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Localization of eigenfunctions in the Dirichlet beaker

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Abstract
We construct the asymptotics of the eigenpairs of the Dirichlet problem for the Laplace operator in a thin-walled beaker and prove the localization effect for the functions near the bottom edge, a smooth closed contour, of the beaker. The main asymptotic terms are described by the eigenpairs of an ordinary differential equation on the edge and by the single eigenvalue belonging to the discrete spectrum of the Dirichlet Laplacian in an L-shaped infinite waveguide. The corresponding eigenfunctions are shown to decay exponentially at some distance from the edge. Also, we find the asymptotics of eigenvalue sequences generated by planar Dirichlet problems on the bottom and walls of the limit beaker of zero thickness. Open questions related to other sequences of eigenvalues are discussed.

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1 | INTRODUCTION

1.1 | Problem statement

Let $\omega \subset \mathbb{R}^2 \ni y = (y_1, y_2)$ be a planar domain enveloped by a simple closed C^∞-smooth contour $\partial\omega$, the length of which is reduced to 2π by rescaling. Denoting by h a small positive parameter, we define a thin beaker $\sqcup^h$ to consist of the bottom $\Omega^h = \omega \times (0, h)$ and the wall

$$\Theta^h = \{x = (y, z) : y \in \theta^h = \omega \cap \mathcal{V}^h, z \in (0, 1)\}, \tag{1.1}$$

where $\mathcal{V}^d \subset \mathbb{R}^2$ is the d-neighborhood of $\partial\omega$.

We consider the spectral Dirichlet problem

$$-\Delta_x u^h(x) = \lambda^h u^h(x), \;\; x \in \sqcup^h = \Omega^h \cup \Theta^h, \tag{1.2}$$

$$u^h(x) = 0, \;\; x \in \partial\sqcup^h, \tag{1.3}$$

and its variational formulation as the integral identity [15]

$$(\nabla_x u^h, \nabla_x \Psi^h)_{\sqcup^h} = \lambda^h (u^h, \Psi^h)_{\sqcup^h} \quad \forall \Psi^h \in H^1_0(\sqcup^h), \tag{1.4}$$

where ∇_x is the gradient, $\Delta_x = \nabla_x \cdot \nabla_x$ is the Laplace operator, $(\, ,\,)_{\sqcup^h}$ is the natural scalar product in the Lebesgue space $L^2(\sqcup^h)$, and $H^1_0(\sqcup^h)$ stands for the Sobolev space of functions satisfying the Dirichlet condition (1.3). Problem (1.4), or (1.2)–(1.3) in its differential form, has the positive monotone unbounded sequence of eigenvalues

$$0 < \lambda^h_1 < \lambda^h_2 \leqslant \lambda^h_3 \leqslant \ldots \leqslant \lambda^h_m \leqslant \ldots \to +\infty, \tag{1.5}$$

and the main goal of this paper is to the describe the behavior of them and the corresponding eigenfunctions, as $h \to 0$. We will prove that the eigenfunctions concentrate near the bottom edge $\Upsilon = \partial\omega \times \{0\}$ and decay at an exponential rate, when the distance from Υ increases. The eigenfunctions are denoted by $u^h_1, u^h_2, u^h_3, \ldots u^h_m \ldots \in H^1_0(\sqcup^h)$, and they are subject to the orthogonality and normalization conditions

$$(u^h_m, u^h_n)_{\sqcup^h} = \delta_{m,n}, \;\; m, n \in \mathbb{N} = \{1, 2, 3, \ldots\}, \tag{1.6}$$

where $\delta_{m,n}$ is the Kronecker symbol.

There exists a vast literature on the localization effects for the eigenfunctions of the Dirichlet problem in thin domains, see the review paper [12]. Let us mention papers closely related to our research. Asymptotics of the spectrum of the Dirichlet problem in thin finite and periodic infinite strips was studied in [3, 10, 11], while multidimensional thin domains were considered in [4] in the case of smooth bases and in [22] for nonsmooth ones. The asymptotics of the eigenvalues and eigenfunctions of the Dirichlet Laplacian in obtuse convex polyhedra were constructed in [18] and in a triangle with an obtuse angle in [25]. The main observation in all of these papers is the localization of the eigenfuntions in the vicinity of certain points on the boundaries, namely, near the maximum points of the curvature of the contour of longitudinal cross-sections. However, the paper [14] contains a study of the mixed boundary value problem for the Laplace operator with Dirichlet conditions on the parallel bases of a thin plate, and localization is proved near the whole Neumann lateral side with long edges. However, this type of localization occurs only in the case of complete rotational symmetry where separation of variables can be applied to reduce the spatial problem to a planar one. The latter is easily investigated with the help of the approach of [5], which deals with same type of problems but in thin cylinders with distorted ends and gives a sufficient condition for the appearance of the effect. There are many other works on the localization effects for the eigenfunctions in Dirichlet or other boundary value problems, see, for example, the reviews and literature references in [12, 21].

The present paper differs from the citations in the sense that here the phenomenon is based on the discrete spectrum of the Dirichlet Laplacian in the union $\mathbb{L}$ of two unit semi-infinite strips

$$\Pi_j = \{\xi = (\xi_1, \xi_2) : \xi_j \in (0, +\infty), \xi_{3-j} \in (0, 1)\}, \quad j = 1, 2. \tag{1.7}$$

The main issue of our asymptotic analysis, which also makes the crucial difference to earlier works, is the derivation of an ordinary differential equation on the bottom edge $\partial\omega \times \{0\} \subset \partial\sqcup^h$ of the beaker: the eigenpairs of this equation form the main asymptotic terms of the eigenpairs $\{\lambda_m^h; u_m^h\}$ of the problem (1.2) and (1.3) in $\sqcup^h$. In this respect, we mention the paper [14], where a mixed boundary value problem in a thin domain also gives rise to a limit ordinary differential equation, posed on a circle.

1.2 | Structure of the paper

In the next two parts of this section, we recall known facts, namely, the results [9, 24] on the spectrum of the Dirichlet Laplacian in the cranked L-shaped waveguide $\mathbb{L}$ and an abstract formulation of the variational problem (1.4), to be applied with the classical Lemma 1.1 on "almost eigenpairs." Based on Lemma 1.1, we will examine in §2 the asymptotic behavior of the eigenpairs $\{\Lambda_m^h(\zeta); U_m^h(\zeta)\}$, when $h \to +0$ and the index $m \in \mathbb{N}$ is fixed. The main results are formulated as Theorems 2.6 and 2.7, whereas the preceding Theorem 2.4 contains the result on the above-described localization effect and Proposition 2.5 on the convergence of the eigenpairs.

In §3, we describe other asymptotic sequences of eigenvalues in the midfrequency range $O(h^{-l}\pi^2)$ of the spectrum. The related eigenfunctions do not share the localization property as they are distributed along the bottom and wall of the beaker $\sqcup^h$. We also formulate some open questions, the solution of which would require more detailed information on the continuous spectrum $\wp_c$ of the problem (1.8), (1.9) than found in [9, 24], cf. Section 1.3.

1.3 | The spectrum of the problem in $\mathbb{L}$

According to [2, Ch.10,§1], the Dirichlet problem

$$-\Delta_\xi w(\xi) = \mu w(\xi), \quad \xi \in \mathbb{L} = \Pi_1 \cup \Pi_2 \subset \mathbb{R}^2, \tag{1.8}$$

$$w(\xi) = 0, \quad \xi \in \partial\mathbb{L}, \tag{1.9}$$

defines a positive definite self-adjoint operator A in $L^2(\mathbb{L})$, the continuous spectrum $\wp_c$ of which consists of the ray $[\pi^2, +\infty)$. It is known, see [9, 24] and also, for example, [8, 17] and others, that the discrete spectrum $\wp_d$ consists of the single eigenvalue $\mu_1 \in (0, \pi^2)$, whose approximate value $\mu_1 \approx 0.93\pi^2$ was computed in [9]. The corresponding eigenfunction $w_1 \in H_0^1(\mathbb{L})$ decays at infinity as $O(e^{-\sqrt{\pi^2-\mu_1}|\xi|})$ but does not belong to $H^2(\mathbb{L})$ because of the singularity $O(|\xi - P|^{2/3})$ at the corner point $P = (1, 1)$ with opening $3\pi/2$. These properties can be easily confirmed by the separation of variables method. We provide a short proof of the fact on the discrete spectrum for the convenience of the reader.

According to the max-min principle, see [2, Thm.10.2.2] and [27, Thm.XIII.3], the eigenvalues in $\wp_d$ can be computed as

$$\mu_j = \max_{\mathcal{E}_j} \inf_{w\in\mathcal{E}_j\setminus\{0\}} \frac{\|\nabla_\xi w; L^2(\mathbb{L})\|^2}{\|w; L^2(\mathbb{L})\|^2} < \pi^2, \tag{1.10}$$

where $\mathcal{E}_j$ is any $(j-1)$-codimensional subspace of $H^1_0(\mathbb{L})$. We cut the unit square $\mathbf{Q} = (0,1)^2$ from $\mathbb{L}$ and write the Friedrichs inequalities by taking into account the Dirichlet condition (1.9),

$$\|\nabla_\xi w; L^2(\Pi_j \setminus \mathbf{Q})\|^2 \geqslant \pi^2 \|w; L^2(\Pi_j \setminus \mathbf{Q})\|^2, \quad j = 1, 2, \tag{1.11}$$

$$\|\nabla_\xi w; L^2(\mathbf{Q})\|^2 \geqslant \frac{\pi^2}{2} \|w; L^2(\mathbf{Q})\|^2. \tag{1.12}$$

Since $\mathcal{E}_1 = H^1_0(\mathbb{L})$, inserting (1.11), (1.12) into (1.10) yields the estimate $\mu_1 \geqslant \pi^2/2$ for the first eigenvalue μ_1, if it exists. As for the second eigenvalue, we consider the 1-codimensional subspace

$$\mathcal{E}_\perp = \left\{ w \in H^1_0(\mathbb{L}) : \int_{\mathbf{Q}} \sin\left(\frac{\pi}{2}\xi_1\right) \sin\left(\frac{\pi}{2}\xi_2\right) w(\xi) d\xi = 0 \right\} \tag{1.13}$$

in (1.10) with $j = 2$. Replacing formula (1.12) by the Poincaré inequality $\|\nabla_\xi w; L^2(\mathbf{Q})\|^2 \geqslant \frac{5}{2}\pi^2 \|w; L^2(\mathbf{Q})\|^2$, which holds for $w \in \mathcal{E}_\perp$, using $\mathbb{L} = (\Pi_1 \setminus \mathbf{Q}) \cup (\Pi_2 \setminus \mathbf{Q}) \cup \mathbf{Q}$ and adding the sum of the inequalities (1.11), we obtain

$$\inf_{w\in\mathcal{E}_\perp\setminus\{0\}} \frac{\|\nabla_\xi w; L^2(\mathbb{L})\|^2}{\|w; L^2(\mathbb{L})\|^2} \geqslant \frac{5}{2}\pi^2.$$

Thus, according to [2, Thm. 10.2.2], there cannot exist a second eigenvalue in the discrete spectrum $\wp_d \subset (0, \pi^2)$. Note that $\pi^2/2$ and $5\pi^2/2$ are the first two eigenvalues of the Dirichlet Laplacian in $\mathbf{Q}$, and the first eigenfunction is the one in the integrand in (1.13).

The max-min-principle (1.10) shows that in order to prove the existence of the eigenvalue $\mu_1 \in \wp_d$, it suffices to present a trial function $\mathbf{w} \in H^1_0(\mathbb{L})$ such that

$$\|\nabla_\xi \mathbf{w}; L^2(\mathbb{L})\|^2 < \pi^2 \|\mathbf{w}; L^2(\mathbb{L})\|^2. \tag{1.14}$$

Applying a trick in [14], we set

$$\mathbf{w}(\xi) = w^\delta(\xi) + \sqrt{\delta} W(\xi),$$

where $W \in C^\infty_0(\mathbf{Q})$ and, for $\delta > 0$ and $j = 1, 2$,

$$\begin{aligned} w^\delta(\xi) &= e^{-\delta(\xi_j - 1)} w^0(\xi), \quad \xi \in \Pi_j \setminus \mathbf{Q}, \quad w^\delta(\xi) = w^0(\xi), \quad \xi \in \mathbf{Q}, \\ w^0(\xi) &= \begin{cases} \sin(\pi\xi_2), & \xi_1 \geqslant \xi_2, \\ \sin(\pi\xi_1), & \xi_2 \geqslant \xi_1. \end{cases} \end{aligned} \tag{1.15}$$

Clearly, $\mathbf{w}(\xi) \in H^1_0(\mathbb{L})$ and

$$\|\mathbf{w}; L^2(\mathbb{L})\|^2 = 2\int_1^{+\infty} e^{-2\delta\xi_1} \int_0^1 \sin^2(\pi\xi_2) d\xi_2 + 2\sqrt{\delta}(w^0, W)_{\mathbf{Q}} + O(\delta),$$

$$\|\nabla_\xi \mathbf{w}; L^2(\mathbb{L})\|^2 = 2\pi^2 \int_1^{+\infty} e^{-2\delta\xi_1} \int_0^1 \cos^2(\pi\xi_1) d\xi_1 + 2\sqrt{\delta}(\nabla_\xi w^0, \nabla_\xi W)_{\mathbf{Q}} + O(\delta).$$

We insert these relations into (1.14), and since the integrals including the sine and cosine coincide, the corresponding terms cancel out so that there remains the estimate

$$0 > \|\nabla_\xi \mathbf{w}; L^2(\mathbb{L})\|^2 - \pi^2\|\mathbf{w}; L^2(\mathbb{L})\|^2 = 2\sqrt{\delta}((\nabla_\xi w^0, \nabla_\xi W)_{\mathbf{Q}} - (\pi^2 w^0, W)_{\mathbf{Q}}) + C_W\delta \tag{1.16}$$

$$= 2\sqrt{\delta}\left(-\int_{\mathbf{Q}} W(\xi)(\Delta_\xi w^0(\xi) + \pi^2 w^0(\xi))d\xi + \int_{\mathrm{D}} W(\xi)[\partial_n w^0](\xi)ds_\xi\right) + C_W\delta.$$

Here, $\mathrm{D} = \{\xi \in \mathbf{Q} : \xi_1 = \xi_2\}$ is the diagonal of the square, Green's formula was applied in $\mathbf{Q}$, and $[\partial_n W]$ is the jump $\sqrt{2}\pi\cos(\pi\xi_1)$ of the normal derivative of the function (1.15) on D. Hence, by noting that $(\Delta_\xi + \pi^2)w^0 = 0$ in $\mathbf{Q} \setminus \mathrm{D}$ and choosing an appropriate function W, we can make the multiplier of $2\sqrt{\delta}$ in (1.16) negative and achieve the inequality (1.14) for a small $\delta > 0$. Consequently, we have proved that the discrete spectrum $\wp_d$ of the problem (1.8), (1.9) is not empty.

1.4 | The abstract formulation of the problem

We equip the Hilbert space $\mathcal{H}^h = H^1_0(\sqcup^h)$ with the scalar product

$$\langle u^h, \psi^h \rangle_h = (\nabla_x u^h, \nabla_x \psi^h)_h \tag{1.17}$$

and define in $\mathcal{H}^h$ the positive, symmetric, and continuous (thus self-adjoint) operator $\mathcal{J}^h$ by the identity

$$\langle \mathcal{J}^h u^h, \psi^h \rangle_h = (u^h, \psi^h)_{\sqcup^h}, \quad \forall u^h, \psi^h \in \mathcal{H}^h. \tag{1.18}$$

The operator is compact and, by [2, Thm. 10.1.5, 10.2.2], [27, Thm. VI.15,16], its essential spectrum consists of the single point $\tau = 0$, and the discrete spectrum is the positive monotone sequence of normal eigenvalues

$$\tau^h_1 \geqslant \tau^h_2 \geqslant \tau^h_3 \geqslant \ldots \tau^h_m \geqslant \ldots \to +0. \tag{1.19}$$

Comparing formulas (1.4), (1.17), and (1.18), we see that the variational formulation of the problem (1.2), (1.3) is equivalent with the abstract equation $\mathcal{J}^h u^h = \tau^h u^h$ in the space $\mathcal{H}^h$, while the eigenvalue sequences (1.5) and (1.19) are related by

$$\tau^h = (\lambda^h)^{-1}. \tag{1.20}$$

The next assertion is known as the lemma on almost eigenvalues and eigenvectors, see its primary source [29] for further details. The result is a consequence of the spectral decomposition of the resolvent (see [2, Ch.6], [27, Ch.VII]).

Lemma 1.1. *Let $U^h \in \mathcal{H}^h$ and $T^h \in \mathbb{R}_+$ be such that*

$$\|U^h; \mathcal{H}^h\| = 1, \ \ \|\mathcal{J}^h U^h - T^h U^h; \mathcal{H}^h\| =: \delta^h \in (0, T^h). \tag{1.21}$$

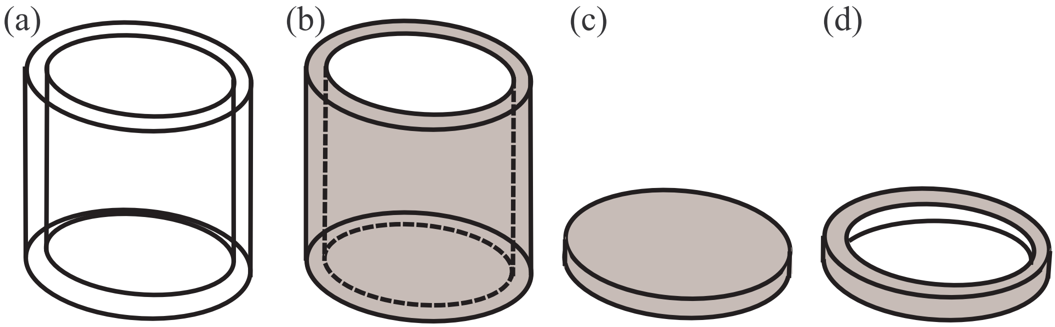

FIGURE 1 (a) The transparent (glass) beaker $\sqcup^h$. (b) The opaque wall $\Theta^h = \theta^h \times (0,1) = (\omega \cap \mathcal{V}^h) \times (0,1)$ with white inner surface. (c) The opaque bottom $\Omega^h = \omega \times (0,h)$. (d) Intersection of the wall and the bottom $\Upsilon^h = \Omega^h \cap \Theta^h = \theta^h \times (0,h)$.

Then, there exists an eigenvalue τ^h_m in (1.19) satisfying the estimate

$$|\tau^h_m - T^h| \leqslant \delta^h. \tag{1.22}$$

Furthermore, for every $\delta^h_* \in (\delta^h, T^h)$, one can find a sequence of coefficients $C^h = (C^h_{N^h}, \dots, C^h_{N^h+X^h-1})$ such that

$$\left\| \mathcal{U}^h - \sum_{n=N^h}^{N^h+X^h-1} C^h_n \mathcal{U}^h_n; \mathcal{H}^h \right\| \leqslant 2\frac{\delta^h}{\delta^h_*}, \quad \|C^h; \mathbb{R}^{X^h}\| = 1, \tag{1.23}$$

where $\tau^h_{N^h}, \dots, \tau^h_{N^h+X^h-1}$ are all eigenvalues of the operator $\mathcal{J}^h$ belonging to the interval $[T^h - \delta^h_*, T^h + \delta^h_*]$ and the corresponding eigenvectors $\mathcal{U}^h_{N^h}, \dots, \mathcal{U}^h_{N^h+X^h-1}$ are subject to the orthogonality and normalization conditions

$$\langle \mathcal{U}^h_m, \mathcal{U}^h_n \rangle_h = \delta_{m,n}. \tag{1.24}$$

2 | ASYMPTOTICS OF THE EIGENVALUES

2.1 | Rescaled coordinates

In the neighborhood $\mathcal{V}^d \subset \mathbb{R}^2$ of the contour $\partial\omega$ (see (1.1) and Figure 1.1), we introduce the intrinsic system (n, s) of curvilinear coordinates, where n is the oriented distance to the contour $\partial\omega$, $n < 0$ in $\omega \cap \mathcal{V}^d$ and s is the arc length measured on $\partial\omega$ counterclockwise. The Laplace operator is written in the coordinates (n, s, z) as

$$\Delta_x = J(n,s)^{-1}\frac{\partial}{\partial n}J(n,s)\frac{\partial}{\partial n} + J(n,s)^{-1}\frac{\partial}{\partial s}J(n,s)^{-1}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial z^2}, \tag{2.1}$$

where $J(n,s) = 1 + n\varkappa(s)$ is the Jacobian and $\varkappa(s)$ is the curvature of the contour at a point $s \in \partial\omega$ with sign, that is, $\varkappa$ is negative for the concave parts of the contour.

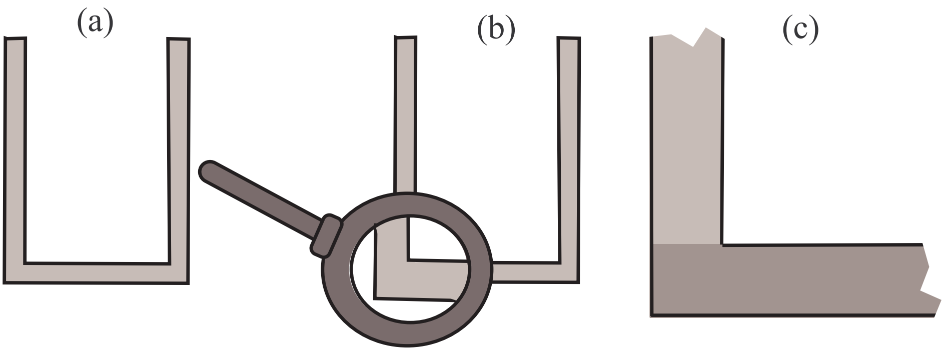

FIGURE 2 (a) Vertical cross-section of the beaker. (b) The corner through a magnifying glass. (c) L-shaped waveguide $\mathbb{L}$ for the description of the boundary later, composed from the semistrips Π_1 (shaded) and Π_2.

The coordinate change

$$n \mapsto \eta = -h^{-1}n,\ z \mapsto \zeta = h^{-1}z,\ \text{ with } \xi = (\xi_1, \xi_2) := (\eta, \zeta) \tag{2.2}$$

and formal setting $h = 0$ turn the domain $\sqcup^h$ into the direct product $\mathbb{L} \times \partial\omega \ni (\xi, s)$ (see Figure 1.3) and the problem (1.2), (1.3) into the problem (1.8), (1.9) depending on the parameter s. Indeed, the operator (2.1) can be decomposed as

$$\Delta_x = \frac{1}{h^2}\Delta_\xi - \frac{1}{h}\varkappa(s)\frac{\partial}{\partial\eta} + \frac{\partial^2}{\partial s^2} + \varkappa(s)^2\eta\frac{\partial}{\partial\eta} + hD(\eta, s, \nabla_\xi, \partial_s), \tag{2.3}$$

where D is a second-order differential operator such that the moduli of its coefficients do not exceed $c(1 + |\eta|)(1 + h|\eta|)$ with a constant c independent of $h \in (0, h_0)$.

2.2 | Asymptotic procedure

The main asypmptotic part of the Laplacian Δ_x becomes $h^{-2}\Delta_\xi$ in the dilated coordinates, and we accept the asymptotic ansätze

$$\lambda^h = h^{-2}\mu_1 + h^{-1}\mu' + h^0\gamma + \cdots, \tag{2.4}$$

$$u^h(x) = v(s)w_1(\xi) + hv(s)w'(\xi, s) + h^2\mathrm{V}(\xi, s) + \cdots, \tag{2.5}$$

where dots indicate higher order terms inessential for our analysis. Notice that since the derivative ∂_s stays unchanged, there is no need to differentiate v in the first correction term. The ansätze are inserted into the differential equation (1.2) and the multipliers of the like powers of the parameter h are collected. Since $\{\mu_1; w_1\}$ is the eigenpair of the problem (1.8), (1.9), the terms of order h^{-2} cancel each other. Taking decomposition (2.1) into account, we obtain the following equation for the second pair $\{\mu'; w'\}$,

$$-\Delta_\xi w'(\xi, s) - \mu_1 w'(\xi, s) = f'(\xi) := \mu' w_1(\xi) - \varkappa(s)\partial_\eta w_1(\xi),\ \xi \in \mathbb{L}. \tag{2.6}$$

Here, we have omitted the common unknown factor $v(s)$. Furthermore, we add the Dirichlet condition

$$w'(\xi, s) = 0, \quad \xi \in \partial\mathbb{L}, \tag{2.7}$$

coming from (1.3). There is only one compatibility condition $(f', w_1)_{\mathbb{L}} = 0$ in problem (2.6), (2.7), which implies, by the definition of f' and the Dirichlet condition (2.7),

$$\mu' = \mu' \|w_1; L^2(\mathbb{L})\|^2 = \kappa(s) \int_{\mathbb{L}} w_1(\xi) \frac{\partial w_1}{\partial \eta}(\xi) d\xi = \frac{\kappa(s)}{2} \int_{\mathbb{L}} \frac{\partial w_1^2}{\partial \xi_1}(\xi) d\xi = 0. \tag{2.8}$$

Thus, the solution $w' \in H_0^1(\mathbb{L})$ exists: it equals

$$w'(\xi, s) = \kappa(s) W(\xi), \tag{2.9}$$

where $W \in H_0^1(\mathbb{L})$ solves the problem

$$-\Delta_\xi W(\xi) - \mu_1 W(\xi) = -\partial_\eta w_1(\xi),\ \xi \in \mathbb{L}, \quad W(\xi) = 0,\ \xi \in \partial\mathbb{L}, \tag{2.10}$$

and decays at infinity as $O = (|\xi| e^{-\sqrt{\pi^2 - \mu_1}|\xi|})$, see, for example, [23, Ch.2]. The orthogonality condition

$$(W, w_1)_{\mathbb{L}} = 0 \tag{2.11}$$

makes the solution W unique.

The next problem in the asymptotic procedure reads as

$$\begin{aligned} -\Delta_\xi V(\xi, s) - \mu_1 V(\xi, s) = {} & \gamma v(s) w_1(\xi) + w_1(\xi) \partial_s^2 v(s) - \kappa(s)^2 v(s) \partial_\eta W(\xi) \\ & + \kappa(s)^2 v(s) \eta \partial_\eta w_1(\xi),\ \xi \in \mathbb{L}, \quad V(\xi, s) = 0,\ \xi \in \partial\mathbb{L}. \end{aligned} \tag{2.12}$$

Now, the following ordinary differential equation on the contour becomes the compatibility condition,

$$-\partial_s^2 v(s) - a\kappa(s)^2 v(s) = \gamma v(s),\ s \in \partial\omega, \tag{2.13}$$

where $a = a_1 + a_2$ and

$$\begin{aligned} a_1 &= -\int_{\mathbb{L}} w_1(\xi) \eta \partial_\eta w_1(\xi) d\xi = \int_{\mathbb{L}} |w_1(\xi)|^2 d\xi + \int_{\mathbb{L}} w_1(\xi) \eta \partial_\eta w_1(\xi) d\xi \Rightarrow a_1 = \frac{1}{2}, \\ a_2 &= \int_{\mathbb{L}} w_1(\xi) \partial_\eta W(\xi) d\xi = -\int_{\mathbb{L}} W(\xi) \partial_\eta w_1(\xi) d\xi \\ &= -\int_{\mathbb{L}} W(\xi)(\Delta_\xi W(\xi) + \mu_1 W(\xi)) d\xi = \int_{\mathbb{L}} (|\nabla_\xi W(\xi)|^2 - \mu_1 |W(\xi)|^2) d\xi. \end{aligned} \tag{2.14}$$

Remark 2.1. The general solution of the problem (2.10) is of the form $W = W_\perp + c w_1$ where $(W_\perp, w_1)_{\mathbb{L}} = 0$. Clearly, changing the coefficient c does not affect the value a_2 in (2.14).

The ordinary differential equation (2.13) is a standard spectral Sturm-Liouville equation (note that the curvature function $\varkappa$ is C^∞-smooth by the geometric assumption on the contour $\partial\omega$ in the very beginning of the paper). Consequently (see, e.g., [28, Thm. 5.11]), it has the monotone unbounded sequence of eigenvalues

$$\gamma_1 < \gamma_2 \leqslant \gamma_3 \leqslant \dots \leqslant \gamma_m \leqslant \dots \to +\infty. \tag{2.15}$$

They and the corresponding eigenfunctions $v_m \in C^\infty(\partial\omega)$ specify the ansätze (2.4) and (2.5). The multiplicities of the eigenvalues (2.15) do not exceed two. The eigenfunctions can be subject to the orthogonality and normalization conditions

$$(v_m, v_n)_{\partial\omega} = \delta_{m,n},\ m, n \in \mathbb{N}. \tag{2.16}$$

Taking the eigenfunctions v_k for v in (2.13) yields the representation

$$V_k(\xi, s) = V_k^0(\xi) v_k(s) \tag{2.17}$$

for a solution of problem (2.12), because the second derivative is a multiple of the function itself. The exact expression for the factor $V_k^0 \in H_0^1(\mathbb{L})$ will not be needed. This solution becomes unique under the orthogonality condition

$$(V_k^0, w_1)_{\mathbb{L}} = 0. \tag{2.18}$$

Remark 2.2. If ω is the unit disk $\{y : |y| < 1\}$, then $\varkappa(s) = 1$ and

$$\gamma_1 = -a, \qquad \gamma_{2k} = \gamma_{2k+1} = \pi^2 k^2 - a,\ k = 1, 2, 3, \dots,$$
$$v_1(s) = (2\pi)^{-1/2}, \quad v_{2k}(s) = 2^{-1/2}\cos(ks), v_{2k+1}(s) = 2^{-1/2}\sin(ks),\ k = 1, 2, 3, \dots$$

2.3 | Asymptotics of an eigenvalue

We take

$$\{T_m^h, U_m^h\} = \{h^2(\mu_1 + h^2\gamma_m)^{-1}, \|\mathcal{W}_m^h; \mathcal{H}^h\|^{-1}\mathcal{W}_m^h\} \tag{2.19}$$

as an “almost eigenpair” of the operator $\mathcal{J}^h$ introduced in Section 1.4. Here, $\{\mu_1; w_1\}$ is the eigenpair of the problem (1.8), (1.9) described in Section 1.3 and

$$\mathcal{W}_m^h(x) = \chi(x) v_m(s)(w_1(\xi) + hW(\xi) + h^2 V_m^0(\xi)), \tag{2.20}$$

where W and V_k^0 are as in (2.9) and (2.17). Moreover, χ is a smooth cut-off function such that $\chi = 1$ in the $d/2$-neighborhood of the edge Υ and $\chi = 0$ outside the d-neighborhood. Let us evaluate the quantity δ_m^h in (1.23) for the pair (2.19). First of all, according to formulas (1.17) and

$$\nabla_x = (\partial_n, \partial_z, J(n,s)^{-1}\partial_s),\ dx = J(n,s)\,dn\,dz\,ds, \tag{2.21}$$

where $|J(n,s) - 1| \leqslant c_\varkappa |n| = c_\varkappa h|\eta|$ so that the coordinate dilation (2.2) and the relation (2.16) yield

$$(\nabla_x \mathcal{W}_m^h, \nabla_x \mathcal{W}_n^h)_{\sqcup^h} = (v_m, v_n)_{\partial\omega}\, \|\nabla_\xi w_1; L^2(\mathbb{L})\|^2 + O(h + e^{-\vartheta/h}),\ \vartheta > 0 \tag{2.22}$$

so that

$$|\langle \mathcal{W}_m^h, \mathcal{W}_n^h\rangle_h - \mu_1\delta_{m,n}| \leqslant c_{m,n}h. \tag{2.23}$$

To see this, recall that $\|\nabla_\xi w_1; L^2(\mathcal{L})\|^2 = \mu_1\|w_1; L^2(\mathcal{L})\|^2$ by the integral identity for the eigenpair $\{\mu_1; w_1\}$ of the problem (1.8), (1.9) and that $w_1(\xi)$ decays exponentially at infinity, which compensate the linear growth of the integrand and leads to corrections of order $O(h)$, while the cut-off function χ causes terms of order $O(e^{-\vartheta/h})$, see Section 1.3. Second, we write

$$\begin{aligned}\delta_m^h &= \sup|\langle \mathcal{J}^h U_m^h - T_m^h U_m^h, \Psi^h\rangle_h| \\ &= T_m^h\|\mathcal{W}_m^h;\mathcal{H}^h\|^{-1}\sup|(\nabla_x\mathcal{W}_m^h, \nabla_x\Psi^h)_{\sqcup^h} - (h^{-2}\mu_1 + \gamma_m)(\mathcal{W}_m^h, \Psi^h)_{\sqcup^h}| \\ &= T_m^h\|\mathcal{W}_m^h;\mathcal{H}^h\|^{-1}\sup|((\Delta_x + h^{-2}\mu_1 + \gamma_m)\mathcal{W}_m^h, \Psi^h)_{\sqcup^h}|,\end{aligned} \tag{2.24}$$

where the supremum is computed over the unit ball in $\mathcal{H}^h$, that is, $\|\Psi^h;\mathcal{H}^h\| = 1$, and hence,

$$\|\Psi^h; L^2(\sqcup^h)\|^2 \leqslant Ch^2\|\nabla_x\Psi^h; L^2(\sqcup^h)\|^2 \leqslant Ch^2. \tag{2.25}$$

Let us estimate the scalar product $(I_m^h, \Psi^h)_{\sqcup^h}$ between the last modulus signs in (2.24). Using representations (2.3) and (2.20), we have

$$\begin{aligned}I_m^h =& [\Delta_x, \chi]\mathcal{W}_m^h + \chi h^{-2}v_m(\Delta_\xi + \mu_1)w_1 + \chi v_n h^{-1}((\Delta_\xi + \mu_1)w' - \varkappa\partial_\eta w_1) \\ &+ \chi h^0(v_m((\Delta_\xi + \mu_1)V_m^0 - \varkappa\partial_\eta w') + (\partial_s^2 + \gamma_n + \varkappa^2\eta\partial_\eta)(w_1 v_k)) \\ &+ \chi h^1((\partial_s^2 + \gamma_m + \varkappa^2\eta\partial_\eta)(w' + hV_m^0)v_m - \varkappa\partial_\eta V_m^0 v_k) + \chi hD\mathcal{W}_m^h.\end{aligned} \tag{2.26}$$

The commutator $[\Delta_x, \chi]$ is a first-order differential operator, and the supports of its coefficients lay outside the $d/2$-neighborhood of the edge Y, where the functions w_1, w', and V_m^0 become smooth and exponentially small. Therefore, using the bound $ce^{-\vartheta/h}$ for the moduli of these functions yields

$$\|[\Delta_x, \chi]\mathcal{W}_m^h; L^2(\sqcup^h)\| \leqslant ce^{-\vartheta/h}, \ \ \vartheta > 0.$$

The multipliers of h^{-2}, h^{-1} and $1 = h^0$ in (2.26) vanish due to our definition of the functions w_1, w', and V_m^0, respectively. Furthermore, recalling the properties of coefficients of the remainder hD in the decomposition of the Laplacian mentioned below (2.3) and using the exponential decay of the functions w_1, w', V_m^0, the sum $\tilde{I}_m^h$ of the last couple of terms in (2.26) can be processed similarly to (2.22). This leads us to the estimate

$$\|\tilde{I}_m^h; L^2(\sqcup^h)\| \leqslant ch\|e^{-\vartheta\,\mathrm{dist}(x,\mathrm{Y})/h}; L^2(\sqcup^h)\| \leqslant ch(h^2)^{1/2} = ch^2.$$

Notice that the factor h^2 is caused by the coordinate change $x \mapsto (\xi, s)$.

The inequalities derived above and the relations $|T_m^h| \leqslant c_m h^2$, $\|\mathcal{W}_m^h;\mathcal{H}^h\| \geqslant \sqrt{\mu_1}/2$, which hold for small h, see (2.19) and (2.23), yield the following estimate of the quantity (2.24):

$$\delta_m^h \leqslant c_m h^2 h^2 h = c_m h^5. \tag{2.27}$$

Lemma 1.1 now gives us an eigenvalue $\tau^h_{m(h)}$ of the operator $\mathcal{J}^h$ such that

$$|\tau^h_{m(h)} - T^h_m| \leqslant c_m h^5. \tag{2.28}$$

Let us verify that in the case Equation (2.13) has a double eigenvalue β_m, there are at least two eigenvalues $\tau_{m(h)}, \tau_{m(h)+1}$ with (2.28) in the sequence (2.15). To this end, we use the second part of Lemma 1.1 where we set $\delta^h = h^5 \max\{c_m, c_{m+1}\}$, $\delta^h_* = \rho^{-1}\delta^h$, $\rho \in (0,1)$, and denote by $C^h_{(j)} \in \mathbb{R}^{X_m(h)}$ and S^h_j with $j = m, m+1$ the columns and the sums in $n = N_m(h), \dots, N_m(h) + X_m(h) - 1$ of eigenvectors given in (1.23). Then, (1.24), (1.23), and (2.23) imply

$$\begin{aligned} |C^h_{(m)} \cdot C^h_{(m+1)}| &= |\langle S^h_{(m)}, S^h_{(m+1)}\rangle_h| \leqslant |\langle S^h_{(m)} - U^h_{(m)}, S^h_{(m+1)}\rangle_h| + |\langle U^h_{(m)}, S^h_{(m+1)} - U^h_{(m+1)}\rangle_h| \\ &\quad + |\langle U^h_{(m)}, U^h_{(m+1)}\rangle_h| \leqslant 2\rho + 2\rho + c_{m,m+1}h. \end{aligned} \tag{2.29}$$

Thus, for small ρ and h, the columns $C^h_{(m)}$ and $C^h_{(m+1)}$ are "almost orthogonal" in $\mathbb{R}^{X_m(h)}$ that may happen only in the case $N_m(h) \geqslant 2$. In other words, fixing an appropriate ρ, we find at least two eigenvalues (1.19) belonging to the interval

$$[T^h_m - \rho^{-1}h^5 \min\{\delta^h_m, \delta^h_{m+1}, C_{m+1}\}, T^h_m + \rho^{-1}h^5 \min\{\delta^h_m, \delta^h_{m+1}, C_{m+1}\}].$$

These eigenvalues $\tau_{m(h)}$ and $\tau_{m(h)+1}$ satisfy the inequality (2.28) with a larger constant C_m in place of c_m. Taking into account (1.20), we can now deduce that

$$\begin{aligned} &|\lambda^h_j - h^{-2}\mu_1 - \gamma_m| \leqslant C_m h^3(\mu_1 + h^2\gamma_m)\lambda^h_j \\ &\Rightarrow \lambda^h_j \leqslant 2(h^{-2}\mu_1 + \gamma_m) \text{ as } C_m h^3(\mu_1 + h^2\gamma_m) \leqslant 1/2 \\ &\Rightarrow |\lambda^h_j - h^{-2}\mu_1 - \gamma_m| \leqslant 2C_m h(\mu_1 + h^2\gamma_m)^2, \end{aligned} \tag{2.30}$$

where $j = m(h), m(h)+1$. Hence, we have verified the following assertion.

Proposition 2.3. *For all $m \in \mathbb{N}$, there exist positive numbers h_m, C_m and an index $m(h) \in \mathbb{N}$ such that*

$$|\lambda^h_{m(h)} - h^{-2}\mu_1 - \gamma_m| \leqslant C_m h \text{ as } h \in (0, h_m]. \tag{2.31}$$

Moreover, $m(h) \neq n(h)$ in the case $m \neq n$.

2.4 | About the convergence

From the last observation in Proposition 2.3, it follows that

$$\lambda^h_m \leqslant h^{-2}\mu_1 + c_m \text{ for } h \in (0, h_m]. \tag{2.32}$$

with some positive c_m and h_m. Hence, we have

$$h^2\lambda^h_m - \mu_1 \to \gamma^0_m \tag{2.33}$$

along a positive sequence $\{h_j\}_{j\in\mathbb{N}}$ converging to null. To find a similar convergence for the eigenfunctions of the problem (1.2), (1.3), we need to prove their localization property.

Theorem 2.4. *Let $m \in \mathbb{N}$ be such that (2.32) holds true for $h \in (0, h_m]$. There exist positive $\mathbf{h}_m$ and $\mathbf{c}_m$ such that the eigenfunction u^h_m normalized in $L^2(\sqcup^h)$, see (1.6), satisfies the weighted estimate*

$$\|\mathcal{R}^h_\alpha \nabla_x u^h_m; L^2(\sqcup^h)\| + h^{-1}\|\mathcal{R}^h_\alpha u^h_m; L^2(\sqcup^h)\| \leqslant \mathbf{c}_m h^{-1} \text{ for } h \in (0, \mathbf{h}_m] \tag{2.34}$$

where $\mathcal{R}^h_\alpha(x) = e^{\alpha \operatorname{dist}(x,\Upsilon)/h}$ and $\alpha > 0$ is a small exponent.

Proof. We insert the test function $\Psi^h = \mathcal{R}^h_\alpha \mathbf{u}^{\mathbf{h}}_{\mathbf{m}}$, where $\mathbf{u}^h_m = \mathcal{R}^h_\alpha u^h_m$, to the integral identity (1.4). A simple calculation yields

$$\|\nabla_x \mathbf{u}^h_m; L^2(\sqcup^h)\|^2 - \|\mathbf{u}^h_m \mathcal{R}^h_{-\alpha} \nabla_x \mathcal{R}^h_\alpha; L^2(\sqcup^h)\|^2 = \lambda^h_m \|\mathbf{u}^h_m; L^2(\sqcup^h)\|^2. \tag{2.35}$$

We make several observations. First,

$$\mathcal{R}^h_{-\alpha}(x)|\nabla_x \mathcal{R}^h_\alpha(x)| = \alpha/h. \tag{2.36}$$

Second, we apply the one-dimensional Friedrichs inequality for the interval $(0, h) \ni z$ in the bottom piece $\Omega^h \setminus \Upsilon^h$, where $\Upsilon^h = \{x \in \sqcup^h : z \in (0, h), n \in (-h, 0)\}$, see Sections 1.1 and 1.2. This gives

$$\|\nabla_x \mathbf{u}^h_m; L^2(\Omega^h \setminus \Upsilon^h)\|^2 \geqslant \|\partial_z \mathbf{u}^h_m; L^2(\Omega^h \setminus \Upsilon^h)\|^2 \geqslant \pi^2 h^{-2} \|\mathbf{u}^h_m; L^2(\Omega^h \setminus \Upsilon^h)\|^2. \tag{2.37}$$

Third, concerning the wall $\Theta^h \setminus \Upsilon^h$, we recall formulas (2.21) and write

$$\begin{aligned} \|\nabla_x \mathbf{u}^h_m; L^2(\Theta^h \setminus \Upsilon^h)\|^2 &\geqslant \int_{\partial\omega} \int_h^1 \int_{-h}^0 (1 + n\varkappa(s))|\partial_n \mathbf{u}^h_m(x)|^2 dn dz ds \\ &\geqslant \pi^2 h^{-2}(1 - c_\omega h)\|\mathbf{u}^h_m; L^2(\Theta^h \setminus \Upsilon^h)\|^2. \end{aligned} \tag{2.38}$$

We now transform (2.35) into

$$\begin{aligned} \lambda^\varepsilon_m \|\mathbf{u}^h_m; L^2(\Upsilon^h)\|^2 + \|\mathbf{u}^h_m \mathcal{R}^h_{-\alpha} \nabla_x \mathcal{R}^h_\alpha; L^2(\Upsilon^h)\|^2 &= \|\nabla_x \mathbf{u}^h_m; L^2(\sqcup^h)\|^2 \\ &\quad - \lambda^\varepsilon \|\mathbf{u}^h_m; L^2(\sqcup^h \setminus \Upsilon^h)\|^2 - \|\mathbf{u}^h_m \mathcal{R}^h_{-\alpha} \nabla_x \mathcal{R}^h_\alpha; L^2(\sqcup^h \setminus \Upsilon^h)\|^2. \end{aligned} \tag{2.39}$$

Formulas (2.32) and (2.36) and the normalization of $u^h_m = \mathcal{R}^h_{-\alpha} \mathbf{u}^h_m$ in $L^2(\sqcup^h)$ show that the left-hand side of (2.39) is less than

$$(h^{-2}\mu_1 + c_m + h^{-2}\alpha^2)e^{2\sqrt{2}\alpha}.$$

Moreover, the inequalities (2.37) and (2.38) provide the following lower bound for the right-hand side with an arbitrary $\sigma \in (0, 1)$:

$$\begin{aligned} \sigma\|\nabla_x \mathbf{u}^h_m; L^2(\sqcup^h)\|^2 &+ \Big((1-\sigma)\frac{\pi^2}{h^2} - \frac{\alpha^2}{h^2} - \lambda^h_m\Big)\|\mathbf{u}^h_m; L^2(\Omega^h \setminus \Upsilon^h)\|^2 \\ &+ \Big((1-\sigma)\frac{\pi^2}{h^2}(1 - c_\omega h) - \frac{\alpha^2}{h^2} - \lambda^h_m\Big)\|\mathbf{u}^h_m; L^2(\Theta^h \setminus \Upsilon^h)\|^2. \end{aligned} \tag{2.40}$$

It suffices to mention that one can choose small positive σ, α, and $\mathbf{h}_m$ such that both Lebesgue norms of $\mathbf{u}_m^h$ on the right of (2.40) exceed $\mathbf{c}h^{-2}$ for some $\mathbf{c} > 0$ and all $h \in (0, \mathbf{h}_m]$. □

We introduce two functions of the variables $s \in \partial\omega$ and $\eta \in \mathbb{L}$:

$$v_m^{h0}(s) = h \int_{\mathbb{L}} w_1(\xi)\chi(x)u_m^h(x)|_{n=-h\eta, z=h\zeta} d\xi, \tag{2.41}$$

$$v_m^{h\perp}(\xi, s) = h\chi(x)u_m(x) - w_1(\xi)v_m^{h0}(s).$$

Since $(v_m^{h\perp}, w_1)_{\mathbb{L}} = 0$ for $s \in \partial\omega$ and $\|w_1; L^2(\mathbb{L})\| = 1$, the information on the spectrum $\wp_d$ of problem (1.8), (1.9) in Section 1.3 gives us

$$\int_{\partial\omega} \|\nabla_\eta v_m^{h\perp}(\cdot, s); L^2(\mathbb{L})\|^2 ds \geqslant \pi^2 \|v_m^{h\perp}(\cdot, s); L^2(\mathbb{L})\|^2 ds. \tag{2.42}$$

Recalling that u_m^ε is normalized in $L^2(\sqcup^h)$ and using the weighted estimate (2.34) in Theorem 2.4 yields

$$\begin{aligned} 1 = \|u_m^h; L^2(\sqcup^h)\|^2 = h^2 \int_{\partial\omega}\int_{\mathbb{L}} |\chi(x)u_m^h(x)|^2 d\xi ds + \int_{\sqcup^h} (1-\chi(x)^2)|u_m^h(x)|^2 dx \\ + \int_{\sqcup^h} (J(n,s)-1)|u_m^h(x)|^2 dx \\ = \|v_m^{h0}; L^2(\partial\omega)\|^2 + \|v_m^{h\perp}; L^2(\mathbb{L}\times\partial\omega)\|^2 + O(e^{-\vartheta/h}) + O(h),\ \vartheta > 0. \end{aligned} \tag{2.43}$$

Here, we also took into account that

$$|J(n,s) - 1| \leqslant c_j|n|, \quad |n|e^{-\alpha \mathrm{dist}(x,\mathrm{Y})/h} \leqslant c_\alpha h \text{ for } x \in \sqcup^h, y \in V^h. \tag{2.44}$$

We now insert the test function $\Psi^h = \chi^2 u_m^h$ into the integral identity (1.4) and get by simple transformations

$$\begin{aligned} h^2 \int_{\partial\omega}\int_{\mathbb{L}} (h^{-2}|\nabla_\xi(\chi u_m^h)|^2 + |\partial_s(\chi u_m^h)|^2 - \lambda_m^h |\chi u_m^h|^2) d\xi ds = \int_{\sqcup^h} |u_m^h \nabla_x \chi|^2 dx \\ + h^2 \int_{\partial\omega}\int_{\mathbb{L}} ((1-J)(h^{-2}|\nabla_\xi(\chi u_m^h)|^2 - \lambda_m^h|\chi u_m^h|^2) + (1-J^{-1})|\partial_s(\chi u_m^h)|^2) d\xi ds. \end{aligned} \tag{2.45}$$

The inequalities (2.34), (2.44), and (2.32) imply that the modulus of the right-hand side of (2.45) does not exceed $c(e^{-\vartheta/h} + h)$.

Since $v_m^{h\perp}$ and $\partial_s v_m^{h\perp}$ are orthogonal to w_1 in $L^2(\mathbb{L})$, we have

$$h^2 \int_{\partial\omega}\int_{\mathbb{L}} |\chi u_m^h|^2 d\xi ds = \|v_m^{h0}; L^2(\partial\omega)\|^2 + \int_{\partial\omega} \|v_m^{h\perp}(\cdot, s); L^2(\mathbb{L})\|^2 ds, \tag{2.46}$$

$$h^2 \int_{\partial\omega}\int_{\mathbb{L}} |\partial_s(\chi u_m^h)|^2 d\xi ds = \|\partial_s v_m^{h0}; L^2(\partial\omega)\|^2 + \int_{\partial\omega} \|\partial_s v_m^{h\perp}(\cdot, s); L^2(\mathbb{L})\|^2 ds,$$

$$\int_{\partial\omega}\int_{\mathbb{L}} |\nabla_\eta(\chi u_m^h)|^2 d\xi ds = h^{-2} \int_{\partial\omega} |v_m^{h0}(s)|^2 ds \|\nabla_\xi w_1; L^2(\mathbb{L})\|^2 \tag{2.47}$$

$$+ h^{-2}\int_{\partial\omega}\int_{\mathbb{L}} |\nabla_\xi v_m^{h\perp}(\xi,s)|^2 d\xi ds + 2h^{-2}\int_{\partial\omega}\int_{\mathbb{L}} v_m^{h0}(s)\nabla_\xi w_1(\eta)\nabla_\xi v_m^{h\perp}(\xi,s) d\xi ds$$
$$\geqslant h^{-2}\mu_1\|v_m^{h0};L^2(\partial\omega)\|^2 + \pi^2 h^{-2}\int_{\partial\omega}\int_{\mathbb{L}} |w_m^{h\perp}(\xi,s)|^2 d\xi ds.$$

The last integral in the middle expression of (2.47) vanished due to an integration by parts, the equation $-\Delta_\xi w_1 = \mu_1 w_1$, and the above-mentioned orthogonality. Two other terms were treated by means of the formula $\|\nabla_\xi w_1;L^2(\mathbb{L})\|^2 = \mu_1$ and the inequality (2.42).

Collecting the estimates derived above yields

$$(\pi^2 h^{-2} - \lambda_m^h)\|v_m^{h\perp};L^2(\mathbb{L}\times\partial\omega)\|^2 + \|\partial_s v_m^{h\perp};L^2(\mathbb{L}\times\partial\omega)\|^2 + \|\partial_s v_m^{h0};L^2(\partial\omega)\|^2 \tag{2.48}$$
$$\leqslant ch + (\lambda_m^h - h^{-2}\mu_1)\|v_m^{h0};L^2(\partial\omega)\|^2.$$

The bound (2.32) and its consequence $\pi^2 h^{-2} - \lambda_m^h \geqslant h^{-2} \geqslant h^{-2}(\pi^2 - \mu_1)/2$ for small $h > 0$ as well as the relation (2.43) allow us to conclude that there exists a subsequence of $\{h_j\}_{j\in\mathbb{N}}$ (we keep the same notation) along which

$$v_m^{\varepsilon 0} \to v_m^{00} \text{ weakly in } H^1(\partial\omega), \tag{2.49}$$
$$v_m^{\varepsilon\perp} \to 0 \text{ strongly in } L^2(\mathbb{L}\times\partial\omega),$$
$$\|v_m^{\varepsilon\perp};L^2(\partial\omega)\| \to 0 \;\Rightarrow\; \|v_m^{00};L^2(\partial\omega)\| = 1.$$

Our next aim is to get an integral identity for the limits v_m^{00} and γ_m^0 in (2.49) and (2.33). Accordingly, we take a function $\psi \in C^\infty(\partial\omega)$ and compose the trial function that imitates the ansätze (2.6):

$$\Psi^h(x) = \chi(x)\psi(s)(w_1(\xi) + \kappa(s)W(\xi)). \tag{2.50}$$

We multiply Equation (1.2) for the eigenpair $\{\lambda_m^h; u_m^h\}$ by Ψ^h and apply Green's formula to write

$$h^{-1}(\chi u_m^h, (\Delta_x + \lambda_m^h)\psi(w_1 + h\kappa W))_{\sqcup^h} = -h^{-1}(u_m^h, [\Delta_x, \chi]\psi(w_1 + h\kappa W))_{\sqcup^h}. \tag{2.51}$$

The coefficients of the commutator $[\Delta_x, \chi]\cdot = \Delta_x(\chi\cdot) - \chi\Delta_x\cdot$ are again first-order differential operators. Their supports lay outside the $d/2$-neighborhood of the edge Υ where u_m^h is exponentially small according to Theorem 2.4. Therefore, the right-hand side of (2.51) tends to zero as $h_j \to 0$.

Furthermore, similarly to our calculation in Section 2.2, we have

$$(\Delta_x + \lambda_m^h)\psi(w_1 + h\kappa W) = \psi h^{-2}(\Delta_x w_1 + \mu_1 w_1) + \psi h^{-1}\kappa(\Delta_\xi W + \mu_1 W - \partial_\eta w_1) + F + h\widetilde{F}, \tag{2.52}$$

$$F = w_1(\partial_s^2\psi + (\lambda_m^h - h^{-2}\mu_1)\psi) + \psi\kappa^2(\eta\partial_\eta w_1 - \partial_\eta W), \tag{2.53}$$

$$\widetilde{F} = D\psi(w_1 + h\kappa W) + (\partial_s^2 + \kappa^2\eta\partial_\eta)\kappa W. \tag{2.54}$$

Owing to the definition of w_1 and W, the multipliers of h^{-2} and h^{-1} in (2.52) vanish. Recalling the properties of the differential operator D mentioned in Section 2.1, we obtain for the function

(2.54) that

$$\|\widetilde{F}; L^2(\{\xi \in \sqcup^h : \operatorname{dist}(x, \mathrm{Y}) < d\})\| \leqslant c_\psi h \;\text{ and }\; h^{-1}|(\chi u_m^h, h\widetilde{F})_{\sqcup^h}| \leqslant c_\psi h.$$

As for the function (2.53), we write

$$h^{-1}(\chi u_m^h, F)_{\sqcup^h} = \int_{\partial\omega} v_m^{h0}(s)(w_1, F(\cdot, s))_{\mathbb{L}}\, ds + (v_m^{h\perp}, F)_{\mathbb{L}\times\partial\omega} =: J_m^{h0}(\psi) + J_m^{h\perp}(\psi).$$

In view of the estimates (2.32) and $\|v_m^{h\perp}; L^2(\mathbb{L}\times\partial\omega)\| \leqslant ch$, see (2.48), we have

$$|J_m^{h\perp}(\psi)| \leqslant c_\psi h.$$

Finally, recalling calculations (2.14) and formula $\|w_1; L^2(\mathbb{L})\| = 1$ yields

$$J_m^{h0}(\psi) = \int_{\partial\omega} v_m^{h0}(s)(\partial_s^2\psi(s) + (\lambda_m^h - h^{-2}\mu_1)\psi(s) + a\kappa(s)^2\psi(s))ds.$$

Passing to the limit $h_j \to +0$ in (2.51), the relations derived above give us the equality

$$(v_m^{00}, \partial_s^2\psi + \gamma_m^0\psi + a\kappa^2\psi)_{\partial\omega} = 0.$$

Thus, v_m^{00} belongs to $C^\infty(\partial\omega)$ and solves the ordinary differential equation (2.13) with $\gamma = \gamma_m^0$.

Proposition 2.5. *The limits in (2.49) and (2.33) form an eigenpair* $\{\gamma_m^0; v_m^{00}\}$ *of Equation (2.13) on the contour* $\partial\omega$, *and the eigenfunction* v_m^{00} *is normalized in* $L^2(\partial\omega)$.

2.5 | Theorems on asymptotics

Everything is prepared for conclusions on the asymptotics of the eigenvalues.

Theorem 2.6. *The entries of eigenvalue sequences* $\{\lambda_j^h\}_{j=1}^\infty$ *and* $\{\gamma_j\}_{j=1}^\infty$, *see (1.5) and (2.15), of the problems (1.2), (1.3), and (2.13), respectively, are in the relationship (2.31) where the index* $m(h)$ *coincides with* m *and* $\mu_1 \in [\pi^2/2, \pi^2)$ *is the only eigenvalue in the discrete spectrum* $\wp_d$ *of the problem (1.8), (1.9).*

Proof. It remains to show that $m(h) = m$ for $h \in (0, h_m]$ and a small $h_m > 0$. Without loss of generality, we may fix m such that $\gamma_m < \gamma_{m+1}$. Our derivation of the inequality (2.32), based on Proposition 2.3, also ensures that $m(h) \geqslant m$. Let us assume that $m(h_j) > m$ for a positive sequence $\{h_j\}_{j\in\mathbb{N}}$ converging to null. Then, we find eigenvalues $\lambda_{m_j}^{h_j} \in (0, h_j^{-2}\mu_1 + \gamma_m]$ such that the corresponding eigenfunction $u_{m_{h_j}}^{h_j}$ is orthogonal to the eigenfunctions $u_1^{h_j}, \dots, u_m^{h_j}$ in $L^2(\sqcup^h)$. Passing to limit as in (2.49) provides an eigenvalue $\widehat{\gamma}^0 \leqslant \gamma_m$ of Equation (2.13) such that the corresponding eigenfunction $\widehat{v}^0$ is orthogonal to the functions $v_1, \dots, v_m$ in $L^2(\partial\omega)$. The last observation contradicts with the method of composing the eigenvalue sequence (2.15). Thus, $m(h) = m$ and the theorem is proved. ☐

We will justify the asymptotics of the eigenfunctions u_m^ε of the problem (1.2), (1.3) only for simple eigenvalues β_m, for example, for the first one. However, the result on asymptotics

(Theorem 2.7) is formulated without the simplicity assumption, because the necessary modifications for the proof of the general case are standard and self-explanatory.

By the second part of Lemma 1.1 and Theorem 2.6, we first observe that there exists $H_m > 0$ such that the interval

$$[h^2(\mu_1 + h^2\gamma_m)^{-1} - h^2 H_m, h^2(\mu_1 + h^2\gamma_m)^{-1} + h^2 H_m]$$

contains only a single eigenvalue $\tau_m^h = (\lambda_m^h)^{-1}$, see (1.20), of the operator $\mathcal{J}^h$. We put $\delta^h = c_m h^5$, see (2.28), and $\delta_*^h = H_m h^2$ in formula (1.23) and thus obtain that $|C_m^h| = 1$ and

$$\|U_m^h - C_m^h \mathcal{U}_m^h; \mathcal{H}^h\| \leqslant 2\delta^h(\delta_*^h)^{-1} = 2c_m H_m^{-1} h^3. \tag{2.55}$$

Then, comparing the normalization conditions (1.24) and (1.6), and recalling formulas (1.17), (1.4), yields the relation $u_m^h = (\lambda_m^h)^{-1/2}\mathcal{U}_m^h$.

Now, straightforward calculations and modifications in the case of multiple eigenvalues lead to the following assertion.

Theorem 2.7. *Let $m \in \mathbb{N}$ and let $\varkappa_m \in \{1,2\}$ be the multiplicity of the eigenvalue γ_m in the sequence (2.15). Then, there exist positive numbers h_m, c_m and an orthogonal matrix C^h of size $\varkappa_m \times \varkappa_m$ with entries C_{pq}^h, $p, q = m, m + \varkappa_m - 1$ such that the eigenfunctions u_p^ε and v_q of problems (1.2), (1.3), and (2.13), respectively, subject to the normalization and orthogonality conditions (1.6) and (2.16), are related by formulas*

$$h\|\nabla_x u_p^\varepsilon - \nabla_x \mathbf{u}_p^\varepsilon; L^2(\sqcup^h)\| + \|u_p^\varepsilon - \mathbf{u}_p^\varepsilon; L^2(\sqcup^h)\| \leqslant c_m h^3 \text{ for } h \in (0, h_m],\, p = m, m + \varkappa_m - 1 \tag{2.56}$$

$$\text{with } \mathbf{u}_p^\varepsilon = \|\mathcal{W}_p^h; L^2(\sqcup^h)\|^{-1}\mathcal{W}_p^h \text{ and } \mathcal{W}_p^h(x) = \chi(x)\sum_{q=m}^{m+\varkappa_m-1} \mathrm{C}_{pq}^h v_q(s)(w_1(\xi) + hW(\xi)),$$

where w_1 and W are functions in $H_0^1(\mathbb{L})$ satisfying the problems (1.8), (1.9), and (2.10), respectively, and subject to the conditions $\|w_1; L^2(\mathbb{L})\| = 1$ and (2.11).

Remark 2.8. The presence of the multiplier $\|\mathcal{W}_p^h; L^2(\sqcup^h)\|^{-1}$ means that formulas (2.56) are not quite explicit. One can derive the following, much more explicit asymptotic formula for a simple eigenvalue γ_m from (2.55):

$$h\|\nabla_x u_m^h - h^{-1}\nabla_x(\chi w_1 v_m); L^2(\sqcup^h)\| + \|u_m^h - h^{-1}\chi w_1 v_m; L^2(\sqcup^h)\| \leqslant c_m h. \tag{2.57}$$

In this estimate, there does not appear the correction term of the asymptotic approximation of the eigenfunction, and the upper bound is larger due to the fact that the solution W of problem (2.10) is not defined uniquely, see Remark 2.1. In Theorem 2.7, this arbitrariness of the choice of W was compensated by the above-mentioned multiplier.

3 | OTHER SEQUENCES OF EIGENVALUES

3.1 | Prelude

At the first glance, the result of Theorem 2.6 seems to give complete information on the asymptotics of the eigenvalues of the problem (1.2), (1.3), for example, because it suggests the convergence

$$h^2\lambda_m^h \;\to\; \gamma_m \text{ as } h \to +0$$

for any fixed $m \in \mathbb{N}$. However, although the relationship (2.31) even gives an estimate for the convergence rate, the numbers h_m and c_m depend on the eigenvalue number m and vanish when $m \to +\infty$ so that the convergence is not uniform in $m \in \mathbb{N}$ for sure. We augment this observation in the present section with an effect related with the particular shape of the thin beaker. Namely, we will find other asymptotic series of eigenvalues

$$\left\{\lambda^h_{p^b_m(h)}\right\}_{m\in\mathbb{N}} \text{ and } \left\{\lambda^h_{p^w_m(h)}\right\}_{m\in\mathbb{N}} \tag{3.1}$$

whose asymptotic behavior is described by two-dimensional limit problems different from the ordinary differential equation (2.13). These problems are obtained by restricting the original problem (1.2), (1.3) onto the bottom Ω^h and the wall Θ^h of the beaker while applying the standard dimension reduction procedure, which is a significant simplification of the procedures, for example, in [4–7, 22]. Another, evident trick will be needed to deal with the thin curved wall, see Section 3.3.

The new asymptotic series (3.1) will possess a feature very different from the intrinsic asymptotic series $\{\lambda_m^h\}_{m\in\mathbb{N}}$ examined in Theorem 2.6, namely, the eigenvalue numbers $p_m^b(h)$ and $p_m^w(h)$ depend on the small parameter h in such a way that both grow unboundedly as $h \to +0$. This fact can be easily explained, according to representations (3.2) and (3.17), below: the entries in (3.1) are situated above the point $h^{-2}\pi^2$ but according to the asymptotic formula (2.31) with $m(h) = m$, see Theorem 2.6, the multiplicity of the spectrum (1.5) of the problem (1.2), (1.3) in the interval $(0, h^{-2}\pi^2)$ tends to infinity as $h \to +0$.

In the next sections, we present new formal asymptotic constructions for approximate eigenvalues, and the justification for the existence of true eigenvalues $\lambda^h_{p^b_m(h)}$ and $\lambda^h_{p^w_m(h)}$ in their vicinity is, of course, based only on the first part of Lemma 1.1, since (see the previous paragraph) it is not possible to prove a convergence result similar to Proposition 2.5, which is the most complicated part in the proof of Theorem 2.6. However, one related point deserves to be mentioned. The bottom and wall pieces cannot be treated independently since the construction of the appropriate approximate eigenfunction of the problem in $\sqcup^h$ requires the continuation of asymptotic structures from one subset of the beaker to the other one. Thus, we make with the help of constructing boundary layers including solutions of the inhomogeneous Dirichlet problem in $\mathbb{L}$. In view of general results in [13], in order to define these boundary layer appropriately, one needs to investigate the threshold resonance phenomenon in the problem (1.8), (1.9). This amounts to finding out whether problem (1.8), (1.9) has a nontrivial bounded solution, when spectral parameter has the threshold value $\mu = \pi^2$. The bounded solution may be either a trapped mode, which decays exponentially at infinity, or an almost standing wave that stabilizes in Π_j to $K_j \sin(\pi\xi_{3-j})$ as $\xi_j \to +\infty$ for both $j = 1, 2$ and with $|K_1| + |K_2| \neq 0$. The absence of the threshold resonance in the waveguide $\mathbb{L}$ is known, see [24], but it also follows from the inequalities derived in Section 1.3 due to either a sufficient condition in [26], or the first criterium in [1].

We emphasize that the asymptotic series (3.1), to be found in the next sections, demonstrate that the spectrum of the problem (1.2), (1.3) somehow “remembers and images” the whole shape of the beaker $\sqcup^h$. Note that not every thin domain has such “memory.” For example, a perturbation of the upper edge of the wall Θ^h may destroy the second series in (3.1) but does not affect Theorem 2.6 at all. Similar observations have been made in [18] and [22] in other geometries.

3.2 | The bottom

We consider the restriction of the problem (1.2), (1.3) to the bottom part $\Omega^h = \omega \times (0,h)$ of the beaker $\sqcup^h$; the Dirichlet condition is extended to the whole boundary $\partial\Omega^h$. Separation of variables gives the following eigenpairs of this auxiliary problem,

$$\{B_p^{bh}, \mathcal{Y}_p^{bh}(x)\} = \{h^{-2}\pi^2 + \beta_p^b, \sin(\pi h^{-1} z) Y_p^b(y)\}, \quad p \in \mathbb{N}, \tag{3.2}$$

where $\{\beta_p^b, Y_p^b\}$ are the eigenpairs of the Dirichlet Laplacian in the cross-section ω, subject to

$$(Y_p^b, Y_q^b)_\omega = \delta_{p,q}, \quad p,q \in \mathbb{N}. \tag{3.3}$$

However, extending the eigenfunction $\mathcal{Y}_p^{bh}$ by zero over the wall $\Theta^h \setminus \Upsilon^h$ does not yield an “almost eigenvector” of the operator $\mathcal{J}^h$, Section 1.4. The reason is the jump of $\partial_z \mathcal{Y}_p^{bh}$ on the thin ring $\{x \in \sqcup^h : z = h\}$. In other words, the discrepancy δ^h in formula (1.21) remains too large, and we cannot apply Lemma 1.1 to find an eigenvalue of the original $\sqcup^h$-problem in a small neighborhood of B_p^h. To obtain an appropriate approximation of an eigenfunction, we construct a boundary layer near the edge.

Proposition 3.1. *The homogeneous problem (1.8), (1.9) with the threshold parameter $\mu = \pi^2$ has the solutions*

$$Z_j(\xi) = X(\xi_j)(\xi_j + K_{jj}) \sin(\pi \xi_{3j}) + X(\xi_{3j}) K_{3-j}(\pi \xi_j) + \widetilde{Z}_j(\xi), \tag{3.4}$$

where $K_{jk} \in \mathbb{R}$, the remainder $\widetilde{Z}_j \in H_0^1(\mathbb{L})$ decays at the rate $O(e^{-|\xi|\pi\sqrt{3}})$ as $|\xi| \to +\infty$ and $X \in C^\infty(\mathbb{R})$ is the cut-off function with

$$X(t) = 1 \textit{ for } t \geqslant 2, \quad X(t) = 0 \textit{ for } t \leqslant 1, \quad 0 \leqslant X \leqslant 1. \tag{3.5}$$

Proof. Following [23, Ch.5], we introduce the following linear waves in the outlets Π_j, $j = 1,2$,

$$z_p^{\pm}(\xi) = (\xi_p \mp i) \sin(\pi \xi_{3-p}), \; p = 1,2, \; i = \sqrt{-1}.$$

According to [23, Ch. 5§3], see also [19] for our Dirichlet problem, there exist two solutions of the problem (1.8), (1.9) with $\mu = \pi^2$ growing linearly at infinity, with the representation

$$\zeta_p(\xi) = X(\xi_p) z_p^-(\xi) + \sum_{q=1,2} \mathrm{S}_{pq} X(\xi_q) z_q^+(\xi) + \widetilde{\zeta}_p(\xi).$$

Here, the remainders decay exponentially and the coefficients $\mathrm{S}_{pq} \in \mathbb{C}$ form a unitary and symmetric 2×2-matrix S, which is the threshold scattering matrix. In addition to the equality $\mathrm{S}_{12} = \mathrm{S}_{21}$, the geometric symmetry implies $\mathrm{S}_{11} = \mathrm{S}_{22}$ and, moreover, $z_2(\xi_1, \xi_2) = z_1(\xi_2, \xi_1)$ so that we set $p = 1$ in the following.

A result of [19], see also [20], assures that the matrix S has no eigenvalue -1 because there does not exist a threshold resonance [13, 16] in problem (1.8), (1.9), see Section 1.3. Thus, the algebraic system

$$\mathrm{Sc} + \mathrm{c} = (1, 0)$$

has a solution $\mathrm{c} = (\mathrm{c}_1, \mathrm{c}_2) \in \mathbb{C}^2$, and by a direct calculation, we find that the linear combination $\mathrm{c}_1\zeta_1 + \mathrm{c}_2\zeta_2$ includes the desired solution (3.4). Since the problem in question has no bounded solutions at the threshold, the coefficients K_{j1} and K_{j2} in (3.4) must be real, because the problem in question is invariant under complex conjugation and the solution with the fixed linear growth in a single outlet Π_j is unique (see the references above). □

We now compose the almost eigenpair of the operator $\mathcal{J}^h$, see (1.18), as follows:

$$\{T_p^{bh}, U_p^{bh}\} = \{h^2(\pi^2 + h^2\beta_p^h)^{-1}, \|W_p^{bh}; \mathcal{H}^h\|^{-1} W_p^h\}, \tag{3.6}$$

$$W_p^{bh}(x) = \mathcal{X}(y)\sin(\pi\zeta)Y_p^b(y) + h\chi(x)\partial_n Y_p^b(0, s)\widehat{Z}_1(\xi). \tag{3.7}$$

Here, $\mathcal{X} \in C_c^\infty(\omega)$ is a cut-off function such that

$$\mathcal{X}(y) = 1 \text{ for } y \in \omega \setminus V^d, \quad \mathcal{X}(y) = X(-h^{-1}n) \text{ for } y \in \omega \cap V^d. \tag{3.8}$$

Notice that the term

$$\widehat{Z}_1(\xi) = Z_1(\xi) - X(\xi_1)\xi_1\sin(\pi\xi_2) = \sum_{j=1,2} K_j X(\xi_j)\sin(\pi\xi_{3j}) + \widetilde{Z}_1(\xi), \tag{3.9}$$

extracted from (3.4), does not decay at infinity and therefore cannot be perfectly regarded as a boundary layer. However, the derivatives $\partial\widehat{Z}_1/\partial\xi_j$ decay exponentially in Π_j as $\xi_j \to +\infty$ and this helps to derive proper estimates. Namely, the cut-off function χ used in §2 and (3.7) can be required to have the properties that

$$\chi \text{ depends on } z \text{ in } \Theta^h \setminus \mathrm{Y}^h \text{ and on } n \text{ in } \Omega^h \setminus \mathrm{Y}^h. \tag{3.10}$$

First of all, we mention that in view of the normalization (3.3) and the Taylor formula

$$Y_p^b(y) = 0 + n\partial_n Y_p^b(0, s) + O(n^2), \ y \in \omega \cap V^d, \tag{3.11}$$

we have

$$\|W_p^{bh}; \mathcal{H}^h\|^2 = \frac{\pi^2}{h^2}\int_0^h \cos^2\left(\pi\frac{z}{h}\right)dz\, \|Y_p^b; L^2(\omega)\|^2 + O(1) = \frac{\pi^2}{2h} + O(1). \tag{3.12}$$

Then, similarly to (2.24), we write

$$\begin{aligned} \delta_p^h &= \sup |\langle \mathcal{J}^h U_p^{bh} - T_p^{bh} U_p^{bh}, \Psi^h\rangle_h| = \\ &= T_p^{bh}\|W_p^{bh}; \mathcal{H}^h\|^{-1} \sup |((\Delta_x + h^{-2}\pi^2 + \beta_p^b)W_p^{bh}, \Psi^h)_{\sqcup^h}|. \end{aligned} \tag{3.13}$$

Furthermore,

$$\begin{aligned}(\Delta_x + h^{-2}\pi^2 + \beta_p^b)W_p^{bh} &= \mathcal{X}^h(\Delta_x + h^{-2}\pi^2 + \beta_p^b)\sin(\pi\zeta)Y_p^b + \sin(\pi\zeta)[\Delta_y, \mathcal{X}^h]Y_p^b \\ &\quad + h\chi(\Delta_x + h^{-2}\pi^2)(\partial_n Y_p^b|_{n=0}\widehat{Z}_1) + h\beta_p^b\chi\partial_n Y_p^b|_{n=0}\widehat{Z}_1 + h[\Delta_x, \chi](\partial_n Y_p^b|_{n=0}\widehat{Z}_1) \\ &=: I_{p1}^{bh} + I_{p2}^{bh} + I_{p3}^{bh} + I_{p4}^{bh} + I_{p5}^{bh}.\end{aligned}$$

The first term I_{p1}^{bh} evidently vanishes. Owing to the Taylor formula (3.11) and the estimate (2.25), we have

$$\begin{aligned}&|(I_{p2}^{bh}, \Psi^h)_{\sqcup^h} - (\sin(\pi\zeta)[\Delta_x, \mathcal{X}^h](n\partial_n Y_p^b|_{n=0}), \Psi^h)_{\sqcup^h}| \\ &\qquad \leqslant ch\,\mathrm{meas}_2\Sigma^h \max_{y\in\Sigma^h}(h^{-4}n^4 + h^{-2}n^2)^{1/2}\|\Psi^h; L^2(\sqcup^h)\| \leqslant ch^2,\end{aligned} \tag{3.14}$$

where $\Sigma^h = \mathrm{supp}|\nabla_y \mathcal{X}^h|$ is a curved ring of width h, see (3.8) and (3.5).

The function (3.9) satisfies the differential equation

$$-\Delta_\xi \widehat{Z}_1(\xi) - \pi^2\widehat{Z}_1(\xi) = \sin(\pi\xi_2)\left[\frac{d^2}{d\xi_1^2}, X(\xi_1)\right]\xi_1,\ \xi \in \mathbb{L},$$

and therefore, performing the inverse coordinates change (2.2) shows that the scalar product $(I_{p3}^{bh}, \Psi^h)_{\sqcup^h}$ coincides with the one that is subtracted on the left-hand side of (3.14). Consequently, the sum of these products has the proper estimate.

Both functions $\partial_n Y_p$ and $\widehat{Z}_1$ are bounded and we have

$$|(I_{p4}^{bh}, \Psi^h)_{\sqcup^h}| \leqslant ch(\mathrm{meas}_3\sqcup^h)^{1/2}\|\Psi^h; L^2(\sqcup^h)\| \leqslant ch^{5/2}.$$

Finally, using the properties of (3.9) and (3.10) gives us

$$|(I_{p5}^{bh}, \Psi^h)_{\sqcup^h}| \leqslant ch(\mathrm{meas}_3 \sqcup^h\,(1 + h^{-1}e^{-\vartheta/h}))^{1/2}\|\Psi^h; L^2(\sqcup^h)\| \leqslant ch^{5/2}.$$

Combining the above estimates yields the bound $\delta_p^h \leqslant c_p h^{3/2}$ for the quantity (3.13), and Lemma 1.1 and an analog of (2.30) lead to the following assertion.

Theorem 3.2. *For all $p \in \mathbb{N}$, there exist positive h_p^b, c_p^b and an index $p^b(h)$ such that*

$$|\lambda_{p^b(h)}^h - h^{-2}\pi^2 - \beta_p^b| \leqslant c_p^b h^{1/2} \ \textit{for}\ h \in (0, h_p^b], \tag{3.15}$$

where $\{\beta_p^b\}_{p\in\mathbb{N}}$ is the monotone sequence of the eigenvalues of the problem

$$-\Delta_y Y^b(y) = \beta^b Y^b(y),\ y \in \omega, \qquad Y^b(y) = 0,\ y \in \partial\omega. \tag{3.16}$$

We will still comment this assertion in Section 3.4

3.3 | The wall

We restrict Equation (1.2) to the thin curved domain (1.1) and impose the Dirichlet condition on the whole boundary $\partial\Theta^h$. Then, we search for the asymptotics of the eigenpairs of the obtained problem in the form

$$\lambda^h = h^{-2}\pi^2 + h^{-1}\beta' + \beta^w + \cdots, \tag{3.17}$$

$$u^h(x) = \sin(\pi\eta)Y^w(s,z) + hY'(s,z;\eta) + h^2Y''(s,z;\eta)\ldots \tag{3.18}$$

Inserting these ansätze into the differential Equation (1.2) in Θ^h and applying the decomposition (2.3) show that the first correction terms in (3.17) and (3.18) must be found from the problem

$$\begin{aligned} &-\partial_\eta^2 Y'(s,z;\eta) - \pi^2 Y'(s,z;\eta) = (\beta'\sin(\pi\eta) - \kappa(s)\pi\cos(\pi\eta))Y^w(s,z),\ \eta\in(0,1),\\ &Y'(s,z;0) = Y'(s,z;1) = 0,\end{aligned}$$

and therefore,

$$\beta' = 0, \quad Y'(s,z;\eta) = \frac{1}{2}\kappa(s)\sin(\pi\eta)Y^w(s,z;\eta). \tag{3.19}$$

At the next step of the asymptotic procedure, we obtain the following problem:

$$\begin{aligned} -\partial_\eta^2 Y''(s,z;\eta) - \pi^2 Y''(s,z;\eta) =& \sin(\pi\eta)(\partial_s^2 + \partial_z^2 + \beta^w)Y^w(s,z) + \kappa(s)^2\eta\pi\cos(\pi\eta))Y^w(s,z)\\ &- \frac{1}{2}\kappa(s)^2\partial_{(}\eta\sin(\pi\eta))Y^w(s,z),\ \eta\in(0,1),\\ Y''(s,z;0) =& Y''(s,z;1) = 0.\end{aligned}$$

The compatibility condition in this problem converts into the partial differential equation

$$-\Delta_{(s,z)}Y^w(s,z) - \frac{1}{4}\kappa(s)^2Y^w(s,z) = \beta^wY^w(s,z),\ (s,z)\in\partial\omega\times(0,1), \tag{3.20}$$

which, owing to the Dirichlet condition (1.3) on $\{x\in\partial\sqcup^h : z = 1\}$ and the absence of the threshold resonance in the problem (1.8), (1.9), we supply with the boundary conditions

$$Y^w(s,0) = 0,\ Y^w(s,1) = 0,\ s\in\partial\omega. \tag{3.21}$$

Repeating the calculation at the end of Section 3.2 with obvious modifications that are due to formulas (3.17)–(3.19) yields the following assertion, which is very similar to Theorem 3.2.

Theorem 3.3. *For all $q\in\mathbb{N}$, there exist positive h_q^w, c_q^w and an index $q^w(h)$ such that*

$$|\lambda^h_{q^w(h)} - h^{-2}\pi^2 - \beta_q^w| \leqslant c_q^w h^{1/2} \text{ for } h\in(0,h_q^w], \tag{3.22}$$

where $\{\beta_q^w\}_{q\in\mathbb{N}}$ is the monotone sequence the eigenvalues of the Dirichlet problem (3.20), (3.21).

3.4 | Final remarks

In the same way as in Section 2.3, by a standard reasoning, one can consider the case when β is an eigenvalue of a problem (3.16) in ω and problem (3.20), (3.21) in $\partial\omega \times (0,1)$ with multiplicities $\varkappa^b$ and $\varkappa^w$, respectively. Namely, the interval $[h^{-2}\pi^2 + \beta - c_\beta, h^{-2}\pi^2 + \beta + c_\beta]$ contains $\varkappa^b + \varkappa^w$ different eigenvalues of the problem (1.2), (1.3). However, the total number of eigenvalues belonging to this interval remains again unknown. Also, all information on the corresponding eigenfunctions in the second part of Lemma 1.1 becomes quite implicit. Thus, we prefer not to prove or even formulate precisely these assertions.

In addition to (3.2), the Dirichlet problem in the isolated bottom Ω^h, discussed in Section 3.2, has the eigenpairs

$$\{h^{-2}k^2\pi^2 + \beta_p^b, \sin(\pi k h^{-1} z) Y_p^h(y)\}, \ p \in \mathbb{N}, \ k = 2, 3, 4, \dots,$$

so that one may try to construct other asymptotic sequences of eigenvalues $\lambda^h_{p^b_k(h)}$. However, to this end, it is necessary to study the threshold resonances in the problem (1.8), (1.9) with the spectral parameter $\mu = \pi^2 k^2$ inside the continuous spectrum, see [20]. This phenomenon must not exist in order to fulfil the Dirichlet condition on $\partial\omega$ in the limit problem, cf. reasoning in [13, 20]. A similar consideration applies to the Dirichlet problem in the isolated wall Θ^h.

Unfortunately, an investigation of the resonances at the higher thresholds $\mu = \pi^2 k^2 > \pi^2$ has not been made yet. Also no example of an eigenvalue $\mu_\bullet$ in the continuous spectrum $\wp_c = [\pi^2, +\infty)$ has been found in problem (1.8), (1.9). Similarly to §2, such embedded eigenvalue may give rise to an asymptotic series of eigenvalues $\lambda^h_{n_\bullet(h)}$, the eigenfunctions of which are located near the edge of the beaker Υ.

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