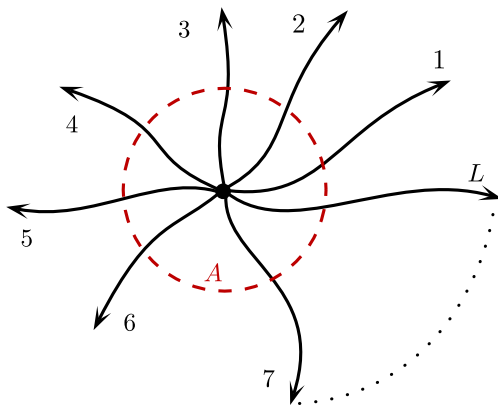


FIG. 2. A graph with L links and one node.

imposed at the node and the properties of the intertwiners. We can write $P_{\{j_l\}}$, the norm of the $\{j_l\}$'s block $\rho_{\{j_l\}}$, as

$$P_{\{j_l\}} \equiv \frac{1}{\mathcal{N}^2} \int_{\text{SU}(2)} dh \prod_l^L [\Delta_{(t_l, J_l)}^{j_l}]^2 \chi_{j_l}[\tilde{g}_l f(L_z) \tilde{g}_l^\dagger h]. \quad (8)$$

In (8) we use that: $f(L_z)$ is given by (4) with $c \equiv 2\pi\gamma$; $\chi_{j_l}(M)$ is the trace over matrix M ; $\hat{\rho}_{\{j_l\}}$ is the reduced density matrix with fixed $\{j_l\}$

$$\hat{\rho}_{\{j_l\}} \equiv \frac{1}{Z_{\{j_l\}}} \int_{\text{SU}(2)} dh \prod_l^L F^{j_l} \cdot D(h) \cdot F^{j_l \dagger}, \quad (9)$$

where F^{j_l} is given by (7); $Z_{\{j_l\}}$ is the normalization such that $\text{Tr} \hat{\rho} = 1$. The integration over SU(2) element h follows from the summation of intertwiner, which is proportional to an integration over Wigner matrices on SU(2)

$$\sum_i \bar{v}_i^{\{p_i\}} v_i^{\{p_i'\}} \propto \int_{\text{SU}(2)} dh \prod_l^L D_{p_l p_l'}^{j_l}(h).$$

The normalization has been absorbed into \mathcal{N} . Then the entanglement entropy

$$S_A = -\text{Tr} \rho_A \ln \rho_A$$

evaluates to

$$S_A = -\sum_{\{j_l\}} P_{\{j_l\}} \text{Tr} \hat{\rho}_{\{j_l\}} \ln \hat{\rho}_{\{j_l\}} - \sum_{\{j_l\}} P_{\{j_l\}} \ln P_{\{j_l\}}.$$

Note that:

- (i) $P_{\{j_l\}}$ can be understood as the probability density of detecting the given boundary configuration $\{j_l\}$, and that

$$\sum_{\{j_l\}} P_{\{j_l\}} = 1;$$

- (ii) it is a subtle issue to define the entanglement entropy in a gauge theory [73,74]—moving from the proposal in [15], the resultant S_A is in fact a SU(2) gauge invariant quantity which only has dependence on the eigenvalues of the Casimir of the boundary links, a choice equivalent to the proposal in [75];
- (iii) the entanglement entropy with a unique boundary link-configuration is given in the first term of (10), and it will be proved later to be the area term in the large spin limit.

To calculate S_A , we first give an explicit expression for $P_{\{j_l\}}$ and $\text{Tr} \hat{\rho}_{\{j_l\}} \ln \hat{\rho}_{\{j_l\}}$. As such, we introduce the coherent states on SU(2) [21], namely

$$D_{mj}^j(n)|j, m\rangle = n|j, j\rangle = n|\uparrow\rangle^{\otimes 2j},$$

$$\langle \dots \rangle \equiv \sum_{\{j\}} (\dots) P_{\{j\}}.$$

with $n \in \text{SU}(2)$. Imposing the $\text{SU}(2)$ coherent states resolution of identity into each χ_{j_l} in (8), $P_{\{j_l\}}$ becomes,

$$P_{\{j_l\}} = \frac{1}{\mathcal{N}^2} \int dh \prod_l^L dn_l [\Delta_{(j_l, J_l)}^t]^2 d_{j_l} \sum_{p_l} \frac{(-e)^{p_l}}{p_l! (2\pi\gamma)^{p_l}} e^{S_p},$$

where the ‘‘action’’ S_p is given by

$$S_p \equiv \sum_l^L 2j_l \ln \langle \uparrow | n_l^\dagger \tilde{g}_l e^{-2\pi\gamma(1+p_l)L_z} \tilde{g}_l^\dagger h n_l | \uparrow \rangle. \quad (10)$$

In order to study the large spin behavior of $P_{\{j_l\}}$, we rescale $j_l \rightarrow \lambda k_l$ and assume $\lambda \gg 1$. It is convenient to perform the asymptotic expansion to get the major contribution of $P_{\{j_l\}}$. The solutions of the equations of motion $\delta_{n_l} S_p = \delta_h S_p = 0$ control the semiclassical behavior of $P_{\{j_l\}}$. For $\delta_{n_l} S_p = 0$, the only possible solutions² are $n_l = \tilde{g}_l, h = \mathbf{1}$. Note that $\delta_h S_p = 0$ is equivalent to the Gauss constraint. Then $P_{\{j_l\}}$, to leading order of the asymptotic expansion and in the limit of (5), becomes

$$P_{\{j_l\}} = \frac{1}{\mathcal{N}^2} \prod_l^L [\Delta_{(j_l, J_l)}^t]^2 \frac{1}{j^{3/2}} \exp\left(-1 - \frac{e^{1-2\pi\gamma j}}{2\pi\gamma}\right). \quad (11)$$

where j is the average of j_l . In order to estimate the term $\text{Tr} \hat{\rho}_{\{j_l\}} \ln \hat{\rho}_{\{j_l\}}$, it is easier to use the replica trick to compute S_n , the Rényi entropy of order n [76]. From

$$\text{Tr} \hat{\rho}_{\{j_l\}}^n = \prod_l^L e^{(n-1)(1-2\pi\gamma j_l)}$$

and taking the $n \rightarrow 1$ limit, we get

$$-\text{Tr} \hat{\rho}_{\{j_l\}} \ln \hat{\rho}_{\{j_l\}} = \sum_l^L 2\pi\gamma j_l - 3/2 \ln \sum_l^L j_l^2.$$

The logarithmic terms is from the gauge invariance [48]. Finally, we have

$$S_A = \sum_l^L \langle 2\pi\gamma j_l \rangle - \frac{3}{2} \left\langle \ln \sum_l^L j_l^2 \right\rangle - \langle \ln P_{\{j_l\}} \rangle, \quad (12)$$

where we have denoted

²Another solution $-e\bar{n}_l = \tilde{g}_l, \text{Re}(\alpha_l) = -2\pi\gamma$ will lead $P_{\{j_l\}} \propto \exp(-e^{2\pi\gamma j})$ and suppressed faster than the solution $n_l = \tilde{g}_l, \text{Re}(\alpha_l) = 2\pi\gamma$ in the large spin limit.

The first term is the area \mathcal{A} of region A 's boundary. The second term has contribution which are proportional to link number L . Disregarding terms independent from the boundary area, when $t = J^{-k}$ the logarithmic correction amounts to $-3k/2 \ln(\mathcal{A}/\ell_p^2)$. So in the semiclassical limit of LQG (5), the entanglement entropy is

$$S_A = \frac{\mathcal{A}}{4\ell_p^2} + \mu L - \frac{3k}{2} \ln \frac{\mathcal{A}}{\ell_p^2}, \quad (13)$$

where the ‘‘chemical potential’’ term now carries only $1/j$ suppressed contributions.

IV. MULTINODE STATES

At this point, we generalize to multinode states. In the following, we show how we can produce a class of many-body states that satisfy all the requirements of area law and gauge invariance spelled above, and that can be the ground states of a local Hamiltonian.

Let us start with the simplest possible multinode state obtained by considering the tensor product of single-node states over all the nodes of the graph and then summing over all the $\{j_l\}, \{i_n\}$ in order to enforce gauge invariance. This state reads

$$\begin{aligned} |\Psi\rangle &\equiv \frac{1}{\mathcal{N}} \sum_{\{j_l\}, \{i_n\}} \prod_{l,n} \Delta_{(j_l, J_l)}^{j_l} F^{j_l} \cdot \bar{v}_{i_n} |\Gamma, j_l, i_n\rangle \\ &= \frac{1}{\mathcal{N}} \sum_{\{j_l\}, \{i_n\}} \prod_{l,n} |\varphi\rangle_{n,l}. \end{aligned} \quad (14)$$

The above many-body state still satisfies the area law, because, although the number of nodes and links contained in \bar{A} scales with its volume, however links that are completely enclosed within the region \bar{A} would not contribute to the entanglement entropy, which will in stead scale with the number of links crossing the boundary $\partial\bar{A}$ yielding the area law,

$$\mathcal{A} = \sum_{l \in \partial\bar{A}} \langle \hat{A}_l \rangle.$$

This statement is confirmed by a series of results within LQG that describe the space of black hole microstates—see e.g. Ref. [47] and formerly Ref. [77]. In particular, in Ref. [78] was argued that for an observer at infinity a black hole can be described by an $\text{SU}(2)$ intertwining operator, effectively corresponding to a ‘‘gigantic’’ single tensor operator intertwining all the links that puncture its horizon.

This procedure corresponds to the large j limit we are interested in for this analysis³

From the many-body point of view, this state is trivial, as there are no quantum fluctuations other than the single-body ones. However, exactly because it is trivial, it is easy to build a local quantum Hamiltonian such that the state in Eq. (14) is its ground state. Let π be the projector onto the gauge-invariant Hilbert space and $\pi_{n,l} = |\varphi\rangle\langle\varphi|_{n,l}$ the projector onto the local state $|\varphi\rangle_{n,l}$. Then the Hamiltonian

$$H(0) = -\pi \left(\sum_{n,l} \pi_{nl} \right) \pi^\dagger$$

is a local Hamiltonian whose ground state is state $|\Psi\rangle$ in Eq. (14).

Moreover, one has to admit that the ansatz Eq. (14), although motivated by the requirements of entanglement and gauge invariance discussed so far, still seems very arbitrary and fine tuned. We have no reason to care about a particular kinematical state, after all. What counts, are the physical states. And yet, physical states should be in the same “equivalence class” of the kinematical state defined above, in the sense that they must retain all the properties we want. Unfortunately, one cannot simply preserve these properties by just writing any state in the kinematical subspace. Indeed, a generic superposition of such kind of states, though, will first of all not respect the area law, and will contain arbitrarily macroscopic superpositions.

In order to form the right equivalence class, we need to deform the trivial state in a way that some many-body entanglement and quantum fluctuations are produced, but without violating the constraints we set up. This can be obtained by the technique of *quasiadiabatic continuation* introduced in [79]. In this way, a whole class of states can be defined, that constitute what is known in condensed matter theory as a quantum phase. This technique allows to continuously deform a quantum state that is the ground state of a local Hamiltonian in order that it is still the ground state of a local Hamiltonian. Let $H(\lambda)$ be a smooth family of *local* Hamiltonians parametrized by $\lambda \in [0, 1]$ such that

$$|\Psi_N\rangle = |\Psi(0)\rangle$$

is the ground state of $H(\lambda = 0)$ just like in Eq. (14). Here, local means that $H(\lambda)$ is the sum of local operators,

$$H(\lambda) = \sum_X \hat{T}_{X\subset\Gamma}(\lambda),$$

where X denotes a subset of the graph Γ and $\hat{T}_X(\lambda)$ means that this operator has only support on X . Also assume that $H(\lambda)$ has a finite gap ΔE between ground and first excited

state for all λ 's. One way to think of $H(\lambda)$ is as the perturbation of the initial Hamiltonian $H(0)$, namely,

$$H(\lambda) = H(0) + \lambda \sum_X \hat{K}_X$$

and \hat{K}_X are any local operators with support on X . Then, following [80], one can define a unitary operator

$$U(\lambda) = \mathcal{T} \exp \left\{ -i \int_0^\lambda \tilde{H}(s) ds \right\}$$

with \mathcal{T} the time ordering operator, and

$$\tilde{H}(s) = i \int dt F(t) e^{iH(s)t} \partial_s H(s) e^{-iH(s)t},$$

with $F(t)$ an appropriate fast decaying smooth function—see [80] for details. The unitary operator $U(\lambda)$ so defined interpolates among ground states of $H(\lambda)$ and it is therefore called *adiabatic continuation* [79]. The ground state of $H(\lambda)$, $|\Psi(\lambda)\rangle$, can be written as the adiabatic continuation of $|\Psi(0)\rangle$, namely

$$|\Psi(\lambda)\rangle = U(\lambda)|\Psi(0)\rangle.$$

Now, the adiabatic continuation preserves the area law [10]. In this way, we can always deform $|\Psi(0)\rangle$ in a way to obtain a new state $|\Psi(\lambda)\rangle$ such that its parent Hamiltonian $H(\lambda)$ is nonintegrable. The above definition is constructive as for any choice of the \hat{K}_X we can construct a family of local Hamiltonians and their gauge-invariant, area-law ground states.

As an application of this method, let us show that the so defined states do not have macroscopic superpositions, like macroscopic space-time Schrödinger cat states. The initial state $|\Psi\rangle$ is by construction not macroscopically entangled. We now show that by adiabatic continuation we also preserve the physicality of entanglement as being microscopic. Adiabatic continuation preserves macroscopic entanglement. So, if one continues a state that is not a Schrödingers' cat, one will not obtain a Schrödingers' cat. On the other hand, continuing a Schrödingers' cat will still yield a macroscopically entangled state. To show this, we apply a result obtained in [59]. The superselection rule is imposed by stating that states with non vanishing mutual information \mathcal{I}_∞ between two distant macroscopic regions must be ruled out. The mutual information is defined as

$$\mathcal{I}(A|B) := S(A) + S(B) - S(AB).$$

Here, S is chosen to be the 2—Rényi entropy

$$S_2 = -\log \text{Tr} \rho_A^2,$$

namely, the logarithm of the purity instead the von Neumann entropy. The technical reason for making this

³We thank the referee for whipping this clarification up.

choice is that one can prove that \mathcal{I}_∞ is preserved by quasi adiabatic continuation. The physical reason, is that this quantity is a physical observable (unlike the Von Neumann entropy) [81–83], but still captures all the entanglement properties that are important in quantum many-body theory [72,84]. As a bonus, starting with a trivial noninteracting Hamiltonian that is the sum of commuting local terms in the tensor product structure of Eq. (14), one can adiabatically continue the very Hamiltonian by means of any other Hamiltonian with a gap, which is the sum of local operators with couplings depending on some parameter λ . This is thus a systematic construction of local Hamiltonians whose ground states satisfy the desired properties we have set forth.

V. SUMMARY AND OUTLOOK

To conclude, we have shown that entanglement can provide stringent guiding principles in selecting states with sensible semiclassical limits in LQG. We obtain these results by imposing that entanglement must be physical (gauge invariant and not macroscopic), deploying methods from quantum information and quantum many-body theory. In perspective, the framework here developed will allow for the study of notions like quantum order or thermalization in a closed quantum system in the context of quantum gravity.

ACKNOWLEDGMENTS

We wish to thank the referee for whipping a clarification up in Sec. IV about the inclusion of multinodes states in our analysis. This work was supported in part by the National Basic Research Program of China Grants No. 2011CBA00300, No. 2011CBA00301 the National Natural Science Foundation of China Grants No. 61033001, No. 61361136003.

APPENDIX A: THE ENTANGLEMENT BETWEEN HALF LINKS

In this section of the appendix we provide an asymptotic analysis of the one-link state entanglement calculated between its two half links. We start by first checking the asymptotic analysis of the normalization of the one-link state.

The normalization of the state can be written as

$$\begin{aligned} \mathcal{N}^2 &\equiv I = d_j \int dn \langle j, j | n^\dagger e^{-cL_z} e^{-\frac{\exp(1-cL_z)}{c}} n | j, j \rangle \\ &= d_j \int dn \langle j, j | n^\dagger e^{-cL_z} \sum_k \frac{(-)^k}{k!c^k} e^{k(1-cL_z)} n | j, j \rangle \\ &= d_j \sum_k \frac{(-)^k}{k!c^k} \int dn \langle j, j | n^\dagger e^{-cL_z} e^{k(1-cL_z)} n | j, j \rangle \\ &= d_j \sum_k \frac{(-)^k e^k}{k!c^k} \int dn e^{S_k}, \end{aligned}$$

where

$$S_k = 2j \ln \langle \uparrow | n^\dagger e^{-c(1+k)L_z} n | \uparrow \rangle.$$

The latter is the ‘‘action’’ defined in Eq. (10) of the manuscript. We search the saddle point of S_k by varying it with respect to n , and obtain

$$\delta_n S_k = 2j \frac{\bar{\eta} \langle \downarrow | n^\dagger e^{-c(1+k)L_z} n | \uparrow \rangle + \eta \langle \uparrow | n^\dagger e^{-c(1+k)L_z} n | \downarrow \rangle}{\langle \uparrow | n^\dagger e^{-c(1+k)L_z} n | \uparrow \rangle} = 0.$$

The latter relation leads to

$$n^\dagger e^{-c(1+k)L_z} n = e^{-\alpha L_z}, \quad (\text{A1})$$

with α a complex number. Equation (A1) has two solutions: one is

$$nL_z n^\dagger = L_z, \quad \alpha^+ = c(1+k);$$

the other one is

$$nL_z n^\dagger = -L_z, \quad \alpha^- = -c(1+k).$$

On the saddle point, we then obtain

$$S_k^\pm = \mp c(1+k)j.$$

One of the leading contribution to the normalization I is given by

$$\begin{aligned} d_j \sum_{\epsilon=\pm} e^{-\epsilon c j} \sum_k \frac{(-)^k}{k!c^k} e^{(1-\epsilon c)k} \\ = d_j \sum_{\epsilon=\pm} \exp \left(-\epsilon c j - \frac{e^{(1-\epsilon c j)}}{c} \right) \equiv d_j \sum_{\epsilon=\pm} e^{S_\epsilon^0}. \end{aligned}$$

Another leading contribution comes from the Hessian. Indeed, the total leading contribution of I can be written as

$$I \sim d_j \sum_{\epsilon=\pm} \frac{e^{S_\epsilon^0}}{\sqrt{\det H^\epsilon}} \equiv \sum_{\epsilon=\pm} I^\epsilon.$$

In order to compute the Hessian, we introduce the variation of n on a spin j representation:

$$\begin{aligned} \delta_n n | j, j \rangle &= \delta_n | j, n \rangle \\ &= \delta_n | n \rangle^{2j} \\ &= \eta \sum_{i=1}^{2j} n \triangleright | \uparrow \cdots \downarrow_i \cdots \uparrow \rangle \\ &= \eta \sqrt{2j} n | j, j-1 \rangle. \end{aligned}$$

The total ‘‘action’’ now reads

$$S = \ln \langle j, j | n^\dagger e^{-cL_z - \frac{e \exp(-cL_z)}{c}} n | j, j \rangle.$$

Moving from its definition, the Hessian is found to be

$$\begin{aligned} -\frac{1}{2} \eta_a H_{ab}^e \eta_b &\equiv -\frac{1}{2} \delta_n^2 S|_e \\ &= -2j \frac{-\bar{\eta} \eta e^{-\epsilon c j - \frac{e \exp(-\epsilon c j)}{c}} + \bar{\eta} \eta e^{-\epsilon c (j-1) - \frac{e \exp(-\epsilon c (j-1))}{c}}}{e^{-\epsilon c j - \frac{e \exp(-\epsilon c j)}{c}}} \\ &\sim 2j \bar{\eta} \eta (-\epsilon c + \exp(1 - \epsilon c j)). \end{aligned}$$

Consequently, the determinant of the Hessian can be expressed in following way:

$$\det H^e = j^2 (-\epsilon c + e^{1 - \epsilon c j})^2.$$

For $\epsilon = 1$ and within the semiclassical limit we are considered in Eq. (4) of the manuscript, we find

$$c \sim \alpha e^{1 - c j}. \quad (\text{A2})$$

In equation (A2), we are assuming α to be $O(1)$, and at the same time $\alpha > 1$. Then we find

$$\det H^+ \sim j^2 [(1 - \alpha) e^{1 - c j}]^2,$$

which entails for I^+ the following expression:

$$I^+ \sim \frac{d_j}{j(\alpha - 1)} \exp\left(-1 - \frac{e^{(1 - c j)}}{c}\right).$$

For $\epsilon = -1$, we find

$$\det H^- \sim j^2 (\exp(1 + c j))^2,$$

and finally

$$I^- \sim \frac{d_j}{j} \exp\left(-1 - \frac{e^{(1 + c j)}}{c}\right).$$

It is obvious that in the semiclassical limit it holds

$$I^+ \gg I^-.$$

Thus the leading contribution of the normalization reads

$$I \sim I^+ \sim \exp\left(-1 - \frac{e^{(1 - c j)}}{c}\right),$$

which coincides with the explicit calculation. For one-link the reduced density matrix ρ_j is found to be

$$\rho_j = \frac{1}{I} \sum_m e^{-cm - \frac{\exp(1 - cm)}{c}} |j, m\rangle \langle j, m|.$$

In order to calculate the entanglement entropy, we use the replica trick. We then consider

$$S_{EE} = \lim_{N \rightarrow 1} \frac{1}{1 - N} \ln \text{tr} \rho_j^N.$$

The trace of ρ_j^N is found to be

$$\text{tr} \rho_j^N = \frac{1}{I^N} \chi_j(e^{-Ncm - N \frac{\exp(1 - cm)}{c}}).$$

Using the same asymptotic analysis we have performed above, it is straightforward to see that

$$\chi_j(e^{-Ncm - N \frac{\exp(1 - cm)}{c}}) \sim \frac{1}{N} e^{-1 + (1 - N)cj - N \frac{\exp(1 - cj)}{c}}.$$

Therefore, we can now calculate the trace of ρ_j^N in the asymptotic limit, and find

$$\begin{aligned} \text{tr} \rho_j^N &\sim \frac{\exp(-1 + (1 - N)cj - N \frac{\exp(1 - cj)}{c})}{N \exp(-N - N \frac{e^{(1 - cj)}}{c})} \\ &= \frac{1}{N} \exp(-(1 - N) + (1 - N)cj). \end{aligned}$$

As a consequence, the entanglement entropy of a one-link state is immediately recovered

$$S_{EE} \sim \lim_{N \rightarrow 1} \frac{1}{1 - N} (-(1 - N) + (1 - N)cj - \ln N) = cj,$$

which is a result that implies the area law.

APPENDIX B: ENTANGLEMENT IN MANY-LINK STATES

For many-link states, we can perform the same analysis deployed in the previous section. Again, we first focus on the normalization of the state:

$$\begin{aligned} I &= \int dh \prod_l d_{j_l} \int dn_l \langle j_l, j_l | n_l^\dagger e^{-cL_{gl} - \frac{\exp(1 - cL_{gl})}{c}} n_l | j_l, j_l \rangle \\ &= \int dh \prod_l d_{j_l} \int dn_l e^{S_l} \\ &= \int dh \prod_l d_{j_l} \sum_{k_l} \frac{(-)^{k_l} e^{k_l}}{k_l! c^{k_l}} \int dn_l e^{S_{k_l}^l}, \end{aligned}$$

where

$$S = \sum_l \ln \langle j_l, j_l | n_l^\dagger e^{-cL_{gl} - \frac{\exp(1 - cL_{gl})}{c}} n_l | j_l, j_l \rangle.$$

In the semiclassical limit, the saddle point solution which gives the leading contribution is expressed by

$$h = 1, \quad n_l = g_l, \quad \sum_l j_l \vec{n}_l = 0.$$

The leading contribution of the action S_l in the saddle point is found to be

$$S_l \sim -c j_l - \frac{\exp(1 - c j_l)}{c}.$$

The Hessian can also be calculated explicitly. Using the parametrization $h = \exp(i\theta_a L^a)$, with $a = 1, 2, 3$, we find three kinds of components for the Hessian,

$$-\frac{1}{2}H = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where

$$A \equiv -\frac{1}{2} \delta_{n_l}^2 S_l \sim 2 j_l \bar{\eta}_l \eta_l (-c + \exp(1 - c j_l)),$$

$$B \equiv \delta_{\theta_a} \delta_{n_l} S_l \\ = -i \bar{\eta}_l \theta_a e^{c - \exp^{1-cj}} j_l \langle \downarrow | n_l^\dagger L^a n_l | \uparrow \rangle - i \eta_l \theta_a j_l \langle \uparrow | n_l^\dagger L^a n_l | \downarrow \rangle,$$

$$C \equiv \delta_{\theta_b} \delta_{\theta_a} S_l = \sum_l \theta_a \theta_b \frac{j_l}{4} ((\delta_{ba} + i\epsilon_{bac} n_l^c) - n_l^b n_l^a).$$

For the third one of these latter relations, when $a = b$ we find

$$\delta_{\theta_a}^2 S_l = \sum_l \theta_a^2 \frac{j_l}{4} (1 - (n_l^a)^2),$$

while when $a \neq b$ we recover

$$\delta_{\theta_b} \delta_{\theta_a} S_l = \sum_l \theta_a \theta_b \frac{j_l}{4} (i\epsilon_{bac} n_l^c - n_l^b n_l^a) \\ \equiv \sum_l \theta_a \theta_b j_l C_{ab}.$$

The determinant for the Hessian can be recast as

$$\det \left(-\frac{1}{2}H \right) = \det A \det (C - B^T A^{-1} B).$$

It is straightforward to convince ourselves that

$$(\det A)^{-1/2} \sim \prod_l j_l \exp(1 - c j_l),$$

and then that

$$(B^T A^{-1} B)_{ab} \sim \sum_l -j_l e^{c j_l} \langle \downarrow | n_l^\dagger L^a n_l | \uparrow \rangle \langle \uparrow | n_l^\dagger L^b n_l | \downarrow \rangle.$$

Furthermore, an easy computation shows that

$$\langle \uparrow | n_l^\dagger L^a n_l | \downarrow \rangle \langle \downarrow | n_l^\dagger L^b n_l | \uparrow \rangle \\ = \langle \uparrow | n_l^\dagger L^a L^b n_l | \uparrow \rangle - \langle \uparrow | n_l^\dagger L^a n_l | \uparrow \rangle \langle \uparrow | n_l^\dagger L^b n_l | \uparrow \rangle \\ = \frac{1}{4} ((\delta_{ab} + i\epsilon_{abc} n_l^c) - n_l^a n_l^b) \\ = \begin{cases} \frac{1}{4} (1 - (n_l^a)^2), & \text{for } a = b, \\ \frac{1}{4} (i\epsilon_{abc} n_l^c - n_l^a n_l^b), & \text{for } a \neq b. \end{cases}$$

As a consequence, we can write

$$(B^T A^{-1} B)_{ab} \sim \sum_l -j_l e^{c j_l} C_{ab}^l,$$

from which it follows that

$$\det (C - B^T A^{-1} B) = \det \left(\sum_l j_l (1 - e^{c j_l}) C_{ab}^l \right) \\ = \det \left(\sum_l j_l (1 - e^{c j_l}) C_{ab}^l \right) \\ \sim j^3 e^{3c j},$$

j denoting the average of j_l . Then the leading term of the normalization is finally recovered to be

$$I \sim \frac{1}{j^{3/2} e^{3c j/2}} \prod_l \exp \left(-1 - \frac{\exp(1 - c j_l)}{c} \right).$$

Using again the replica trick, we can calculate the entropy of the reduced density matrix, the latter being

$$\rho_{\{j\}} = \frac{1}{I} \int dh \prod_l \sum_{m_l n_l} D_{m_l n_l}^{j_l} \left(e^{-c L_{g_l} - \frac{\exp(1 - c L_{g_l})}{c}} h \right) |j_l, m_l\rangle \langle j_l, n_l|.$$

As done in the previous section, the trace of N copies of $\rho_{\{j\}}$ is expressed as

$$\text{tr} \rho_{\{j\}}^N = \frac{1}{I^N} \int \prod_s dh_s \prod_l \chi^{j_l} \left(\prod_s \left(e^{-c L_{g_l}^s - \frac{\exp(1 - c L_{g_l}^s)}{c}} h_s \right) \right) \\ \equiv \frac{1}{I^N} \prod_l d_{j_l}^N \int \prod_s dh_s \int dn_l^s e^S,$$

where

$$S = \sum_l \sum_s \ln \langle j_l, j_l | n_l^{s\dagger} e^{-c L_{g_l} - \frac{\exp(1 - c L_{g_l})}{c}} h^s n_l^{s+1} | j_l, j_l \rangle.$$

The saddle point solution of the ‘‘action’’ is read out of the relations

$$h_s = 1, \quad n_l^s = g_l, \quad \sum_l j_l \bar{n}_l^s = 0.$$

The leading term of the action is

$$S = \sum_l -N c j_l - N \frac{\exp(1 - c j_l)}{c}.$$

The components of Hessian H are

$$\delta_{n_l^s}^2 S = -4 j_l \eta_l^s \bar{\eta}_l^s,$$

$$\delta_{n_l^{s+1}} \delta_{n_l^s} S \sim 2 j_l \bar{\eta}_l^s \eta_l^{s+1} e^{c - \exp(1 - c j_l)},$$

$$\delta_{\theta_a^s} \delta_{n_l^s} S = i \theta_a^s \bar{\eta}_l^s e^{c - \exp(1 - c j_l)} 2 j_l \langle \downarrow | n_l^\dagger L_s^a n_l | \uparrow \rangle,$$

$$\delta_{\theta_a^{s-1}} \delta_{n_l^s} S = i \theta_a^{s-1} \eta_l^s 2 j_l \langle \uparrow | n_l^\dagger L_{s-1}^a n_l | \downarrow \rangle,$$

$$\delta_{\theta_a^s}^2 S = - \sum_l \theta_b \theta_a \frac{j_l}{2} ((\delta_{ba} + i \epsilon_{bac} n_l^c) - n_l^b n_l^a).$$

As we have noticed for the normalization part, the Hessian too has three components, which we denote as A , B and C .

A contains $\delta_{n_l^s}^2 S$ and $\delta_{n_l^{s+1}} \delta_{n_l^s} S$, B contains $\delta_{\theta_a^s} \delta_{n_l^s} S$ and $\delta_{\theta_a^{s-1}} \delta_{n_l^s} S$, while $\delta_{\theta_a^s}^2 S$ enters C . The determinant of the Hessian is then cast as

$$\det H \sim \det A \det (C - B^T A^{-1} B).$$

The determinant of A is found to be

$$\det A = \prod_l (j_l \exp(1 - c j_l))^{2N}.$$

To compute the determinant of $C - B^T A^{-1} B$ is highly nontrivial. However remind that the entanglement entropy counts the number of microstates $\Omega(j_l)$ on the boundary, and that the gauge invariance introduces a reduction on $\Omega(j_l)$ by a factor $(\sum_l j_l^2)^{-3/2}$ [48]. This entails a logarithmic correction to the entanglement entropy. So finally, for a L -link state, the entanglement entropy is recovered to be

$$S_{EE} = \sum_l c j_l - \frac{3}{2} \ln \left(\sum_l j_l^2 \right).$$

-
- [1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Rev. Mod. Phys.* **80**, 517 (2008).
- [2] X. Wen and Q. Niu, *Phys. Rev. B* **41**, 9377 (1990).
- [3] X.-G. Wen, *Phys. Rev. Lett.* **90**, 016803 (2003).
- [4] W. X. Gang, *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, New York, 2004).
- [5] X.-G. Wen, *Phys. Lett. A* **300**, 175 (2002).
- [6] A. Hamma, R. Ionicioiu, and P. Zanardi, *Phys. Lett. A* **337**, 22 (2005).
- [7] A. Hamma, R. Ionicioiu, and P. Zanardi, *Phys. Rev. A* **71**, 022315 (2005).
- [8] A. Kitaev and J. Preskill, *Phys. Rev. Lett.* **96**, 110404 (2006).
- [9] M. Levin and X.-G. Wen, *Phys. Rev. Lett.* **96**, 110405 (2006).
- [10] J. Eisert, M. Cramer, and M. B. Plenio, *Rev. Mod. Phys.* **82**, 277 (2010).
- [11] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, *Phys. Rev. D* **34**, 373 (1986).
- [12] M. Srednicki, *Phys. Rev. Lett.* **71**, 666 (1993).
- [13] V. P. Frolov and I. Novikov, *Phys. Rev. D* **48**, 4545 (1993).
- [14] S. Ryu and T. Takayanagi, *Phys. Rev. Lett.* **96**, 181602 (2006).
- [15] E. Bianchi and R. C. Myers, *Classical Quantum Gravity* **31**, 214002 (2014).
- [16] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004).
- [17] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2007).
- [18] A. Perez, *Living Rev. Relativity* **16**, 3 (2013).
- [19] T. Thiemann, *Classical Quantum Gravity* **18**, 2025 (2001).
- [20] T. Thiemann and O. Winkler, *Classical Quantum Gravity* **18**, 2561 (2001).
- [21] E. R. Livine and S. Speziale, *Phys. Rev. D* **76** (2007).
- [22] E. Bianchi, E. Magliaro, and C. Perini, *Phys. Rev. D* **82**, 024012 (2010).
- [23] F. Conrady and L. Freidel, *Phys. Rev. D* **78**, 104023 (2008).
- [24] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, H. Gomes, and F. Hellmann, *J. Math. Phys. (N.Y.)* **50**, 112504 (2009).
- [25] J. W. Barrett, R. Dowdall, W. J. Fairbairn, F. Hellmann, and R. Pereira, *Classical Quantum Gravity* **27**, 165009 (2010).
- [26] M. Han and M. Zhang, *Classical Quantum Gravity* **30**, 165012 (2013).
- [27] M. Han and M. Zhang, *Classical Quantum Gravity* **29**, 165004 (2012).
- [28] M. Han and T. Krajewski, *Classical Quantum Gravity* **31**, 015009 (2014).
- [29] M. Han, *Classical Quantum Gravity* **31**, 015004 (2014).
- [30] M. Han, *Phys. Rev. D* **88**, 044051 (2013).
- [31] M. Han, *Phys. Rev. D* **89**, 124001 (2014).
- [32] C. Rovelli, *Phys. Rev. Lett.* **97**, 151301 (2006).
- [33] E. Bianchi, E. Magliaro, and C. Perini, *Nucl. Phys.* **B822**, 245 (2009).

- [34] C. Rovelli and M. Zhang, *Classical Quantum Gravity* **28**, 175010 (2011).
- [35] E. Bianchi and Y. Ding, *Phys. Rev. D* **86**, 104040 (2012).
- [36] A. Riello, *Phys. Rev. D* **89**, 064021 (2014).
- [37] M. Bojowald and A. Tsobanjan, *Classical Quantum Gravity* **27**, 145004 (2010).
- [38] M. Bojowald, *Phys. Rev. D* **75**, 123512 (2007).
- [39] E. Bianchi, C. Rovelli, and F. Vidotto, *Phys. Rev. D* **82**, 084035 (2010).
- [40] E. Magliaro, A. Marciano, and C. Perini, *Phys. Rev. D* **83**, 044029 (2011).
- [41] C. Rovelli, *Phys. Rev. Lett.* **77**, 3288 (1996).
- [42] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, *Phys. Rev. Lett.* **80**, 904 (1998).
- [43] R.K. Kaul and P. Majumdar, *Phys. Lett. B* **439**, 267 (1998).
- [44] A. Ashtekar, J. C. Baez, and K. Krasnov, *Adv. Theor. Math. Phys.* **4**, 1 (2000).
- [45] M. Domagala and J. Lewandowski, *Classical Quantum Gravity* **21**, 5233 (2004).
- [46] A. Corichi, [arXiv:0901.1302](https://arxiv.org/abs/0901.1302).
- [47] J. Engle, A. Perez, and K. Noui, *Phys. Rev. Lett.* **105**, 031302 (2010).
- [48] E. Bianchi, *Classical Quantum Gravity* **28**, 114006 (2011).
- [49] E. Bianchi, [arXiv:1204.5122](https://arxiv.org/abs/1204.5122).
- [50] M. Han, [arXiv:1402.2084](https://arxiv.org/abs/1402.2084).
- [51] J.B. Achour, A. Mouchet, and K. Noui, *J. High Energy Phys.* **06** (2015) 145.
- [52] E. R. Livine and D. R. Terno, *Nucl. Phys.* **B794**, 138 (2008).
- [53] E. R. Livine and D. R. Terno, *Nucl. Phys.* **B806**, 715 (2009).
- [54] A. Dasgupta, *Classical Quantum Gravity* **23**, 635 (2006).
- [55] W. Donnelly, *Phys. Rev. D* **77**, 104006 (2008).
- [56] W. Donnelly, *Phys. Rev. D* **85**, 085004 (2012).
- [57] N. Bodendorfer, *Classical Quantum Gravity* **31**, 214004 (2014).
- [58] P. Zanardi, D. A. Lidar, and S. Lloyd, *Phys. Rev. Lett.* **92**, 060402 (2004).
- [59] A. Hamma, S. Giampaolo, and F. Illuminati, *Phys. Rev. A* **93**, 012303 (2016).
- [60] J. Engle, R. Pereira, and C. Rovelli, *Phys. Rev. Lett.* **99**, 161301 (2007).
- [61] J. Engle, E. Livine, R. Pereira, and C. Rovelli, *Nucl. Phys.* **B799**, 136 (2008).
- [62] L. Freidel and K. Krasnov, *Classical Quantum Gravity* **25**, 125018 (2008).
- [63] W. Kaminski, M. Kisielowski, and J. Lewandowski, *Classical Quantum Gravity* **27**, 095006 (2010).
- [64] S. Speziale and W.M. Wieland, *Phys. Rev. D* **86**, 124023 (2012).
- [65] J.F. Plebanski, *J. Math. Phys. (N.Y.)* **18**, 2511 (1977).
- [66] R. Capovilla, T. Jacobson, J. Dell, and L. Mason, *Classical Quantum Gravity* **8**, 41 (1991).
- [67] W. Wieland, *Ann. Henri Poincare* **13**, 425 (2012).
- [68] L. Freidel and S. Speziale, *Phys. Rev. D* **82**, 084040 (2010).
- [69] L. Freidel and S. Speziale, *Phys. Rev. D* **82**, 084041 (2010).
- [70] E. Bianchi, P. Dona, and S. Speziale, *Phys. Rev. D* **83**, 044035 (2011).
- [71] G. Chirco, C. Rovelli, and P. Ruggiero, *Classical Quantum Gravity* **32**, 035011 (2015).
- [72] S. T. Flammia, A. Hamma, T.L. Hughes, and X.-G. Wen, *Phys. Rev. Lett.* **103**, 261601 (2009).
- [73] P. Buividovich and M. Polikarpov, *Phys. Lett. B* **670**, 141 (2008).
- [74] H. Casini, M. Huerta, and J. A. Rosabal, *Phys. Rev. D* **89**, 085012 (2014).
- [75] L.-Y. Hung and Y. Wan, *J. High Energy Phys.* **04** (2015) 122.
- [76] P. Calabrese and J. Cardy, *J. Stat. Mech.* (2004) P06002.
- [77] L. Smolin, *J. Math. Phys. (N.Y.)* **36**, 6417 (1995).
- [78] K. Krasnov and C. Rovelli, *Classical Quantum Gravity* **26**, 245009 (2009).
- [79] M. B. Hastings and X.-G. Wen, *Phys. Rev. B* **72**, 045141 (2005).
- [80] S. Bravyi, M. B. Hastings, and F. Verstraete, *Phys. Rev. Lett.* **97**, 050401 (2006).
- [81] A. K. Ekert, C. M. Alves, D. K. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, *Phys. Rev. Lett.* **88**, 217901 (2002).
- [82] D. A. Abanin and E. Demler, *Phys. Rev. Lett.* **109**, 020504 (2012).
- [83] A. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, *Phys. Rev. Lett.* **109**, 020505 (2012).
- [84] S. V. Isakov, M. B. Hastings, and R. G. Melko, *Nat. Phys.* **7**, 772 (2011).