LOWER SEMICONTINUITY AND RELAXATION FOR FREE DISCONTINUITY FUNCTIONALS WITH NON-STANDARD GROWTH

STEFANO ALMI, DARIO REGGIANI, AND FRANCESCO SOLOMBRINO

ABSTRACT. A lower semicontinuity result and a relaxation formula for free discontinuity functionals with non-standard growth in the bulk energy are provided. Our analysis is based on a non-trivial adaptation of the blow-up [8] and of the global method for relaxation [17] to the setting of generalized special function of bounded variation with Orlicz growth. Key tools developed in this paper are an integral representation result and a Poincaré inequality under non-standard growth.

1. INTRODUCTION

Integral functionals with non-standard growth first appeared in the works of Zhikov [67, 66] for modeling composite materials characterized by a strongly anisotropic behavior. The non-standard character of such functionals is typically expressed in terms of a point-dependent integrability of the deformation gradient, which may be captured, in a functional setting, in terms of variable exponents spaces [53] or, more in general, in Orlicz type of spaces (see, e.g., [47]). As relevant examples of bulk energies undergoing non-standard growth we report here the so-called *variable exponent* and the *double-phase* case

$$|\xi|^{p(x)} \qquad \text{and} \qquad |\xi|^p + a(x)|\xi|^q \qquad \text{for } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times m}, \tag{1.1}$$

for suitable choices of the exponent function $p\colon \mathbb{R}^d \to (1,+\infty)$, of the exponents $1 < p < q < +\infty$, and of the weight function $a\colon \mathbb{R}^d \to [0,+\infty)$. We refer to Section 4.3 for a list of relevant examples in the literature.

In a Sobolev/Orlicz setting, the study of integral functionals with non-standard growth has attracted an increasing attention in the last decades. Lower semicontinuity, relaxation, integral representation, and regularity of minimizers have been tackled in a number of papers: we mention the works [1, 18, 21, 43, 44, 45, 54, 55], dealing with the role of convexity and quasiconvexity in the *gap problem* for functionals characterized by a (p,q)-growth, i.e., different growth rates from above and below. Further results on relaxation in the gap problem have been presented in [61], where some explicit examples of concentration effects for bulk densities of double-phase type (1.1) are discussed, pointing out the importance of the Hölder regularity of $a(\cdot)$ in the relaxation procedure. The variable exponent setting has been dealt with in [31] under the so called log-Hölder continuity assumption on the exponent function $p(\cdot)$. Further instances of non-standard growth in partial differential equations may be found in [15, 16, 23, 32, 39, 60, 63], while applications ranging from optimal design to electro-rheological fluids and homogeneization appeared in [13, 14, 64, 69, 70]. Let us also mention the somehow related topic of regularity of minimizers of functionals with non-standard growth, which have been thoroughly discussed in [2, 4, 30, 40, 41] for the variable exponent, in, e.g., [11, 12, 28, 29, 36, 37, 59] for the double phase, and in [51] in a unified and generalized framework.

When dealing with composite materials, it is rather natural to account for failure phenomena, such as fracture, which can not be captured by a mere bulk energy defined on Sobolev or Orlicz spaces. This leads us to the extension of the above non-standard growth functionals to a Free Discontinuity setting, where singularities may appear in the form of jump discontinuities. Besides Materials Science, applications of a variable exponent in the setting of functions of bounded variation already appeared in image reconstruction [24] (see also [48, 49]), where an intermediate regime between Total Variation and the isotropic diffusion away from the edges was proposed.

2020 *Mathematics Subject Classification.* 49J45, 46E30, 49M20.

Key words and phrases. Free discontinuity functionals, non-standard growth, integral representation, lower semicontinuity, relaxation.

The aim of this paper it to provide a *unified framework* for lower semicontinuity and relaxation of functionals of the form

$$\mathcal{G}(u) := \int_\Omega f(x, \nabla u)\, dx + \int_{J_u} g(x, [u](x), \nu_u)\, d\mathcal{H}^{d-1}\,, \tag{1.2}$$

focusing on the role played by the non-standard growth condition of volume integrand $f(x,\xi)$. In formula (1.2), $\Omega \subseteq \mathbb{R}^d$ is an open bounded subset of $\mathbb{R}^d$, $u \in GSBV(\Omega;\mathbb{R}^m)$ is a function of *generalized special bounded variation* (see [10, Section 4.5]), J_u denotes the jump set of u, ν_u stands for the approximate unit normal to u, and $[u] := u^+ - u^-$ represents the jump of u, that is, the difference between the traces u^+ and u^- of u on J_u, defined according to the orientation of ν_u. For the sole variable exponent case, lower semicontinuity results for $\mathcal{G}$ have been obtained in [35], while Γ-convergence and relaxation issues have been recently considered in [65]. The key assumptions in the mentioned papers are superlinearity of $p(\cdot)$ (meaning $\min_\Omega p(\cdot) > 1$), which allows for a separation of scales in the Γ-convergence and relaxation processes, and a log-Hölder continuity of the exponent, necessary to avoid the Lavrentiev's phenomenon, as originally observed in [68].

Our primary interest is in providing suitable conditions which either imply the lower semicontinuity of the functional $\mathcal{G}$ in (1.2) or guarantee an explicit formula for its lower semicontinuous envelope, while assuming BV-ellipticity [10, Chapter 5] of g and non-standard growth for f. In this regard, we will suppose that there exists a superlinear *generalized* Φ*-function* (see also Definitions 4.3 and 4.13) $\psi\colon \Omega\times[0,+\infty) \to [0,+\infty)$ such that

$$a\psi(x,|\xi|) \le f(x,\xi) \le b(1+\psi(x,|\xi|)) \qquad \text{for } (x,\xi) \in \Omega\times\mathbb{R}^{d\times m} \tag{1.3}$$

for some $0 < a < b < +\infty$. The superlinearity of ψ (and thus of f) is expressed by the conditions $(\mathrm{Inc})_p$ and $(\mathrm{Dec})_q$ for some $p, q \in (1,+\infty)$ (see also Definition 4.12), meaning that for a.e. $x \in \Omega$ the maps $t \mapsto \frac{\psi(x,t)}{t^p}$ and $t \mapsto \frac{\psi(x,t)}{t^q}$ are monotone increasing and monotone decreasing, respectively. The third basic hypothesis on ψ, common to all our results, is (A0). Loosely speaking, such condition does not allow for a too degenerate behavior on small balls contained in Ω of the functions

$$\psi^+_B(t) := \sup_{x\in B} \psi(x,t) \qquad \psi^-_B(t) := \inf_{x\in B} \psi(x,t) \qquad \text{for } t \in [0,+\infty) \text{ and } B \subseteq \Omega. \tag{1.4}$$

We remark that $(\mathrm{Inc})_p$, $(\mathrm{Dec})_q$, and (A0) are well-suited for a blow-up argument, which is at the core of our proof's strategy, and are rather standard in the theory of generalized Orlicz spaces [47].

Let us discuss our results in more details. In Theorem 3.3 we prove the lower semicontinuity of the functional $\mathcal{G}$ in the space $GSBV^\psi(\Omega;\mathbb{R}^m)$, the space of functions $u \in GSBV(\Omega;\mathbb{R}^m)$ with L^ψ-integrable approximate gradient ∇u. We refer to Definition 4.28 and Section 4.2 for more details on such space. Besides $(\mathrm{Inc})_p$, $(\mathrm{Dec})_q$, and (A0), in Theorem 3.3 we ask for the quasiconvexity of f and for a mild continuity property of $\psi(x,t)$ with respect to $x \in \Omega$ (see (3.6) for a precise statement). We notice that the last condition allows for improvements in the existing theory of generalized Orlicz spaces. For instance, in the variable exponent case the log-Hölder continuity of $p(\cdot)$ is not necessary for (3.6), as pointed out in Section 4.3. The proof of Theorem 3.3 is based on the approximation strategy of [58] and on a localization technique which leads us to study the asymptotic behavior of $\mathcal{G}$ around Lebesgue points of the limit function, as first done in [8] in the $GSBV$-setting (see also [10, Theorem 5.29]). Our assumptions are designed in such a way that the maximal operator for SBV^p-functions, exploited in [8] for the construction of more regular approximating sequences, can be replaced by the maximal operator in Orlicz spaces, for which continuity estimates have been obtained in [52] (see also Theorem 4.35).

In Theorem 3.4 we show a relaxation formula of the functional $\mathcal{G}$ of the form

$$\overline{\mathcal{G}}(u) := \int_\Omega \mathcal{Q}f(x,\nabla u)\, dx + \int_{J_u} \mathcal{R}g(x,[u],\nu_u) d\mathcal{H}^{d-1}, \tag{1.5}$$

which therefore maintains the original structure (1.2). In particular, in (1.5) the functions $\mathcal{Q}f\colon \Omega\times\mathbb{R}^{m\times d} \to [0,+\infty)$ and $\mathcal{R}g\colon \Omega\times\mathbb{R}^m\times\mathbb{S}^{d-1} \to [0,+\infty)$ denote the quasiconvex and the BV-elliptic envelope of f and g, respectively. When dealing with relaxation, we have to strengthen condition (3.6). Thus, in Theorem 3.4 we replace the latter with the stronger assumption (adA1), which allows to estimate ψ^+_B with ψ^-_B on small balls $B \subseteq \Omega$ with a fixed control rate. We remark that condition (adA1) is weaker than the more traditional (A1), introduced in the framework of Orlicz spaces (see, e.g., [47]). In the variable exponent case, (adA1) still requires log-Hölder continuity of $p(\cdot)$, while in the double-phase case it calls for a Hölder continuity of the weight $a(x)$, which may be however weaker than the one considered in regularity theory in Sobolev/Orlicz setting (see, for instance, [12, 29]). Further examples are discussed in Section 4.3.

The proof of Theorem 3.4 follows the well-established strategy of the global method of relaxation [17, 19], which needs to be adapted to the non-standard growth framework (1.3). On the one hand, we follow closely this classical path, which is based on the localization procedure described below, combined with technical machinery for generalized Orlicz spaces recalled in Section 4.1. On the other hand, there are at least two relevant points where we have to enhance the existing tools and significantly refine the available estimates, namely

- the introduction of a new Poincaré inequality for SBV functions with Orlicz growth for the gradient (Section 5).
- the construction of the cell energy optimizing sequences, which allow one to detect the relaxed energy density at Lebesgue and surface points (Lemmas 6.6 and 6.9).

The crucial step towards the relaxation formula (1.5) is the integral representation of the lower semicontinuous envelope $\overline{\mathcal{G}}$ of $\mathcal{G}$ in $GSBV^{\psi}(\Omega;\mathbb{R}^m)$ (see Theorem 3.1 and Corollary 3.2). This ensures that $\overline{\mathcal{G}}$ can still be written as the sum of a bulk energy depending on the approximate gradient $\nabla u \in L^{\psi}(\Omega;\mathbb{R}^{m\times d})$ and of a surface term obtained by integrating over the jump set J_u of u a suitable function depending on x, $[u]$, and ν_u. A fundamental tool to characterize the behavior of the blow-up on jump points and on approximate differentiability points of u is a Poincaré inequality. In our case, it has to be established for special functions of bounded variation with L^{φ}-integrable approximate gradient, where φ denotes a (generalized) Φ-function in the sense of Definition 4.3. This is quite a delicate issue, as the classical approach of [38] cannot be directly adapted to our general framework (up to the special case of variable exponents in [65]). We have to rely, instead, on fine properties of rearrangments of BV-functions , and the techniques of [5, 25, 26, 27]. We are able to prove a Poincaré inequality for our context in Theorems 5.1 and 5.9. Let us stress our firm belief that the tool is of independent mathematical interest and useful for many other applications than relaxation problems, exactly as it happens for its p-growth counterpart.

The crucial role played by the Poincaré inequality is evident in Lemmas 6.6 and 6.9 and enables us to recover the densities of the lower semicontinuous envelope $\overline{\mathcal{G}}$ as blow-up limits of cell minimization formulas. In both lemmas, the Poincaré inequality is exploited to replace the optimal blow-up sequences with more regular functions, without excessively increasing the energy. Since the only control we have on the energy is in terms of a nonstandard growth, this is another point where our analysis significantly departs from, e.g., [8, 17]. A crucial issue is that, due to the possible non-homogeneity of φ in space, the truncations of u we construct can not be controlled only in term of ∇u, as some remainder term appears. Such a problem does not appear, when a p-growth is assumed. Estimating the remainder terms along the blow-up procedure is a delicate point, where we heavily exploit (adA1).

Outlook and open problems. Our paper presents a general framework for lower semicontinuity and relaxation of free discontinuity functionals under non-standard growth of the bulk functional. The generalized Φ-functions we consider cover most of the examples appeared in the literature, and in particular among them the variable exponent and the double phase (1.1). Because of the superlinear assumption $(\mathrm{Inc})_p$ for $p > 1$, however, we can not handle an $\ell \log \ell$-kind of behavior, which has been studied in [57] for a lower-semicontinuity problem in SBV without x-dependence on the integrand function in (1.2). Hence, an extension of our results in the above direction will be considered in a forthcoming research. Furthermore, regularity issues for minimizers of the functional $\mathcal{G}$ may be investigated, in the spirit of [10, Chapter 6], and could lead to stronger assumptions on the growth function ψ (see, e.g., [2, 3, 4] for the variable exponent). Also the investigation of lower semicontinuity and relaxation issues for free discontinuity functionals with bulk energies having mixed (p,q)-growth condition, relevant for the modeling of determinant constraints, is not fully covered by our theory and deserves further analysis.

Plan of the paper. In Section 2 we introduce the basic notation of the paper. In Section 3 we state the main results of our paper: the integral representation in Theorem 3.1 and Corollary 3.2, the lower semicontinuity result in Theorem 3.3, and the relaxation formula in Theorem 3.4. Section 4 is devoted to some preliminaries on (generalized) Φ-functions and on $(G)SBV$-spaces, as well as to the discussion of the main assumptions of the above theorems. In particular, in Section 4.3 we report a number of generalized Φ-functions known in the literature, and discuss how they fall into our theory. In Section 5 we state and prove the Poincaré inequality for SBV-functions with L^{φ}-integrable approximate gradient (see Theorems 5.1 and 5.9). Finally, in Sections 6–8 we proceed with the proof of Theorems 3.1, 3.3, and 3.4, respectively.

2. Notations

Throughout the paper we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary and that $d \geq 2$. We denote by $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ the family of open sets and the family of Borel measurable sets contained in Ω, respectively. For every $x \in \mathbb{R}^d$ and $\varepsilon > 0$ we indicate with $B_\varepsilon(x)$ the ball centered in x with radius ε, if $x = 0$ we write B_ε. Given $x \in \mathbb{R}^d$ we indicate with $|x|$ its Euclidean norm. The set $\mathbb{R}^{m\times d}$ is the set of $m \times d$ matrices with real coefficients, $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d \colon |x| = 1\}$, and $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$. The Lebesgue measure of the d-dimensional unit ball is indicated ω_d. We denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the d-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively. The space $L^0(\Omega)$ stands for the space of all measurable functions in Ω. Given $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$, for any set $A \subset \mathbb{R}^d$ we set

$$A_{\varepsilon,x_0} := x_0 + \varepsilon(A - x_0). \tag{2.1}$$

The closure of a set A is indicated with $\overline{A}$, the diameter with $\mathrm{diam}(A)$. Given two sets $A_1, A_2 \subset \mathbb{R}^d$ we denote their symmetric difference with $A_1 \Delta A_2$. We write χ_A for the characteristic function of any set $A \subset \mathbb{R}^d$. If A is a set of finite perimeter we indicate with $\partial^M A$ its essential boundary (the points of the boundary which do not have density zero nor one) and with $\partial^* A$ its reduced boundary (the points for which a "normal" can be defined). For a Carathéodory function $\varphi \colon \Omega \times \mathbb{R}^+ \to \mathbb{R}$ and a ball $B \subset \mathbb{R}^d$ we define

$$\varphi^-_B(t) := \operatorname*{ess\,inf}_{x\in B\cap\Omega} \varphi(x,t) \quad \text{and} \quad \varphi^-_B(t) := \operatorname*{ess\,sup}_{x\in B\cap\Omega} \varphi(x,t). \tag{2.2}$$

for every $t \geq 0$. For a monotone function φ on the real line, the customary notations $\varphi(t-)$ and $\varphi(t+)$ are used to denote left and right limits in t, respectively. For an increasing coercive function φ, setting for simplicity $\varphi(+\infty) := +\infty$, we denote by $\varphi^{-1} \colon [0,+\infty] \to [0,+\infty]$ the left inverse of φ defined as

$$\varphi^{-1}(s) := \inf\{t \geq 0 \colon \varphi(t) \geq s\}.$$

3. Main Results

The paper will be concerned with integral representation, lower semicontinuity, and relaxation of functionals defined on the space $GSBV^{\psi}(\Omega,\mathbb{R}^m)$ of generalized special functions of bounded variation with ψ-growth on the gradient. Here ψ is a suitable (generalized) Orlicz function ψ. We will consider functionals

$$\mathcal{F} : GSBV^{\psi}(\Omega,\mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0,+\infty)$$

satisfying the following assumptions:

- (H1) $\mathcal{F}(u,\cdot)$ is a Borel measure for any $u \in GSBV^{\psi}(\Omega,\mathbb{R}^m)$;
- (H2) $\mathcal{F}(\cdot,A)$ is lower semicontinuous with respect to convergence in measure in Ω for any $A \in \mathcal{A}(\Omega)$;
- (H3) $\mathcal{F}(\cdot,A)$ is local for every $A \in \mathcal{A}(\Omega)$, that is, if $u, v \in GSBV^{\psi}(\Omega,\mathbb{R}^m)$ satisfy $u = v$ $\mathcal{L}^d$-a.e. in A then $\mathcal{F}(u,A) = \mathcal{F}(v,A)$;
- (H4) there exist $0 < a < b$ such that for every $u \in GSBV^{\psi}(\Omega,\mathbb{R}^m)$ and every $B \in \mathcal{B}(\Omega)$ it holds

$$a\left(\int_B \psi(x,|\nabla u|)dx + \mathcal{H}^{d-1}(J_u \cap B)\right) \leq \mathcal{F}(u,B) \leq b\left(\int_B (1+\psi(x,|\nabla u|))dx + \mathcal{H}^{d-1}(J_u \cap B)\right).$$

In (H4), as we said, we consider a generalized Orlicz function ψ. The assumptions we make on ψ will be detailed in Section 4 (see (aInc), (aDec) and the so-called weight condition (A0)). These properties are standard in the theory of generalized Orlicz functions. We also assume that ψ satisfies

(adA1) For every ball $B \subset \Omega$ with $\mathrm{diam}(B) \leq 1$ there exists $\beta \in (0,1)$ such that

$$\psi^+_B(\beta t) \leq \psi^-_B(t) \quad \text{for all } t \in \left[\sigma, (\psi^-_B)^{-1}\left(\tfrac{1}{\mathrm{diam}(B)}\right)\right],$$

where $\sigma \geq 1$ is the constant in the weight condition (A0). We remark that this condition is a *weaker variant* of the standard assumption (A1) (see Definition 4.21), called local continuity condition and usually considered for generalized Orlicz spaces. All the main results of the paper will hold, if one requires that (A0), (adA1), (aInc) and (aDec) are satisfied on Ω. They encompass a broad range of applications, with some relevant examples that will be discussed in Section 4.3.

Our first result concerns the integral representation, and requires some notation to be fixed. For every $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$ and every $A \in \mathcal{A}(\Omega)$ we define

$$\mathbf{m}_{\mathcal{F}}(u,A) := \inf_{v \in GSBV^\psi(\Omega,\mathbb{R}^m)} \{\mathcal{F}(v,A)\colon v = u \text{ in a neighborhood of } \partial A\}. \tag{3.1}$$

Moreover, given $x_0 \in \Omega$, $u_0 \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^{m\times d}$, we define the affine function $\ell_{x_0,u_0,\xi}\colon \mathbb{R}^d \to \mathbb{R}^m$ as

$$\ell_{x_0,u_0,\xi} := u_0 + \xi(x - x_0).$$

Given $x_0 \in \Omega$, $\mathfrak{a},\mathfrak{b} \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{d-1}$ we also introduce $u_{x_0,\mathfrak{a},\mathfrak{b},\nu}\colon \mathbb{R}^d \to \mathbb{R}^m$ as

$$u_{x_0,\mathfrak{a},\mathfrak{b},\nu} = \begin{cases} \mathfrak{a} & \text{if } (x-x_0)\cdot\nu > 0,\\ \mathfrak{b} & \text{if } (x-x_0)\cdot\nu < 0.\end{cases}$$

Our main result concerning the integral representation is the following. Below, $\Phi_w(\Omega)$ denotes the class of weak generalized Φ-functions, whose definition is recalled in Definition 4.13.

Theorem 3.1 (Integral representation in $GSBV^\psi$)**.** *Let $\psi \in \Phi_w(\Omega)$ satisfy (A0), (adA1), (aInc) and (aDec) on Ω. Assume that $\mathcal{F} : GSBV^\psi(\Omega,\mathbb{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty)$ satisfies assumptions (H1)–(H4). Then, for all $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$ and all $A \in \mathcal{A}(\Omega)$*

$$\mathcal{F}(u,A) = \int_A f(x,u(x),\nabla u(x))\,dx + \int_{J_u\cap A} g(x,u^+(x),u^-(x),\nu_u(x))\,d\mathcal{H}^{d-1},$$

where

$$f(x_0,u_0,\xi) := \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\ell_{x_0,u_0,\xi},B_\varepsilon(x_0))}{\omega_d\varepsilon^d} \quad \textit{for } (x_0,u_0,\xi_0) \in \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d}, \tag{3.2}$$

$$g(x_0,\mathfrak{a},\mathfrak{b},\nu) := \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u_{x_0,\mathfrak{a},\mathfrak{b},\nu},B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \quad \textit{for all } x_0 \in \Omega,\ \mathfrak{a},\mathfrak{b} \in \mathbb{R}^m \textit{ and } \nu \in \mathbb{S}^{d-1}. \tag{3.3}$$

If translation invariance with respect to u is assumed on $\mathcal{F}$, namely

(H5) $\mathcal{F}(u+c,A) = \mathcal{F}(u,A)$ for every $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$, every $A \in \mathcal{B}(\Omega)$ and every $c \in \mathbb{R}^m$,

we have the following result.

Corollary 3.2. *Let $\psi \in \Phi_w(\Omega)$ be as in Theorem 3.1 and suppose that $\mathcal{F} : GSBV^\psi(\Omega,\mathbb{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty)$ satisfies assumptions (H1)–(H5). Then, for all $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$ and all $A \in \mathcal{A}(\Omega)$*

$$\mathcal{F}(u,A) = \int_A f(x,\nabla u(x))\,dx + \int_{J_u\cap A} g(x,[u](x),\nu_u(x))\,d\mathcal{H}^{d-1},$$

where f and g are as in (3.2) and (3.3), respectively.

Our next result concerns the lower semicontinuity in $GSBV^\psi(\Omega,\mathbb{R}^m)$ of variational functionals $\mathcal{G} : L^0(\Omega,\mathbb{R}^m)\times\mathcal{A}(\Omega) \to [0,+\infty]$ of the form

$$\mathcal{G}(u,A) = \int_A f(x,\nabla u(x)) + \int_{J_u} g(x,[u](x),\nu_u(x))\,d\mathcal{H}^{d-1}, \tag{3.4}$$

In (3.4) the function $f : \Omega\times\mathbb{R}^{m\times d} \to [0,+\infty)$ satisfies the following assumptions:

(f1) f is a Carathéodory function;

(f2) there exist two constants $a, b > 0$ such that

$$a\psi(x,|\xi|) \leq f(x,\xi) \leq b(1+\psi(x,|\xi|)) \tag{3.5}$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^{m\times d}$.

On the other hand, the function $g\colon \Omega\times\mathbb{R}^m_0\times\mathbb{S}^{d-1} \to [0,+\infty)$ satisfies the following assumptions:

(g1) g is a Borel measurable function lower semicontinuous in x and continuous in the remaining variables;

(g2) there exist $\alpha_1, \alpha_2 > 0$ such that for every $x \in \Omega$, $\zeta \in \mathbb{R}^m_0$ and $\nu \in \mathbb{S}^{d-1}$

$$\alpha_1 \le g(x, \zeta, \nu) \le \alpha_2.$$

For the result below, we can further weaken (adA1). We will namely assume that ψ complies with the following property: for $\sigma \ge 1$ being the constant in the weight condition (A0), and for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ there exists $C = C(x_0) > 0$ such that

$$\begin{gathered} \text{given } \theta > \sigma, \text{ we can find } \varepsilon_0 > 0 \text{ such that for every } \varepsilon \le \varepsilon_0 \text{ and every } t \in [\sigma, \theta], \\ \psi^+_{B_\varepsilon(x_0)}(t) \le C \psi^-_{B_\varepsilon(x_0)}(t). \end{gathered} \tag{3.6}$$

As we will discuss in Section 4, if $\psi \in \Phi_w(\Omega)$ satisfies (A0), (aDec) and (adA1) on Ω, then it also satisfies (3.6). Hence, the assumptions for the lower semicontinuity Theorem below are weaker than those in Theorem 3.1.

Theorem 3.3. *Let $\psi \in \Phi_w(\Omega)$ satisfying (A0), (aInc) and (aDec) on Ω. Assume also that ψ satisfies (3.6). Consider a functional $\mathcal{G}\colon L^0(\Omega, \mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty]$ as in (3.4). Let $f\colon \Omega \times \mathbb{R}^{m\times d} \to [0, +\infty)$ be a function satisfying (f1)-(f2) and such that $z \mapsto f(x, z)$ is quasiconvex in $\mathbb{R}^{m\times d}$ for every $x \in \Omega$. Let $g\colon \Omega \times \mathbb{R}^m_0 \times \mathbb{S}^{d-1} \to [0, +\infty)$ be a function satisfying (g1)–(g2) and such that $(\zeta, \nu) \mapsto g(x, \zeta, \nu)$ is BV-elliptic for every $x \in \Omega$. Then, for every $A \in \mathcal{A}(\Omega)$, we have*

$$\mathcal{G}(u, A) \le \liminf_{k\to+\infty} \mathcal{G}(u_k, A)$$

for every sequence $\{u_k\}_k \subset GSBV^\psi(A, \mathbb{R}^m)$ converging to a function $u \in GSBV^\psi(A, \mathbb{R}^m)$ in measure.

The third main result is a relaxation result, which requires both the use of Theorems 3.1 and 3.3. Given $\mathcal{G}$ as in (3.4), for every $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$ and every $A \in \mathcal{A}(\Omega)$ we denote the lower semicontinuous envelope of the functional $\mathcal{G}$ as

$$\overline{\mathcal{G}}(u, A) := \inf \left\{ \liminf_{k\to+\infty} \mathcal{G}(u_k, A)\colon \{u_k\}_k \subset GSBV^\psi(A, \mathbb{R}^m) \text{ and } u_k \to u \text{ in measure on } A \right\}.$$

We assume that $g\colon \Omega \times \mathbb{R}^m_0 \times \mathbb{S}^{d-1} \to [0, +\infty)$ satisfies also the following additional properties:

(g3) there exists $c > 0$ such that for every $x \in \Omega$ and every $\nu \in \mathbb{S}^{d-1}$ it holds

$$g(x, \zeta_1, \nu) \le g(x, \zeta_2, \nu) \quad \text{for every } \zeta_1, \zeta_2 \in \mathbb{R}^m_0 \text{ with } c|\zeta_1| \le |\zeta_2|;$$

(g4) for every $x \in \Omega$, $\zeta \in \mathbb{R}^m_0$ and $\nu \in \mathbb{S}^{d-1}$

$$g(x, \zeta, \nu) = g(x, -\zeta, -\nu).$$

Theorem 3.4. *Let $\psi \in \Phi_w(\Omega)$ satisfying (A0), (adA1), (aInc) and (aDec) on Ω. Let $\mathcal{G}$ be as in (3.4) and $f\colon \Omega \times \mathbb{R}^{m\times d} \to [0, +\infty)$ satisfying (f1) and (f2). Assume also that $g\colon \Omega \times \mathbb{R}^m_0 \times \mathbb{S}^{d-1} \to [0, +\infty)$ is a continuous function satisfying (g1)–(g4). Then, for every $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$ and every $A \in \mathcal{A}(\Omega)$,*

$$\overline{\mathcal{G}}(u, A) = \int_A \mathcal{Q}f(x, \nabla u(x))\, dx + \int_{J_u \cap A} \mathcal{R}g(x, [u](x), \nu_u(x))\, d\mathcal{H}^{d-1},$$

where $\mathcal{Q}f$ is the quasiconvex envelope of f and $\mathcal{R}g$ is the BV-elliptic envelope of g.

4. Preliminaries

In this preliminary section we recall some basic facts about (generalized) Φ-functions and Orlicz spaces, and on $GSBV$ functions. Finally, in Subsection 4.3 we give a list of non-standard growth functions ψ which fit into the scope of our results.

4.1. **(Generalized) Φ-functions and Orlicz spaces.** We begin by collecting some basic definitions and useful facts about Φ-functions and generalized Orlicz spaces. For a complete treatment of the topic (and the proofs of the statements below which are left without proof) we refer to [47].

Definition 4.1. A function $g\colon (0, +\infty) \to \mathbb{R}$ is almost increasing (resp. almost decreasing) if there exists a constant $a \ge 1$ such that $g(s) \le a g(t)$ (resp $a g(s) \ge g(t)$) for every $0 < s < t$.

Increasing and decreasing functions are included in the above definition if $a = 1$.

Definition 4.2. Let $f\colon (0, \infty) \to \mathbb{R}$ and $p, q > 0$. We say that f satisfies

$(\mathrm{Inc})_p$ if $\frac{f(t)}{t^p}$ is increasing;
$(\mathrm{aInc})_p$ if $\frac{f(t)}{t^p}$ is almost increasing;
$(\mathrm{Dec})_q$ if $\frac{f(t)}{t^q}$ is decreasing;
$(\mathrm{aDec})_q$ if $\frac{f(t)}{t^q}$ is almost decreasing;

We say that f satisfies (aInc), (Inc), (aDec) or (Dec) if there exist $p > 1$ or $q < \infty$ such that f satisfies $(\mathrm{Inc})_p$, $(\mathrm{aInc})_p$, $(\mathrm{Dec})_q$ or $(\mathrm{aDec})_q$, respectively.

Definition 4.3. Let $\varphi : [0, +\infty) \to [0, +\infty]$ be increasing with $\varphi(0) = 0 = \lim_{t\to 0^+} \varphi(t)$ and $\lim_{t\to +\infty} \varphi(t) = +\infty$. Such φ is called a Φ-prefunction. We say that a Φ-prefunction φ is a

- weak Φ-function if it satisfies $(\mathrm{aInc})_1$ on $(0, +\infty)$;
- convex Φ-function if it is left-continuous and convex;
- strong Φ-function if it is continuous in the topology of $[0, +\infty]$ and convex.

The set of weak, convex and strong Φ-functions are denoted by Φ_w, Φ_c and Φ_s, respectively.

It follows by definition that $\Phi_s \subset \Phi_c \subset \Phi_w$. If a function $\varphi \in \Phi_c$ satisfies (aDec) then $\varphi \in \Phi_s$.

Definition 4.4. Two functions φ and ψ are called equivalent, $\varphi \simeq \psi$, if there exists $L \geq 1$ such that $\varphi(t/L) \leq \psi(t) \leq \varphi(Lt)$ for all $t \geq 0$. The notation "$\approx$" is instead used with the following meaning: $\varphi \approx \psi$ if and only if there exist two constants $c_1, c_2 > 0$ such that $c_1\varphi \leq \psi \leq c_2\varphi$.

Lemma 4.5. *Let $\varphi, \psi \colon [0, +\infty) \to [0, +\infty]$ be increasing with $\varphi \simeq \psi$. Then, the following facts hold:*

(a) *if φ is a Φ-prefunction, then ψ is a Φ-prefunction;*
(b) *if φ satisfies $(\mathrm{aInc})_p$, then ψ satisfies $(\mathrm{aInc})_p$;*
(c) *if φ satisfies $(\mathrm{aDec})_q$, then ψ satisfies $(\mathrm{aDec})_q$.*

Lemma 4.6. *If $\varphi \in \Phi_w$ satisfies $(\mathrm{aInc})_p$ with $p \geq 1$, then there exists $\psi \in \Phi_c$ equivalent to φ such that $\psi^{1/p}$ is convex. In particular, ψ satisfies $(\mathrm{Inc})_p$.*

Theorem 4.7. *Every weak Φ-function is equivalent to a strong Φ-function. Moreover, if $\varphi \in \Phi_w$ is finite valued and satisfies $(\mathrm{Inc})_1$, then we can find $\psi \in \Phi_s$ such that*

$$\psi(t) \leq \varphi(t) \leq \psi(2t) \quad t \geq 0.$$

Now, we recall the concept of doubling functions and its equivalence with (aDec).

Definition 4.8. We say that a function $\varphi \colon [0, +\infty) \to [0, +\infty]$ satisfies Δ_2, or that it is doubling, if there exists a constant $K \geq 2$ such that

$$\varphi(2t) \leq K\varphi(t) \quad \text{for all } t \geq 0.$$

Lemma 4.9. *The following statements hold.*

(a) *If $\varphi \in \Phi_w$, then Δ_2 is equivalent to (aDec).*
(b) *If $\varphi \in \Phi_c$, then Δ_2 is equivalent to (Dec).*

Remark 4.10. By the previous Lemma it follows that if $\varphi \in \Phi_c$ satisfies $(\mathrm{aDec})_q$, then it satisfies $(\mathrm{Dec})_{q_2}$ for some possibly larger q_2. This and Lemma 4.5(c) imply that if φ satisfies Δ_2 and $\psi \simeq \varphi$, then ψ satisfies Δ_2.

Theorem 4.11. *Let $\psi, \varphi \in \Phi_w$. Then $\varphi \simeq \psi$ if and only if $\varphi^{-1} \approx \psi^{-1}$. In particular, if $\psi(t/L) \leq \varphi(t) \leq \psi(Lt)$ for some $L \geq 1$ then,*

$$\psi^{-1}(t)/L \leq \varphi^{-1}(t) \leq L\psi^{-1}(t).$$

We now come to the generalized setting, where explicit dependence on the space variable x is allowed.

Definition 4.12. Let (A, Σ, μ) be a complete, σ-finite measure space. Let $\varphi \colon A \times [0, +\infty) \to [0, +\infty]$ and $p, q > 0$. We say that φ satisfies $(\mathrm{aInc})_p$ or $(\mathrm{aDec})_q$ if there exists a constant $a \geq 1$ such that $t \mapsto \varphi(x, t)$ satisfies $(\mathrm{aInc})_p$ or $(\mathrm{aDec})_q$ respectively, for μ-a.e. $x \in A$. When $a = 1$ we use the notation $(\mathrm{Inc})_p$ and $(\mathrm{Dec})_q$. For (Inc) and (Dec) the definition is analogous.

Definition 4.13. Let (A, Σ, μ) be a complete, σ-finite measure space. A function $\varphi\colon A \times [0, +\infty) \to [0, +\infty]$ is said to be a generalized Φ-prefunction on (A, Σ, μ) if $x \mapsto \varphi(x, |f(x)|)$ is measurable for any $f \in L^0(A, \mu)$ and $\varphi(x, \cdot)$ is a Φ-prefunction for μ-a.e. $x \in A$. We say that the generalized Φ-prefunction φ is

- a weak generalized Φ-function if $\varphi(x, \cdot)$ satisfies $(\mathrm{Inc})_1$ on $(0, +\infty)$ for a.e. $x \in A$;
- a convex generalized Φ-function if $\varphi(x, \cdot) \in \Phi_c$ for μ-a.e. $x \in A$;
- a strong generalized Φ-function if $\varphi(x, \cdot) \in \Phi_s$ for μ-a.e. $x \in A$.

If φ is a generalized weak Φ-function on (A, Σ, μ), we write $\varphi \in \Phi_w(A, \mu)$. Similarly we define $\varphi \in \Phi_c(A, \mu)$ and $\varphi \in \Phi_s(A, \mu)$. If $A \subset \mathbb{R}^d$ is an open set and $\mu = \mathcal{L}^d$, we omit the measure dependence and simply write $\Phi_w(A)$, $\Phi_c(A)$ or $\Phi_s(A)$ and we say that φ is a generalized Φ-function on A.

Proposition 4.14. *Let $\varphi\colon A \times [0, +\infty) \to [0, +\infty]$, $x \mapsto \varphi(x, t)$ be measurable for every $t \geq 0$ and $t \mapsto \varphi(x, t)$ be increasing and left-continuous for μ-a.e. $x \in A$. If $f \in L^0(A, \mu)$. Then $x \mapsto \varphi(x, |f(x)|)$ is measurable.*

Properties of Φ-functions are generalized point-wise uniformly to the generalized Φ-function case.

Definition 4.15. Two function $\varphi, \psi\colon A \times [0, +\infty) \to [0, +\infty]$ are called equivalent, $\varphi \simeq \psi$, if there exists $L \geq 1$ such that

$$\varphi(x, t/L) \leq \psi(x, t) \leq \varphi(x, Lt) \quad \text{for } \mu\text{-a.e. } x \in A \text{ and for all } t \geq 0.$$

Lemma 4.16. *Let $\varphi, \psi\colon A \times [0, +\infty) \to [0, +\infty]$, be increasing with respect to the second variable, such that $\varphi \simeq \psi$, and $x \mapsto \varphi(x, |f(x)|)$ and $x \mapsto \psi(x, |f(x)|)$ be measurable for any $f \in L^0(A, \mu)$. Then*

(a) *if φ is a generalized Φ-prefunction, then ψ is a generalized Φ-prefunction;*
(b) *if φ satisfies $(\mathrm{aInc})_p$, then ψ satisfies $(\mathrm{aInc})_p$;*
(c) *if φ satisfies $(\mathrm{aDec})_q$, then ψ satisfies $(\mathrm{aDec})_q$.*

Lemma 4.17. *If $\varphi \in \Phi_w(A, \mu)$ satisfies $(\mathrm{aInc})_p$ with $p \geq 1$, then there exists $\psi \in \Phi_c(A, \mu)$ equivalent to φ such that $\psi^{1/p}$ is convex. In particular, ψ satisfies $(\mathrm{Inc})_p$.*

We can now discuss our main assumptions on the growth function ψ, starting from the weight condition (A0) and the local continuity condition (A1).

Definition 4.18. Let $\varphi \in \Phi_w(\Omega)$. We say that φ satisfies (A0) if there exists a constant $\sigma \geq 1$ such that

$$\varphi\left(x, \frac{1}{\sigma}\right) \leq 1 \leq \varphi(x, \sigma) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega.$$

Proposition 4.19. *Given $\varphi, \psi \in \Phi_w(\Omega)$, if $\varphi \simeq \psi$ and φ satisfies (A0), then ψ satisfies (A0).*

If $\varphi \in \Phi_w(\Omega)$ satisfies (A0), then $\varphi_B^-, \varphi_B^+ \in \Phi_w$ for every $B \Subset \Omega$ (see [47, Lemma 2.5.16]).

Definition 4.20. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0). We say that φ satisfies (A1) if for every ball $B \subset \Omega$ with $\mathcal{L}^d(B) \leq 1$ there exists $\beta \in (0, 1)$ such that

$$\varphi_B^+(\beta t) \leq \varphi_B^-(t) \quad \text{for all } t \in \left[\sigma, (\varphi_B^-)^{-1}\left(\tfrac{1}{\mathcal{L}^d(B)}\right)\right]. \tag{4.1}$$

Notice that the above definition explicitly depends on the space dimension d. To our purposes, an adimensional (and weaker) version of (A1) will be sufficient. We state it below.

Definition 4.21. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0). We say that φ satisfies (adA1) if for every ball $B \subset \Omega$ with $\mathrm{diam}(B) \leq 1$ there exists $\beta \in (0, 1)$ such that

$$\varphi_B^+(\beta t) \leq \varphi_B^-(t) \quad \text{for all } t \in \left[\sigma, (\varphi_B^-)^{-1}\left(\tfrac{1}{\mathrm{diam}(B)}\right)\right]. \tag{4.2}$$

Lemma 4.22. *Let $\varphi, \psi \in \Phi_w(\Omega)$ and $\varphi \simeq \psi$. If φ satisfies (A0) and (A1) (resp. (adA1)), then ψ does as well.*

Remark 4.23. Let $\varphi \in \Phi_s(B_\varepsilon)$. Define $\varphi_\varepsilon(x, \cdot) := \varphi(\varepsilon x, \cdot)$ for every $x \in B_1$. If φ satisfies (A0) with a constant $\sigma \geq 1$ then ψ satisfies (A0) with the same constant σ of φ. If φ satisfies $(\mathrm{Inc})_p$ with $p \in [1, \infty)$ (resp. $(\mathrm{Dec})_q$ with $q \in [1, \infty)$), then ψ satisfies $(\mathrm{Inc})_p$ with the same p (resp. $(\mathrm{Dec})_q$ with the same q).

The next two results show that the validity of our main assumptions is unchanged by passing to equivalent Φ-functions. will be used to simplify our proofs while remaining in full generality.

Lemma 4.24. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (adA1), (aInc) and (aDec) on Ω. Then, there exist $\psi \in \Phi_s(\Omega)$ which satisfies (A0), (adA1), (Inc) and (Dec) on Ω and such that $\psi \simeq \varphi$ and $\psi \approx \varphi$.*

Proof. By assumption we have that φ satisfies (aDec) on Ω and thus is finite valued. Therefore, using Lemma 4.17, there exists $\psi \in \Phi_c(\Omega)$ and hence in $\Phi_s(\Omega)$ such that $\psi \simeq \varphi$ and satisfying (Inc) on Ω. Moreover, by Lemma 4.16 and Lemma 4.9, we have that ψ satisfies (Dec) on Ω. Finally, properties (A0) and (adA1) come from Lemma 4.19 and 4.22, respectively, while the fact that $\psi \approx \varphi$ is a consequence of $\psi \simeq \varphi$ and the fact that both functions are doubling. □

Lemma 4.25. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0), and (aDec) on Ω and (3.6) $\mathcal{L}^d$-a.e. in Ω. Given $\psi \in \Phi_w(\Omega)$ such that $\varphi \simeq \psi$ then ψ satisfies (3.6) $\mathcal{L}^d$-a.e. in Ω.*

Proof. We have that ψ satisfies (A0) with some $\sigma \geq 1$, and it is doubling on Ω with a constant $K \geq 2$. Let $x_0 \in \Omega$ such that (3.6) holds for φ, we want to prove that (3.6) holds for ψ in x_0 as well. We have that there exists $C_\varphi = C_\varphi(x_0) > 0$ such that given $\theta > \sigma$, we can find $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$ and every $t \in [\sigma, \theta]$

$$\varphi^+_{B_\varepsilon(x_0)}(t) \leq C_\varphi \varphi^-_{B_\varepsilon(x_0)}(t).$$

There exists $L \geq 1$ with $\varphi(x, t/L) \leq \psi(x,t) \leq \varphi(x, Lt)$ for every $t \geq 0$. Observe that ψ satisfies (A0) with $L\sigma$. Then, for every $\theta \geq L\sigma$ we deduce that there exists $\varepsilon_0 = \varepsilon_0(L\theta) > 0$ such that for every $\varepsilon \leq \varepsilon_0$

$$\psi^+_{B_\varepsilon(x_0)}(t) \leq \varphi^+_{B_\varepsilon(x_0)}(Lt) \leq C_\varphi \varphi^-_{B_\varepsilon(x_0)}(Lt) \leq C_\varphi \psi^-_{B_\varepsilon(x_0)}(L^2 t) \leq C_\varphi K^{2\log_2(L)+1} \psi^-_{B_\varepsilon(x_0)}(t), \quad t \in [L\sigma, \theta],$$

where we have used property (3.6) of φ in x_0 and the fact that ψ is doubling. Therefore ψ satisfies property (3.6) in x_0 with the constant $C_\varphi K^{2\log_2(L)+1}$. □

Finally, in the next elementary Lemma we observe that (adA1), together with (A0), (aDec), is stronger than condition (3.6) on Ω.

Lemma 4.26. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (aDec) and (adA1). Then φ satisfies (3.6) on every $x_0 \in \Omega$.*

Proof. Let $x_0 \in \Omega$, let σ be the constant of (A0) and let K be the doubling constant of ψ. Fix $\theta > \sigma$. We take $\varepsilon_0 > 0$ such that $\theta \leq \varphi^-_{B_{\varepsilon_0}(x_0)}\left(\frac{1}{2\varepsilon_0}\right)$. Indeed, since $\varphi(x, \cdot)$ satisfies $(\mathrm{Inc})_1$ for a.e. $x \in \Omega$, we have that $\varphi^-_{B_\varepsilon(x_0)}\left(\frac{1}{2\varepsilon}\right) \to +\infty$ as $\varepsilon \to 0$. By (adA1) there exists $\beta \in (0,1)$ such that for every $0 < \varepsilon \leq \varepsilon_0$

$$\varphi^+_{B_\varepsilon(x_0)}(\beta t) \leq \varphi^-_{B_\varepsilon(x_0)}(t), \qquad \text{for every } t \in [\sigma, \theta] \subseteq \left[\sigma, \varphi^-_{B_\varepsilon(x_0)}\left(\tfrac{1}{2\varepsilon}\right)\right].$$

This together with the doubling property of φ implies that (3.6) holds for every $x_0 \in \Omega$ with the uniform constant $K^{1-\log_2(\beta)}$. □

We recall the concept of generalized Orlicz space and report a brief list of useful definitions and properties of such spaces. Given $\varphi \in \Phi_w(\Omega)$ and $f \in L^0(\Omega, \mathbb{R}^m)$ we define the *modular* $\rho_\varphi(f)$ as

$$\rho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|)\, dx.$$

The set

$$L^\varphi(\Omega, \mathbb{R}^m) := \{f \in L^0(\Omega) \colon \rho_\varphi(\lambda f) < +\infty \quad \text{for some } \lambda > 0\},$$

is called a generalized Orlicz space.

Lemma 4.27. *Let $\varphi \in \Phi_w(\Omega)$. Then*

(a) $L^\varphi(\Omega, \mathbb{R}^m) = \{f \in L^0(\Omega, \mathbb{R}^m) \colon \lim_{\lambda \to 0^+} \rho_\varphi(\lambda f) = 0\}$;

(b) *if φ additionally satisfies (aDec) we have*

$$L^\varphi(\Omega, \mathbb{R}^m) = \{f \in L^0(\Omega, \mathbb{R}^m) \colon \rho_\varphi(f) < +\infty\}.$$

Next we define the (quasi)-norm associated to generalized Orlicz spaces.

Definition 4.28. Let $\varphi \in \Phi_w(\Omega)$. For $f \in L^0(\Omega, \mathbb{R}^m)$ we define

$$\|f\|_{L^\varphi(\Omega,\mathbb{R}^m)} := \inf\left\{\lambda > 0\colon \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Most of the time we will abbreviate the notation writing $\|f\|_\varphi$. Now we present some properties linking the modular with the Orlicz norm.

Lemma 4.29. *If $\varphi \in \Phi_w(\Omega)$ then $\|\cdot\|_\varphi$ is a quasi-norm. If $\varphi \in \Phi_c(\Omega)$ then $\|\cdot\|_\varphi$ is a norm.*

Proposition 4.30. *Let $\varphi, \psi \in \Phi_w(\Omega)$. If $\varphi \simeq \psi$, then $L^\varphi(\Omega, \mathbb{R}^m) = L^\psi(\Omega, \mathbb{R}^m)$ and the two norms are equivalent.*

Remark 4.31. Lemmas 4.24, 4.25 and 4.30 entail, in particular, that it suffices to prove all our results under the stronger assumption that $\psi \in \Phi_s(\Omega)$ satisfying (A0), (Inc), (Dec) and (adA1) (or (3.6), depending on the need) on Ω. The extension to a $\psi \in \Phi_w(\Omega)$ is indeed immediate, upon noticing that, if we take equivalent Orlicz functions, then the generalized Orlicz space remains the same.

Lemma 4.32 (Unit ball property)**.** *Let $\varphi \in \Phi_w(\Omega)$. Given $f \in L^0(\Omega, \mathbb{R}^m)$ we have*

$$\|f\|_\varphi < 1 \quad \Rightarrow \quad \rho_\varphi(f) \le 1 \quad \Rightarrow \quad \|f\|_\varphi \le 1.$$

If in addition φ is left continuous then $\rho_\varphi(f) \le 1 \Leftrightarrow \|f\|_\varphi \le 1$. Moreover, the following properties hold:

(a) *if $\|f\|_\varphi < 1$, then $\rho_\varphi(f) \le \|f\|_\varphi$;*
(b) *if $\|f\|_\varphi > 1$, then $\|f\|_\varphi \le \rho_\varphi(f)$;*
(c) *in any case, $\|f\|_\varphi \le \rho_\varphi(f) + 1$.*

In generalized Orlicz spaces we also have a generalization of the concept of Lebesgue points.

Proposition 4.33. *Let $\varphi \in \Phi_w(\Omega)$ satisfying (A0) and (aDec). Then, for every $f \in L^0(\Omega, \mathbb{R}^m)$ such that $\rho_\varphi(f) < +\infty$, we have*

$$\lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon(x_0)} \varphi(x, |f(x) - f(x_0)|)\, dx = 0 \quad \textit{for } \mathcal{L}^d\textit{-a.e. } x_0 \in \Omega.$$

Proof. The proof can be carried out exactly as the proof of Theorem 3.1 in [46] with some minor changes, so we omit it. □

We present some properties of the maximal operator in Orlicz spaces (see [47, Chapter 4] or [52]). Since the proof of Proposition 7.3 only needs the estimate for Φ-functions independent of x, we give the statement only for non generalized Φ-functions.

Definition 4.34. Given $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ we define the restricted maximal operator to Ω as

$$Mf(x) := \sup_{r>0} \frac{1}{\omega_d r^d} \int_{B_r(x)\cap\Omega} |f(y)|\, dy.$$

Analogously, for a non-negative, finite Radon measure μ on Ω one can define

$$M\mu(x) := \sup_{r>0} \frac{\mu(B_r(x) \cap \Omega)}{\omega_d r^d}.$$

Theorem 4.35. *Let $\varphi \in \Phi_w$ finite valued satisfy $(\mathrm{Inc})_\gamma$ with $\gamma > 1$. Then, the restricted maximal operator $M\colon L^\varphi(\Omega, \mathbb{R}^m) \to L^\varphi(\Omega, \mathbb{R}^m)$ is bounded. In particular, we have that if $\|f\|_\varphi \ne 0$, setting $\varepsilon := \frac{1}{2\|f\|_\varphi}$, there exists $c = c(\varphi^{-1}(1), d, \gamma)$ such that $\rho_\varphi(c\varepsilon Mf) \le 1$.*

Proof. The proof builds upon the key estimate [47, Theorem 4.3.3]. In particular, let $c_1 = c_1(d, \gamma)$ be the constant such that $\|Mf\|_{L^\gamma(\Omega)} \le c_1 \|f\|_{L^\gamma(\Omega)}$. Then, the constant c is given by

$$c := \frac{(\varphi^{-1}(1))^3}{16 c_1}.$$

□

Corollary 4.36. *Let $\varphi \in \Phi_w$ finite valued satisfy* (Inc)$_\gamma$ *with $\gamma > 1$. Assume in addition that it satisfies* (Dec)$_q$ *with $1 < q < \infty$. Then, there exists $C = C(\varphi^{-1}(1), d, \gamma, q) > 0$ such that for every $f \in L^\varphi(\Omega, \mathbb{R}^m)$*

$$\rho_\varphi(Mf) \leq C(\rho_\varphi(f) + 1)^q. \tag{4.3}$$

Proof. Using Theorem 4.7, we can find $\psi \in \Phi_s$ such that $\psi(t) \leq \varphi(t) \leq \psi(2t)$ for every $t \geq 0$. Moreover, by Lemma 4.5, ψ satisfies (Inc)$_\gamma$ and (Dec)$_q$. Let us prove (4.3) for ψ. If $f \equiv 0$ then there is nothing to prove. Assume that f is not identically zero on Ω. Since $\|\cdot\|_\psi$ is a norm by Lemma 4.29, we have that $\|f\|_\psi \neq 0$. Let $\varepsilon := \frac{1}{2\|f\|_\psi}$ and $c = c(\psi^{-1}(1), d, \gamma) = c(\varphi^{-1}(1), d, \gamma) > 0$ be the constant from Theorem 4.35. If $c\varepsilon \leq 1$ by (Dec)$_q$ and Theorem 4.35 we have

$$1 \geq \int_\Omega \psi(c\varepsilon Mf(x))\,dx \geq (c\varepsilon)^q \int_\Omega \psi(x, Mf(x))\,dx.$$

Therefore, using also Lemma 4.32(c) we get

$$\int_\Omega \psi(Mf(x))\,dx \leq c^{-q}(2\|f\|_\psi)^q \leq 2^q c^{-q}\left(1 + \int_\Omega \psi(|f(x)|)\,dx\right)^q,$$

If $c\varepsilon > 1$, then it is enough to observe that since ψ is increasing

$$\int_\Omega \psi(Mf(x))\,dx \leq \int_\Omega \psi(c\varepsilon Mf(x))\,dx \leq 1 \leq 1 + \|f\|_\psi \leq 2 + 2\rho_\psi(f) \leq 2(1 + \rho_\psi(f))^q.$$

Setting $C := \max\{2^q c^{-q}, 2\}$ we conclude (4.3) for ψ. Finally, for φ we have

$$\rho_\varphi(Mf/2) \leq \rho_\psi(Mf) \leq C(\rho_\psi(f) + 1)^q \leq C(\rho_\varphi(f) + 1)^q,$$

using again (Dec)$_q$, we conclude (4.3) for φ as well with $C = C(\varphi^{-1}(1), d, \gamma, q)$. ∎

4.2. **(Generalized) Special functions of Bounded Variation.** We now discuss the definition and some useful properties of $GSBV$ and SBV function. For a complete treatment of the topic we refer to [10].

Let $A \in \mathcal{A}(\mathbb{R}^d)$ and $x \in A$ such that

$$\limsup_{r \searrow 0} \frac{\mathcal{L}^d(A \cap B(x, r))}{r^d} > 0,$$

Given $u \in L^0(A, \mathbb{R}^m)$, we say that $a \in \mathbb{R}^m$ is the approximate limit of u at x if

$$\lim_{r \searrow 0} \frac{\mathcal{L}^d(\Omega \cap B(x, r) \cap \{|a - v(x)| > \varepsilon\})}{r^d} = 0 \quad \text{for every } \varepsilon > 0.$$

In that case we write

$$\operatorname*{ap-lim}_{y \to x} u(y) = a.$$

We say that $x \in A$ is an approximate jump point of u, and we write $x \in J_u$, if there exists $a, b \in \mathbb{R}^m$ with $a \neq b$ and $\nu \in \mathbb{S}^{d-1}$ such that

$$\operatorname*{ap-lim}_{\substack{y \to x \\ (y-x)\cdot\nu > 0}} u(y) = a \quad \text{and} \quad \operatorname*{ap-lim}_{\substack{y \to x \\ (y-x)\cdot\nu < 0}} u(y) = b.$$

In particular, for every $x \in J_u$ the triple (a, b, ν) is uniquely determined up to a change of sign of ν and a permutation of a and b. We indicate such triple by $(u^+(x), u^-(x), \nu_u(x))$. The jump of u at $x \in J_u$ is defined as $[u](x) := u^+(x) - u^-(x) \in \mathbb{R}^m_0$. The set J_u is countably rectifiable and has ν as an approximate unit normal vector at $\mathcal{H}^{d-1}$-every point (see [62]).

The space $\mathrm{BV}(A, \mathbb{R}^m)$ of functions of bounded variation is the set of $u \in L^1(A; \mathbb{R}^m)$ whose distributional gradient Du is a bounded Radon measure on A with values in $\mathbb{R}^{m\times d}$. Given $u \in \mathrm{BV}(A, \mathbb{R}^m)$ we can write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular w.r.t. $\mathcal{L}^d$. The density $\nabla u \in L^1(A, \mathbb{R}^{m\times d})$ of $D^a u$ w.r.t. $\mathcal{L}^d$ coincides a.e. in A with the approximate gradient of u. That is, for a.e. $x \in A$ it holds

$$\operatorname*{ap-lim}_{y \to x} \frac{u(y) - u(x) - \nabla u(x)\cdot(y - x)}{|x - y|} = 0.$$

The space $SBV(A,\mathbb{R}^m)$ of special functions of bounded variation is defined as the set of all $u \in \mathrm{BV}(A,\mathbb{R}^m)$ such that $|D^s u|(A\setminus J_u)=0$. Moreover, we denote by $SBV_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ the space of functions belonging to $SBV(U,\mathbb{R}^m)$ for every $U \Subset A$ open. For $p\in[1,+\infty)$, $SBV^p(A,\mathbb{R}^m)$ stands for the set of functions $u\in SBV(A,\mathbb{R}^m)$, with approximate gradient $\nabla u\in L^p(A,\mathbb{R}^{m\times d})$ and $\mathcal{H}^{d-1}(J_u)<+\infty$, that is

$$SBV^p(A,\mathbb{R}^m):=\{u\in SBV(A,\mathbb{R}^m)\colon \nabla u\in L^p(\Omega,\mathbb{R}^m), \mathcal{H}^{d-1}(J_u)<+\infty\}.$$

We say that $u\in GSBV(A,\mathbb{R}^m)$ if $X(u)\in SBV_{\mathrm{loc}}(A,\mathbb{R}^m)$ for every $X\in C^1_c(\mathbb{R}^m,\mathbb{R}^m)$. Also for $u\in GSBV(A,\mathbb{R}^m)$ the approximate gradient ∇u exists $\mathcal{L}^d$-a.e. in A and the jump set J_u is countably $\mathcal{H}^{d-1}$-rectifiable with approximate unit normal vector ν_u. Finally, for $p\in[1,+\infty)$, we define as before

$$GSBV^p(A,\mathbb{R}^m):=\{u\in GSBV(A,\mathbb{R}^m)\colon \nabla u\in L^p(\Omega,\mathbb{R}^m), \mathcal{H}^{d-1}(J_u)<+\infty\};$$

it is known that $GSBV^p(A,\mathbb{R}^m)$ is a vector space (see e.g. [34, pg. 172]).

Moreover, given a generalized Orlicz function $\varphi\in\Phi_w(\Omega)$ we denote with $SBV^\varphi(\Omega,\mathbb{R}^m)$ the space of functions $u\in SBV(\Omega,\mathbb{R}^m)$ with $\mathcal{H}^{d-1}(J_u)<+\infty$ and $\nabla u\in L^\varphi(\Omega,\mathbb{R}^{m\times d})$. The definition of $GSBV^\varphi(\Omega,\mathbb{R}^m)$ is analogous.

The following three theorems can be found in [38] for the scalar case and in [22] for vector valued SBV functions.

Theorem 4.37 ([38, Lemma 2.6])**.** *Let $p\geq 1$ and $u\in SBV(\Omega,\mathbb{R}^m)$ be such that*

$$\int_K |\nabla u|^p\,dx+\mathcal{H}^{d-1}(J_u\cap K)<+\infty$$

for every compact $K\Subset\Omega$. Then

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{d-1}}\left(\int_{B_\varepsilon(x_0)}|\nabla u(x)|^p\,dx+\mathcal{H}^{d-1}(B_\varepsilon(x_0)\cap J_u)\right)=0 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x_0\in\Omega\setminus J_u.$$

For the other two theorems we need first to fix some notations. Given $a=(a_1,\dots,a_m)$ and $b=(b_1,\dots,b_m)$ vectors in $\mathbb{R}^m$, we set

$$a\wedge b:=(\min(a_1,b_1),\dots,\min(a_m,b_m)) \quad\text{and}\quad a\vee b:=(\max(a_1,b_1),\dots,\max(a_m,b_m)).$$

Let B be a ball in $\mathbb{R}^d$. For every measurable function $u\colon B\to\mathbb{R}^m$, given $0\leq s\leq\mathcal{L}^d(B)$, we define

$$u_*(s;B):=((u_1)_*(s;B),\dots,(u_m)_*(s;B)),$$

where, for every $i=1,\dots,m$,

$$(u_i)_*(s;B):=\inf\left\{t\in\mathbb{R}\colon \mathcal{L}^d(\{x\in B\colon\ u_i(x)<t\})\geq s\right\}.$$

Moreover, we define $\mathrm{med}(u;B):=u_*\left(\frac{\mathcal{L}^d(B)}{2};B\right)$.

Let γ_{iso} be the dimensional constant in the relative isoperimetric inequality in balls. For every $u\in GSBV(\Omega,\mathbb{R}^m)$ such that

$$\left(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u\cap B)\right)^{\frac{d}{d-1}}\leq\frac12\mathcal{L}^d(B),$$

we define

$$\begin{aligned}\tau'(u;B)&:=u_*\left(\left(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u\cap B)\right)^{\frac{d}{d-1}};B\right),\\ \tau''(u;B)&:=u_*\left(\mathcal{L}^d(B)-\left(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u\cap B)\right)^{\frac{d}{d-1}};B\right)\end{aligned}\tag{4.4}$$

and the truncation operator in B

$$T_Bu(x):=(\tau'(u;B)\vee u(x))\wedge\tau''(u;B).\tag{4.5}$$

Notice that it holds

$$\mathcal{L}^d(\{T_Bu\neq u\}\cap B)\leq 2\left(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u\cap B)\right)^{\frac{d}{d-1}}.\tag{4.6}$$

Theorem 4.38 ([38, Theorem 3.1 and Remark 3.2])**.** *Let $B \subset \mathbb{R}^d$ be a ball and $u \in SBV(B,\mathbb{R}^m)$ satisfy*

$$\left(2\gamma_{\rm iso}\mathcal{H}^{d-1}(J_u \cap B)\right)^{\frac{d}{d-1}} \le \frac{1}{2}\mathcal{L}^d(B). \tag{4.7}$$

If $1 \le p < d$ then,

$$\left(\int_B |T_B u - \mathrm{med}(u,B)|^{p^*}\,dx\right)^{1/p^*} \le \frac{2\gamma_{\rm iso}p(d-1)}{d-p}\left(\int_B |\nabla u|^p\,dx\right)^{1/p}, \tag{4.8}$$

and

$$\mathcal{L}^d(\{T_B u \neq u\} \cap B) \le 2\left(2\gamma_{\rm iso}\mathcal{H}^{d-1}(J_u \cap B)\right)^{\frac{d}{d-1}}. \tag{4.9}$$

If $p \ge d$ instead, for every $q \ge 1$ we have

$$\|T_B u - \mathrm{med}(u,B)\|_{L^q(B,\mathbb{R}^m)} \le \frac{2\gamma_{\rm iso}q(d-1)}{d}\mathcal{L}^d(B)^{\frac{1}{d}+\frac{1}{q}-\frac{1}{p}}\|\nabla u\|_{L^p(B,\mathbb{R}^m)}. \tag{4.10}$$

Theorem 4.39 ([38, Theorem 3.6])**.** *Let $u \in SBV_{\rm loc}(\Omega,\mathbb{R}^m)$ and $p > 1$ and let $x_0 \in \Omega$. If*

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{d-1}}\left(\int_{B_\varepsilon(x_0)}|\nabla u(x)|^p\,dx + \mathcal{H}^{d-1}(B_\varepsilon(x_0)\cap J_u)\right) = 0,$$

then $x_0 \neq J_u$ and

$$\lim_{\varepsilon\to 0^+}\mathrm{med}(u,B_\varepsilon(x_0)) = \mathop{\text{ap-}\lim}_{x\to x_0} u(x) \in \mathbb{R}^m.$$

Moreover, there exists $\varepsilon_0 > 0$ such that for $\delta \in (3/4,1)$ and $\varepsilon,\rho > 0$ with $\delta\varepsilon < \rho < \varepsilon < \varepsilon_0$,

$$\tau'(u,B_\rho(x_0)) \le \mathrm{med}(u,B_\varepsilon(x_0)) \le \tau''(u,B_\rho(x_0)),$$
$$\tau'(u,B_\varepsilon(x_0)) \le \mathrm{med}(u,B_\rho(x_0)) \le \tau''(u,B_\varepsilon(x_0)).$$

4.3. **Examples.** In this last subsection of preliminaries, we present some examples of generalized Orlicz functions to which our theory applies. We consider the following Orlicz functions:

(I) **Variable exponent**: $\psi(x,t) = t^{p(x)}$. For the variable exponent case, the semicontinuity was already addressed in [35], while integral representation and Γ-convergence are studied in [65].
(II) **Perturbed Orlicz**: $\psi(x,t) = a(x)\varphi(t)$.
(III) **Double phase**: $\psi(x,t) = t^p + a(x)t^q$.
(IV) **Degenerate double phase**: $\psi(x,t) = t^p + a(x)t^p\log(\mathrm{e}+t)$.
(V) **Triple phase**: $\psi(x,t) = t^p + a(x)t^q + b(x)t^r$.
(VI) **Variable exponent double phase**: $\psi(x,t) = t^{p(x)} + a(x)t^{q(x)}$. This type of generalized Orlicz function was studied recently in [32].

We say that a function $p\colon \Omega \to [1,+\infty)$ is log-Hölder continuous on Ω if

$$\exists\, C > 0 \quad \text{such that} \quad |p(x)-p(y)| \le \frac{C}{-\log|x-y|} \quad \text{for every } x,y \in \Omega \text{ with } |x-y| \le \frac{1}{2}.$$

We use the usual shortcut $p \in C^{\log}(\Omega)$ to denote a log-Hölder continuous variable exponent on Ω. In the Table below we collect conditions that are sufficient (sometimes necessary) for properties (aInc), (aDec), (A0), (3.6) and (adA1) in the special cases above. The usual notations C^0 and $C^{0,\beta}$ are used for continuous and Hölder continuous functions with exponent β, respectively. In each line, the checkmark is used to denote that a property needs no new assumption to be satisfied than the previously considered ones.

We remark that in the case of variable exponent, the sole continuity assumption will be enough to obtain (3.6) and thus Theorem 3.3. This represents an improvement with respect to the result of [35], where log–Hölder continuity has been assumed.

We briefly discuss how our conditions can be checked in all the aforementioned cases. Checking the validity of (aInc) and (aDec) is straightforward, while we refer to Table 1 in [50] for condition (A0). Concerning (3.6), it follows from the following general claim: given $\varphi \in \Phi_w(\Omega)$ satisfying (A0) on Ω, we have that if $\varphi \in C^0(\Omega \times [0,+\infty))$, then it satisfies property (3.6). Now, assume that φ satisfies (A0) with $\sigma \ge 1$ and let for simplicity $x_0 = 0$. Let $\zeta > \sigma$ fixed. Since φ is uniformly continuous on

	(aInc)	(aDec)	(A0)	(3.6)	(adA1)
(I)	$\operatorname{ess\,inf}_\Omega p > 1$	$\operatorname{ess\,sup}_\Omega p < +\infty$	✓	$p \in C^0$	$p \in C^{\log}$
(II)	φ is (aInc)	φ is (aDec)	$c_1 \leq a \leq c_2$	✓	✓
(III)	$a \geq 0,\ p > 1$	$q < +\infty$	$a \in L^\infty$	$a \in C^0$	$a \in C^{0,q/p-1}$
(IV)	$a \geq 0,\ p > 1$	$p < +\infty$	$a \in L^\infty$	$a \in C^0$	$a \in C^{\log}$
(V)	$a, b \geq 0,\ p > 1$	$q \leq r < +\infty$	$a, b \in L^\infty$	$a, b \in C^0$	$a \in C^{0,q/p-1}$, $b \in C^{0,r/p-1}$
(VI)	$\operatorname{ess\,inf}_\Omega p > 1$	$\operatorname{ess\,sup}_\Omega q < +\infty$	$a \in L^\infty$	$a, p, q \in C^0$	$p \in C^{\log}$, $a \in C^{0,\alpha}$, $q \in C^{0,\alpha/q^-}$, $\frac{q}{p} \leq 1+\alpha$

$B_\delta \times [\sigma, \zeta]$ for some $\delta > 0$ small enough, we observe that there exists a uniform modulus of continuity $\omega\colon [0,+\infty) \to [0,+\infty)$ such that

$$|\varphi(x,t) - \varphi(y,t)| \leq \omega(|x-y|) \quad \text{for every } x, y \in B_\delta \text{ and every } t \in [\sigma, \zeta].$$

Let $\varepsilon_0 \in (0,\delta)$ such that $\omega(2\varepsilon_0) \leq 1$. Thanks to (A0) and the fact that $\varphi(y,\cdot)$ is increasing, we estimate

$$\varphi(x,t) \leq \varphi(y,t) + \omega(|x-y|) \leq \varphi(y,t) + 1 \leq 2\varphi(y,t), \quad \text{for every } x, y \in B_\varepsilon \text{ and every } t \in [\sigma, \zeta]\,.$$

Thus, property (3.6) is satisfied for every $x_0 \in \Omega$ with $C = 2$.
As for condition (adA1), we discuss it here for example in the double phase case. Namely, we prove that if $\psi(x,t) = |t|^p + a(x)|t|^q$ with $a \in C^{0,\alpha}(\overline{\Omega})$, $a(x) \geq 0$ and $1 < p < q$ with $\frac{q}{p} \leq 1+\alpha$ for some $\alpha \in (0,1]$, then ψ satisfies (adA1).

Note that $\psi(x,t) \approx \max\{t^p, a(x)t^q\}$. Let $B \Subset \Omega$ a ball with $\operatorname{diam}(B) \leq 1$, denote $a_B^- := \min_B a(\cdot)$ and $a_B^+ := \max_B a(\cdot)$. We need to show that $\max\{t^p, a_B^+ t^q\} \lesssim \max\{t^p, a_B^- t^q\}$ whenever $\psi_B^-(t) \leq \frac{1}{\operatorname{diam}(B)}$. In view of the fact that $a \in C^{0,\alpha}(\overline{\Omega})$ and $\frac{q}{p} \leq 1+\alpha$, we have

$$a_B^+ \lesssim \max\{\operatorname{diam}(B)^{\frac{q-p}{p}}, a_B^-\}.$$

Now, when $t^p \leq \frac{1}{\operatorname{diam}(B)}$, one has in particular $\psi_B^-(t) \leq \frac{1}{\operatorname{diam}(B)}$, so let us assume that $t^p \leq \frac{1}{\operatorname{diam}(B)}$. Hence, $a_B^+ \lesssim \max\{t^{p-q}, a_B^-\}$, so that $a_B^+ t^q \lesssim \max\{t^p, a_B^- t^q\}$. From this, we conclude.

Some further comment on the meaning and on the role of our assumptions are collected in the subsection below.

4.4. **On the role of our assumptions.** The relaxation result, Theorem 3.4, is proved under assumptions (A0), (adA1), (aInc) and (aDec). Let us stress that all these assumptions are by now well-established in the theory of variational integrals with ψ-growth [47, 51] (in the case without discontinuities), up to (adA1), which is weaker than similar conditions considered in the literature, such as (A1) (or even (VA1) in [51]). The need for stronger assumptions is indeed often connected to further issues than relaxation (e.g., regularity of minimizers): we chose to prove our results in a weaker setting, which has nevertheless the same flavour as the customary ones and contains all of them. A reader not willing to go into such details may also assume that (A1) is satisfied, our results being working *verbatim* in the same way also in that case, which is also not covered by previous literature.

Let us also stress that, due to the presence of a discontinuity set, condition (aInc) is a crucial growth assumption. It is namely needed to separate scales throughout relaxation; namely, to avoid the interplay between bulk and surface effects (an issue which is already well known in the p-growth case).

Condition (aDec) ensures the doubling condition Δ_2 and the "equivalence" between the modular $\rho_\psi(\cdot)$ and the norm $\|\cdot\|_{L^\psi}$ (see Lemma 4.27). This is heavily used in our proofs (see e.g. Theorem 5.27 and Lemma 6.9). Indeed, if we were to not assume (aDec), we may have that $\psi(\cdot, |\nabla u|)$ is integrable on the reference set Ω, while $\psi(\cdot, 2|\nabla u|)$ is not.

Condition (A0) ensures that the functions ψ_B^- and ψ_B^+ (see (2.2)) have a non-degenerate behavior for every ball $B \subset \Omega$ and it is needed to define (adA1). In particular, combining (A0) with (aDec) gives that constant functions belong to L^ψ. We further observe that conditions (A0) and (adA1) comply very well with localization and blow-up methods which are typical for relaxation methods involving quasiconvexification. For instance, (A0) together with (aDec) ensures the Lebesgue point property for functions in L^ψ (see Proposition 4.33). On the other hand, (adA1) is fundamental to derive a Poincaré inequality for generalized Orlicz function (see Theorem 5.9) and it is used in the blow-up Lemmas 6.6 and 6.9.

As we mentioned before, conditions of the type (adA1) are already known in relaxation theory and regularity of minima for functionals of the form

$$\int_\Omega f(x, \nabla u)\, dx \qquad \text{in } W^{1,\psi}, \tag{4.11}$$

where f satisfies the growth conditions in (1.3). In particular, (A1) and a stronger counterpart thereof (called (VA1) in [51]), which are stronger versions than (adA1), have been shown to be optimal for the regularity of minima in $W^{1,\psi}$ for functionals defined as in (4.11) (see [51]). On the one hand, our framework extends functionals of the type (4.11) to the fractured setting and reduces to (4.11) any time minimizers have no discontinuity set: hence, a reasonable relaxation result must comply with the customary framework for (4.11). In particular, notice that in [61], a Poincaré inequality is *assumed* in order to prove a relaxation result in the Sobolev-Orlicz case, while we are able to deduce it from our assumptions. On the other hand, when one is only interested in the relaxation of a functional and not on the regularity of the minima, weaker hypotheses may be sufficient. This is the spirit under which we use condition (adA1) in our proofs. Coming to technical details, the relevance of (adA1) is evident from Lemmas 6.6 and 6.9, where we have to truncate the functions in order to obtain more regular sequences. In turn, we cannot truncate too much as, after matching the cell boundary conditions, this could create a jump set which is too big and it may interfere with the blow-up procedure. Condition (adA1) gives exactly the level at which we can truncate our functions without modifying the energy in an uncontrolled way (see the proof of Lemmas 6.6 and 6.9 for more details).

Finally, for the lower semicontinuity Theorem 3.3, condition (adA1) can be further weakened (see condition 3.6). The case study of $p(\cdot)$-growth is illuminating to understand the improvements. In that case, continuity of $p(\cdot)$ is sufficient to have lower semicontinuity, refining the result of [35] where instead log-Hölder continuity has been assumed. While well-posedness of a free-discontinuity problem in this context can now, thanks to Theorem 3.3, regularity of the corresponding minimizers requires instead the so-called strongly log-Hölder continuity condition, as proved in [56].

5. Poincaré inequality in SBV^φ

In this section we present a fundamental ingredient we need in the proofs of our main results and which is in our opinion of independent interest. Namely, a Poincaré inequality for functions belonging to SBV^φ with small jump set a là De Giorgi-Carriero-Leaci. We start with the case of Φ-functions, while the extension to the generalized setting will be given in Subsection 5.3.

Theorem 5.1 (Poincaré inequality in SBV^φ)**.** *Let $\varphi \in \Phi_w$ be a finite valued Orlicz function satisfying* $(\mathrm{Inc})_1$ *and let $B_r(x)$ be a ball in $\mathbb{R}^d$ with radius r and centred in $x \in \mathbb{R}^d$. Set $\phi(t) := \varphi^{\frac{d}{d-1}} t^{-\frac{1}{d-1}}$ and let $u \in SBV^\varphi(B_r(x))$ be such that*

$$\big(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u \cap B_r(x))\big)^{\frac{d}{d-1}} \le \frac{1}{2}\mathcal{L}^d(B_r). \tag{5.1}$$

Then, there exists a dimensional constant $C = C(d)$ such that the following inequality holds

$$\phi^{-1}\left(\fint_{B_r(x)} \phi\left(\frac{|T_{B_r(x)}u(y) - \mathrm{med}(u, B_r(x))|}{r}\right) dy\right) \le C\varphi^{-1}\left(\fint_{B_r(x)} \varphi(C|\nabla u(y)|)\, dy\right), \tag{5.2}$$

where $T_{B_r(x)}u$ is defined in (4.5).

Notice that since we have a truncation of a scalar SBV function, the result above applies also to real valued $GSBV$ functions. Also observe that if φ has linear growth, then the inequality in (5.2) reduces to the standard homogeneous Poincaré inequality with $p = 1$ for SBV functions.

Remark 5.2. By translation invariance and rescaling, it is enough to verify inequality (5.2) only on B_1. Indeed, let $u\colon B_r(x) \to \mathbb{R}^m$, define $v\colon B_1 \to \mathbb{R}^m$ as $v(y) := u(x+ry)/r$. We have $u_*(s; B_r) = rv_*(s/r^d; B_1)$

for every $s \in [0, \mathcal{L}^d(B_r)]$. Thus,

$$\tau'(v, B_1) = \frac{\tau'(u, B_r)}{r}, \quad \tau''(v, B_1) = \frac{\tau''(u, B_r)}{r}, \quad \mathrm{med}(v, B_1) = \frac{\mathrm{med}(u, B_r)}{r},$$

and $T_{B_1}v(y) := T_{B_r(x)}u(x + ry)/r$ for every $y \in B_1$.

Then, if the Poincaré inequality (5.2) holds on B_1 for v, we deduce

$$\begin{aligned}\phi^{-1}\left(\fint_{B_r(x)} \phi\left(\frac{|T_{B_r(x)}u(y) - \mathrm{med}(u, B_r(x))|}{r}\right) dy\right) &= \phi^{-1}\left(\fint_{B_1} \phi\left(|T_{B_1}v(z) - \mathrm{med}(v, B_1)|\right) dz\right) \\ &\le C\varphi^{-1}\left(\fint_{B_1} \varphi(C|\nabla v(z)|)\, dz\right) \\ &= C\varphi^{-1}\left(\fint_{B_r} \varphi(C|\nabla u(y)|)\, dx\right).\end{aligned}$$

The proof of Theorem 5.1 will be given in Subsection 5.2, after recalling and proving some preliminary material.

5.1. **Rearrangements of BV functions.** We are going to make use of the theory of rearrangements for BV functions. Since they are used only in this section, we report here their main definitions and properties. For a complete treatment of the topic see [27]. Let $d \in \mathbb{N}$. We say that two functions u and v are equi-measurable if $\mathcal{L}^d(\{u > t\}) = \mathcal{L}^d(\{v > t\})$ for every $t \in \mathbb{R}$.

For simplicity, let us denote by B the unit ball in $\mathbb{R}^d$ with $\mathcal{L}^d(B) = \omega_d$. Let $u\colon B \to \mathbb{R}$ be a measurable function. We define the distribution function $\mu_u\colon \mathbb{R} \to [0, \omega_d]$ of u as

$$\mu_u(t) := \mathcal{L}^d(\{x \in B\colon u(x) > t\}) \quad t \in \mathbb{R}. \tag{5.3}$$

We have that μ_u is right-continuous and non-increasing. By definition, $\mu_u(\operatorname{ess\,sup} u) = 0$. Moreover, $\mu_u(t-) = \mathcal{L}^d(\{u \ge t\})$ and $\mu_u(t-) - \mu_u(t) = \mathcal{L}^d(\{u = t\})$. Thus, μ_u is continuous in t if and only if $\mathcal{L}^d(\{u = t\}) = 0$. We can now define also the signed decreasing rearrangement of u as a function $u^0\colon [0, \omega_d] \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$u^0(s) := \sup\{t \in \mathbb{R}\colon \mu_u(t) > s\} \quad s \in [0, \omega_d]. \tag{5.4}$$

We have that u^0 is right-continuous, non-increasing and it is equi-measurable with u. By definition, $u^0(0) = \operatorname{ess\,sup} u$. Moreover, notice that

$$\{s \in [0, \omega_d]\colon u^0(s) > t\} = [0, \mu_u(t)) \quad \text{for every } t \in \mathbb{R}.$$

Therefore, u^0 and u are equi-distributed, that is $\nu_u = \nu_{u^0}$. We also have

$$u^0(\mu_u(t)) \le t \;\text{ for every } t \in \mathbb{R} \quad \text{and} \quad t \le u^0(\mu_u(t)-) \;\text{ for every } t \in (-\infty, \operatorname{ess\,sup} u) \tag{5.5}$$

and

$$\mu_u(u^0(s)) \le s \;\text{ for every } s \in [0, \omega_d] \quad \text{and} \quad s \le \mu_u(u^0(s)-) \;\text{ for every } s \in [0, \omega_d). \tag{5.6}$$

To summarize, u^0 is a monotone real valued function on $(0, \omega_d)$ and thus $u^0 \in BV_{\mathrm{loc}}((0, \omega_d))$. Moreover, u^0 is constant in the interval (s_1, s_2) if and only if there exists $t \in \mathbb{R}$ where μ_u jumps and $(s_1, s_2) \subset (\mu_u(t), \mu_u(t-))$. Vice versa, u^0 jumps at some point $s_0 \in (0, \omega_d)$ and $(t_1, t_2) \subset (u^0(s_0), u^0(s_0-))$ if and only if μ_u is constant on (t_1, t_2). Therefore, a countable family of left-open intervals $(\alpha_i, \beta_i]$ exists such that

$$(\operatorname{ess\,inf} u, \operatorname{ess\,sup} u) \setminus \bigcup_{i \in I} (\alpha_i, \beta_i] \subseteq u^0((0, \omega_d)) \subseteq [\operatorname{ess\,inf} u, \operatorname{ess\,sup} u] \setminus \bigcup_{i \in I} (\alpha_i, \beta_i]. \tag{5.7}$$

Finally, if u is a non-negative function on B, we have

$$\mu_u(t) = \mathcal{L}^d(\{x \in B\colon u(x) > t\}), \quad t \in [0, +\infty), \tag{5.8}$$

and the decreasing rearrangement as $u^*\colon [0, \omega_d] \to [0, +\infty]$ is given by

$$u^*(s) := \sup\{t \in [0, +\infty)\colon \mu_u(t) > s\}, \quad s \in [0, \omega_d]. \tag{5.9}$$

The above definitions can be adapted to the case of rescaled balls with due changes.

We start by proving the following fact.

Proposition 5.3. *Let $\psi \in \Phi_s$ be a finite valued Orlicz function satisfying $(\mathrm{Inc})_1$. Then, for every $u \in SBV(B)$ such that (5.1) holds, we have that $(T_B u)^0$ is absolutely continuous in $(0, \omega_d)$ and*

$$\int_0^{\omega_d} \psi\left(c(d)\min\{s, \omega_d - s\}^{\frac{d-1}{d}}\left(-\frac{d(T_B u)^0}{ds}(s)\right)\right) ds \le \int_B \psi\left(2|\nabla u(x)|\right) dx, \tag{5.10}$$

where $c(d)$ is a positive constant depending only on the dimension.

In order to prove Proposition 5.3, we need the following two results, the first one is [27, Equation (3.21)] and the second is contained in [5, Proposition 2.1].

Lemma 5.4. *Let $u \in BV(B)$ and set $C_u \subseteq B$ the set of points in B where u is approximately continuous. We have that*

$$\partial^M\{u > t\} \cap C_u = \emptyset \quad \textit{for a.e. } t \in (\operatorname{ess\,inf} u, \operatorname{ess\,sup} u) \setminus u^0((0, \omega_d)).$$

Proof. The proof can be carried out exactly as the proof of [27, Equation (3.21)]. □

Lemma 5.5. *Let $f, g \in L^1(0, \omega_d)$ such that $f(x) \ge 0$ and $g(x) \ge 0$ for $\mathcal{L}^1$-a.e. $x \in (0, \omega_d)$ and let $F \colon [0, +\infty) \to [0, +\infty)$ be a convex function with $F(0) = 0$. Assume also that*

$$\int_0^t f^*(s)\, ds \le \int_0^t g^*(s)\, ds \quad \textit{for every } t \in [0, \omega_d], \qquad \int_0^{\omega_d} F(g^*(s))\, ds < +\infty.$$

Then,

$$\int_0^{\omega_d} F(f(x))\, dx \le \int_0^{\omega_d} F(g(x))\, dx. \tag{5.11}$$

Proof. The result is essentially proved in [5, Proposition 2.1], which actually requires F to be also Lipschitz. However, an inspection of the proof reveals that we need F to be Lipschitz only on the set $\{f^*(s) \colon s \in [0, \omega_d]\}$. Since F is convex and thus locally Lipschitz, fixing $\eta > 0$ arbitrarily small, we can apply [5, Proposition 2.1] to $f \wedge f^*(\eta)$ and g getting

$$\int_\eta^{\omega_d} F(f^*(s))\, ds \le \int_0^{\omega_d} F(g^*(s))\, ds.$$

Letting $\eta \to 0^+$, by monotone convergence theorem and using the fact that F is strictly increasing, we deduce

$$\int_0^{\omega_d} (F(f))^*(s)\, ds = \int_0^{\omega_d} F(f^*(s))\, ds \le \int_0^{\omega_d} F(g^*(s))\, ds = \int_0^{\omega_d} (F(g))^*(s)\, ds.$$

Finally, using the equi-measurability of rearrangements we conclude (5.11). □

We are now in a position to prove Proposition 5.3.

Proof of Proposition 5.3. In order to ease the notation we set $(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} =: \Lambda$. We split the proof into three steps.

Step 1: u^0 is continuous in $[\Lambda, \omega_d - \Lambda]$. We actually prove that u^0 is continuous in

$$\left(\left(\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}}, \omega_d - \left(\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}}\right). \tag{5.12}$$

Assume this is false. Then, we can find s belonging to the interval in (5.12) such that u^0 jumps in s and thus $(u^0(s), u^0(s-)) \neq \emptyset$. Hence, owning Lemma 5.4 and the fact that $u \in SBV(B)$, we can find $t \in (u^0(s), u^0(s-))$ such that $\partial^M\{u > t\} \cap C_u = \emptyset$ and the set $\{u > t\}$ has finite perimeter. This implies $\partial^*\{u > t\} \subset J_u$. Moreover, recall that $\mathcal{H}^{d-1}(\partial^M\{u > t\} \setminus \partial^*\{u > t\}) = 0$. By definition of u^0 we have that $\mathcal{L}^d(\{u > t\}) = s$. Thus, both $\{u > t\}$ and $B \setminus \{u > t\}$ have measure strictly greater than $(\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$. Recalling the isoperimetric inequality this gives

$$\left(\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(\partial^*\{u > t\})\right)^{\frac{d}{d-1}} = \left(\gamma_{\mathrm{iso}}P(\{u > t\}, B)\right)^{\frac{d}{d-1}} > \left(\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}}.$$

This means that $\mathcal{H}^{d-1}(J_u) < \mathcal{H}^{d-1}(\partial^*\{u > t\})$, a contradiction. Hence, u^0 is continuous in the interval in (5.12) and thus in $[\Lambda, \omega_d - \Lambda]$. Since u^0 admits a continuous representative in $[\Lambda, \omega_d - \Lambda]$, let us define

$$\tau' := u^0(\omega_d - \Lambda) \quad \text{and} \quad \tau'' := u^0(\Lambda).$$

Notice that τ' and τ'' by definition of u^0 and Λ coincide with $\tau'(u,B)$ and $\tau''(u,B)$, respectively, defined in (4.4).

Step 2: u^0 is absolutely continuous in $(\Lambda, \omega_d - \Lambda)$. We shall use some arguments from [25, Theorem 6.5 and Lemma 6.6]. Assume $\mathcal{H}^{d-1}(J_u) > 0$, otherwise the absolute continuity of u^0 follows by [25, Lemma 6.6]. Let

$$h_B(s) := \frac{1}{\gamma_{\text{iso}}} \min\{s, \omega_d - s\}^{\frac{d-1}{d}}.$$

Fix $\varepsilon > 0$. Since $|\nabla u| \in L^1(B)$, by the standard theory of rearrangements we have $|\nabla u|^* \in L^1((0,\omega_d))$. Thus, we can find $\delta > 0$ such that

$$\int_0^\delta |\nabla u|^*(s)\, ds \le \varepsilon. \tag{5.13}$$

Let $\alpha, \beta \in u^0((\Lambda, \omega_d - \Lambda))$ with $\alpha \le \beta$. We have that $\alpha, \beta \in (\tau', \tau'')$ by Step 1. Set $u_\alpha^\beta \in SBV(B)$ as $u_\alpha^\beta := \alpha \vee u \wedge \beta$. Using the coarea formula we deduce

$$\int_\alpha^\beta P(\{u > t\}, B)\, dt = \int_{-\infty}^{+\infty} P(\{u_\alpha^\beta > t\}, B)\, dt = |Du_\alpha^\beta|(B) \le \int_B |\nabla u_\beta^\alpha(x)|\, dx + (\beta - \alpha)\mathcal{H}^{d-1}(J_u). \tag{5.14}$$

On the other hand, by definition of h_B and the isoperimetric inequality in B we infer

$$\int_\alpha^\beta P(\{u > t\}, B) dt \ge \int_\alpha^\beta h_B(\mu_u(t))\, dt. \tag{5.15}$$

Since $t \in (\alpha, \beta) \subset (\tau', \tau'') = (u^0(\omega_d - \Lambda), u^0(\Lambda))$, using (5.6) we have that

$$\lim_{t \to u^0(\Lambda)^-} \mu_u(t) = \mu_u(u^0(\Lambda)^-) \ge \Lambda \quad \text{and} \quad \lim_{t \to u^0(\omega_d - \Lambda)^+} \mu_u(t) = \mu_u(u^0(\omega_d - \Lambda)) \le \omega_d - \Lambda.$$

Recalling that μ_u is non-increasing, this gives that $\mu_u(t) \in (\Lambda, \omega_d - \Lambda)$ for every $t \in (\alpha, \beta)$. Hence, for every $t \in (\alpha, \beta)$,

$$h_B(\mu_u(t)) \ge h_B(\Lambda) = h_B\left(\left(2\gamma_{\text{iso}}\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}}\right) = 2\mathcal{H}^{d-1}(J_u). \tag{5.16}$$

Combining (5.14)–(5.16) gives

$$\int_\alpha^\beta h_B(\mu_u(t))\, dt \le \int_B |\nabla u_\alpha^\beta(x)|\, dx + \frac{1}{2}\int_\alpha^\beta h_B(\mu_u(t))\, dt,$$

and finally,

$$\int_\alpha^\beta h_B(\mu_u(t))\, dt \le 2\int_{\{\alpha \le u \le \beta\}} |\nabla u(x)|\, dx. \tag{5.17}$$

Consider now a finite family of disjoint open intervals contained in $(\Lambda, \omega_d - \Lambda)$, namely (a_i, b_i) for $i = 1, \dots, N$, such that $\sum_{i=1}^N (b_i - a_i) \le \delta$ where $\delta > 0$ is as in (5.13). Now consider inequality (5.17) with $\alpha := u^0(b_i) \le u^0(a_i) =: \beta$ for every $i = 1, \dots, N$. If we add all these inequalities we get

$$\int_{\cup_{i=1}^N (u^0(b_i), u^0(a_i))} h_B(\mu_u(t))\, dt \le 2\int_{\cup_{i=1}^N \{u^0(b_i) < u < u^0(a_i)\}} |\nabla u(x)|\, dx, \tag{5.18}$$

where we have also used the fact that $\int_{\{u=t\}} |\nabla u|\, dx = 0$ for every $t \in \mathbb{R}$ (see [10, Proposition 3.73 (c)]). Recalling (5.6), for every $i = 1, \dots, N$ it holds

$$\lim_{t \to u^0(a_i)^-} \mu_u(t) = \mu_u(u^0(a_i)-) \ge a_i \quad \text{and} \quad \lim_{t \to u^0(b_i)^+} \mu_u(t) = \mu_u(u^0(b_i)) \le b_i. \tag{5.19}$$

Hence, by definition of μ_u we have that $\mathcal{L}^d(\{u^0(b_i) < u < u^0(a_i)\}) = \mu_u(u^0(b_i)) - \mu_u(u^0(a_i)-) \le b_i - a_i$ for every $i = 1, \dots, N$. For every non-negative measurable function f and every Borel set E, by the Hardy-Littlewood inequality, it holds

$$\int_E f(x)\, dx \le \int_0^{\mathcal{L}^d(E)} f^*(t)\, dt.$$

Combining this result with inequality (5.18) gives

$$\begin{aligned}\int_{\cup_{i=1}^N(u^0(b_i),u^0(a_i))} h_B(\mu_u(t))\,dt &\le 2\int_0^{\sum_{i=1}^N[\mu_u(u^0(b_i))-\mu_u(u^0(a_i))]} |\nabla u|^*\,dt \le 2\int_0^{\sum_{i=1}^N(b_i-a_i)} |\nabla u|^*\,dt \\ &\le 2\int_0^\delta |\nabla u|^*\,dt \le 2\varepsilon.\end{aligned} \tag{5.20}$$

Again by (5.19) we get that $\mu_u(t)\in(\Lambda,\omega_d-\Lambda)$ for every $t\in\cup_{i=1}^N(a_i,b_i)$. Therefore, arguing as before,

$$h_B(\mu_u(t)) \ge 2\mathcal{H}^{d-1}(J_u)$$

for every $t\in(a_i,b_i)$, $i=1,\dots,N$. Thus,

$$2\mathcal{H}^{d-1}(J_u)\sum_{i=1}^N (u^0(a_i)-u^0(b_i)) \le 2\varepsilon,$$

and this gives the absolute continuity of u^0 on $(\Lambda,\omega_d-\Lambda)$ since the family of sub-intervals and $\varepsilon>0$ where arbitrary.

Step 3: Conclusion. By the previous steps we have that u^0 is uniformly continuous and absolutely continuous in $[\Lambda,\omega_d-\Lambda]$ and $(\Lambda,\omega_d-\Lambda)$, respectively. Hence, by definition of τ',τ'', the function $\tau'\vee u^0\wedge\tau''$ is absolutely continuous in $(0,\omega_d)$. Observe that by definition of u^0 and T_Bu (see (4.5)), we have that

$$\tau'\vee u^0\wedge\tau'' = (\tau'\vee u\wedge\tau'')^0 = (T_Bu)^0.$$

Let h_B be as in Step 2, and let us set $v:=T_Bu$. Recalling inequalities (5.18) and (5.20), since v^0 is constant in $(0,\omega_d)\setminus(\Lambda,\omega_d-\Lambda)$, without any additional effort, we can deduce that for every family of disjoint intervals $\{(a_i,b_i)\}_{i=1}^N$ in $(0,\omega_d)$, it holds

$$\int_{\cup_{i=1}^N(v^0(b_i),v^0(a_i))} h_B(\nu_v(t))\,dt \le 2\int_0^{\sum_{i=1}^N(b_i-a_i)} |\nabla u|^*\,dt.$$

Moreover, since v^0 is absolutely continuous on $(0,\omega_d)$, by a change of variables we get

$$\int_{\cup_{i=1}^N(a_i,b_i)} -\frac{dv^0}{dr}(r)h_B(r)\,dr \le 2\int_0^{\sum_{i=1}^N(b_i-a_i)} |\nabla u|^*\,dt,$$

Observe that by approximation the inequality holds even if the union is countable. Since every open subset of $(0,\omega_d)$ is the union of disjoint open intervals contained in $(0,\omega_d)$, given $\ell\in(0,\omega_d)$, it holds

$$\sup_{A\in\mathcal{A}((0,\omega_d)),\ \mathcal{L}^1(A)=\ell} \int_A -\frac{dv^0}{dr}(r)h_B(r)\,dr \le 2\int_0^\ell |\nabla u|^*\,dt.$$

In particular, taking $A=(0,\omega_d)$ we deduce that $-\frac{dv^0}{dr}(r)h_B(r)\in L^1((0,\omega_d))$. Thus, since every Borel set E can be approximated arbitrarily well in the sense of the Lebesgue measure by open sets, by the absolute continuity of the integral we conclude that for every $\ell\in(0,\omega_d)$

$$\sup_{E\in\mathcal{B}((0,\omega_d)),\ \mathcal{L}^1(E)=\ell} \int_E -\frac{dv^0}{dr}(r)h_B(r)\,dr \le 2\int_0^\ell |\nabla u|^*\,dt.$$

In particular, this implies that for every $\ell\in(0,\omega_d)$

$$\int_0^\ell \left(-\frac{dv^0}{dr}(r)h_B(r)\right)^* dr \le \int_0^\ell |2\nabla u|^*\,dt. \tag{5.21}$$

Finally, since $\psi\in\Phi_s$ is convex and (5.21) holds for every $\ell\in(0,\omega_d)$, we can apply Lemma 5.5 to the functions $-\frac{dv^0}{dr}h_B$ and $|2\nabla u|^*$ to infer

$$\int_0^{\omega_d} \psi\left(-\frac{dv^0}{dr}(r)h_B(r)\right) dr \le \int_0^{\omega_d} \psi(|2\nabla u|^*(t))\,dt = \int_0^{\omega_d} (\psi(|2\nabla u|))^*(t)\,dt,$$

where we have also used that ψ is strictly increasing. Using the equi-measurability of rearrangements and recalling the definition of h_B, we get (5.10) with $c(d):=1/\gamma_{\text{iso}}$. This concludes the proof. □

5.2. **Proof of Theorem 5.1.** To prove Theorem 5.1 we need the following Hardy type inequality proven in [26].

Lemma 5.6 ([26, Lemma 2.2])**.** *Let $\ell \in (0, +\infty)$. Let $\varphi \in \Phi_w$ be a finite valued Orlicz function satisfying* $(\mathrm{Inc})_1$*, set $\phi := \varphi^{\frac{d}{d-1}} t^{-\frac{1}{d-1}}$. Then, there exists a constant $c_1 := c_1(d, \ell) > 0$ such that for every measurable functions $h\colon (0, \ell) \to [0, +\infty)$ it holds*

$$\phi^{-1}\left(\int_0^\ell \phi\left(\int_s^\ell h(t)\,dt\right) ds\right) \le \varphi^{-1}\left(\int_0^\ell \varphi(c_1 s^{\frac{d-1}{d}} h(s))\,ds\right).$$

Using the definitions of u^0, $T_B u$ and $\mathrm{med}(u, B)$ we have:

$$\begin{aligned}
&u^0\left(\frac{\omega_d}{2}\right) = \mathrm{med}(u, B) =: m(u),\\
&((T_B u - m(u))_+)^*(s) = ((T_B u)^0(s) - m(u))_+ \quad \text{for a.e. } s \in (0, \omega_d),\\
&((T_B u - m(u))_-)^*(s) = ((T_B u)^0(\omega_d - s) - m(u))_- = (m(u) - (T_B u)^0(\omega_d - s))_+ \quad \text{for a.e. } s \in (0, \omega_d).
\end{aligned} \tag{5.22}$$

Now we are in a position to prove Theorem 5.1. In this proof we proceed similarly to the proof of [26, Theorem 1.2].

Proof of Theorem 5.1. We begin by recalling that since φ satisfies $(\mathrm{Inc})_1$,

$$\lambda\varphi(t) \le \varphi(\lambda t) \;\text{ and }\; \varphi^{-1}(s+t) \le 2(\varphi^{-1}(s) + \varphi^{-1}(t)) \quad \text{for every } \lambda \ge 1,\ s, t \ge 0. \tag{5.23}$$

The same properties hold for ϕ by the very same definition of ϕ. Moreover, in virtue of Theorems 4.7 and 4.11, we have that there exists $\psi \in \Phi_s$ such that

$$\psi(t) \le \varphi(t) \le \psi(2t) \;\text{ and }\; \psi^{-1}(t) \le 2\varphi^{-1}(t) \le 2\psi^{-1}(t) \quad \text{for every } t \ge 0. \tag{5.24}$$

We have the following chain of inequalities for the left hand side term in (5.2)

$$\begin{aligned}
\phi^{-1}\left(\int_B \phi(|T_B u(x) - m(u)|)\ dx\right) &= \phi^{-1}\left(\int_B \phi((T_B u(x) - m(u))_+)\,dx + \int_B \phi((T_B u(x) - m(u))_-)\,dx\right)\\
&\le 2\phi^{-1}\left(\int_0^{\omega_d/2} \phi((T_B u - m(u))_+^*(s))\,ds\right) + 2\phi^{-1}\left(\int_0^{\omega_d/2} \phi((T_B u - m(u))_-^*(s))\,ds\right)\\
&= 2\phi^{-1}\left(\int_0^{\omega_d/2} \phi\left(\int_s^{\omega_d/2} -\frac{d(T_B u)^0}{dr}(r)\,dr\right) ds\right)\\
&\quad + 2\phi^{-1}\left(\int_0^{\omega_d/2} \phi\left(\int_s^{\omega_d/2} -\frac{d(T_B u)^0}{dr}(\omega_d - r)\,dr\right) ds\right),
\end{aligned} \tag{5.25}$$

where in the first inequality we have used the equi-measurability of the rearrangement, the fact that ϕ is strictly increasing and (5.23) for ϕ, in the last equality we have used (5.22).

Now we estimate the right hand side term. Let $c = c(d)$ be the constant from Proposition 5.3 and $c_1 = c_1(d, \omega_d/2) = c_1(d)$ be the constant from Lemma 5.6. We have

$$\begin{aligned}
\varphi^{-1}\left(\int_B \varphi\left(4\frac{c_1}{c}|\nabla u(x)|\right)\,dx\right) &\geq \frac{1}{2}\psi^{-1}\left(\int_B \psi\left(4\frac{c_1}{c}|\nabla u(x)|\right)\,dx\right)\\
&\geq \frac{1}{2}\psi^{-1}\left(\int_0^{\omega_d} \psi\left(2c_1\min\{s, \omega_d - s\}^{\frac{d-1}{d}}\left(-\frac{d(T_B u)^0}{ds}(s)\right)\right)\,ds\right)\\
&\geq \frac{1}{4}\psi^{-1}\left(\int_0^{\omega_d/2} \psi\left(2c_1 s^{\frac{d-1}{d}}\left(-\frac{d(T_B u)^0}{ds}(s)\right)\right)\,ds\right)\\
&\quad+\frac{1}{4}\psi^{-1}\left(\int_0^{\omega_d/2} \psi\left(2c_1 s^{\frac{d-1}{d}}\left(-\frac{d(T_B u)^0}{ds}(\omega_d - s)\right)\right)\,ds\right)\\
&\geq \frac{1}{4}\varphi^{-1}\left(\int_0^{\omega_d/2} \varphi\left(c_1 s^{\frac{d-1}{d}}\left(-\frac{d(T_B u)^0}{ds}(s)\right)\right)\,ds\right)\\
&\quad+\frac{1}{4}\varphi^{-1}\left(\int_0^{\omega_d/2} \varphi\left(c_1 s^{\frac{d-1}{d}}\left(-\frac{d(T_B u)^0}{ds}(\omega_d - s)\right)\right)\,ds\right),
\end{aligned} \tag{5.26}$$

where we have used in this order: (5.24), Proposition 5.3, the fact that ψ^{-1} is increasing and, for the last inequality, we used (5.24) together with (5.23).

Finally, combining (5.25) and (5.26) and using Lemma 5.6, we infer that

$$8\varphi^{-1}\left(\int_B \varphi\left(\frac{4c_1}{c}|\nabla u(x)|\right)\,dx\right) \geq \phi^{-1}\left(\int_B \phi(|T_B u(x) - m(u)|)\,dx\right),$$

which is (5.2) up to taking $C = C(d)$ larger in order to obtain an averaged integral in the expression above. In view of Remark 5.2 we thus conclude. $\square$

We will also make use of the following weaker version of the Poincaré inequality in (5.2) for functions which are also doubling.

Theorem 5.7. *Let $\varphi \in \Phi_w$ be an Orlicz function satisfying $(\mathrm{Inc})_1$ and (aDec). Then, there exists $C = C(d, K) > 0$, where K is the doubling constant of φ, such that for every ball $B_r(x) \subset \mathbb{R}^d$ and every $u \in SBV(B_r(x))$ satisfying (5.1), the following inequality holds*

$$\int_{B_r(x)} \varphi\left(\frac{|T_{B_r(x)}u(y) - \mathrm{med}(u, B_r(x))|}{r}\right)\,dy \leq C\int_{B_r(x)} \varphi(|\nabla u(y)|)\,dy. \tag{5.27}$$

In order to prove Theorem 5.7 we need the following elementary Lemma.

Lemma 5.8. *Let $\varphi, \phi \in \Phi_s$ be two finite valued Orlicz functions such that*

$$\lim_{t\to 0^+}\phi(\varphi^{-1}(t)) = 0, \quad t \mapsto \frac{\phi(t)}{\varphi(t)} \quad \textit{is increasing.} \tag{5.28}$$

Then, for every ball $B_r \subset \mathbb{R}^d$ and every measurable function $u \in L^1(B_r)$ we have that

$$\varphi^{-1}\left(\fint_{B_r} \varphi(|u(x)|)\,dx\right) \leq \phi^{-1}\left(\fint_{B_r} \phi(2|u(x)|)\,dx\right) \tag{5.29}$$

Proof. Let us set $\psi := \phi(\varphi^{-1})$. We claim that $\psi \in \Phi_w$ satisfies $(\mathrm{Inc})_1$. Indeed, since φ is continuous, convex and finite valued, then it is a bijection, thus, φ^{-1} is its inverse which is continuous. This gives that ψ is continuous and finite valued on $(0, +\infty)$. Moreover, by the assumptions in (5.28) and definition of ψ we have that $\psi(0) = 0 = \lim_{t\to 0^+}\psi(t)$ and $t \mapsto \psi(t)/t$ is increasing. Therefore, by Theorem 4.7, there exists $\tilde{\psi} \in \Phi_s$ such that

$$\tilde{\psi}(t) \leq \psi(t) \leq \tilde{\psi}(2t) \quad t \geq 0.$$

Since $\tilde{\psi}$ is convex, by Jensen inequality we get for every $v \in L^1(B_r)$

$$\psi\left(\fint_{B_r}|v(x)|\,dx\right) \leq \tilde{\psi}\left(2\fint_{B_r}|v(x)|\,dx\right) \leq \fint_{B_r}\tilde{\psi}(2|v(x)|)\,dx \leq \fint_{B_r}\psi(2|v(x)|)\,dx.$$

By substituting back the expression of ψ with $v(x) = \varphi(|u(x)|)$ and recalling that $\varphi^{-1}(2t) \leq 2\varphi^{-1}(t)$ we get (5.29). $\square$

Proof of Theorem 5.7. By translation invariance it is not restrictive to consider $x = 0$. Using Proposition 4.7 we can find $\tilde{\varphi} \in \Phi_s$ such that $\tilde{\varphi}(t) \leq \varphi(t) \leq \tilde{\varphi}(2t)$. Setting $\tilde{\phi} := \tilde{\varphi}^{\frac{d}{d-1}} t^{-\frac{1}{d-1}}$ we have that $\tilde{\varphi}$ and $\tilde{\phi}$ satisfy all the assumptions of Lemma 5.8. Using Lemma 5.8 we thus infer

$$\tilde{\varphi}^{-1}\left(\fint_{B_r} \tilde{\varphi}\left(\frac{|T_{B_r}u(x) - \operatorname{med}(u, B_r)|}{r}\right) dx\right) \leq \tilde{\phi}^{-1}\left(\fint_{B_r} \tilde{\phi}\left(2\frac{|T_{B_r}u(x) - \operatorname{med}(u, B_r)|}{r}\right) dx\right);$$

which gives

$$\varphi^{-1}\left(\fint_{B_r} \varphi\left(\frac{|T_{B_r}u(x) - \operatorname{med}(u, B_r)|}{r}\right) dx\right) \leq 2\phi^{-1}\left(\fint_{B_r} \phi\left(8\frac{|T_{B_r}u(x) - \operatorname{med}(u, B_r)|}{r}\right) dx\right).$$

By Theorem 5.1 this implies that for every $u \in SBV(B_r)$ satisfying (5.1) then

$$\varphi^{-1}\left(\fint_{B_r} \varphi\left(\frac{|T_{B_r}u(x) - \operatorname{med}(u, B_r)|}{r}\right) dx\right) \leq C\varphi^{-1}\left(\fint_{B_r} \varphi(C|\nabla u(x)|)\, dx\right)$$

with $C = C(d)$. Using the fact that φ is doubling and φ^{-1} is increasing, we get (5.7) with $C = C(d, K)$ where $K \geq 2$ is the doubling constant of φ. $\square$

5.3. **Poincaré inequality for generalized Orlicz spaces.** We now extend the Poincaré inequality for $SBV(B_r(x))$ functions with small jump set to the cases where $\varphi \in \Phi_s(B_r(x))$ is a generalized Orlicz function satisfying (A0), (adA1), $(\mathrm{Inc})_1$ and (Dec) on $B_r(x)$. Notice that the statement below requires the introduction of a further truncation, depending on φ and on the ball $B_r(x)$. Indeed, thanks to this truncation we can exploit (adA1) reducing ourselves to consider only an Orlicz function non dependent on x. In order to ease the notation, we only consider the case $x = 0$.

Theorem 5.9. *Let $\varphi \in \Phi_s(B_r)$ be a generalized Orlicz function satisfying (A0), (adA1), $(\mathrm{Inc})_1$ and (Dec) on B_r. Let $\phi(x,t) := \varphi(x,t)^{\frac{d}{d-1}} t^{-\frac{1}{d-1}} \in \Phi_s(B_r)$. Moreover, let K be the doubling constant of φ, σ be the constant from (A0) and β be the constant appearing in (adA1). For $u \in SBV^{\varphi}(B_r)$, set*

$$\mathrm{u}_r^{\varphi}(x) := \left(\operatorname{med}(u, B_r) - r(\varphi_{B_r}^-)^{-1}\left(\frac{1}{2r}\right)\right) \vee T_{B_r}u(x) \wedge \left(\operatorname{med}(u, B_r) + r(\varphi_{B_r}^-)^{-1}\left(\frac{1}{2r}\right)\right). \tag{5.30}$$

Then, there exists $C = C(d, K, \beta)$ such that for every $u \in SBV^{\varphi}(B_r)$ satisfying (5.1), the following inequalities hold

$$(\phi_{B_r}^-)^{-1}\left(\fint_{B_r} \phi_{B_r}^+\left(\left|\frac{\mathrm{u}_r^{\varphi} - \operatorname{med}(u, B_r)}{r}\right|\right) dx\right) \leq C(\varphi_{B_r}^-)^{-1}\left(\fint_{B_r} \varphi_{B_r}^-(|\nabla u|)\, dx\right) + C\sigma, \tag{5.31}$$

$$\int_{B_r} \varphi_{B_r}^+\left(\left|\frac{\mathrm{u}_r^{\varphi} - \operatorname{med}(u, B_r)}{r}\right|\right) dx \leq C\int_{B_r}\left(\varphi_{B_r}^-(|\nabla u|) + \varphi_{B_r}^+\left(\left|\frac{u - \operatorname{med}(u, B_r)}{r}\right| \wedge \sigma\right)\right) dx. \tag{5.32}$$

Proof. By assumption we have that $\varphi_{B_r}^-$ satisfies $(\mathrm{Inc})_1$. Therefore inequality (5.2) hold with φ replaced by $\varphi_{B_r}^-$ and ϕ replaced by $\phi_{B_r}^-$. By definition of u_r^{φ}, we have that

$$\frac{|\mathrm{u}_r^{\varphi}(x) - \operatorname{med}(u, B_r)|}{r} \leq (\varphi_{B_r}^-)^{-1}\left(\frac{1}{2r}\right) \quad \text{and} \quad |\mathrm{u}_r^{\varphi}(x) - \operatorname{med}(u, B_r)| \leq |T_{B_r}u(x) - \operatorname{med}(u, B_r)|. \tag{5.33}$$

Hence, observing that $\phi_{B_r}^-$ is increasing and

$$\phi_{B_r}^+(\beta t) = (\varphi_{B_r}^+(\beta t))^{\frac{d}{d-1}}(\beta t)^{-\frac{1}{d-1}} \leq \frac{1}{\beta}(\varphi_B^-(t))^{\frac{d}{d-1}} t^{-\frac{1}{d-1}} = \frac{1}{\beta}\phi_{B_r}^-(t) \quad \text{for } t \in \left(\sigma, (\varphi_{B_r}^-)^{-1}\left(\frac{1}{2r}\right)\right),$$

inequality (5.31) is a consequence of (5.33) and (adA1).

Analogously, we have that inequality (5.32) is a consequence of the fact that (5.27) holds with φ replaced by $\varphi_{B_r}^-$, (5.33) and (adA1). $\square$

6. Integral Representation

This section is devoted to the proof of Theorem 3.1 and its Corollary 3.2. Throughout this section, ψ stands for a function in $\Phi_s(\Omega)$ satisfying (A0), (adA1), (Inc) and (Dec) on Ω. Notice that the additional requirements that $\psi \in \Phi_s(\Omega)$ instead of $\Phi_w(\Omega)$ and that (Inc) and (Dec) replace (aInc) and (aDec), respectively, cause no restriction, as we discussed in Remark 4.31.

6.1. **Fundamental estimate in $GSBV^\psi$.** We begin by proving an important tool for integral representation, that is the fundamental estimate.

Lemma 6.1 (Fundamental estimate in $GSBV^\psi$)**.** *Let $\eta > 0$ and $B \Subset \Omega$ a ball, let $D', D'', E \in \mathcal{A}(B)$ with $D' \Subset D''$. For every functional $\mathcal{F}$ satisfying* (H1)*,* (H3) *and* (H4) *with ψ, for every $u \in GSBV^\psi(D'', \mathbb{R}^m)$ and every $v \in GSBV^\psi(E, \mathbb{R}^m)$ there exists a function $\varphi \in C^\infty(\mathbb{R}^d, [0,1])$ such that the function $w := \varphi u + (1-\varphi)v \in GSBV^\psi(E \cup D', \mathbb{R}^m)$ satisfies*

(i) *$w = u$ on D' and $w = v$ on $E \setminus D''$;*

(ii) $\mathcal{F}(w, D' \cup E) \leq (1+\eta)(\mathcal{F}(u, D'') + \mathcal{F}(v, E)) + M \int_F \psi\left(x, \frac{|u-v|}{\delta}\right) dx + \eta \mathcal{L}^d(D' \cup E),$

with $\delta := \frac{1}{2}\mathrm{dist}(D', \partial D'')$, where $F := (D'' \setminus D') \cap E$ and $M = M(D', D'', E, \psi, \eta) > 0$ is independent of u and v. Moreover, given $\varepsilon > 0$ and $x_0 \in \mathbb{R}^d$ such that $D'_{\varepsilon,x_0}, D''_{\varepsilon,x_0}, E_{\varepsilon,x_0} \Subset \Omega$, then

$$M(D'_{\varepsilon,x_0}, D''_{\varepsilon,x_0}, E_{\varepsilon,x_0}, \psi, \eta) = M(D', D'', E, \psi, \eta)$$

and the integral term in (ii) *becomes: $M \int_{F_{\varepsilon,x_0}} \psi\left(x, \frac{|u-v|}{\varepsilon\delta}\right) dx$.*

Proof. Let K be the doubling constant of ψ, take $k \in \mathbb{N}$ such that $k \geq \max\left\{\frac{b(K+1)^2}{a\eta}, \frac{b}{\eta}\right\}$. We set $D_1 := D'$ and, for $i = 1, \dots, k$,

$$D_{i+1} := \left\{x \in D'' : \mathrm{dist}(x, D') < \frac{i\delta}{k}\right\}.$$

We thus have $D' =: D_1 \Subset D_2 \Subset \cdots \Subset D_{k+1} \Subset D''$. Let $\varphi_i \in C_0^\infty(D_{i+1}, [0,1])$ with $\varphi_i = 1$ in a neighborhood U_i of $\overline{D_i}$ and such that $\|\nabla \varphi_i\|_{L^\infty} \leq \frac{2k}{\delta}$.

Let $u \in GSBV^\psi(D'', \mathbb{R}^m)$ and $v \in GSBV^\psi(E, \mathbb{R}^m)$ such that $u - v \in L^\psi(F, \mathbb{R}^m)$, otherwise the thesis is trivial. We define the function $w_i := \varphi_i u + (1-\varphi_i)v$. We notice that $w_i \in GSBV^\psi(D' \cup E, \mathbb{R}^m)$, this follows from properties (A0) and (Dec) of ψ, with u and v extended arbitrarily outside D'' and E respectively. Set $I_i := E \cap (D_{i+1} \setminus \overline{D_i})$. Using (H1) and (H3) we infer

$$\begin{aligned} \mathcal{F}(w_i, D' \cup E) &\leq \mathcal{F}(u, (D' \cup E) \cap U_i) + \mathcal{F}(v, E \setminus \mathrm{Supp}\varphi_i) + \mathcal{F}(w_i, I_i) \\ &\leq \mathcal{F}(u, D'') + \mathcal{F}(v, E) + \mathcal{F}(w_i, I_i). \end{aligned} \tag{6.1}$$

We now estimate the last term of (6.1) using assumption (H4) and properties (A0), (Dec) (i.e. doubling) of ψ and the fact that ψ is convex since it belongs to the class Φ_s. We have

$$\begin{aligned} \mathcal{F}(w_i, I_i) &\leq b\mathcal{L}^d(I_i) + b(K+1)\int_{I_i} \psi(x, |\nabla u - \nabla v|)dx + b(K+1)\int_{I_i} \psi(x, |(u-v) \otimes \nabla\varphi|)dx \\ &\quad + b\mathcal{H}^{d-1}(J_u \cap I_i) + b\mathcal{H}^{d-1}(J_v \cap I_i) \\ &\leq b\mathcal{L}^d(I_i) + b(K+1)^2 \int_{I_i} \psi(x, |\nabla u|)\, dx + b(K+1)^2 \int_{I_i} \psi(x, |\nabla v|)\, dx \\ &\quad + b(K+1)K^{\log_2(k)+2} \int_{I_i} \psi\left(x, \frac{|u-v|}{\delta}\right) dx + b\mathcal{H}^{d-1}(J_u \cap I_i) + b\mathcal{H}^{d-1}(J_v \cap I_i) \\ &\leq \frac{b}{a}(K+1)^2(\mathcal{F}(u, I_i) + \mathcal{F}(v, I_i)) + b(K+1)K^{\log_2(k)+2}\int_{I_i} \psi\left(x, \frac{|u-v|}{\delta}\right) dx + b\mathcal{L}^d(I_i). \end{aligned}$$

By our initial choice of k and using (H1) we can find $i_0 \in \{1, \dots, k\}$ such that

$$\mathcal{F}(w_{i_0}, I_{i_0}) \leq \frac{1}{k}\sum_{i=1}^k \mathcal{F}(w_i, I_i) \leq \eta(\mathcal{F}(u, D'') + \mathcal{F}(v, E)) + \eta\mathcal{L}^d(D' \cup E) + M\int_F \psi\left(x, \frac{|u-v|}{\delta}\right) dx$$

with $M := b(K+1)K^{\log_2(k)+2}k^{-1}$. This combined with (6.1) concludes the proof by setting $w := w_{i_0}$. For the second statement it is enough to consider cut-off functions $\varphi_i^\varepsilon \in C_0^\infty((D_{i+1})_{\varepsilon,x_0},[0,1])$ defined as $\varphi_i^\varepsilon(x) := \varphi_i(x_0 + (x-x_0)/\varepsilon)$, for $i = 1,\dots,k$ and proceed as before. □

6.2. **Proof of the integral representation theorem.** Now we proceed with the proof of Theorem 3.1. We begin by showing that the Radon-Nikodym derivatives of $\mathcal{F}$ and $\mathbf{m}_\mathcal{F}$ with respect to the measure

$$\mu := \mathcal{L}^d|_\Omega + \mathcal{H}^{d-1}|_{J_u \cap \Omega} \tag{6.2}$$

coincide. To this aim, we state the following Lemma.

Lemma 6.2. *Let $\mathcal{F}$ satisfy (H1)–(H4), let $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$ and let μ be as in (6.2). Then, for μ-a.e. $x_0 \in \Omega$ it holds*

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))}.$$

We postpone the proof of this lemma at the end of this section. As a second step we prove that asymptotically as $\varepsilon \to 0^+$ the two minimization problems $\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))$ and $\mathbf{m}_\mathcal{F}(\overline{u}^{\mathrm{bulk}}_{x_0},B_\varepsilon(x_0))$ coincide for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$, where $\overline{u}^{\mathrm{bulk}}_{x_0} := \ell_{x_0,u(x_0),\nabla u(x_0)}$.

Lemma 6.3. *Let $\mathcal{F}$ satisfy (H1) and (H3)-(H4) and let $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$. Then, for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ we have*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\omega_d\varepsilon^d} = \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(\overline{u}^{\mathrm{bulk}}_{x_0},B_\varepsilon(x_0))}{\omega_d\varepsilon^d}. \tag{6.3}$$

We defer the proof of this lemma to Section 6.3. As a third and final step we prove that asymptotically as $\varepsilon \to 0^+$ the two minimization problems $\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))$ and $\mathbf{m}_\mathcal{F}(\overline{u}^{\mathrm{surf}}_{x_0},B_\varepsilon(x_0))$ coincide for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$, where $\overline{u}^{\mathrm{surf}}_{x_0} := u_{x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)}$.

Lemma 6.4. *Let $\mathcal{F}$ satisfy (H1) and (H3)-(H4) and let $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ we have*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} = \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(\overline{u}^{\mathrm{surf}}_{x_0},B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \tag{6.4}$$

We defer the proof of this lemma to Section 6.4. We are now ready to prove Theorem 3.1 and Corollary 3.2.

Proof of Theorem 3.1. In view of the Besicovitch derivation theorem, we need to prove that

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{L}^d}(x_0) = f(x_0,u(x_0),\nabla u(x_0)), \quad \text{for } \mathcal{L}^d\text{-a.e. } x_0 \in \Omega, \tag{6.5}$$

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{H}^{d-1}|_{J_u}}(x_0) = g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)), \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x_0 \in J_u, \tag{6.6}$$

where f and g are defined in (3.2) and (3.3), respectively.

Using Lemma 6.2 and that $\lim_{\varepsilon\to 0^+}(\omega_d\varepsilon^d)^{-1}\mu(B_\varepsilon(x_0)) = 1$ for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$, we infer that for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{L}^d}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\omega_d\varepsilon^d} < +\infty.$$

Then, (6.5) follows by definition of f ((3.2)) and Lemma 6.3.

By Lemma 6.2 and the fact that $\lim_{\varepsilon\to 0^+}(\omega_{d-1}\varepsilon^{d-1})^{-1}\mu(B_\varepsilon(x_0)) = 1$ for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ we infer that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{H}^{d-1}|_{J_u}}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u,B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} < +\infty,$$

and (6.6) follows by definition of g ((3.3)) and Lemma 6.4. □

Proof of Corollary 3.2. In view of Theorem 3.1 we have that for all $u \in GSBV^{\psi}(\Omega, \mathbb{R}^m)$ and all $A \in \mathcal{A}(\Omega)$

$$\mathcal{F}(u,A) = \int_A \tilde{f}(x, u(x), \nabla u(x))\, dx + \int_{J_u \cap A} \tilde{g}(x, u^+(x), u^-(x), \nu_u(x))\, d\mathcal{H}^{d-1},$$

where $\tilde{f}$ and $\tilde{g}$ are defined in (3.2) and (3.3), respectively. Recalling the definition of $\mathbf{m}_{\mathcal{F}}$, since $\mathcal{F}$ satisfies assumption (H5), by definition of $\tilde{f}$ and $\tilde{g}$ we deduce that for every $c \in \mathbb{R}^m$

$$\tilde{f}(x, u(x) + c, \nabla u(x)) = \tilde{f}(x, 0, \nabla u(x)), \qquad \tilde{g}(x, u^+(x) + c, u^-(x) + c, \nu_u(x)) = \tilde{g}(x, [u](x), 0, \nu_u(x)).$$

Hence, $\tilde{f}(x, u, \nabla u) =: f(x, \nabla u)$ and $\tilde{g}(x, u^+(x), u^-(x), \nu_u(x)) =: g(x, [u](x), \nu_u(x))$. $\square$

In the remaining part of the section we prove Lemma 6.2 following the lines of [17]. We start by fixing some notations. Given $\delta > 0$ and $A \in \mathcal{A}(\Omega)$ we define

$$\mathbf{m}^{\delta}_{\mathcal{F}}(u,A) := \inf \left\{ \sum_{i=1}^{\infty} \mathbf{m}_{\mathcal{F}}(u, B_i) \colon B_i \subset A \text{ pairwise disjoint balls, } \operatorname{diam}(B_i) < \delta,\ \mu\left(A \setminus \cup_{i=1}^{\infty} B_i\right) = 0 \right\} \tag{6.7}$$

where μ is the measure introduced in (6.2). Since $\mathbf{m}^{\delta}_{\mathcal{F}}(u,A)$ is decreasing in δ we can also introduce

$$\mathbf{m}^{*}_{\mathcal{F}}(u,A) := \lim_{\delta \to 0^+} \mathbf{m}^{\delta}_{\mathcal{F}}(u,A). \tag{6.8}$$

In the next lemma we prove that under assumptions (H1)–(H4) $\mathcal{F}$ and $\mathbf{m}^*_{\mathcal{F}}$ coincide.

Lemma 6.5. *Let $\mathcal{F}$ satisfy (H1)–(H4) and let $u \in GSBV^{\psi}(\Omega, \mathbb{R}^m)$. Then, for all $A \in \mathcal{A}(\Omega)$ we have $\mathcal{F}(u,A) = \mathbf{m}^*_{\mathcal{F}}(u,A)$.*

Proof. We mainly follow the lines of [19, Lemma 3.3]. We start by proving the inequality $\mathcal{F}(u,A) \geq \mathbf{m}^*_{\mathcal{F}}(u,A)$. For every ball $B \subset A$ we have that $\mathbf{m}_{\mathcal{F}}(u,B) \leq \mathcal{F}(u,B)$ by definition. Using (H1) we infer $\mathcal{F}(u,A) \geq \mathbf{m}^{\delta}_{\mathcal{F}}(u,A)$ for all $\delta > 0$. Thus the desired inequality follows by definition of $\mathbf{m}^*_{\mathcal{F}}$ in (6.8).

We now prove the reverse inequality. Fix $\delta > 0$ and $A \in \mathcal{A}(\Omega)$ and let $\{B^{\delta}_i\}_i$ be a family of balls as in (6.7) such that

$$\sum_{i=1}^{\infty} \mathbf{m}_{\mathcal{F}}(u, B^{\delta}_i) \leq \mathbf{m}^{\delta}_{\mathcal{F}}(u,A) + \delta. \tag{6.9}$$

By definition of $\mathbf{m}_{\mathcal{F}}$ we can find $v^{\delta}_i \in GSBV^{\psi}(B^{\delta}_i, \mathbb{R}^m)$ such that $v^{\delta}_i = u$ in a neighborhood of ∂B^{δ}_i and

$$\mathcal{F}(v^{\delta}_i, B^{\delta}_i) \leq \mathbf{m}_{\mathcal{F}}(u, B^{\delta}_i) + \delta \mathcal{L}^d(B^{\delta}_i). \tag{6.10}$$

We define

$$v^{\delta,n} := \sum_{i=1}^{n} v^{\delta}_i \chi_{B^{\delta}_i} + u \chi_{N^{\delta,n}_0} \quad n \in \mathbb{N}, \qquad v^{\delta} := \sum_{i=1}^{\infty} v^{\delta}_i \chi_{B^{\delta}_i} + u \chi_{N^{\delta}_0}, \tag{6.11}$$

where $N^{\delta,n}_0 := \Omega \setminus \cup_{i=1}^{n} B^{\delta}_i$ and $N^{\delta}_0 := \Omega \setminus \cup_{i=1}^{\infty} B^{\delta}_i$. By construction, we have that $v^{\delta,n} \in GSBV^{\psi}(\Omega, \mathbb{R}^m)$ for every $n \in \mathbb{N}$ and, using (6.9), (6.10) and (H4),

$$\sup_{n \in N} \left(\int_{\Omega} \psi\left(x, |\nabla v^{\delta,n}(x)|\right)\, dx + \mathcal{H}^{d-1}(J_{v^{\delta,n}}) \right) < +\infty. \tag{6.12}$$

Moreover, we have that $v^{\delta,n} \to v^{\delta}$ pointwise $\mathcal{L}^d$-a.e. and in measure on Ω by construction. Recalling that ψ satisfies $(\mathrm{Inc})_{\gamma}$ for some $\gamma > 1$, using [7, Theorem 2.2] (see also [6] and [9]) together with the compactness in L^0 of $\{v^{\delta,n}\}_n$ gives that $v^{\delta} \in GSBV^{\gamma}(\Omega, \mathbb{R}^m)$ and $\nabla v^{\delta,n} \rightharpoonup \nabla v^{\delta}$ weakly in L^{γ}. Now using Ioffe's theorem and (6.12), we infer that

$$\int_{\Omega} \psi\left(x, |\nabla v^{\delta}(x)|\right)\, dx \leq \liminf_{n \to +\infty} \int_{\Omega} \psi\left(x, |\nabla v^{\delta,n}(x)|\right)\, dx < +\infty.$$

Therefore, $v^{\delta} \in GSBV^{\psi}(\Omega, \mathbb{R}^m)$. Observe that it holds

$$\begin{aligned} \mathcal{F}(v^{\delta}, A) &= \sum_{i=1}^{\infty} \mathcal{F}(v^{\delta}_i, B^{\delta}_i) + \mathcal{F}(u, A \cap N^{\delta}_0) \leq \sum_{i=1}^{\infty} (\mathbf{m}_{\mathcal{F}}(u, B^{\delta}_i) + \delta \mathcal{L}^d(B^{\delta}_i)) \\ &\leq \mathbf{m}^{\delta}_{\mathcal{F}}(u,A) + \delta(1 + \mathcal{L}^d(A)), \end{aligned} \tag{6.13}$$

where we have used that $\mu(A \cap N_0^\delta) = \mathcal{F}(u, A \cap N_0^\delta) = 0$ by definition of $\{B_i^\delta\}_i$ and (H4). Thanks to $(\mathrm{Inc})_\gamma$ we have $\psi_\Omega^+(1)t^\gamma \le \psi(x,t)$ for $\mathcal{L}^d$-a.e. $x \in \Omega$ and every $t \ge 1$. Therefore, (6.13) together with (H4) implies that

$$\int_A |\nabla v^\delta|^\gamma \, dx + \mathcal{H}^{d-1}(J_{v^\delta} \cap A) \le \frac{1}{a\psi_\Omega^+(1)} \left(\mathbf{m}_\mathcal{F}^\delta(u,A) + \delta(1+\mathcal{L}^d(A))\right) + \mathcal{L}^d(A). \tag{6.14}$$

We now claim that

$$w^\delta := u - v^\delta \to 0 \quad \text{in measure on } A. \tag{6.15}$$

Notice that if (6.15) holds, then (H2), (6.8) and (6.13) imply that $\mathbf{m}_\mathcal{F}^*(u,A) \ge \mathcal{F}(u,A)$ in the limit $\delta \to 0^+$.

In order to prove (6.15), we first notice that $w^\delta|_{B_i^\delta} \in GSBV^\gamma(B_i^\delta, \mathbb{R}^m)$ has zero trace on ∂B_i^δ. Setting $w^{\delta,M} := -M \vee w^\delta \wedge M$ with $M > 0$, by the homogeneous Poincaré inequality in BV we have

$$\|w^{\delta,M}\|_{L^1(B_i^\delta)} \le C\delta |Dw^{\delta,M}|(B_i^\delta).$$

Therefore, by definition of $\{B_i^\delta\}_i$ we deduce that

$$\|w^{\delta,M}\|_{L^1(A)} \le C\delta |Dw^{\delta,M}|(\cup_{i=1}^\infty B_i^\delta) \le C\delta |Dw^{\delta,M}|(A).$$

The quantity $|Dw^{\delta,M}|(A)$ is bounded in view of (6.14) since $u \in GSBV(A, \mathbb{R}^m)$. Indeed,

$$|Dw^{\delta,M}|(A) \le \int_A |\nabla v^\delta| \, dx + \int_A |\nabla u| \, dx + 2M \left(\mathcal{H}^{d-1}(J_u \cap A) + \mathcal{H}^{d-1}(J_{v^\delta} \cap A)\right) < +\infty.$$

This implies $w^{\delta,M} \to 0$ in $L^1(A, \mathbb{R}^m)$ as $\delta \to 0^+$ and thus in measure on A for every $M > 0$. Observe that if we take $M = 1$ then we have that for every $\varepsilon \in (0,1)$

$$\{x \in A \colon |w^\delta(x)| > \varepsilon\} \subseteq \{x \in A \colon |w^{\delta,1}(x)| > \varepsilon\},$$

and this gives (6.15). □

We finally prove Lemma 6.2.

Proof of Lemma 6.2. The proof follows the same arguments of [17, Lemma 5 and Lemma 6]. It essentially relies on Lemma 6.5 and on the assumptions on $\mathcal{F}$ but it is not hinged on the growth conditions. Hence we omit it. □

It remains to prove Lemmas 6.3 and 6.4. This is the subject of the following two sections.

6.3. **Lebesgue points.** This section is devoted to the proof of Lemma 6.3. We now proceed with the bulk part of the energy, that is, we analyze the blow up at points where the approximate gradient exists.

Lemma 6.6. *Let $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ and $\mathcal{L}^1$-a.e. $\lambda \in (0,1)$ there exists a sequence $u_\varepsilon \in GSBV^\psi(B_\varepsilon(x_0))$ such that the following properties hold:*

$$\begin{aligned}
&(i) \quad u_\varepsilon = u \text{ on } B_\varepsilon(x_0) \setminus \overline{B_{\lambda\varepsilon}(x_0)}, \qquad \lim_{\varepsilon \to 0^+} \varepsilon^{-(d+1)} \mathcal{L}^d(\{u_\varepsilon \neq u\} \cap B_\varepsilon(x_0)) = 0,\\
&(ii) \quad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^d} \int_{B_{\lambda\varepsilon}(x_0)} \psi\left(x, \frac{|u_\varepsilon(x) - u(x_0) - \nabla u(x_0)(x - x_0)|}{\varepsilon}\right) dx = 0;\\
&(iii) \quad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^d} \mathcal{H}^{d-1}(J_{u_\varepsilon} \cap B_\varepsilon(x_0)) = 0.
\end{aligned} \tag{6.16}$$

Proof. Without loss of generality we can assume $m = 1$. Take $x_0 \in \Omega$ satisfying

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^d} \int_{B_\varepsilon(x_0)} \psi(x, |\nabla u(x) - \nabla u(x_0)|) \, dx = 0; \tag{6.17a}$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^d} \mathcal{H}^{d-1}(J_u \cap B_\varepsilon(x_0)) = 0; \tag{6.17b}$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^d} \mathcal{L}^d\left(\left\{x \in B_\varepsilon(x_0) \colon \frac{|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|}{|x - x_0|} > \rho\right\}\right) = 0, \quad \rho > 0. \tag{6.17c}$$

Notice that (6.17a)–(6.17c) are satisfied by $\mathcal{L}^d$-a.e $x_0 \in \Omega$ by Proposition 4.33. Set for brevity $T_\varepsilon := T_{B_\varepsilon(x_0)}$. Observe that for ε small enough, (6.17b) implies (5.1). For every $x \in B_\varepsilon(x_0)$ we can thus define the truncated functions

$$\overline{u}_\varepsilon(x) := T_\varepsilon\left(u(x) - u(x_0) - \nabla u(x_0)(x - x_0)\right)$$

and, similarly as (5.30),

$$\mathfrak{u}^\psi_\varepsilon(x) := \left(\mathrm{med}(\overline{u}_\varepsilon(x), B_\varepsilon(x_0)) - \varepsilon(\psi^-_{B_\varepsilon(x_0)})^{-1}\left(\frac{1}{2\varepsilon}\right)\right) \vee \overline{u}_\varepsilon(x) \wedge \left(\mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0)) + \varepsilon(\psi^-_{B_\varepsilon(x_0)})^{-1}\left(\frac{1}{2\varepsilon}\right)\right).$$

Finally, we set $v_\varepsilon(x) := u(x_0) + \nabla u(x_0)(x - x_0) + \mathfrak{u}^\psi_\varepsilon(x)$.

We now want to estimate the quantity $\mathcal{L}^d(\{v_\varepsilon \neq u\} \cap B_\varepsilon(x_0))$. We have

$$\begin{aligned}\mathcal{L}^d(\{v_\varepsilon \neq u\} \cap B_\varepsilon(x_0)) &\leq \mathcal{L}^d(\{\mathfrak{u}^\psi_\varepsilon \neq \overline{u}_\varepsilon\} \cap B_\varepsilon(x_0)) \\ &\quad + \mathcal{L}^d(\{\overline{u}_\varepsilon \neq u(x) - u(x_0) - \nabla u(x_0)(x - x_0)\} \cap B_\varepsilon(x_0)).\end{aligned} \tag{6.18}$$

By definition of T_ε and (4.6), we can estimate the second quantity on the right hand side of (6.18) with

$$\mathcal{L}^d(\{\overline{u}_\varepsilon \neq u(x) - u(x_0) - \nabla u(x_0)(x - x_0)\} \cap B_\varepsilon(x_0)) \leq \left(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u \cap B_\varepsilon(x_0))\right)^{\frac{d}{d-1}}.$$

For the first term of (6.18) recall that by Theorem 4.7 and Lemma 4.5 we have that for every $t \geq 0$

$$\psi^-_{B_\varepsilon(x_0)}\left((\psi^-_{B_\varepsilon(x_0)})^{-1}(t)\right) \geq \frac{t}{2K},$$

where $K \geq 2$ is the doubling constant of ψ.

We set $m_\varepsilon := \mathrm{med}(u - \nabla u(x_0)(\cdot - x_0), B_\varepsilon(x_0))$. Using the definition of $\mathfrak{u}^\psi_\varepsilon$, Chebychev inequality and Theorem 5.7,

$$\begin{aligned}\mathcal{L}^d\left(\{\mathfrak{u}^\psi_\varepsilon \neq \overline{u}_\varepsilon\} \cap B_\varepsilon(x_0)\right) &\leq \mathcal{L}^d\left(B_\varepsilon(x_0) \cap \left\{|\overline{u}_\varepsilon - \mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0))| \geq \varepsilon(\psi^-_{B_\varepsilon(x_0)})^{-1}\left(\frac{1}{2\varepsilon}\right)\right\}\right) \\ &\leq \mathcal{L}^d\left(B_\varepsilon(x_0) \cap \left\{\psi^-_{B_\varepsilon(x_0)}\left(\frac{|\overline{u}_\varepsilon - \mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0))|}{\varepsilon}\right) \geq \frac{1}{4K\varepsilon}\right\}\right) \\ &\leq 4K\varepsilon \int_{B_\varepsilon(x_0)} \psi^-_{B_\varepsilon(x_0)}\left(\frac{|\overline{u}_\varepsilon(x) - \mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0))|}{\varepsilon}\right) dx \\ &= 4K\varepsilon \int_{B_\varepsilon(x_0)} \psi^-_{B_\varepsilon(x_0)}\left(\frac{|T_\varepsilon(u(x) - \nabla u(x_0)(x - x_0)) - m_\varepsilon|}{\varepsilon}\right) dx \\ &\leq \varepsilon C \int_{B_\varepsilon(x_0)} \psi^-_{B_\varepsilon(x_0)}(|\nabla u(x) - \nabla u(x_0)|)\, dx \\ &\leq \varepsilon C \int_{B_\varepsilon(x_0)} \psi(x, |\nabla u(x) - \nabla u(x_0)|)\, dx.\end{aligned}$$

where $C = C(d, K)$. Thus, using Fubini Theorem and (6.17a)–(6.17b), we deduce that

$$\begin{aligned}\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^d} \int_0^1 \mathcal{H}^{d-1}(\{v_\varepsilon \neq u\} \cap \partial B_{\lambda\varepsilon}(x_0)) d\lambda &= \limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{d+1}} \mathcal{L}^d(\{v_\varepsilon \neq u\} \cap B_\varepsilon(x_0)) \\ &\leq \limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{d+1}} \left(\varepsilon C \int_{B_\varepsilon(x_0)} \psi(x, |\nabla u(x) - \nabla u(x_0)|)\, dx + \left(2\gamma_{\mathrm{iso}}\mathcal{H}^{d-1}(J_u \cap B_\varepsilon(x_0))\right)^{\frac{d}{d-1}}\right) \\ &= 0.\end{aligned} \tag{6.19}$$

Therefore, since $d \geq 2$, for every sequence $\varepsilon \to 0$, one can find a subsequence such that for $\mathcal{L}^1$-a.e. $\lambda \in (0, 1)$ it holds

$$\begin{aligned}&\mathcal{H}^{d-1}(\partial B_{\lambda\varepsilon}(x_0) \cap J_{v_\varepsilon}) = 0, \\ &\lim_{\varepsilon \to 0^+} \varepsilon^{-d}\mathcal{H}^{d-1}(\{v_\varepsilon \neq u\} \cap \partial B_{\lambda\varepsilon}(x_0)) = 0.\end{aligned} \tag{6.20}$$

Now let us fix a subsequence $\varepsilon \to 0$ (not relabelled) and $\lambda \in (0,1)$ such that (6.20) holds. We set

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } x \in B_{\lambda\varepsilon}(x_0), \\ u(x) & \text{if } x \in B_\varepsilon(x_0) \setminus \overline{B_{\lambda\varepsilon}(x_0)}. \end{cases}$$

From the definition of u_ε and the estimates in (6.19), we deduce (6.16)(i) and (6.16)(iii). We are left to prove (6.16)(ii).

We actually will prove a slightly stronger statement, namely

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^d} \int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|v_\varepsilon(x) - u(x_0) - \nabla u(x_0)(x - x_0)|}{\varepsilon}\right) dx = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^d} \int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|\mathfrak{u}^\psi_\varepsilon(x)|}{\varepsilon}\right) dx = 0. \tag{6.21}$$

Let $\varepsilon_0 > 0$ be fixed. Set $\psi^-_\varepsilon := \psi^-_{B_\varepsilon(x_0)}$ and $\psi^+_\varepsilon := \psi^+_{B_\varepsilon(x_0)}$ for brevity. Using property (adA1) and the fact that ψ satisfies (aDec), by definition of $\mathfrak{u}^\psi_\varepsilon$ and by inequality (5.32) we estimate, for every $\varepsilon < \varepsilon_0$,

$$\begin{aligned} &\int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|\mathfrak{u}^\psi_\varepsilon(x) - \mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0))|}{\varepsilon}\right) dx \\ &\le C \int_{B_\varepsilon(x_0)} \psi^-_\varepsilon\left(|\nabla u(x) - \nabla u(x_0)|\right) dx + \int_{B_\varepsilon(x_0)} \psi^+_{\varepsilon_0}\left(\frac{|\overline{u}_\varepsilon(x) - \mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0))|}{\varepsilon} \wedge \sigma\right) dx. \end{aligned}$$

with $C = C(d, K, \beta)$, where σ and β are the constants appearing in (A0) and (adA1), respectively. By (6.17a), in order to prove (6.21) we only need to show that

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \frac{\mathrm{med}(\overline{u}_\varepsilon, B_\varepsilon(x_0))}{\varepsilon} = \lim_{\varepsilon\to 0^+} \mathrm{med}\left(\frac{u - u(x_0) - \nabla u(x_0)(\cdot - x_0)}{\varepsilon}, B_\varepsilon(x_0)\right) = 0, \\ &\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^d} \int_{B_\varepsilon(x_0)} \psi^+_{\varepsilon_0}\left(\frac{|\overline{u}_\varepsilon(x)|}{\varepsilon} \wedge \sigma\right) dx = 0. \end{aligned}$$

The first limit follows by (6.17c) The second limit follows again by (6.17c) and dominate convergence theorem after rescaling. □

We can now prove Lemma 6.3 which will actually follow as a consequence of the next two Lemmas.

Lemma 6.7. *Let $\mathcal{F}$ satisfy (H1) and (H3)-(H4) and let $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$. Then, for $\mathcal{L}^d$-a.e $x_0 \in \Omega$*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\omega_d \varepsilon^d} \le \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(\overline{u}^{\mathrm{bulk}}_{x_0}, B_\varepsilon(x_0))}{\omega_d \varepsilon^d}. \tag{6.22}$$

Proof. We will prove the assertion for all the points $x_0 \in \Omega$ for which the statement of Lemma 6.6 holds, $\lim_{\varepsilon\to 0^+}(\omega_d\varepsilon^d)^{-1}\mu(B_\varepsilon(x_0)) = 1$ and

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u, B_\varepsilon(x_0))}{\omega_d \varepsilon^d} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\omega_d \varepsilon^d} < +\infty. \tag{6.23}$$

By Lemma 6.2, property (6.23) hold for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$. Let $(u_\varepsilon)_\varepsilon$ be the sequence given by Lemma 6.6 and fix $\lambda \in (0,1)$ such that (6.16)(ii) holds. We write $\lambda = 1 - \theta$ for some $\theta \in (0,1)$.

Given $z_\varepsilon \in GSBV^\psi(B_{(1-3\theta)\varepsilon}(x_0), \mathbb{R}^m)$ such that $z_\varepsilon = \overline{u}^{\mathrm{bulk}}_{x_0}$ in a neighborhood of $\partial B_{(1-3\theta)\varepsilon}(x_0)$ and

$$\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0)) \le \mathbf{m}_\mathcal{F}(\overline{u}^{\mathrm{bulk}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0)) + \omega_d \varepsilon^{d+1}, \tag{6.24}$$

we extend it to $z_\varepsilon \in GSBV(B_\varepsilon(x_0), \mathbb{R}^m)$ by setting $z_\varepsilon = \overline{u}^{\mathrm{bulk}}_{x_0}$ outside of $B_{(1-3\theta)\varepsilon}(x_0)$. We further set $C_{\varepsilon,\theta}(x_0) := B_\varepsilon(x_0) \setminus \overline{B_{(1-4\theta)\varepsilon}(x_0)}$. Now we use Lemma 6.1 with u and v replaced by z_ε and u_ε, respectively, and with the sets

$$D'_{\varepsilon,x_0} := B_{(1-2\theta)\varepsilon}(x_0), \quad D''_{\varepsilon,x_0} := B_{(1-\theta)\varepsilon}(x_0), \quad E_{\varepsilon,x_0} := C_{\varepsilon,\theta}(x_0). \tag{6.25}$$

Notice that $C_{\varepsilon,\theta}(x_0) = (C_{1,\theta}(x_0))_{\varepsilon,x_0}$ according to the notation introduced in (2.1), where $C_{1,\theta}(x_0) := B_1(x_0) \setminus \overline{B_{(1-4\theta)}(x_0)}$. Moreover, we observe that $\mathcal{L}^d(C_{1,\theta}(x_0)) = \omega_d(1 - (1 - 4\theta)) \to 0$ as $\theta \to 0$. For an arbitrarily fixed $\eta > 0$, we find $w_\varepsilon \in GSBV^\psi(B_\varepsilon(x_0), \mathbb{R}^m)$ such that $w_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0) \setminus B_{(1-\theta)\varepsilon}(x_0)$ and

$$\begin{aligned}\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) &\le (1+\eta)(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))) + \eta\mathcal{L}^d(B_\varepsilon(x_0)) \\ &\quad + M\int_{B_{(1-\theta)\varepsilon}(x_0)\setminus B_{(1-2\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|z_\varepsilon - u_\varepsilon|}{\varepsilon}\right) dx\end{aligned} \tag{6.26}$$

Recalling (6.16)(i) in Lemma 6.6, we have that $w_\varepsilon = u_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$. Moreover, since $z_\varepsilon = \overline{u}^{\mathrm{bulk}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$, using (6.16)(ii) we infer that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^d}\int_{B_{(1-\theta)\varepsilon}(x_0)\setminus B_{(1-2\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|z_\varepsilon - u_\varepsilon|}{\varepsilon}\right) dx \\ &\qquad \le \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^d}\int_{B_{(1-\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|\overline{u}^{\mathrm{bulk}}_{x_0} - u_\varepsilon|}{\varepsilon}\right) dx = 0.\end{aligned} \tag{6.27}$$

From (6.26) and (6.27) we deduce that there exists a non-negative infinitesimal sequence $(\rho_\varepsilon)_\varepsilon$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))) + \varepsilon^d\rho_\varepsilon + \eta\omega_d\varepsilon^d. \tag{6.28}$$

Then, using the fact that $z_\varepsilon = \overline{u}^{\mathrm{bulk}}_{x_0}$ in $B_\varepsilon(x_0) \setminus B_{(1-3\theta)\varepsilon}(x_0) \subset C_{\varepsilon,\theta}(x_0)$, (H1), (H4) and (6.24), setting $\psi^+_\varepsilon(t) := \psi^+_{B_\varepsilon(x_0)}(t)$ for $t > 0$, we deduce that there exists $\varepsilon_0 > 0$ small enough such that for every $\varepsilon \in (0, \varepsilon_0)$

$$\frac{\mathcal{F}(\overline{u}^{\mathrm{bulk}}_{x_0}, C_{\varepsilon,\theta}(x_0))}{\varepsilon^d} \le b\mathcal{L}^d(C_{1,\theta}(x_0))(1 + \psi^+_{\varepsilon_0}(|\nabla u(x_0)|)). \tag{6.29}$$

Thus,

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0))}{\varepsilon^d} &\le \limsup_{\varepsilon\to 0^+}\left(\frac{\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0))}{\varepsilon^d} + \frac{\mathcal{F}(\overline{u}^{\mathrm{bulk}}_{x_0}, C_{\varepsilon,\theta}(x_0))}{\varepsilon^d}\right) \\ &\le \limsup_{\varepsilon\to 0^+}\left(\frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\mathrm{bulk}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0))}{\varepsilon^d} + b\mathcal{L}^d(C_{1,\theta}(x_0))(1 + \psi^+_{\varepsilon_0}(|\nabla u(x_0)|))\right) \\ &\le (1-3\theta)^d \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\mathrm{bulk}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0))}{(1-3\theta)^d\varepsilon^d} \\ &\quad + \limsup_{\varepsilon\to 0^+} b\mathcal{L}^d(C_{1,\theta}(x_0))(1 + \psi^+_{\varepsilon_0}(|\nabla u(x_0)|)).\end{aligned} \tag{6.30}$$

On the other hand, using again (H4) we also deduce

$$\mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0)) \le b\mathcal{L}^d(C_{\varepsilon,\theta}(x_0)) + b\int_{C_{\varepsilon,\theta}(x_0)} \psi(x, |\nabla u_\varepsilon(x)|)\,dx + b\mathcal{H}^{d-1}(J_{u_\varepsilon} \cap C_{\varepsilon,\theta}(x_0)).$$

Now, using (6.16)(iii), we get

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))}{\varepsilon^d} \le b\mathcal{L}^d(C_{1,\theta}(x_0)) + \limsup_{\varepsilon\to 0^+} \frac{b}{\varepsilon^d}\int_{C_{\varepsilon,\theta}(x_0)} \psi(x, |\nabla u_\varepsilon(x)|)\,dx. \tag{6.31}$$

By exploiting the construction in Lemma 6.6, we notice that $|\nabla u_\varepsilon| \le |\nabla u|$ $\mathcal{L}^d$-a.e. in $B_\varepsilon(x_0)$ for any $\varepsilon > 0$. Thus, using (6.17a) and the fact that ψ is doubling with constant $K \ge 2$, we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0^+} \frac{b}{\varepsilon^d}\int_{C_{\varepsilon,\theta}(x_0)} \psi(x, |\nabla u_\varepsilon|)\,dx \\ &\le \limsup_{\varepsilon\to 0^+} \frac{b}{\varepsilon^d}\int_{C_{\varepsilon,\theta}(x_0)} \psi(x, |\nabla u|)\,dx \\ &\le \limsup_{\varepsilon\to 0^+} (1+K)\left(\frac{b}{\varepsilon^d}\int_{B_\varepsilon(x_0)} \psi(x, |\nabla u(x) - \nabla u(x_0)|)\,dx + b\mathcal{L}^d(C_{\varepsilon,\theta}(x_0))\psi^+_{\varepsilon_0}(|\nabla u(x_0)|)\right) \\ &\le b(1+K)\mathcal{L}^d(C_{1,\theta}(x_0))\psi^+_{\varepsilon_0}(|\nabla u(x_0)|).\end{aligned} \tag{6.32}$$

Combining (6.31) and (6.32) we infer

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))}{\varepsilon^d} \le b\mathcal{L}^d(C_{1,\theta}(x_0))\left(1+(1+K)\psi^+_{\varepsilon_0}(|\nabla u(x_0)|)\right). \tag{6.33}$$

Recalling that $w_\varepsilon = u_\varepsilon$ in a neighborhood of $\partial B_\varepsilon(x_0)$, by (6.28), (6.30) and (6.33) we finally get

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_d \varepsilon^d} &\le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\omega_d\varepsilon^d} \\ &\le (1+\eta)(1-3\theta)^d \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\rm bulk}_{x_0}, B_\varepsilon(x_0))}{\omega_d\varepsilon^d} \\ &\quad + (1+\eta)b\omega_d^{-1}\mathcal{L}^d(C_{1,\theta}(x_0))\left(1+(1+K)\psi^+_{\varepsilon_0}(|\nabla u(x_0)|)\right)+\eta.\end{aligned}$$

Letting $\eta\to 0$ and $\theta\to 0$ we get (6.22).

□

Lemma 6.8. *Let $\mathcal{F}$ satisfy (H1) and (H3)-(H4) and let $u\in GSBV^\psi(\Omega,\mathbb{R}^m)$. Then, for $\mathcal{L}^d$-a.e $x_0\in\Omega$ we have*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_d\varepsilon^d} \ge \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\rm bulk}_{x_0}, B_\varepsilon(x_0))}{\omega_d\varepsilon^d}. \tag{6.34}$$

Proof. We prove the statement for points $x_0\in\Omega$ considered in Lemma 6.7. Let $\eta>0$ and $\lambda = 1-\theta$ be fixed as in the proof of Lemma 6.7, and let $(u_\varepsilon)_\varepsilon$ be the sequence given by Lemma 6.6. Using Fubini's Theorem (as in (6.19) and (6.20)), we have that for every $\varepsilon>0$ we can find $s\in(1-4\theta,1-3\theta)$ such that

$$\begin{aligned}&\mathcal{H}^{d-1}(\partial B_{s\varepsilon}(x_0)\cap(J_u\cup J_{u_\varepsilon}))=0 \quad \text{for every } \varepsilon>0,\\ &\lim_{\varepsilon\to 0^+}\varepsilon^{-d}\mathcal{H}^{d-1}(\{u_\varepsilon\ne u\}\cap\partial B_{s\varepsilon}(x_0))=0.\end{aligned} \tag{6.35}$$

From now on, we will follow mainly the arguments of the proof of Lemma 6.7. We choose a sequence $z_\varepsilon\in GSBV^\psi(B_{s\varepsilon}(x_0),\mathbb{R}^m)$ such that $z_\varepsilon = u$ in a neighborhood of $\partial B_{s\varepsilon}(x_0)$ and

$$\mathcal{F}(z_\varepsilon, B_{s\varepsilon}(x_0)) \le \mathbf{m}_{\mathcal{F}}(u, B_{s\varepsilon}(x_0)) + \omega_d\varepsilon^{d+1}. \tag{6.36}$$

Setting $z_\varepsilon = u_\varepsilon$ outside $B_{s\varepsilon}(x_0)$ we extend it to $z_\varepsilon\in GSBV^\psi(B_\varepsilon(x_0),\mathbb{R}^m)$. We now use Lemma 6.1 applied with u and v replaced by z_ε and $\overline{u}^{\rm bulk}_{x_0}$, respectively, and the same choice of sets $D'_{\varepsilon,x_0}, D''_{\varepsilon,x_0}, E_{\varepsilon,x_0}$ as in (6.25).

Hence, we find $w_\varepsilon\in GSBV^\psi(B_\varepsilon(x_0),\mathbb{R}^m)$ such that $w_\varepsilon = \overline{u}^{\rm bulk}_{x_0}$ on $B_\varepsilon(x_0)\setminus B_{(1-\theta)\varepsilon}(x_0)$ and

$$\begin{aligned}\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le &(1+\eta)\left(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(\overline{u}^{\rm bulk}_{x_0}, C_{\varepsilon,\theta}(x_0))\right) + \eta\mathcal{L}^d(B_\varepsilon(x_0)) +\\ &+ M\int_{B_{(1-\theta)\varepsilon}(x_0)\setminus B_{(1-2\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|z_\varepsilon - \overline{u}^{\rm bulk}_{x_0}|}{\varepsilon}\right)dx.\end{aligned}$$

Since from the initial choice of s we have that $z_\varepsilon = u_\varepsilon$ outside $B_{(1-3\theta)\varepsilon}(x_0)$, by arguing as in the proof of Lemma 6.7 (see (6.27) and (6.28)), we find an non-negative infinitesimal sequence $(\rho_\varepsilon)_\varepsilon$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))) + \varepsilon^d\rho_\varepsilon + \eta\omega_d\varepsilon^d. \tag{6.37}$$

We now estimate the terms in (6.37). Using that $z_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0)\setminus B_{s\varepsilon}(x_0)\subset C_{\varepsilon,\theta}(x_0)$, (H1), (H4) and (6.36) we get

$$\begin{aligned}\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) \le &\mathbf{m}_{\mathcal{F}}(u, B_{s\varepsilon}(x_0)) + \omega_d\varepsilon^{d+1} + b\mathcal{H}^{d-1}(\partial B_{s\varepsilon}(x_0)\cap(J_{u_\varepsilon}\cup J_u))\\ &+ \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0)).\end{aligned} \tag{6.38}$$

Now, recalling (6.33) and using (6.35) together with the fact that $s\varepsilon\le(1-3\theta)\varepsilon$, we estimate

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\frac{\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0))}{\varepsilon^d} &\le s^d\limsup_{\varepsilon\to 0^+}\frac{\mathbf{m}_{\mathcal{F}}(u, B_{s\varepsilon}(x_0))}{s^d\varepsilon^d} + b\mathcal{L}^d(C_{1,\theta}(x_0))\left(1+C\psi^+_\varepsilon(|\nabla u(x_0)|)\right) \qquad (6.39)\\ &\le (1-3\theta)^d\limsup_{\varepsilon\to 0^+}\frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\varepsilon^d} + b\mathcal{L}^d(C_{1,\theta}(x_0))\left(1+C\psi^+_\varepsilon(|\nabla u(x_0)|)\right)\end{aligned}$$

where $C>0$ is a constant depending only on ψ.

The estimate for $\mathcal{F}(\overline{u}_{x_0}^{\rm bulk}, C_{\varepsilon,\theta}(x_0))$ in (6.29) together with the estimates (6.37) and (6.39) gives, recalling that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\varepsilon^d} \le&(1+\eta)(1-3\theta)^d \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\varepsilon^d}\\ &+ 2(1+\eta)b\mathcal{L}^d(C_{1,\theta})\left(1 + C\psi_\varepsilon^+(|\nabla u(x_0)|)\right) + \omega_d\eta.\end{aligned}$$

Finally, letting $\eta \to 0$ and $\theta \to 0$ and recalling that $w_\varepsilon = \overline{u}_{x_0}^{\rm bulk}$ in a neighborhood of $\partial B_\varepsilon(x_0)$, we deduce

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}_{x_0}^{\rm bulk}, B_\varepsilon(x_0))}{\omega_d \varepsilon} &\le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\omega_d\varepsilon}\\ &\le \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_d\varepsilon} = \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_d\varepsilon},\end{aligned}$$

this proves (6.34). □

6.4. **Surface points.** This section is devoted to the proof of Lemma 6.3. We can now perform a similar analysis for the jump points.

Lemma 6.9. *Let $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$. Then, for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ and $\mathcal{L}^1$-a.e. $\lambda \in (0,1)$ there exists a sequence $u_\varepsilon \in GSBV^\psi(B_\varepsilon(x_0))$ such that the following properties hold*

$$\begin{aligned}&(i)\quad u_\varepsilon = u \text{ on } B_\varepsilon(x_0)\setminus \overline{B_{\lambda\varepsilon}(x_0)}, \qquad \lim_{\varepsilon\to 0^+} \varepsilon^{-d}\mathcal{L}^d(\{u_\varepsilon \neq u\}\cap B_\varepsilon(x_0)) = 0;\\ &(ii)\quad J_{u_\varepsilon}\setminus J_u \subset \partial B_{\lambda\varepsilon}(x_0), \qquad \lim_{\varepsilon\to 0^+} \varepsilon^{-(d-1)}\mathcal{H}^{d-1}(J_{u_\varepsilon}\setminus J_u) = 0;\\ &(iii)\quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}}\int_{B_\varepsilon(x_0)} \psi(x, |\nabla u_\varepsilon(x)|)\,dx = 0;\\ &(iv)\quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}}\int_{B_{\lambda\varepsilon}(x_0)} \psi\left(x, \frac{|u_\varepsilon(x) - u_{x_0,\nu}(x)|}{\varepsilon}\right)dx = 0.\end{aligned} \tag{6.40}$$

Proof. As before we assume without loss of generality $m = 1$. Since $\rho_\psi(\nabla u) < +\infty$ we have that for $\mathcal{H}^{d-1}$-a.e $x_0 \in J_u$ it holds

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}}\int_{B_\varepsilon(x_0)} \psi(x, |\nabla u(x)|)\,dx = 0; \tag{6.41}$$

see e.g. [42, Section 2.4.3, Theorem 2.10]. Since J_u is $(d-1)$-rectifiable, there exists a sequence of compact sets $K_j \subset \Omega$ and a set $N \subset \Omega$ such that

$$J_u = \bigcup_{j=1}^\infty K_j \cup N, \qquad \mathcal{H}^{d-1}(N) = 0;$$

and each K_j is a subset of a C^1-hypersurface. Then, taking $\varepsilon_0 > 0$ small enough, in a neighborhood $B_{\varepsilon_0}(y)$ of each point $y \in K_j$, up to rotations, we may find a C^1-function $\Gamma_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$K_j \cap B_{\varepsilon_0}(y) \subseteq \{x = (x', x_d) \in B_{\varepsilon_0}(x_0) \colon x_d = \Gamma_j(x')\}.$$

Since $u \in GSBV(\Omega)$, if we restrict it to the Lipschitz set $\Omega_j := \{(x', x_d) \in \Omega \colon x_d > \Gamma_j(x')\}$ then it has unique measurable trace on its boundary, that is, for $\mathcal{H}^{d-1}$-a.e. $x_0 \in \partial\Omega_j$ there exists $\mathrm{tr}(u)(x_0) \in \mathbb{R}$ such that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-d}\mathcal{L}^d\left(\Omega_j \cap B_\varepsilon(x_0) \cap \{|u - \mathrm{tr}(u)(x_0)| > \rho\}\right) = 0 \quad \text{for all } \rho > 0; \tag{6.42}$$

see e.g. [33, Theorem 5.5]. Up to taking ε_0 smaller, we can define a function $w \in GSBV^\psi(B_{\varepsilon_0}(y))$ as

$$w(x) := \begin{cases} u(x', x_d) & \text{if } x_d > \Gamma_j(x'),\\ u(x', -x_d + 2\Gamma_j(x')) & \text{if } x_d < \Gamma_j(x').\end{cases}$$

By construction we have that $|\nabla w| \le C|\nabla u|$ $\mathcal{L}^d$-a.e. on $B_{\varepsilon_0}(y)$, thus indeed $w \in GSBV^\psi(B_{\varepsilon_0}(y))$. Moreover, $J_w \cap B_{\varepsilon_0}(y) \subset B_{\varepsilon_0}(y)\setminus K_j$. Now we claim that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in B_{\varepsilon_0}(y) \cap K_j$ we have either

$w(x_0) = u^+(x_0)$ or $w(x_0) = u^-(x_0)$. Indeed, by definition of w and (6.42) we infer that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in B_{\varepsilon_0}(y) \cap K_j$ it holds

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-d}\mathcal{L}^d\left(B_\varepsilon(x_0) \cap \{|w - w(x_0)| > \rho\}\right) = 0 \quad \text{for all } \rho > 0. \tag{6.43}$$

Thus, combining (6.42) and (6.43) we deduce the claim, being $\mathrm{tr}(u)(x_0)$ either $u^+(x_0)$ or $u^-(x_0)$.

Thanks to Theorem 4.37, we have that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in K_j \cap B_{\varepsilon_0}(y)$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}}\mathcal{H}^{d-1}(J_w \cap B_\varepsilon(x_0)) = 0. \tag{6.44}$$

Moreover, by (6.41) and definition of w, we have that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in K_j \cap B_{\varepsilon_0}(y)$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi_\varepsilon^-(|\nabla w(x)|)\,dx = 0, \tag{6.45}$$

where we recall that $\psi_\varepsilon^- := \inf_{B_\varepsilon(x_0)} \psi(x,\cdot)$.

Let us fix $x_0 \in K_j \cap B_{\varepsilon_0}(y)$ such that properties (6.43)–(6.45) hold and assume without loss of generality that $w(x_0) = u^+(x_0)$. We fix $\eta > 0$ arbitrarily small as before, by (6.44) and (6.45), for every $\varepsilon > 0$ small enough we have

$$\int_{B_\varepsilon(x_0)} \psi_\varepsilon^-(|\nabla w(x)|)\,dx + \mathcal{H}^{d-1}(J_w \cap B_\varepsilon(x_0)) < \eta\varepsilon^{d-1}. \tag{6.46}$$

Set $T_\varepsilon := T_{B_\varepsilon(x_0)}$ and $m_\varepsilon := \mathrm{med}(w, B_\varepsilon(x_0))$. We define similarly to (5.30)

$$\mathfrak{v}_\varepsilon^\psi = \left(m_\varepsilon - \varepsilon(\psi^-_{B_\varepsilon(x_0)})^{-1}\left(\frac{1}{2\varepsilon}\right)\right) \wedge T_\varepsilon w(x) \vee \left(m_\varepsilon + \varepsilon(\psi^-_{B_\varepsilon(x_0)})^{-1}\left(\frac{1}{2\varepsilon}\right)\right). \tag{6.47}$$

Using property (adA1) of ψ, Remark 4.23 and inequality (5.32), we infer that there exist $\varepsilon_0 > 0$ and $C > 0$ depending on the dimension and on ψ but not on ε such that, for every $\varepsilon < \varepsilon_0$,

$$\int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|\mathfrak{v}_\varepsilon^\psi - m_\varepsilon|}{\varepsilon}\right) dx \le C \int_{B_\varepsilon(x_0)} \psi_\varepsilon^-\left(|\nabla w(x)|\right)\,dx + \omega_d \varepsilon^d \psi_{\varepsilon_0}^+(\sigma).$$

Hence, keeping in mind (6.45),

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|\mathfrak{v}_\varepsilon^\psi - m_\varepsilon|}{\varepsilon}\right) dx = 0. \tag{6.48}$$

We now want to show that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi_\varepsilon^+\left(\frac{|m_\varepsilon - u^+(x_0)|}{\varepsilon}\right) dx = \lim_{\varepsilon\to 0^+} \varepsilon\omega_d\psi_\varepsilon^+\left(\frac{|m_\varepsilon - u^+(x_0)|}{\varepsilon}\right) = 0. \tag{6.49}$$

To this aim observe that condition (6.43) and the fact that $w(x_0) = u^+(x_0)$ imply that

$$\lim_{\varepsilon\to 0^+} \mathrm{med}(w, B_\varepsilon(x_0)) = u^+(x_0). \tag{6.50}$$

Assume that $|m_\varepsilon - u^+(x_0)| \ge \varepsilon\sigma$ for every $\varepsilon > 0$ small enough, otherwise the thesis is obvious. The function $\psi_\varepsilon^- \in \Phi_w$ satisfies $(\mathrm{Inc})_\gamma$ with the same $\gamma > 1$ of ψ and it is doubling with the same constant $K \ge 2$ as ψ. Therefore, by Theorem 4.7 and Lemma 4.5, for every $\varepsilon > 0$ there exists an Orlicz function $\Psi_\varepsilon \in \Phi_s$ which satisfies $(\mathrm{Inc})_\gamma$, it is doubling with the same constant K and

$$\Psi_\varepsilon(t) \le \psi_\varepsilon^-(t) \le \Psi_\varepsilon(2t), \qquad \Psi_\varepsilon^{-1}(t) \le 2(\psi_\varepsilon^-)^{-1}(t) \le 2\Psi_\varepsilon^{-1}(t) \quad \text{ for every } t \ge 0.$$

In particular, by continuity, $\Psi_\varepsilon(\Psi_\varepsilon^{-1}(t)) = \Psi_\varepsilon^{-1}(\Psi_\varepsilon(t)) = t$ for every $t \ge 0$. We claim that for every $\delta \in (0,1)$ and for every $\varepsilon > 0$ it holds

$$(\Psi_\varepsilon)^{-1}\left(\frac{t}{\delta}\right) \le \delta^{-\frac{1}{\gamma}}(\Psi_\varepsilon)^{-1}(t) \quad \text{for every } t > 0. \tag{6.51}$$

Indeed, this can be checked using the relation $\Psi_\varepsilon(\lambda s) \ge \lambda^\gamma \Psi_\varepsilon(s)$ for $\lambda \ge 1$ given by the property $(\mathrm{Inc})_\gamma$, with $\lambda := \delta^{-\frac{1}{\gamma}}$ and $s := (\Psi_\varepsilon)^{-1}(t)$. Fix $\delta \in (3/4, 1)$ and $\delta\varepsilon < \rho < \varepsilon$. We set

$$\widehat{w} := (w \wedge \tau''(w, B_\varepsilon(x_0)) \wedge \tau''(w, B_\rho(x_0))) \vee (\tau'(w, B_\varepsilon(x_0)) \vee \tau'(w, B_\rho(x_0))).$$

By Theorem 4.39, the following inequalities hold

$$|\widehat{w}-m_\varepsilon| \le |T_\varepsilon w - m_\varepsilon| \qquad |\widehat{w}-m_\rho| \le |T_\rho w - m_\rho|. \tag{6.52}$$

Using property $(\mathrm{Inc})_\gamma$, the inequalities in (6.52), Theorem 5.7 applied to Ψ_ε and (6.46), we get that

$$\begin{aligned}
\Psi_\varepsilon\left(\frac{|m_\varepsilon-m_\rho|}{\varepsilon}\right) &= \fint_{B_\rho(x_0)} \Psi_\varepsilon\left(\frac{|m_\varepsilon-m_\rho|}{\varepsilon}\right)\,dx \\
&\le K\fint_{B_\rho(x_0)} \Psi_\varepsilon\left(\frac{|\widehat{w}-m_\rho|}{\varepsilon}\right)\,dx + K\fint_{B_\rho(x_0)} \Psi_\varepsilon\left(\frac{|\widehat{w}-m_\varepsilon|}{\varepsilon}\right)\,dx \\
&\le K\fint_{B_\rho(x_0)} \Psi_\varepsilon\left(\frac{|\widehat{w}-m_\rho|}{\rho}\right)\,dx + \frac{K}{\delta^d}\fint_{B_\varepsilon(x_0)} \Psi_\varepsilon\left(\frac{|\widehat{w}-m_\varepsilon|}{\varepsilon}\right)\,dx \\
&\le C\fint_{B_\rho(x_0)} \Psi_\varepsilon\left(|\nabla w(x)|\right)\,dx + \frac{C}{\delta^d}\fint_{B_\varepsilon(x_0)} \Psi_\varepsilon\left(|\nabla w(x)|\right)\,dx \\
&\le C\eta\varepsilon^{-1}.
\end{aligned}$$

where $C = C(d,\psi)$. Reasoning analogously one also gets that there exists $C>0$ not depending on ε and on $k\ge 1$ such that it holds

$$\Psi_\varepsilon\left(\frac{|m_{\delta^k\varepsilon}-m_{\delta^{k+1}\varepsilon}|}{\delta^k\varepsilon}\right) \le C\eta(\delta^k\varepsilon)^{-1}.$$

Let ρ_k be an infinitesimal sequence such that $\delta^{k+1}\varepsilon \le \rho_k \le \delta^k\varepsilon$ for every $k\ge 1$. Combining the estimates above, we have that

$$\begin{aligned}
|m_\varepsilon - m_{\rho_k}| &\le |m_{\delta^k\varepsilon} - m_{\rho_k}| + \sum_{j=0}^{k-1} |m_{\delta^{j+1}\varepsilon} - m_{\delta^j\varepsilon}| \\
&\le \delta^k\varepsilon\Psi_\varepsilon^{-1}\left(C\eta(\delta^k\varepsilon)^{-1}\right) + \sum_{j=0}^{k-1}\delta^j\varepsilon\Psi_\varepsilon^{-1}\left(C\eta(\delta^j\varepsilon)^{-1}\right).
\end{aligned} \tag{6.53}$$

In the following $C>0$ will denote a generic constant depending only on the dimension d and on ψ. Let $k\to+\infty$ in (6.53), keeping in mind (6.50) and (6.51), we have for every $\varepsilon>0$ small enough

$$\begin{aligned}
|m_\varepsilon - u^+(x_0)| &\le \sum_{j=0}^{\infty}\delta^j\varepsilon\Psi_\varepsilon^{-1}\left(C\eta(\delta^j\varepsilon)^{-1}\right) \le C\sum_{j=0}^{\infty}\delta^j\varepsilon\Psi_\varepsilon^{-1}\left(\eta(\delta^j\varepsilon)^{-1}\right) \\
&\le \varepsilon C\sum_{j=0}^{\infty}(\delta^j)^{1-\frac{1}{\gamma}}\Psi_\varepsilon^{-1}\left(\eta\varepsilon^{-1}\right) = \varepsilon C\frac{1}{1-\delta^{(\gamma-1)/\gamma}}\Psi_\varepsilon^{-1}\left(\eta\varepsilon^{-1}\right) \\
&\le \varepsilon C\Psi_\varepsilon^{-1}\left(\eta\varepsilon^{-1}\right).
\end{aligned} \tag{6.54}$$

Taking η smaller if necessary, we get that

$$C\Psi_\varepsilon^{-1}(\eta\varepsilon^{-1}) \le \frac{1}{2}\Psi_\varepsilon^{-1}(2^{-1}\varepsilon^{-1}) \le (\psi_\varepsilon^-)^{-1}(2^{-1}\varepsilon^{-1}). \tag{6.55}$$

Combining (6.54) and (6.55) we obtain that

$$\frac{|m_\varepsilon - u^+(x_0)|}{\varepsilon} \le (\psi_\varepsilon^-)^{-1}\left(\frac{1}{2\varepsilon}\right).$$

Thus, by property (adA1) of ψ and using that Ψ_ε is doubling, we deduce that

$$\psi_\varepsilon^+\left(\frac{|m_\varepsilon-u^+(x_0)|}{\varepsilon}\right) \le C\psi_\varepsilon^-\left(\frac{|m_\varepsilon-u^+(x_0)|}{\varepsilon}\right) \le C\Psi_\varepsilon\left(2\frac{|m_\varepsilon-u^+(x_0)|}{\varepsilon}\right) \le C\Psi_\varepsilon\left(\frac{|m_\varepsilon-u^+(x_0)|}{\varepsilon}\right).$$

Therefore,

$$\begin{aligned}
\lim_{\varepsilon\to0^+}\varepsilon\psi_\varepsilon^+\left(\frac{|m_\varepsilon-u^+(x_0)|}{\varepsilon}\right) &\le \lim_{\varepsilon\to0^+}\varepsilon C\Psi_\varepsilon\left(\frac{|m_\varepsilon-u^+(x_0)|}{\varepsilon}\right) \\
&\le \lim_{\varepsilon\to0^+}\varepsilon C\Psi_\varepsilon\left(C\Psi_\varepsilon^{-1}\left(\frac{\eta}{\varepsilon}\right)\right) \le C\eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrarily small, letting $\eta \to 0^+$ we infer (6.49). Combining (6.48) and (6.49) we deduce that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|\mathfrak{v}_\varepsilon^\psi(x) - u^+(x_0)|}{\varepsilon}\right) dx = 0. \tag{6.56}$$

We repeat the above arguments for the function defined as

$$z(x) := \begin{cases} u(x', x_d) & \text{if } x_d < \Gamma_j(x'), \\ u(x', -x_d + 2\Gamma_j(x')) & \text{if } x_d > \Gamma_j(x'). \end{cases}$$

We have that (6.44) and (6.45) are satisfied with w replaced by z. Moreover, let $\mathfrak{z}_\varepsilon^\psi$ be defined as in (6.47) with w replaced by z, reasoning analogously we deduce

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|\mathfrak{z}_\varepsilon^\psi(x) - u^-(x_0)|}{\varepsilon}\right) dx = 0. \tag{6.57}$$

We set

$$v_\varepsilon(x) := \begin{cases} \mathfrak{v}_\varepsilon^\psi(x) & \text{if } x_d > \Gamma_j(x'), \\ \mathfrak{z}_\varepsilon^\psi(x) & \text{if } x_d < \Gamma_j(x'). \end{cases}$$

Using (6.41), since $|\nabla v_\varepsilon| \le |\nabla u|$ on $B_\varepsilon(x_0)$ by definition, we get

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi(x, |\nabla v_\varepsilon(x)|)\, dx = 0. \tag{6.58}$$

From (6.56) and (6.57) we also infer that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x_0)} \psi\left(x, \frac{|v_\varepsilon(x) - u_{x_0,\nu}(x)|}{\varepsilon}\right) dx = 0. \tag{6.59}$$

By definition of $\mathfrak{v}_\varepsilon^\psi$, $\mathfrak{z}_\varepsilon^\psi$, by Chebychev inequality, by (4.6) and by Theorem 5.7, we estimate

$$\begin{aligned} \mathcal{L}^d\left(\{\mathfrak{v}_\varepsilon^\psi \neq w\} \cup \{\mathfrak{z}_\varepsilon^\psi \neq z\}\right) &\le \mathcal{L}^d\left(\{\mathfrak{v}_\varepsilon^\psi \neq T_\varepsilon w\}\right) + \mathcal{L}^d\left(\{\mathfrak{z}_\varepsilon^\psi \neq T_\varepsilon z\}\right) + \mathcal{L}^d\left(\{T_\varepsilon w \neq w\}\right) + \mathcal{L}^d\left(\{T_\varepsilon z \neq z\}\right) \\ &\le \varepsilon C \int_{B_\varepsilon(x_0)} \psi_\varepsilon^-(|\nabla w(x)|)\, dx + \varepsilon C \int_{B_\varepsilon(x_0)} \psi_\varepsilon^-(|\nabla z(x)|)\, dx \\ &\quad + \left(2\gamma_{\mathrm{iso}} \mathcal{H}^{d-1}(J_w \cap B_\varepsilon(x_0))\right)^{\frac{d}{d-1}} + \left(2\gamma_{\mathrm{iso}} \mathcal{H}^{d-1}(J_z \cap B_\varepsilon(x_0))\right)^{\frac{d}{d-1}}, \end{aligned}$$

where $C = C(d, K)$. Thus, by definition of v_ε, (6.44) and (6.45) for w and z, we infer that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-d} \mathcal{L}^d(\{x \in B_\varepsilon(x_0) \colon v_\varepsilon(x) \neq u(x)\}) = 0. \tag{6.60}$$

Now, an analogous argument as in (6.19) shows that for every sequence $\varepsilon \to 0^+$ one can find a subsequence (not relabeled) such that for $\mathcal{L}^1$-a.e. $\lambda \in (0,1)$

$$\mathcal{H}^{d-1}(\partial B_{\lambda\varepsilon}(x_0) \cap J_{v_\varepsilon}) = 0 \qquad \lim_{\varepsilon\to 0^+} \varepsilon^{1-d} \mathcal{H}^{d-1}(\{v_\varepsilon \neq u\} \cap \partial B_{\lambda\varepsilon}(x_0)) = 0. \tag{6.61}$$

We finally define

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } x \in B_{\lambda\varepsilon}(x_0), \\ u(x) & \text{if } x \in B_\varepsilon(x_0) \setminus \overline{B_{\lambda\varepsilon}(x_0)}. \end{cases}$$

By definition it is clear that $u_\varepsilon \in GSBV^\psi(B_\varepsilon(x_0))$. Properties (6.40)(i)–(ii) follow by definition of u_ε, (6.60) and (6.61). Property (6.40)(iii) follows by (6.41) and (6.58). Finally, property (6.40)(iv) is a consequence of the definition of u_ε and (6.59). $\square$

With this approximation tool, we can now prove Lemma 6.4 which will follow as a consequence of Lemma 6.10 and Lemma 6.11.

Lemma 6.10. *Let $\mathcal{F}$ satisfy (H1) and (H3)-(H4) and let $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$. Then, for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ it holds*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \le \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}_{x_0}^{\mathrm{surf}}, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}}. \tag{6.62}$$

Proof. Let $x_0 \in J_u$ such that the statement of Lemma 6.9 holds and

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} \le \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} < +\infty. \tag{6.63}$$

By Lemma 6.2 and recalling the definition of μ, (6.63) holds $\mathcal{H}^{d-1}$-a.e. in $J_u \cap \Omega$. Let $(u_\varepsilon)_\varepsilon$ be the sequence given as in Lemma 6.9; in the following we set $\nu := \nu_u(x_0)$ for brevity, where we recall that $\nu_u(x_0)$ is the normal unit vector to J_u in x_0. Fix $\eta > 0$, take $\lambda \in (0,1)$ such that (6.40) holds, and set $\lambda = 1-\theta$ with $\theta \in (0,1)$. Take a sequence $z_\varepsilon \in GSBV^\psi(B_{(1-3\theta)\varepsilon}(x_0), \mathbb{R}^m)$ with $z_\varepsilon = \overline{u}^{\mathrm{surf}}_{x_0}$ in a neighborhood of $\partial B_{(1-3\theta)\varepsilon}(x_0)$ and such that

$$\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0)) \le \mathbf{m}_{\mathcal{F}}(\overline{u}^{\mathrm{surf}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0)) + \omega_{d-1}\varepsilon^d. \tag{6.64}$$

We extend z_ε to a function in $GSBV^\psi(B_\varepsilon(x_0), \mathbb{R}^m)$ by setting $z_\varepsilon = \overline{u}^{\mathrm{surf}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$. Now we apply Lemma 6.1 with u and v replaced by z_ε and u_ε, respectively, and D'_{ε,x_0}, D''_{ε,x_0}, E_{ε,x_0} defined as in (6.25). Thus, we find $w_\varepsilon \in GSBV^\psi(B_\varepsilon(x_0), \mathbb{R}^m)$ such that $w_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0) \setminus B_{(1-\theta)\varepsilon}(x_0)$ and

$$\begin{aligned}\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le &(1+\eta)\big(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))\big) + \eta\mathcal{L}^d(B_\varepsilon(x_0)) \\ &+ M\int_{B_{(1-\theta)\varepsilon}(x_0)\setminus B_{(1-2\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|z_\varepsilon - u_\varepsilon|}{\varepsilon}\right) dx.\end{aligned} \tag{6.65}$$

By (6.40)(i) we have that $w_\varepsilon = u_\varepsilon = u$ on a neighborhood of $\partial B_\varepsilon(x_0)$. Moreover, using the fact that $z_\varepsilon = \overline{u}^{\mathrm{surf}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$, by (6.40)(iv) we deduce

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_{(1-\theta)\varepsilon}(x_0)\setminus B_{(1-2\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|z_\varepsilon - u_\varepsilon|}{\varepsilon}\right) dx \\ &\qquad \le \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_{B_{(1-\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|u_\varepsilon - \overline{u}^{\mathrm{surf}}_{x_0}|}{\varepsilon}\right) dx = 0.\end{aligned}$$

Recall that $C_{\varepsilon,\theta}(x_0) := B_\varepsilon(x_0) \setminus \overline{B_{(1-4\theta)\varepsilon}(x_0)}$ as defined in Lemma 6.7. Together with (6.65) we infer that there exists a non-negative infinitesimal sequence $(\rho_\varepsilon)_\varepsilon$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)\big(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))\big) + \varepsilon^{d-1}\rho_\varepsilon + \eta\omega_d\varepsilon^d. \tag{6.66}$$

We now estimate the terms in (6.66). Let Π_0 be the hyperplane passing through x_0 with normal ν. Using that $z_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0) \setminus B_{(1-3\theta)\varepsilon}(x_0) \subset C_{\varepsilon,\theta}(x_0)$ together with (H1), (H4) and (6.64) we get

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(\overline{u}^{\mathrm{surf}}_{x_0}, C_{\varepsilon,\theta}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \le b(1-(1-4\theta)^{d-1}), \tag{6.67}$$

and

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(\overline{u}^{\mathrm{surf}}_{x_0}, C_{\varepsilon,\theta}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \\ &\le \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\mathrm{surf}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + \frac{b}{\omega_{d-1}}\mathcal{H}^{d-1}(C_{1,\theta}(x_0)\cap\Pi_0) \\ &\le (1-3\theta)^{d-1}\limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + \frac{b}{\omega_{d-1}}(1-(1-4\theta)^{d-1}).\end{aligned} \tag{6.68}$$

Notice that by rectifiability of J_u and (6.40)(ii) it holds

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{H}^{d-1}(J_{u_\varepsilon}\cap C_{\varepsilon,\theta}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \le \lim_{\varepsilon\to 0^+} \frac{\mathcal{H}^{d-1}(J_u\cap C_{\varepsilon,\theta}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} = \frac{\mathcal{H}^{d-1}(\Pi_0\cap C_{1,\theta}(x_0))}{\omega_{d-1}} \tag{6.69}$$

Therefore, using (6.40)(iii) and (H4) again we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to 0^+} \frac{b}{\omega_{d-1}\varepsilon^{d-1}}\left(\int_{C_{\varepsilon,\theta}(x_0)} (1+\psi(x,|\nabla u_\varepsilon|))\,dx + \mathcal{H}^{d-1}(J_{u_\varepsilon}\cap C_{\varepsilon,\theta}(x_0))\right) \\ &\le b(1-(1-4\theta)^{d-1}).\end{aligned} \tag{6.70}$$

Finally, recalling that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$, that $w_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$, and using (6.66), (6.68) and (6.70), we deduce

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \\ &\le (1+\eta)(1-3\theta)^{d-1} \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\rm surf}_{x_0}, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + 2b(1+\eta)(1-(1-4\theta)^{d-1}).\end{aligned}$$

Letting $\eta \to 0$ and $\theta \to 0$ we get (6.62). □

Lemma 6.11. *Let $\mathcal{F}$ satisfy (H1) and (H3)-(H4) and let $u \in GSBV^\psi(\Omega, \mathbb{R}^m)$. Then, for $\mathcal{H}^d$-a.e $x_0 \in J_u$ we have*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \ge \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\rm surf}_{x_0}, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}}. \tag{6.71}$$

Proof. Again, we prove the assertion for the points $x_0 \in J_u$ considered in the proof of Lemma 6.10. Fix $\eta > 0$ and let $(u_\varepsilon)_\varepsilon$ be the sequence given by Lemma 6.9. By (6.40)(i) and Fubini's Theorem it follows that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{d-1}} \int_0^1 \mathcal{H}^{d-1}(\{u_\varepsilon \neq u\} \cap \partial B_{\lambda\varepsilon}(x_0))\, d\lambda = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^d} \mathcal{L}^d(\{u_\varepsilon \neq u\} \cap B_\varepsilon(x_0)) = 0.$$

Thus, given $\theta \in (0,1)$, for every $\varepsilon > 0$ there exists $\lambda \in (1-4\theta, 1-3\theta)$ such that

$$\begin{aligned}&\mathcal{H}^{d-1}((J_{u_\varepsilon} \cup J_u) \cap \partial B_{\lambda\varepsilon}(x_0)) = 0 \quad \text{for every } \varepsilon > 0, \\ &\lim_{\varepsilon\to 0^+} \varepsilon^{-(d-1)} \mathcal{H}^{d-1}(\{u_\varepsilon \neq u\} \cap \partial B_{\lambda\varepsilon}(x_0)) = 0.\end{aligned} \tag{6.72}$$

Take a sequence $z_\varepsilon \in GSBV^\psi(B_{\lambda\varepsilon}(x_0), \mathbb{R}^m)$ with $z_\varepsilon = u$ in a neighborhood of $\partial B_{\lambda\varepsilon}(x_0)$ and such that

$$\mathcal{F}(z_\varepsilon, B_{\lambda\varepsilon}(x_0)) \le \mathbf{m}_{\mathcal{F}}(u, B_{\lambda\varepsilon}(x_0)) + \omega_{d-1}\varepsilon^d. \tag{6.73}$$

Again, we can extend z_ε to a function in $GSBV^\psi(B_\varepsilon(x_0), \mathbb{R}^m)$ by setting $z_\varepsilon = u$ outside $B_{\lambda\varepsilon}(x_0)$. We apply Lemma 6.1 with u and v replaced by z_ε and $\overline{u}^{\rm surf}_{x_0}$ respectively, with the sets D'_{ε,x_0}, D''_{ε,x_0} and E_{ε,x_0} defined as in (6.25). We thus find $w_\varepsilon \in GSBV^\psi(B_\varepsilon(x_0), \mathbb{R}^m)$ such that $w_\varepsilon = \overline{u}^{\rm surf}_{x_0}$ in a neighborhood of $\partial B_\varepsilon(x_0)$ and

$$\begin{aligned}\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le &(1+\eta)\left(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(\overline{u}^{\rm surf}_{x_0}, C_{\varepsilon,\theta}(x_0))\right) + \eta\mathcal{L}^d(B_\varepsilon(x_0)) \\ &+ M \int_{B_{(1-\theta)\varepsilon}(x_0)\setminus B_{(1-2\theta)\varepsilon}(x_0)} \psi\left(x, \frac{|z_\varepsilon - \overline{u}^{\rm surf}_{x_0}|}{\varepsilon}\right) dx.\end{aligned}$$

By our initial choice of λ then, $z_\varepsilon = u_\varepsilon$ outside of $B_{(1-3\theta)\varepsilon}(x_0)$. Therefore, using (6.40)(iv), there exists a non-negative infinitesimal sequence $(\rho_\varepsilon)_\varepsilon$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)\left(\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) + \mathcal{F}(\overline{u}^{\rm surf}_{x_0}, C_{\varepsilon,\theta}(x_0))\right) + \rho_\varepsilon\varepsilon^{d-1} + \eta\omega_d\varepsilon^d. \tag{6.74}$$

Now what is left is to estimate the terms in (6.74). Recalling that by our choice of λ we have $z_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0) \setminus B_{\lambda\varepsilon}(x_0)$ and using (H1), (H4) and (6.73) we deduce

$$\begin{aligned}\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0)) \le &\mathbf{m}_{\mathcal{F}}(u, B_{\lambda\varepsilon}(x_0)) + \omega_{d-1}\varepsilon^d + \mathcal{F}(u_\varepsilon, C_{\varepsilon,\theta}(x_0)) \\ &+ b\mathcal{H}^{d-1}\left((\{u_\varepsilon \neq u\} \cup J_{u_\varepsilon} \cup J_u) \cap \partial B_{\lambda\varepsilon}(x_0)\right).\end{aligned} \tag{6.75}$$

Since $\lambda \le 1-3\theta$, using the estimate in (6.70) and (6.72) we get

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(z_\varepsilon, B_{(1-\theta)\varepsilon}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_{\lambda\varepsilon}(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + b(1-(1-4\theta)^{d-1}) \\ &\le (1-3\theta)^{d-1} \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + b(1-(1-4\theta)^{d-1}).\end{aligned} \tag{6.76}$$

Thus, collecting the estimates (6.67), (6.74) and (6.76) we obtain

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \le (1+\eta)\left((1-3\theta)^{d-1} \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} + 2b(1-(1-4\theta)^{d-1})\right).$$

Finally, since $w_\varepsilon = \overline{u}^{\rm surf}_{x_0}$ in a neighborhood of $\partial B_\varepsilon(x_0)$ and letting $\eta \to 0$ and $\theta \to 0$ we infer that

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\overline{u}^{\rm surf}_{x_0}, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} &\leq \limsup_{\varepsilon\to 0^+} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \leq \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}} \\ &= \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\omega_{d-1}\varepsilon^{d-1}}.\end{aligned}$$

This gives (6.71). □

7. Lower Semicontinuity

Recalling the assumptions on $\psi \in \Phi_w(\Omega)$ required by Theorem 3.3, and Remark 4.31, throughout this section we will assume wiht no loss of generality that ψ is a function in $\Phi_s(\Omega)$ satisfying (A0), (Inc), (Dec) and (3.6) on Ω. Before proving the lower semicontinuity, we need some preliminary results. We start with a truncation Lemma which is a generalization of [20, Lemma 4.1] where the authors deal with the case $\psi(x,t) = t^p$ for $p \in (1,\infty)$. The adaptation of the proof of [20, Lemma 4.1] to our setting requires only some minor changes and it is hinged on the fact that $\psi(x,\cdot)$ is strictly increasing for every $x \in \Omega$.

Lemma 7.1. *Let $\mathcal{G}$ be as in (3.4) where $f\colon \Omega \times \mathbb{R}^{m\times d} \to [0,+\infty)$ satisfies (f1)–(f2) and $g\colon \Omega \times \mathbb{R}^m_0 \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfies (g1)–(g2). Let $\eta, \lambda > 0$. There exists $\mu > \lambda$ depending on η, λ, a, b such that the following holds: for every $A \in \mathcal{A}(\Omega)$ and every $u \in L^0(\mathbb{R}^d, \mathbb{R}^m)$ such that $u|_A \in GSBV^\psi(A, \mathbb{R}^m)$, there exists $\tilde{u} \in L^\infty(\mathbb{R}^d, \mathbb{R}^m)$ such that $\tilde{u}|_A \in SBV^\psi(A, \mathbb{R}^m)$ and*

(i) *$|\tilde{u}| \leq \mu$ on $\mathbb{R}^d$;*
(ii) *$\tilde{u} = u$ $\mathcal{L}^d$-a.e. in $\{|u| \leq \lambda\}$ and $J_{\tilde{u}} \subseteq J_u$;*
(iii) *$\mathcal{G}(\tilde{u}, A) \leq (1+\eta)\mathcal{G}(u, A) + b\mathcal{L}^d(A \cap \{|u| \geq \lambda\})$.*

The next lemma concerns the approximation of SBV^ψ functions with Lipschitz functions in the unit ball.

Lemma 7.2 (Lusin approximation in SBV^ψ)**.** *For every $u \in SBV^\psi(B_1, \mathbb{R}^m)$ and every $\lambda > 0$ there exists a Lipschitz function $v : B_1 \to \mathbb{R}^m$ satisfying* $\mathrm{Lip}(v) \leq \tau\lambda$ *with $\tau = \tau(d,m)$, such that $v = u$ in $\{M(Du) \leq \lambda\}$ and*

$$\mathcal{L}^d(A \cap \{M(|Du|) > \lambda\}) \leq \frac{\tau}{\lambda}\int_{J_u} |u^+ - u^-|\, d\mathcal{H}^{d-1} + \frac{1}{\psi^-_A(\lambda)}\int_{A\cap\{M(|\nabla u|)>\lambda\}} \psi(x, M(|\nabla u|))\, dx, \quad (7.1)$$

for any $A \in \mathcal{B}(B_1)$, where M is the restricted maximal function to B_1 (see Definition 4.34).

Proof. The proof of this fact is no different than the standard Lusin approximation on the whole space $\mathbb{R}^d$ for SBV^p and builds upon the fact that

$$\inf_{x,x'\in B_1,\ \rho=|x-x'|} \frac{\mathcal{L}^d(B_\rho(x) \cap B_\rho(x') \cap B_1)}{\rho^d} > 0,$$

see e.g. [10, Theorem 5.34 and Theorem 5.36]. □

We are finally ready to prove the lower semicontinuity. In our proof we will combine some arguments contained in [8] with some ideas from [58] about the approximation of the integrand from below.

Proposition 7.3. *Let $\psi \in \Phi_s(B_1)$ satisfy (A0), (Inc), (Dec) on B_1 and property (3.6) in $x_0 = 0$. Consider $\{\varepsilon_k\}_k$ an infinitesimal sequence and set $\psi_k : B_1 \times [0,+\infty) \to [0,+\infty)$ as*

$$\psi_k(x,t) := \psi(\varepsilon_k x, t).$$

Let $a, b > 0$ and $\{f_k\}_k$ be a sequence of Carathéodory functions such that for each $k \geq 1$

$$a\psi_k(x,|\xi|) \leq f_k(x,\xi) \leq b(1+\psi_k(x,|\xi|)) \quad (x,\xi) \in B_1 \times \mathbb{R}^{m\times d}. \quad (7.2)$$

Assume also that there exists a quasi-convex function f such that

$$\lim_{k\to+\infty} f_k(x,\xi) = f(\xi) \quad (7.3)$$

locally uniformly in $\mathbb{R}^{m\times d}$ *for a.e.* $x \in B_1$. *Then,*

$$\liminf_{k\to+\infty} \int_{B_1} f_k(x, \nabla u_k)\, dx \geq \int_{B_1} f(\nabla u)\, dx \tag{7.4}$$

for any sequence $u_k \in GSBV(B_1, \mathbb{R}^m)$ *converging in measure to a linear function* $u\colon B_1 \to \mathbb{R}^m$ *and satisfying* $\mathcal{H}^{d-1}(J_{u_k} \cap B_1) \to 0$ *as* $k \to +\infty$.

Proof. Assume that ψ satisfies $(\mathrm{Inc})_\gamma$ and $(\mathrm{Dec})_q$ with $\gamma > 1$ and $q \in (1, \infty)$. Taking into account Remark 4.23, we have that for every $k \geq 1$, $\psi_k \in \Phi_s(B_1)$ satisfies properties (A0) with $\sigma = \sigma(\psi) \geq 1$, $(\mathrm{Inc})_\gamma$ and $(\mathrm{Dec})_q$ on B_1.

Assuming the liminf in (7.4) to be finite, from (7.2) we deduce that

$$\sup_{k\geq 1} \int_{B_1} \psi_k(x, |\nabla u_k(x)|)\, dx < +\infty.$$

Hence, $u_k \in GSBV^{\psi_k}(B_1, \mathbb{R}^m)$.

We first use the truncation argument in Lemma 7.1 applied to u_k with $\eta > 0$ small and $\theta > 1$ large enough, in order to find $\{v_k\}_k \subset SBV^{\psi_k}(B_1, \mathbb{R}^m)$ and $\mu > \theta$, such that $v_k \to u$ in measure, $\|v_k\|_{L^\infty} \leq \mu$ and, for every $k \geq 1$,

$$\int_{B_1} f_k(x, \nabla v_k(x))\, dx \leq (1+\eta) \int_{B_1} f_k(x, \nabla u_k(x)) + b\mathcal{L}^d(\{|u_k| \geq \theta\}). \tag{7.5}$$

Notice that for θ large enough $\mathcal{L}^d(\{|u_k| \geq \theta\}) \to 0$ as $k \to +\infty$, since u_k converges to a linear bounded function in measure. Therefore, if we prove the liminf inequality with v_k in place of u_k, the thesis follows from (7.5) sending $\eta \to 0$. From now on $C > 0$ will indicate a positive constant depending only on ψ. Taking into account (7.2) and (7.5), we have

$$\sup_{k\geq 1} \int_{B_1} \psi_k(x, |\nabla v_k(x)|)\, dx < +\infty. \tag{7.6}$$

Notice also that $|D^s v_k|(B_1) \leq 2\mu\mathcal{H}^{d-1}(J_{v_k}) \leq 2\mu\mathcal{H}^{d-1}(J_{u_k}) \to 0$ as $k \to +\infty$ by (i) and (ii) of Lemma 7.1. Moreover, by (ii) of Lemma 7.1, we have that for $\theta > 0$ large enough, $v_k \to u$ in measure on B_1.

Let $\psi_k^-(t) := \inf_{B_1} \psi_k(\cdot, t)$ and $\psi_k^+(t) := \sup_{B_1} \psi_k(\cdot, t)$. By Remark 4.23 and Corollary 4.36 applied to ψ_k^-, given a function $w \in L^{\psi_k}(B_1, \mathbb{R}^m)$, we have that there exists $C = C(d, \sigma, \gamma, q) > 0$ such that for every $k \geq 1$

$$\int_{B_1} \psi_k^-\,(M(w)(x))\, dx \leq C \left(\int_{B_1} \psi_k^-\,(|w(x)|)\, dx + 1 \right)^q. \tag{7.7}$$

Thus, by (7.6) and (7.7) we get that the sequence $\{\psi_k^-(M(|\nabla v_k|)(x))\}_k$ is bounded in $L^1(B_1)$. Hence, by Chacon biting lemma, we can find a sequence of sets $E_h \in \mathcal{B}(B_1)$ such that $\mathcal{L}^d(E_h) \to 0$ as $h \to +\infty$ and $\{\psi_k^-(M(|\nabla v_k|))\chi_{B_1\setminus E_h}\}_k$ is equiintegrable for every $h \geq 1$. Let

$$\Lambda_h(s) := \sup\left\{ \limsup_{k\to+\infty} \int_F \psi_k^-(M(|\nabla v_k|)(x))\, dx : \ F \in \mathcal{B}(B_1),\ F \subset B_1 \setminus E_h,\ \mathcal{L}^d(F) \leq s \right\}.$$

Due to equiintegrability, we deduce that $\Lambda_h(s) \to 0$ as $s \searrow 0$ for every $h \geq 1$.

Let $\lambda > 0$. Thanks to Lemma 7.2 applied to v_k, we find functions $u_k^\lambda : B_1 \to \mathbb{R}^m$ and $\mathcal{L}^d$-measurable sets E_k^λ such that

$$\mathrm{Lip}(u_k^\lambda) \leq \tau\lambda \quad \text{and} \quad u_k^\lambda = v_k \text{ in } B_1 \setminus E_k^\lambda, \tag{7.8}$$

where $\tau = \tau(d, m)$. Using (7.1) we deduce that for every $\lambda > 0$ large enough and every $A \in \mathcal{B}(B_1)$ it holds

$$\mathcal{L}^d(E_k^\lambda \cap A) \leq \frac{\tau}{\lambda} 2\mu\mathcal{H}^{d-1}(J_{v_k}) + \frac{1}{\psi_k^-(\lambda)} \int_{\{M(|\nabla v_k|)>\lambda\}\cap A} \psi_k^-(M(|\nabla v_k|))\, dx. \tag{7.9}$$

Since $|v_k| \leq \mu$ on $\mathbb{R}^d$ for every $k \geq 1$ and Lemma 7.2 uses Kirszbraun extension for Lipschitz functions, we can assume without loss of generality that $\|u_k^\lambda\|_{L^\infty} \leq m\mu$ for every $k \geq 1$ and every $\lambda > 0$.

Observe that for every $k \geq 1$ and every $\lambda > 0$ large enough, using Chebychev inequality, (7.6) and (7.7), we get

$$\mathcal{L}^d(\{M(|\nabla v_k|) > \lambda\}) \leq \frac{C}{\lambda^\gamma} \int_{B_1} \psi_k^-(M(|\nabla v_k|)(x))\, dx \leq \frac{C}{\lambda^\gamma},$$

where in the first inequality we have also used that ψ satisfies (A0) and $(\mathrm{Inc})_\gamma$ on B_1 with $\gamma > 1$. Hence, keeping in mind the fact that $|D^s v_k|(B_1) \to 0$ as $k \to +\infty$, taking $A = \Omega \setminus E_h$ in (7.9), we obtain

$$\limsup_{k\to+\infty} \left(\psi_k^-(\lambda)\ \mathcal{L}^d(E_k^\lambda \setminus E_h)\right) \leq \Lambda_h \left(\frac{C}{\lambda^\gamma}\right), \tag{7.10}$$

for every $\lambda > 0$ large enough and every $h \geq 1$.

For every fixed $\lambda > 0$ large enough, the sequence $\{u_k^\lambda\}_k$ is equibounded and equicontinuous. Therefore, it converges uniformly in $\overline{B}_1$ as $k \to +\infty$ to a function $u_\lambda \in C(B_1, \mathbb{R}^m)$. Moreover, by the lower semicontinuity under convergence in measure of the map

$$w \mapsto \mathcal{L}^d(\{x \in B_1 \setminus E_h :\ w(x) \neq 0\}),$$

for a fixed $h \geq 1$, using (7.10) we have that for every $\lambda > 0$ large enough

$$\lambda^\gamma \mathcal{L}^d(\{x \in B_1 \setminus E_h :\ u_\lambda(x) \neq u(x)\}) \leq \limsup_{k\to+\infty} \lambda^\gamma \mathcal{L}^d(\{x \in B_1 \setminus E_h :\ u_k^\lambda(x) \neq v_k(x)\}| \leq \Lambda_h \left(\frac{C}{\lambda^\gamma}\right).$$

Hence, if we set $L_\lambda := \{x \in B_1 : u_\lambda(x) \neq u(x)\}$, we have $\mathcal{L}^d(L_\lambda \setminus E_h) \to 0$ as $\lambda \to +\infty$ for every $h \geq 1$.

We are finally in a position to conclude. Since $\mathcal{L}^d(E_h) \to 0$ as $h \to +\infty$, we need to prove that for every $h \geq 1$

$$\liminf_{k\to+\infty} \int_{B_1} f_k(x, \nabla v_k)\, dx \geq \mathcal{L}^d(B_1 \setminus E_h) f(\nabla u). \tag{7.11}$$

We now fix $\zeta > \tau\lambda > \sigma$. Let $g \in C^\infty([0,+\infty); [0,1])$ such that $g(s) = 1$ for $s \leq \zeta$ and $g(s) = 0$ for $s \geq 2\zeta$. We define for every $k \geq 1$ the Carathéodory function

$$f_k^\zeta(x,\xi) := g(|\xi|) f_k(x,\xi) + a(1 - g(|\xi|))\psi_k^-(|\xi|),$$

where $a > 0$ is the constant appearing in (7.2). Notice that in view of (7.2) and property (3.6) of ψ in 0, there exists $k_0 \geq 1$ such that for every $k \geq k_0$ it holds

$$a\psi_k^-(|\xi|) \leq f_k^\zeta(x,\xi) \leq 2\mathfrak{C}b(1 + \psi_k^-(|\xi|)), \quad (x,\xi) \in B_1 \times \mathbb{R}^{m\times d}, \tag{7.12}$$

where $\mathfrak{C}$ does not depend on ζ nor on k. Using (7.8) and (7.12), for every $k \geq k_0$ we have

$$\begin{aligned}
\int_{B_1} f_k^\zeta(x, \nabla v_k)\, dx &\geq \int_{B_1 \setminus (E_h \cup E_k^\lambda)} f_k^\zeta(x, \nabla v_k)\, dx \\
&= \int_{B_1 \setminus (E_h \cup E_k^\lambda)} f_k^\zeta(x, \nabla u_k^\lambda)\, dx \\
&\geq \int_{B_1 \setminus E_h} f_k^\zeta(x, \nabla u_k^\lambda)\, dx - \int_{E_k^\lambda \setminus E_h} 2\mathfrak{C}b(1 + \psi_k^-(\lambda))\, dx.
\end{aligned} \tag{7.13}$$

By (7.13), the fact that $\zeta > \tau\lambda$, the locally uniform convergence of f_k to f for $\mathcal{L}^d$-a.e. $x \in B_1$ and the quasiconvexity of f, we deduce

$$\begin{aligned}
\liminf_{k\to+\infty} \int_{B_1} f_k(x, \nabla v_k)\, dx &\geq \liminf_{k\to+\infty} \int_{B_1} f_k^\zeta(x, \nabla v_k)\, dx \\
&\geq \liminf_{k\to+\infty} \int_{B_1 \setminus E_h} f(\nabla u_k^\lambda) + \left(f_k^\zeta(x, \nabla u_k^\lambda) - f(\nabla u_k^\lambda)\right)\, dx \\
&\qquad - \int_{E_k^\lambda \setminus E_h} 2\mathfrak{C}b(1 + \psi_k^-(\lambda))\, dx \\
&\geq \liminf_{k\to+\infty} \int_{B_1 \setminus E_h} f(\nabla u_k^\lambda)\, dx - 4\mathfrak{C}b \int_{E_k^\lambda \setminus E_h} \psi_k^-(\lambda)\, dx, \\
&\geq \int_{B_1 \setminus E_h} f(\nabla u_\lambda)\, dx - 4\mathfrak{C}b \limsup_{k\to+\infty} \left(\psi_k^-(\lambda)\ \mathcal{L}^d(E_k^\lambda \setminus E_h)\right).
\end{aligned}$$

Therefore, using (7.10), we have that for every $\lambda > 0$ large enough it holds

$$\liminf_{k\to+\infty} \int_{B_1} f_k(x, \nabla v_k)\, dx \geq \int_{B_1 \setminus E_h} f(\nabla u_\lambda)\, dx - 4\mathfrak{C}b\Lambda_h \left(\frac{C}{\lambda^\gamma}\right). \tag{7.14}$$

For the first term in the right hand side of (7.14) we estimate

$$\int_{B_1\setminus E_h} f(\nabla u_\lambda)\,dx \geq \int_{B_1\setminus(E_h\cup L_\lambda)} f(\nabla u)\,dx \geq (\mathcal{L}^d(B_1\setminus E_h)-\mathcal{L}^d(L_\lambda\setminus E_h))f(\nabla u). \tag{7.15}$$

Thus, for every $h\geq 1$ fixed, sending $\lambda\to+\infty$ from (7.14) and (7.15) we conclude (7.11). □

We now prove the lower semicontinuity for the bulk energy.

Proposition 7.4. *Let $f\colon\Omega\times\mathbb{R}^{m\times d}\to[0,+\infty)$ be a Borel measurable function satisfying (f1)–(f2) with $a,b>0$ and such that $z\mapsto f(x,z)$ is quasiconvex in $\mathbb{R}^{m\times d}$ for every $x\in\Omega$. Let $\psi\in\Phi_s(\Omega)$ satisfy (A0), (Inc), (Dec) and property (3.6) for $\mathcal{L}^d$-a.e $x_0\in\Omega$. Given $A\in\mathcal{A}(\Omega)$,*

$$\liminf_{k\to+\infty}\int_A f(x,\nabla u_k)\,dx\geq\int_A f(x,\nabla u)\,dx$$

for every sequence $\{u_k\}_k\subset GSBV^\psi(A,\mathbb{R}^m)$ converging to a function $u\in GSBV^\psi(A,\mathbb{R}^m)$ in measure and such that $\sup_k\mathcal{H}^{d-1}(J_{u_k}\cap A)<+\infty$.

Proof. The proof is standard and we only sketch it. Possibly extracting a subsequence we can assume that u_k converges to u $\mathcal{L}^d$-a.e. in A and that $\mathcal{H}^{d-1}(J_{u_k})$ and $f(x,\nabla u_k(x))\mathcal{L}^d$ weakly* converge in A to Radon measures μ and λ respectively. By Besicovitch derivation theorem, in order to prove the statement, we need to show that for a.e. $x_0\in A$ it holds

$$\limsup_{\varepsilon\searrow 0}\frac{\lambda(B_\varepsilon(x_0))}{\varepsilon^d}\geq\omega_d f(x_0,\nabla u(x_0)). \tag{7.16}$$

Take $x_0\in\Omega$ such that property (3.6) holds and let $\{\varepsilon_k\}_k$ be such that $\varepsilon_k\searrow 0$ and $\lambda(\partial B_{\varepsilon_k}(x_0))=0$ for every $k\geq 1$. Take ε_1 suitably small so that $B_{\varepsilon_1}(x_0)\Subset A$ and define $\psi_1: B_1\times[0,+\infty)\to[0,+\infty)$ as

$$\psi_1(x,t):=\psi\left(\varepsilon_1 x+x_0,t\right)$$

for every $x\in B_1$ and every $t\geq 0$. By Remark 4.23 we have that $\psi_1\in\Phi_s(B_1)$ satisfies the same properties of ψ with the same constants in B_1 and property (3.6) in $0\in B_1$.

Define $f_k(x,\xi):=f(x_0+\varepsilon_k x,\xi)$ for any $x\in B_1$. We have that

$$\lim_{k\to+\infty}f_k(x,\xi)=f(x_0,\xi)$$

locally uniformly in $\mathbb{R}^{m\times d}$ for $\mathcal{L}^d$-a.e. $x\in B_1$. We can also find a sequence $\{w_k\}_k\subset GSBV(B_1,\mathbb{R}^m)$ such that w_k converges in measure to the map $y\mapsto\nabla u(x_0)y$ in B_1, $\mathcal{H}^{d-1}(J_{w_k})\to 0$ as $k\to+\infty$ and

$$\int_{B_1}f_k(x,\nabla w_k)\,dx\leq\frac{\lambda(B_{\varepsilon_k}(x_0))}{\varepsilon_k^d}+\varepsilon_k. \tag{7.17}$$

Since $a\psi(x_0+\varepsilon_k x,|\xi|)\leq f(x_0+\varepsilon_k x,\xi)\leq b(1+\psi(x_0+\varepsilon_k x,|\xi|))$ by (f2), using Proposition 7.3 with ψ_1, f_k and w_k, we deduce that

$$\liminf_{k\to+\infty}\int_{B_1}f_k(x,\nabla w_k)\,dx\geq\omega_d f(x_0,\nabla u(x_0)).$$

This together with (7.17) gives (7.16). □

We conclude with the surface part. The following result is contained in [7, Theorem 3.3].

Theorem 7.5. *Let $g\colon\Omega\times\mathbb{R}^m_0\times\mathbb{S}^{d-1}\to[0,+\infty)$ satisfy (g1)–(g2) and such that $(\zeta,\nu)\to g(x,\zeta,\nu)$ is BV-elliptic for every $x\in\Omega$. Let $A\in\mathcal{A}(\Omega)$. Then, for every $u\in GSBV^p(A,\mathbb{R}^m)$ and any sequence $u_k\in GSBV^p(A,\mathbb{R}^m)$ converging in measure to u in A and such that*

$$\sup_k\int_A|\nabla u_k|^p\,dx<+\infty \tag{7.18}$$

for some $p>1$, the following inequality holds

$$\int_{J_u\cap A}g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}\leq\liminf_{k\to+\infty}\int_{J_{u_k}\cap A}g(x,[u_k],\nu_{u_k})\,d\mathcal{H}^{d-1}.$$

Using Proposition 7.4 and Theorem 7.5 we are now ready to conclude the proof of Theorem 3.3.

Proof of Theorem 3.3. Thanks to the assumptions on ψ, can apply Proposition 7.4 and Theorem 7.5 once we notice that $\sup_k \int_A \psi(x,|\nabla u_k|)\,dx < +\infty$ and that ψ satisfies $(\mathrm{Inc})_\gamma$ with $\gamma > 1$. Therefore, condition (7.18) holds. $\square$

8. Relaxation

Using the integral representation and the lower semicontinuity result for quasiconvex and BV-elliptic integrands, we can now prove the relaxation Theorem 3.4.

Proof of Theorem 3.4. Using Theorem 3.3 and taking the infimum over all the sequences, for every function $u \in GSBV^\psi(A,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ we get

$$\int_A \mathcal{Q}f(x,\nabla u(x))\,dx + \int_{J_u\cap A} \mathcal{R}g(x,[u](x),\nu_u(x))\,d\mathcal{H}^{d-1} \le \overline{\mathcal{G}}(u,A). \tag{8.1}$$

We now want to show that $\overline{\mathcal{G}}$ satisfies (H1)–(H5). By definition of $\overline{\mathcal{G}}$ we deduce immediately that it satisfies (H2), (H3) and (H5). The upper bound of (H4) is obvious since we have by definition $\overline{\mathcal{G}}(u,A) \le \mathcal{G}(u,A)$ for every $u \in GSBV^\psi(\Omega,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$. On the other hand, the lower bound of (H4) for $\overline{\mathcal{G}}$ can be deduced observing that by Ioffe's Theorem the functional

$$u \mapsto \int_A \psi(x,|\nabla u(x)|)\,dx + \mathcal{H}^{d-1}(J_u \cap A)$$

is lower semicontinuous with respect to the convergence in measure on A. Finally, the proof of (H1) for $\overline{\mathcal{G}}$ is standard and builds upon the fundamental estimate (see e.g. [17] proof of Theorem 10), we omit it.

Hence, by Corollary 3.2 there exist two Borel functions $\overline{f}\colon \Omega\times\mathbb{R}^{m\times d} \to [0,+\infty)$ and $\overline{g}\colon \Omega\times\mathbb{R}^m_0\times\mathbb{S}^{d-1} \to [0,+\infty)$ such that, for every $u \in GSBV^\psi(A,\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$,

$$\overline{\mathcal{G}}(u,A) = \int_A \overline{f}(x,\nabla u(x))\,dx + \int_{J_u\cap A} \overline{g}(x,[u](x),\nu_u(x))\,d\mathcal{H}^{d-1}.$$

We now proceed as in [17, Proof of Theorem 4] and show that

$$\overline{f}(x,\xi) \le \mathcal{Q}f(x,\xi) \quad \text{for } \mathcal{L}^d\text{ -a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^{m\times d}. \tag{8.2}$$

Let $\eta > 0$, since f is a Carathéodory function, by the Scorza-Dragoni theorem there exists a compact set $K \subset \Omega$ with $\mathcal{L}^d(\Omega\setminus K) \le \eta$, such that the function

$$f\colon K \times \mathbb{R}^{m\times d} \to [0,+\infty)$$

is continuous. Let K^1 be the set of points with density one for K. Fix $(x,\xi) \in K^1 \times \mathbb{R}^{m\times d}$ and let $\varphi \in C_0^\infty(B,\mathbb{R}^m)$ be such that

$$\mathcal{Q}f(x,\xi) + \eta \ge \int_B f(x,\xi+\nabla\varphi(y))\,dy. \tag{8.3}$$

For any $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$, set $v_\varepsilon(y) := \varepsilon\varphi((y-x)/\varepsilon)$. Then, $v_\varepsilon \in W_0^{1,\infty}(B_\varepsilon(x),\mathbb{R}^m)$ and, by definition of $\mathbf{m}_{\mathcal{G}}$ and $\overline{f}$ (see (3.1) and (3.2)),

$$\overline{f}(x,\xi) \le \limsup_{\varepsilon\searrow 0} \frac{\mathbf{m}_{\mathcal{G}}(\xi(\cdot - x) + v_\varepsilon, B_\varepsilon(x))}{\varepsilon^d} \le \limsup_{\varepsilon\searrow 0} \frac{1}{\varepsilon^d}\int_{B_\varepsilon(x)} f(y,\xi+\nabla v_\varepsilon(y))\,dy. \tag{8.4}$$

Using the modulus of uniform continuity for f we get that for ε small enough

$$|f(x,\xi+\nabla v_\varepsilon(y)) - f(y,\xi+\nabla v_\varepsilon(y))| \le \eta \tag{8.5}$$

for every $y \in B_\varepsilon(x)$. Thus, by (8.3), (8.4), (8.5) and recalling properties (A0) and (Dec) of ψ together with (f2),

$$\begin{aligned}
\overline{f}(x,\xi) &\le \limsup_{\varepsilon\searrow 0}\left(\frac{1}{\varepsilon^d}\int_{B_\varepsilon(x)\cap K} f(y,\xi+\nabla v_\varepsilon(y))\,dy + \frac{1}{\varepsilon^d}\int_{B_\varepsilon(x)\setminus K} b(1+\psi(x,|\xi|+\|\nabla\varphi\|_{L^\infty}))\,dy\right)\\
&\le \limsup_{\varepsilon\searrow 0}\frac{1}{\varepsilon^d}\int_{B_\varepsilon(x)\cap K} f(x,\xi+\nabla v_\varepsilon(y))\,dy + \eta\\
&\le \int_B f(x,\xi+\nabla\varphi(y))\,dy + \eta \le \mathcal{Q}f(x,\xi) + 2\eta,
\end{aligned}$$

where we have used also that $x \in K^1$. Letting $\eta \to 0$, gives (8.2).

We now deal the surface part, showing that, for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$ and every $(\zeta, \nu) \in \mathbb{R}^m_0 \times \mathbb{S}^{d-1}$,

$$\overline{g}(x, \zeta, \nu) \leq \mathcal{R}g(x, \zeta, \nu). \tag{8.6}$$

Take $\eta > 0$ and fix $(x, \zeta, \nu) \in \Omega \times \mathbb{R}^m_0 \times \mathbb{S}^{d-1}$. Since g satisfies (g1)–(g4), we can find a function $w \in SBV^\psi(Q_\nu, \mathbb{R}^m) \cap L^\infty(B, \mathbb{R}^m)$ such that $\nabla w = 0$ a.e. in B, $w = u_{x,\zeta,0,\nu}$ in a neighborhood of ∂B and

$$\mathcal{R}g(x, \zeta, \nu) + \eta \geq \int_B g(x, [w](y), \nu_w(y))\, d\mathcal{H}^{d-1}(y). \tag{8.7}$$

We recall that

$$u_{x,\zeta,0,\nu}(y) = \begin{cases} \zeta & \text{if } (y - x) \cdot \nu > 0, \\ 0 & \text{if } (y - x) \cdot \nu \leq 0. \end{cases}$$

For any $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$, set $v_\varepsilon(y) := w((y - x)/\varepsilon)$. Then, by definition of $\mathbf{m}_\mathcal{G}$ and $\overline{g}$ (see (3.1) and (3.3)),

$$\overline{g}(x, \zeta, \nu) \leq \limsup_{\varepsilon \searrow 0} \frac{\mathbf{m}_\mathcal{G}(v_\varepsilon, B_\varepsilon(x))}{\varepsilon^{d-1}} \leq \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x) \cap J_{v_\varepsilon}} g(y, [v_\varepsilon](y), \nu_{v_\varepsilon}(y))\, d\mathcal{H}^{d-1}(y). \tag{8.8}$$

Using the modulus of uniform continuity of g we get that for ε small enough, for every $y \in B_\varepsilon(x)$ it holds

$$|g(x, [v_\varepsilon](y), \nu_{v_\varepsilon}(y)) - g(y, [v_\varepsilon](y), \nu_{v_\varepsilon}(y))| \leq \eta. \tag{8.9}$$

Combining (8.7), (8.8) and (8.9) we get

$$\begin{aligned} \overline{g}(x, \zeta, \nu) &\leq \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x) \cap J_{v_\varepsilon}} g(y, [v_\varepsilon](y), \nu_{v_\varepsilon}(y))\, d\mathcal{H}^{d-1}(y) \\ &\leq \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{d-1}} \int_{B_\varepsilon(x) \cap J_{v_\varepsilon}} g(x, [v_\varepsilon](y), \nu_{v_\varepsilon}(y))\, d\mathcal{H}^{d-1}(y) + \eta \\ &= \int_{B \cap J_w} g(x, [w](y), \nu_w(y))\, d\mathcal{H}^{d-1}(y) + \eta \leq \mathcal{R}g(x, \zeta, \nu) + 2\eta. \end{aligned}$$

Letting $\eta \to 0$ gives (8.6). □

Acknowledgments

The work of S. Almi was funded by the FWF Austrian Science Fund through the Projects ESP-61 and P35359-N.
The work of F. Solombrino is part of the MIUR - PRIN 2017, project Variational Methods for Stationary and Evolution Problems with Singularities and Interfaces, No. 2017BTM7SN. He also acknowledges support by project Starplus 2020 Unina Linea 1 "New challenges in the variational modeling of continuum mechanics" from the University of Naples Federico II and Compagnia di San Paolo, and by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA-INdAM).

Data Availability statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflict-of-interest statement

The authors declare no conflict of interest.

References

[1] E. Acerbi, G. Bouchitté, and I. Fonseca, *Relaxation of convex functionals: the gap problem*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 20 (2003), pp. 359–390.

[2] E. Acerbi and G. Mingione, *Regularity results for a class of functionals with non-standard growth*, Arch. Rational Mech. Anal., 156 (2001), pp. 121–140.

[3] E. Acerbi and G. Mingione, *Regularity results for a class of quasiconvex functionals with nonstandard growth*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 30 (2001), pp. 311–339.

[4] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Rational Mech. Anal., 164 (2002), pp. 213–259.

[5] A. Alvino, G. Trombetti, and P.-L. Lions, *On optimization problems with prescribed rearrangements*, Nonlinear Anal., 13 (1989), pp. 185–220.

[6] L. Ambrosio, *A compactness theorem for a new class of functions of bounded variation*, Boll. Un. Mat. Ital. B (7), 3 (1989), pp. 857–881.

[7] ———, *Existence theory for a new class of variational problems*, Arch. Rational Mech. Anal., 111 (1990), pp. 291–322.

[8] ———, *On the lower semicontinuity of quasiconvex integrals in* $\mathrm{SBV}(\Omega, \mathbf{R}^k)$, Nonlinear Anal., 23 (1994), pp. 405–425.

[9] ———, *A new proof of the SBV compactness theorem*, Calc. Var. Partial Differential Equations, 3 (1995), pp. 127–137.

[10] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[11] P. Baroni, M. Colombo, and G. Mingione, *Nonautonomous functionals, borderline cases and related function classes*, Algebra i Analiz, 27 (2015), pp. 6–50.

[12] P. Baroni, M. Colombo, and G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations, 57 (2018), pp. Paper No. 62, 48.

[13] A. C. Barroso and E. Zappale, *Relaxation for optimal design problems with non-standard growth*, Appl. Math. Optim., 80 (2019), pp. 515–546.

[14] ———, *An optimal design problem with non-standard growth and no concentration effects*, Asymptot. Anal., 128 (2022), pp. 385–412.

[15] V. Bögelein, F. Duzaar, and P. Marcellini, *Parabolic equations with* p, q*-growth*, J. Math. Pures Appl. (9), 100 (2013), pp. 535–563.

[16] ———, *Parabolic systems with* p, q*-growth: a variational approach*, Arch. Ration. Mech. Anal., 210 (2013), pp. 219–267.

[17] G. Bouchitté, I. Fonseca, G. Leoni, and L. Mascarenhas, *A global method for relaxation in* $W^{1,p}$ *and in* SBV_p, Arch. Ration. Mech. Anal., 165 (2002), pp. 187–242.

[18] G. Bouchitté, I. Fonseca, and J. Malý, *The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent*, Proc. Roy. Soc. Edinburgh Sect. A, 128 (1998), pp. 463–479.

[19] G. Bouchitté, I. Fonseca, and L. Mascarenhas, *A global method for relaxation*, Arch. Rational Mech. Anal., 145 (1998), pp. 51–98.

[20] F. Cagnetti, G. Dal Maso, L. Scardia, and C. I. Zeppieri, Γ*-convergence of free-discontinuity problems*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 36 (2019), pp. 1035–1079.

[21] M. Carozza, J. Kristensen, and A. Passarelli di Napoli, *Higher differentiability of minimizers of convex variational integrals*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 28 (2011), pp. 395–411.

[22] M. Carriero and A. Leaci, S^k*-valued maps minimizing the* L^p*-norm of the gradient with free discontinuities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 18 (1991), pp. 321–352.

[23] M. Cencelj, V. D. Rădulescu, and D. D. Repovš, *Double phase problems with variable growth*, Nonlinear Anal., 177 (2018), pp. 270–287.

[24] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., 66 (2006), pp. 1383–1406.

[25] A. Cianchi, D. E. Edmunds, and P. Gurka, *On weighted Poincaré inequalities*, Math. Nachr., 180 (1996), pp. 15–41.

[26] A. Cianchi and N. Fusco, *Gradient regularity for minimizers under general growth conditions*, J. Reine Angew. Math., 507 (1999), pp. 15–36.

[27] ———, *Functions of bounded variation and rearrangements*, Arch. Ration. Mech. Anal., 165 (2002), pp. 1–40.

[28] M. Colombo and G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal., 218 (2015), pp. 219–273.

[29] ———, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal., 215 (2015), pp. 443–496.

[30] A. Coscia and G. Mingione, *Hölder continuity of the gradient of* $p(x)$*-harmonic mappings*, C. R. Acad. Sci. Paris Sér. I Math., 328 (1999), pp. 363–368.

[31] A. Coscia and D. Mucci, *Integral representation and* Γ*-convergence of variational integrals with* $p(x)$*-growth*, ESAIM Control Optim. Calc. Var., 7 (2002), pp. 459–519.

[32] A. Crespo-Blanco, L. Gasiński, P. Harjulehto, and P. Winkert, *A new class of double phase variable exponent problems: existence and uniqueness*, J. Differential Equations, 323 (2022), pp. 182–228.

[33] G. Dal Maso, *Generalised functions of bounded deformation*, J. Eur. Math. Soc. (JEMS), 15 (2013), pp. 1943–1997.

[34] G. Dal Maso, G. A. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal., 176 (2005), pp. 165–225.

[35] V. De Cicco, C. Leone, and A. Verde, *Lower semicontinuity in SBV for integrals with variable growth*, SIAM J. Math. Anal., 42 (2010), pp. 3112–3128.

[36] C. De Filippis, *Higher integrability for constrained minimizers of integral functionals with (p,q)-growth in low dimension*, Nonlinear Anal., 170 (2018), pp. 1–20.
[37] C. De Filippis and J. Ok, *Regularity for multi-phase variational problems*, J. Differential Equations, 267 (2019), pp. 1631–1670.
[38] E. De Giorgi, M. Carriero, and A. Leaci, *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rational Mech. Anal., 108 (1989), pp. 195–218.
[39] L. Diening, D. Lengeler, B. Stroffolini, and A. Verde, *Partial regularity for minimizers of quasi-convex functionals with general growth*, SIAM J. Math. Anal., 44 (2012), pp. 3594–3616.
[40] L. Diening and S. Schwarzacher, *Global gradient estimates for the $p(\cdot)$-Laplacian*, Nonlinear Anal., 106 (2014), pp. 70–85.
[41] M. Eleuteri and A. Passarelli di Napoli, *Lipschitz regularity of minimizers of variational integrals with variable exponents*, Nonlinear Analysis: Real World Applications, 71 (2023), p. 103815.
[42] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Textbooks in Mathematics, CRC Press, Boca Raton, FL, revised ed., 2015.
[43] I. Fonseca, G. Leoni, and S. Müller, *$\mathscr{A}$-quasiconvexity: weak-star convergence and the gap*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 21 (2004), pp. 209–236.
[44] I. Fonseca and J. Malý, *Relaxation of multiple integrals below the growth exponent*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 14 (1997), pp. 309–338.
[45] F. Gmeineder and J. Kristensen, *Quasiconvex functionals of (p,q)-growth and partial regularity of relaxed minimizers*, Preprint arXiv, arxiv.org/abs/2209.01613.
[46] P. Harjulehto and P. Hästö, *Lebesgue points in variable exponent spaces*, Ann. Acad. Sci. Fenn. Math., 29 (2004), pp. 295–306.
[47] ———, *Orlicz spaces and generalized Orlicz spaces*, vol. 2236 of Lecture Notes in Mathematics, Springer, Cham, 2019.
[48] P. Harjulehto, P. Hästö, and V. Latvala, *Minimizers of the variable exponent, non-uniformly convex Dirichlet energy*, J. Math. Pures Appl. (9), 89 (2008), pp. 174–197.
[49] P. Harjulehto, P. Hästö, V. Latvala, and O. Toivanen, *Critical variable exponent functionals in image restoration*, Appl. Math. Lett., 26 (2013), pp. 56–60.
[50] P. Harjulehto and A. Karppinen, *Sharp growth conditions for boundedness of maximal function in generalized orlicz spaces; [tarkat kasvuehdot maksimaalifunktion ra joittuneisuudelle yleistetyissä orliczin avaruuksissa]*, Annales Fennici Mathematici, 47 (2022), p. 489 – 505.
[51] P. Hästö and J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. (JEMS), 24 (2022), pp. 1285–1334.
[52] P. A. Hästö, *The maximal operator on generalized Orlicz spaces*, J. Funct. Anal., 269 (2015), pp. 4038–4048.
[53] O. Kovácik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., 41 (1991), pp. 592–618.
[54] J. Kristensen, *Lower semicontinuity in Sobolev spaces below the growth exponent of the integrand*, Proc. Roy. Soc. Edinburgh Sect. A, 127 (1997), pp. 797–817.
[55] ———, *Lower semicontinuity of quasi-convex integrals in BV*, Calc. Var. Partial Differential Equations, 7 (1998), pp. 249–261.
[56] C. Leone, G. Scilla, F. Solombrino, and A. Verde, *Regularity of minimizers for free-discontinuity problems with $p(\cdot)$-growth*, ESAIM: COCV, 10.1051/cocv/2023062 (2023).
[57] C. Leone and A. Verde, *Lower semicontinuity in SBV for functionals with nearly linear growth*, J. Convex Anal., 24 (2017), pp. 459–476.
[58] P. Marcellini, *Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals*, Manuscripta Math., 51 (1985), pp. 1–28.
[59] ———, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations, 90 (1991), pp. 1–30.
[60] A. Menovschikov, A. Molchanova, and L. Scarpa, *An extended variational theory for nonlinear evolution equations via modular spaces*, SIAM J. Math. Anal., 53 (2021), pp. 4865–4907.
[61] G. Mingione and D. Mucci, *Integral functionals and the gap problem: sharp bounds for relaxation and energy concentration*, SIAM J. Math. Anal., 36 (2005), pp. 1540–1579.
[62] G. D. Nin, *Rectifiability of the jump set of locally integrable functions*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 22 (2021), p. 1233 – 1240.
[63] V. D. Rădulescu, *Nonlinear elliptic equations with variable exponent: old and new*, Nonlinear Anal., 121 (2015), pp. 336–369.
[64] M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, vol. 1748 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.
[65] G. Scilla, F. Solombrino, and B. Stroffolini, *Integral representation and Γ-convergence for free-discontinuity problems with $p(\cdot)$-growth*, Calc. Var. Partial Differential Equations, 62 (2023), pp. Paper No. 213, 55.
[66] V. V. Zhikov, *Problems of convergence, duality, and averaging for a class of functionals of the calculus of variations*, (Russian) Dokl. Akad. Nauk SSSR, 267 (1982), pp. 524–528.
[67] ———, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat., 50 (1986), pp. 675–710.
[68] ———, *On Lavrentiev's phenomenon*, Russian J. Math. Phys., 3 (1995), pp. 249–269.
[69] ———, *Meyer-type estimates for solving the nonlinear Stokes system*, Differ. Uravn., 33 (1997), pp. 107–114, 143.

[70] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994.

(Stefano Almi) Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, via Cintia, 80126 Napoli, Italy.
E-mail address: stefano.almi@unina.it

(Dario Reggiani) Scuola Superiore Meridionale, Universitá di Napoli Federico II Largo San Marcellino 10, 80138 Napoli, Italy.
E-mail address: dario.reggiani@unina.it

(Francesco Solombrino) Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, via Cintia, 80126 Napoli, Italy.
E-mail address: francesco.solombrino@unina.it