



# Second Order Regularity for a Linear Elliptic System Having BMO Coefficients

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**Abstract.** We consider linear elliptic systems whose prototype is

$$\operatorname{div} \Lambda [\exp(-|x|) - \log |x|] I Du = \operatorname{div} F + g \text{ in } B. \quad (0.1)$$

Here  $B$  denotes the unit ball of  $\mathbb{R}^n$ , for  $n > 2$ , centered in the origin,  $I$  is the identity matrix,  $F$  is a matrix in  $W^{1,2}(B, \mathbb{R}^{n \times n})$ ,  $g$  is a vector in  $L^2(B, \mathbb{R}^n)$  and  $\Lambda$  is a positive constant. Our result reads that the gradient of the solution  $u \in W_0^{1,2}(B, \mathbb{R}^n)$  to Dirichlet problem for system (0.1) is weakly differentiable provided the constant  $\Lambda$  is not large enough.

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## 1. Introduction

We consider the linear elliptic system

$$\operatorname{div} A(x) Du(x) = \operatorname{div} F(x) + g \quad (1.1)$$

in a bounded domain  $\Omega \subset \mathbb{R}^n$ ,  $n > 2$ , with  $A(x) = (A_{ij}(x))$  symmetric, positive definite matrix with measurable coefficients,  $F$  given matrix field in  $W^{1,2}(\Omega, \mathbb{R}^{n \times n})$  and  $g$  given vector field in  $L^2(\Omega, \mathbb{R}^n)$ . A vector field  $u$  in the Sobolev space  $W_0^{1,2}(\Omega, \mathbb{R}^n)$  is a weak solution of the Dirichlet problem:

$$\begin{cases} \operatorname{div} A(x) Du = \operatorname{div} F + g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

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if it verifies

$$\int_{\Omega} \langle A(x)Du(x), D\varphi(x) \rangle dx = \int_{\Omega} \langle F(x), D\varphi(x) \rangle dx + \int_{\Omega} \langle g(x), \varphi \rangle dx$$

$$\forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^n). \quad (1.3)$$

We assume that the entries of the matrix  $A$  lie in the John - Nirenberg space  $BMO$  and that for a.e.  $x \in \Omega$  the following condition holds

$$|A(x + he_i) - A(x)| \leq K(x)|h|, \quad i = 1, \dots, n,$$

with  $K(x) \in L^{n,\infty}(\Omega)$ , and  $h \in \mathbb{R}$  such that  $x + he_i \in \Omega$ . Thanks to a characterization of the Sobolev functions due to Hajlasz [20], the function  $K$  above plays the role of the derivative  $D_x A$ . In fact this condition describes a weak form of continuity since the function  $K$  may blow up at some points.

In the account of the typical functions of  $BMO$  and  $L^{n,\infty}$  respectively, it's obvious that the matrix

$$A(x) = \Lambda (e^{-|x|} - \log|x|) I, \quad (1.4)$$

with  $x \in B(0, 1) = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$ , satisfies the assumptions above. Here  $\Lambda$  is a positive constant and  $I$  denotes the identity matrix.

Note that the hypothesis  $A(x)$  in  $BMO$  guarantees that the problem (1.1) admits a unique solution (see Theorem 1.1 of [28]).

The regularity results for linear systems with continuous coefficients can be considered classical. The first remarkable contribution is due to Agmon, Douglis and Nirenberg (see [2] and [3]). Later regularity results of Schauder type in the class of Hölderian functions are proved by Campanato [8] and Morrey [26]. See also [9]. A full discussion can be found in [15] and [16].

The aim of this paper is to study the second order regularity of the solution of (1.1). More precisely, we prove the following:

**Theorem 1.1.** *Let  $\Omega$  be a regular Lipschitz domain. There exists  $\varepsilon_0 > 0$ , depending on  $n$ , such that, if*

$$\mathcal{D}_K \equiv \text{dist}_{L^{n,\infty}}(K(x), L^\infty) < \varepsilon_0, \quad (1.5)$$

then  $u \in W_{loc}^{2,2}(\Omega, \mathbb{R}^n)$  and

$$\int_{B_R} |D^2 u|^2 dx \leq c \int_{B_{2R}} \left( \left(1 + \frac{1}{R^2}\right) |Du|^2 + \frac{1}{R^2} |F|^2 + |DF|^2 + |g|^2 \right) dx,$$

for every ball  $B_{2R} \subset \subset \Omega$  and for a constant  $c$ , depending on  $n$ ,  $\mathcal{D}_K$  and the  $BMO$ -norm of  $A$ .

For the definition of regular domain, see the Sect. 2 below. Anyway, balls and cubes of  $\mathbb{R}^n$  are regular domains.

The condition (1.5) on the distance  $\mathcal{D}_K$  of  $K(x)$  to  $L^\infty$  is clearly satisfied if the derivatives of  $A(x)$  belong to any subspace of  $L^{n,\infty}$  in which  $L^\infty$  is dense, and then, in particular, if they belong to  $L^{n,q}$  with  $1 < q < \infty$ , since their distances to  $L^\infty$  are null. On the contrary,  $L^\infty$  is not dense in  $L^{p,\infty}$  for any  $p > 1$ . We point out that the condition (1.5) does not imply the smallness of the norm of  $K(x)$  in  $L^{n,\infty}$ . In fact, if  $A(x)$  is the matrix in (1.4), an elementary calculation shows that

it reduces to consider the constant  $\Lambda < \varepsilon_0 \omega_n^{-1/n}$ , where  $\omega_n$  denotes the measure of unit ball in  $\mathbb{R}^n$ . A value of  $\varepsilon_0$  is given in (3.13). It follows that assuming (1.5) is more general than considering a condition on the norm and allows us to present different settings of our result in a unified way. We explicitly remark that, thanks to the embedding theorem 2.2, our result applies if the entries of  $A(x)$  lie in  $W^{1,n}$ . The boundedness of the coefficients in a system of the type (1.1) is sometimes too restrictive in applications, as for example in physical process of diffusion or in mathematical finance.

The novelty of Theorem 1.1 is to consider systems with *BMO* coefficients, which feature is that they are allowed to be very irregular. In this case the energy functional

$$\int_{\Omega} \langle A(x) Du, Du \rangle dx$$

could not be bounded, then a priori we cannot use test functions in (1.3) proportional to the solution  $u$ . The Hodge decomposition and a generalization of Coifman, Rochberg and Weiss commutator result [12] allow us to establish an a priori estimate. Then the result follows by considering regularized approximating problems. If the boundary of  $\Omega$  is more regular, a global version of Theorem 1.1 is also available (Proposition 4.1).

The study of the second order regularity of solutions to linear equations with discontinuous coefficients goes back to C. Miranda [25], who considered the case of coefficients in  $W^{1,n}$ . Then a significant improvement has been given in [5] and in [10, 11]. Linear equations having coefficients in *BMO* with small norm have been addressed in [19]. More recently, a condition similar to (1.5) has been considered in [16] to study the  $L^p$  - regularity of a linear Dirichlet problem. In connection with the regularity of minimizers of functionals of the Calculus of Variations [1], the study of the regularity theory for systems had a remarkable development in last years. Recently in [13] linear systems with coefficients having in some directions locally small mean oscillation have been studied. We refer to [23, 24] and references therein for an almost complete recent treatment.

## 2. Preliminaries

This section is devoted to notation and preliminary results useful for our aims.

### 2.1. BMO Spaces

**Definition 2.1** ([7, 22]). Let  $\Omega$  be a cube or the entire space  $\mathbb{R}^n$ . The *BMO*( $\Omega$ ) space consists of all functions  $b$  which are integrable on every cube  $Q \subset \Omega$  with sides parallel to those of  $\Omega$  and satisfy:

$$\|b\|_* = \sup_Q \left\{ \frac{1}{|Q|} \int_Q |b - b_Q| dx \right\} < \infty,$$

where  $b_Q = \frac{1}{|Q|} \int_Q b(y) dy$  and  $|Q|$  denotes the Lebesgue measure of  $Q$ .

It is clear that the functional  $\|\cdot\|_*$  does not define a norm since it vanishes on constant functions. However  $BMO$  becomes a Banach space provided we identify functions which differ a. e. by a constant.

Bounded functions  $f$  are in  $BMO$ . On the other hand,  $BMO$  contains unbounded functions. The standard example of  $BMO$  function is

$$f(x) = \log |x|, \quad x \in B(0, 1).$$

We also recall the following inclusion

**Theorem 2.2** ([7]). *For any cube  $Q \subset \mathbb{R}^n$  the following inclusion holds with continuous embedding:*

$$W^{1,n}(Q) \hookrightarrow BMO(Q).$$

**2.2. Hodge Decomposition**

We shall now discuss briefly the Hodge decomposition of vector fields; for a more complete treatment see [21]. For a given vector field  $F = (f^1, \dots, f^n) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ ,  $1 < p < \infty$ , the Poisson equation  $\Delta u = \operatorname{div} F$  can be solved by using the Riesz transforms in  $\mathbb{R}^n$ ,  $\mathcal{R} = (R_1, \dots, R_n)$ ,

$$\nabla u = -(\mathcal{R} \otimes \mathcal{R})(F) =: \mathcal{H}(F).$$

Here the tensor product operator  $\mathcal{H} = -\mathcal{R} \otimes \mathcal{R} = -[R_{ij}]$  is the  $n \times n$  matrix of the second order Riesz transforms  $R_{ij} = R_i \circ R_j$ ,  $i, j = 1, \dots, n$ . Notice that the range of the operator

$$\mathcal{H} := Id - \mathcal{H} : L^p(\mathbb{R}^n, \mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^n)$$

consists of the divergence free vector fields. We then arrive at the familiar Hodge decomposition of  $F$

$$F = \nabla u + H, \quad \operatorname{div} H = 0.$$

Hence,  $L^p$ -estimates for Riesz transform yield an uniform estimate

$$\|\nabla u\|_{L^p} + \|H\|_{L^p} \leq c(p)\|F\|_{L^p}.$$

Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $G = G(x, y)$  the Green’s function. For  $h \in C_0^\infty(\Omega)$  the integral

$$u(x) = \int_{\Omega} G(x, y)h(y) dy$$

defines a solution of the Poisson equation  $\Delta u = h$  with  $u$  vanishing on the boundary of  $\Omega$ . If  $h$  has a divergence form, say  $h = \operatorname{div} F$  with  $F = (f^1, \dots, f^n) \in C_0^\infty(\Omega, \mathbb{R}^n)$ , then integration by parts yields

$$u(x) = - \int_{\Omega} \nabla_y G(x, y)F(y) dy.$$

Hence the gradient of  $u$  is expressed by a singular integral

$$\nabla u(x) = - \int_{\Omega} \nabla_x \nabla_y G(x, y)F(y) dy =: (\mathcal{H}_\Omega F)(x).$$

The continuity of  $\mathcal{H}_\Omega : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathbb{R}^n)$  is easily established by interpreting  $\mathcal{H}_\Omega F$  as the orthogonal projection of  $F$  into gradient fields.

Let  $\mathcal{D}^p(\Omega, \mathbb{R}^n)$  denote the closure of the range of the gradient operator  $\nabla : C_0^\infty(\Omega) \rightarrow L^p(\Omega, \mathbb{R}^n)$ ,  $1 < p < \infty$ . If  $\Omega$  is smooth, then  $\mathcal{K}_\Omega$  extends continuously to all  $L^p(\Omega, \mathbb{R}^n)$  spaces. Consequently the formula  $\nabla u = \mathcal{K}_\Omega F$  extends to all  $F \in L^p(\Omega, \mathbb{R}^n)$  giving a solution with  $\nabla u \in \mathcal{D}^p(\Omega, \mathbb{R}^n)$ ,  $1 < p < \infty$ .

**Definition 2.3** ([21]). A domain  $\Omega \subset \mathbb{R}^n$  will be called regular if the operator  $\mathcal{K}_\Omega$  acts boundedly in all  $L^p(\Omega, \mathbb{R}^n)$ -spaces, for  $1 < p < \infty$ .

For  $\Omega$  a regular domain we introduce, as before, the operator

$$\mathcal{H}_\Omega := Id - \mathcal{K}_\Omega : L^p(\Omega, \mathbb{R}^n) \rightarrow L^p(\Omega, \mathbb{R}^n).$$

Obviously, the range of  $\mathcal{H}_\Omega$  consists of the divergence free vector fields on  $\Omega$ . We have the Hodge decomposition of  $F \in L^p(\Omega, \mathbb{R}^n)$ ,

$$F = \nabla u + H, \quad \text{div } H = 0, \quad \nabla u \in \mathcal{D}^p(\Omega, \mathbb{R}^n). \tag{2.1}$$

We also have the uniform estimate

$$\|\nabla u\|_{L^p} + \|H\|_{L^p} \leq c(p, \Omega)\|F\|_{L^p}.$$

We now turn to commutators. We need the following definition.

**Definition 2.4** ([28]). Let  $k(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . We will call  $k$  a Calderon-Zygmund kernel (CZ kernel) if  $k$  satisfies the following properties

1.  $k(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ ,
2.  $k(x)$  is homogeneous of degree  $-n$ ,
3.  $\int_\Sigma k(x) d\sigma_x = 0$  where  $\Sigma$  is the unit sphere of  $\mathbb{R}^n$ .

Given such a kernel, one can define a bounded operator in  $L^p$ , called Calderon-Zygmund singular operator, as follows

$$Kf(x) = P.V.(k \star f)(x) := P.V. \int_{\mathbb{R}^n} k(x-y)f(y) dy.$$

Let  $\varphi \in BMO(\mathbb{R}^n)$  and  $k$  a CZ kernel. Following [12], we define, for  $f \in L^p(\mathbb{R}^n)$  ( $1 < p < \infty$ ), the commutator of  $\varphi$  and  $k$  as the principal value

$$C[\varphi, f] = \varphi(x)P.V.k \star f(x) - P.V.k \star (\varphi f)(x).$$

**Theorem 2.5** ([12]). Under the previous assumptions on  $\varphi$  and  $k$ ,  $C[\varphi, f]$  is well defined for  $f \in L^p$ . Moreover  $C[\varphi, f]$  is a bounded operator in  $L^p(\mathbb{R}^n)$ , i.e. for some constant  $c = c(n, p, \|k\|_{L^2})$  we have

$$\|C[\varphi, f]\|_{L^p} \leq c\|\varphi\|_*\|f\|_{L^p}.$$

We will state a generalization of Theorem 2.5 in finite - dimensional normed spaces. Let  $E$  be a finite-dimensional normed space of dimension  $m$ ,

$$T : L^p(\mathbb{R}^n, E) \mapsto L^p(\mathbb{R}^n, E), \quad 1 < p < \infty,$$

a Calderon - Zygmund integral operator. Fixed a basis in  $E$ , we can associate to the operator  $T$  a  $m \times m$  matrix of Calderon-Zygmund operators

$$T_{ij} : L^p(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n).$$

Given  $A \in BMO(\mathbb{R}^n, Aut(E))$ , where  $Aut(E)$  denotes the space of linear maps from  $E$  to  $E$ , and

$$\|A\|_* = \sup_Q \frac{1}{|Q|} \int_Q \|A - A_Q\| dx$$

is the  $BMO$  norm, we consider the range of  $A$ :

$$\mathcal{A} = \{A(z); z \in \mathbb{R}^n\} \subset Aut(E).$$

Let us state the theorem:

**Theorem 2.6** ([28]). *Given  $f \in L^p(\mathbb{R}^n, E)$  such that commutes with the range of  $A$ ,*

$$T[A(z)f](x) = A(z)Tf(x) \tag{2.2}$$

for almost every  $z \in \mathbb{R}^n$ , then

$$\|T(Af) - ATf\|_{L^p} \leq c\|A\|_*\|f\|_{L^p}.$$

Since we would apply Theorem 2.6 to the projection  $T$  onto divergence free matrix fields, we need to identify the range of the compatibility condition (2.2). Take now  $A^0$  a  $n \times n$ -matrix,  $A^0$  induces a linear transformation given by  $A^0(X) = A^0X$ , the row by column product of matrices. It is easy to verify that  $A^0$  commutes with the operator  $T$ ; in fact, multiplying the decomposition of  $F$  by  $A^0$  we get

$$A^0F = A^0Du + A^0H = D(A^0u) + A^0H \quad \text{and} \quad div(A^0H) = 0$$

in the sense of distributions:

$$\int \langle A^0H, D\varphi \rangle = \int \langle H, A^{0t}D\varphi \rangle = \int \langle H, D(A^{0t}\varphi) \rangle,$$

where  $A^{0t}$  denotes the transpose of  $A^0$ . It can be shown a local version of Theorem 2.5 in the case of Hodge decomposition of matrix fields. Therefore, given any  $BMO$ -matrix field  $A(x)$ , we get:

**Lemma 2.7.** ([28]) *For  $\Omega = \mathbb{R}^n$  or  $\Omega$  regular domain, let  $T$  be the projection onto divergence free matrix fields, and  $A(x)$  a  $BMO$  - matrix field. Then*

$$\|TA(F) - AT(F)\|_{L^p} \leq c\|A\|_*\|F\|_{L^p}$$

for every  $F \in L^p(\Omega, R^{n \times n})$ .

### 2.3. Lorentz Spaces

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . Given  $1 < p, q < \infty$ , the Lorentz space  $L^{p,q}(\Omega)$  consists of all measurable functions  $g$  defined on  $\Omega$  for which the quantity

$$\|g\|_{L^{p,q}}^q = p \int_0^\infty |\Omega_t(g)|^{\frac{q}{p}} t^{q-1} dt$$

is finite, where  $\Omega_t(g) = \{x \in \Omega : |g(x)| > t\}$  and  $|\Omega_t|$  is the Lebesgue measure of  $\Omega_t$ . Note that  $\|\cdot\|_{L^{p,q}}$  is equivalent to a norm and  $L^{p,q}$  becomes a Banach space when endowed with it. For  $p = q$ , the Lorentz space  $L^{p,p}$  reduces to the standard

Lebesgue space  $L^p$ . For  $q = \infty$ , the class  $L^{p,\infty}$  consists of all measurable functions  $g$  defined on  $\Omega$  such that

$$\|g\|_{L^{p,\infty}}^p = \sup_{t>0} t^p |\Omega_t(g)| < \infty$$

and it coincides with the Marcinkiewicz class, weak- $L^p$ . For Lorentz spaces the following inclusions hold

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega),$$

whenever  $1 \leq q < p < r \leq \infty$

Fundamental to us will be the Sobolev embedding theorem in Lorentz spaces (see [4]).

**Theorem 2.8.** *Let us assume that  $1 < p < n$ ,  $1 \leq q \leq p$ , then any function  $u \in W_0^{1,1}(\Omega)$  such that  $|\nabla u| \in L^{p,q}(\Omega)$  actually belongs to  $L^{p^*,q}(\Omega)$  and*

$$\|u\|_{L^{p^*,q}} \leq S_{n,p} \|\nabla u\|_{L^{p,q}}.$$

Here  $p^* = \frac{np}{n-p}$  and  $S_{n,p} = \omega_n^{-\frac{1}{n}} \frac{p}{n-p}$ , where  $\omega_n$  is the Lebesgue measure of the unit ball in  $\mathbb{R}^n$ .

We define the distance of a given  $f \in L^{p,\infty}$  to  $L^\infty$  as

$$dist_{L^{p,\infty}}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g\|_{L^{p,\infty}}.$$

To find a formula for the distance, we consider the truncation operator. For  $k > 0$  and  $y \in \mathbb{R}$ , we set

$$T_k(y) = \min\{k, \max\{-k, y\}\}.$$

Then

$$dist_{L^{p,\infty}}(f, L^\infty) = \lim_{k \rightarrow \infty} \|f - T_k f\|_{L^{p,\infty}}.$$

Indeed,  $\forall g \in L^\infty, \forall k \geq \|g\|_{L^\infty}$ , we have for almost every  $x \in \Omega$ ,

$$|f(x) - g(x)| \geq |f(x) - T_k f(x)|.$$

Let  $\Omega$  be the unit ball of  $\mathbb{R}^n$ . The function

$$f(x) = \frac{1}{|x|}$$

belongs to  $L^{n,\infty}$ . Setting  $\omega_n = |\Omega|$ , we have

$$\|f - T_k f\|_{L^{p,\infty}} = \omega_n^{1/n}$$

and it does not depend on  $k$ . For more details, see [15].

We recall the following relevant properties.

**Lemma 2.9** ([6]). *If  $E \in L^{p,\infty}(\mathbb{R}^n)$ ,  $1 < p < \infty$ , and  $f \in L^1(\mathbb{R}^n)$ , then  $E \star f \in L^{p,\infty}(\mathbb{R}^n)$  and*

$$\|E \star f\|_{L^{p,\infty}} \leq \|E\|_{L^{p,\infty}} \|f\|_{L^1}.$$

**Theorem 2.10.** (Hölder’s inequality in Lorentz spaces, [27]) *If  $0 < p_1, p_2, p < \infty$  and  $0 < q_1, q_2, q \leq \infty$  obey  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$  and  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$  then*

$$\|fg\|_{L^{p,q}} \leq \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}$$

whenever the right-hand side norms are finite.

**2.4. Difference Quotients**

**Definition 2.11** ([18]). Let  $f(x)$  be a function defined in an open set  $\Omega \subset \mathbb{R}^n$ , and let  $h$  be a real number. We call the difference quotient of  $f$  with respect to  $x_s$  the function

$$\Delta_{s,h}f(x) = \frac{f(x + he_s) - f(x)}{h} \equiv \frac{\tau_{s,h} f(x)}{h},$$

where  $e_s$  denotes the direction of the  $x_s$  axis and  $\tau_{s,h}$  is the finite difference operator.

When no confusion can arise, we shall omit the index  $s$ , and we shall write simply  $\Delta_h$  instead of  $\Delta_{s,h}$ .

The function  $\Delta_{s,h}f$  is defined in the set

$$\Delta_{s,h}\Omega := \{x \in \Omega : x + he_s \in \Omega\},$$

and hence in the set

$$\Omega_{|h|} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > |h|\}.$$

The following properties of the difference quotients are immediate:

- If  $f \in W^{1,p}(\Omega)$ , then  $\Delta_h f \in W^{1,p}(\Omega_{|h|})$ , and

$$D_i(\Delta_h f) = \Delta_h(D_i f).$$

- If at least one of the functions  $f$  or  $g$  has support contained in  $\Omega_{|h|}$ , then

$$\int_{\Omega} f \Delta_h g \, dx = - \int_{\Omega} g \Delta_{-h} f \, dx. \tag{2.3}$$

- We have

$$\Delta_h(fg)(x) = f(x + he_s)\Delta_h g(x) + g(x)\Delta_h f(x).$$

*Remark 2.12.* It follows immediately from (2.3) that the derivatives  $D_s g$  of a Lipschitz-continuous function  $g$ , which exist almost everywhere as limits of the difference quotient  $\Delta_{s,h}g$ , coincide with its weak derivatives. In fact, if  $f$  is a test function, we can pass to the limit in (2.3), getting

$$\int f D_s g \, dx = - \int g D_s f \, dx.$$

In other words, we have  $Lip(\Omega) = W^{1,\infty}(\Omega)$ .

For  $R > 0$  and  $x_0 \in \mathbb{R}^n$ , we define

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

but in the case no ambiguity arises, we shall use the short notation  $B_R$ .



**Lemma 2.13.** ([18]) *There exists a constant  $c(n)$  such that if  $v \in W^{1,p}(\Omega)$ ,  $\Sigma \subset\subset \Omega$ ,  $1 < p < \infty$ ,  $s \in \{1, \dots, n\}$  and  $|h| < h_0 = \frac{1}{10\sqrt{n}} \text{dist}(\Sigma, \partial\Omega)$*

$$\|\Delta_{s,h}v\|_{L^p(\Sigma)} \leq c\|D_s v\|_{L^p(\Omega)}.$$

Moreover, if  $0 < \varrho < R$ ,  $|h| < R - \varrho$ ,

$$\int_{B_\varrho} |v(x + he_s)|^p dx \leq c(n,p) \int_{B_R} |v(x)|^p dx.$$

Finally we recall the following useful lemma:

**Lemma 2.14.** ([17]) *For  $R_0 < R_1$ , consider a bounded function  $f : [R_0, R_1] \rightarrow [0, \infty)$  with*

$$f(s) \leq \vartheta f(t) + \frac{A}{(s-t)^\alpha} + B \quad \text{for all } R_0 < s < t < R_1,$$

where  $A$ ,  $B$  and  $\alpha$  denote non-negative constants and  $\vartheta \in (0, 1)$ . Then we have

$$f(R_0) \leq c(\alpha, \vartheta) \left( \frac{A}{(R_1 - R_0)^\alpha} + B \right).$$

### 3. A Priori Estimate

**Theorem 3.1.** *Let  $\Omega$  be a regular Lipschitz domain. If the solution  $u$  of (1.2) is in  $W_{loc}^{2,2}(\Omega, \mathbb{R}^n)$ , then there exists  $\varepsilon_0 > 0$ , depending only on  $n$ , such that, if  $\mathcal{D}_K < \varepsilon_0$ , the following estimate holds*

$$\int_{B_R} |D^2 u|^2 dx \leq c \int_{B_{2R}} \left( \left(1 + \frac{1}{R^2}\right) |Du|^2 + \frac{1}{R^2} |F|^2 + |DF|^2 + |g|^2 \right) dx,$$

for all  $B_{2R} \subset\subset \Omega$  and for a constant  $c$  depending on  $n$ ,  $\mathcal{D}_K$  and the BMO-norm of  $A$ .

*Proof.* If  $u \in W_0^{1,2}(\Omega)$  is the solution of (1.2), then for every  $\varepsilon > 0$  we have that  $A(x)Du \in L^{2-\varepsilon}(\Omega, \mathbb{R}^{n \times n})$  since  $A$  belongs to  $L^p$  for every  $1 < p < \infty$ . We decompose as in (2.1)

$$ADu = D\Psi + H, \tag{3.1}$$

with  $D\Psi \in L^{2-\varepsilon}(\Omega, \mathbb{R}^{n \times n})$  and  $H \in L^{2-\varepsilon}(\Omega, \mathbb{R}^{n \times n})$  divergence free vector field. Since  $u$  solves problem (1.2), we get

$$\text{div} D\Psi = \Delta\Psi = \text{div} F + g,$$

then, by the classical theory,  $D\Psi \in L^2(\Omega, \mathbb{R}^{n \times n})$  and

$$\|D\Psi\|_{L^2} \leq \|F\|_{L^2} + \|g\|_{L^2}.$$

Let us examine the other term of the Hodge decomposition:  $H$  in (3.1) is a commutator with BMO-matrix of a gradient field in  $L^2$ ; using Lemma 2.7 we conclude that  $H \in L^2$  and

$$\|H\|_{L^2} \leq c\|A\|_* \|Du\|_{L^2},$$

where  $c = c(n)$ . Finally from (3.1) we deduce that  $A(x)Du$  belongs to  $L^2(\Omega, \mathbb{R}^{n \times n})$  and we get

$$\|A(x)Du\|_{L^2} \leq c(n)(\|A\|_* \|Du\|_{L^2} + \|F\|_{L^2} + \|g\|_{L^2}).$$

Now for a fixed ball  $B_{2R} \subset\subset \Omega$  and radii  $R < s < t < 2R$  with  $R$  small enough, consider a function  $\xi \in C_0^\infty(B_t)$ ,  $0 \leq \xi \leq 1$ ,  $\xi = 1$  on  $B_s$ ,  $|\nabla \xi| \leq \frac{1}{t-s}$  and set  $\psi = \xi^2 \tau_h u$  for sufficiently small  $h$ . Since  $u$  is a weak solution of (1.1), we are able to use  $\varphi = \tau_{-h} \psi$  as test function in (1.3). Then

$$\int_{B_t} \langle A(x)Du, D\varphi \rangle dx = \int_{B_t} \langle F(x), D\varphi \rangle dx + \int_{B_t} \langle g(x), \varphi \rangle dx,$$

and by virtue of the properties of difference quotients

$$\int_{B_t} \langle \tau_h(A(x)Du), D(\xi^2 \tau_h u) \rangle dx = \int_{B_t} \langle \tau_h F(x), D\psi \rangle dx - \int_{B_t} \langle g(x), \varphi \rangle dx.$$

It follows that

$$\begin{aligned} & \int_{B_t} \xi^2 \langle \tau_h(A(x)Du), D\tau_h u \rangle dx + 2 \int_{B_t} \xi \langle \tau_h(A(x)Du), \nabla \xi \otimes \tau_h u \rangle dx \\ &= \int_{B_t} \langle \tau_h F, D\psi \rangle dx - \int_{B_t} \langle g, \tau_{-h}(\xi^2 \tau_h u) \rangle dx. \end{aligned} \tag{3.2}$$

We remark that

$$\tau_h(A(x)Du) = A(x + he_i)D\tau_h u + (\tau_h A(x))Du.$$

Then from (3.2) we get

$$\begin{aligned} & \int_{B_t} \xi^2 \langle A(x + he_i)D\tau_h u, D\tau_h u \rangle dx \\ &= - \int_{B_t} \xi^2 \langle (\tau_h A(x))Du, D\tau_h u \rangle dx \\ &\quad - 2 \int_{B_t} \xi \langle A(x + he_i)D\tau_h u, \nabla \xi \otimes \tau_h u \rangle dx \\ &\quad - 2 \int_{B_t} \xi \langle (\tau_h A(x))Du, \nabla \xi \otimes \tau_h u \rangle dx \\ &\quad + \int_{B_t} \langle \tau_h F, D\psi \rangle dx \\ &\quad - \int_{B_t} \langle g, \tau_{-h}(\xi^2 \tau_h u) \rangle dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.3}$$

Now let  $K_0 \in L^\infty(\Omega)$ . The use of Hölder’s inequality in Lorentz spaces (Theorem 2.10), together with Young’s inequality with a constant  $\nu \in (0, 1)$  that will be chosen later, yields

$$\begin{aligned}
 |I_1| &\leq \int_{B_t} \xi^2 |h| K(x) |Du| |D\tau_h u| dx \\
 &\leq \frac{\nu}{2} \int_{B_t} \xi^2 |D\tau_h u|^2 dx + \frac{1}{2\nu} \int_{B_t} |h|^2 |K(x) - K_0|^2 |\xi Du|^2 dx \\
 &\quad + \int_{B_t} \xi^2 \|K_0\|_{L^\infty} |h| |Du| |D\tau_h u| dx \\
 &\leq \frac{\nu}{2} \int_{B_t} \xi^2 |D\tau_h u|^2 dx + \frac{|h|^2}{2\nu} \|K(x) - K_0\|_{L^{n,\infty}(B_t)}^2 \|\xi Du\|_{L^{2^*,2}(B_t)}^2 \\
 &\quad + \frac{\nu}{2} \int_{B_t} \xi^2 |D\tau_h u|^2 dx + \frac{|h|^2}{2\nu} \int_{B_t} \xi^2 \|K_0\|_{L^\infty}^2 |Du|^2 dx.
 \end{aligned}$$

Finally by Theorem 2.8

$$\begin{aligned}
 |I_1| &\leq \nu \int_{B_t} \xi^2 |D\tau_h u|^2 dx + \\
 &\quad + \frac{|h|^2}{2\nu} S_{2,n}^2 \|K(x) - K_0\|_{L^{n,\infty}}^2 \|D(\xi Du)\|_{L^2(B_t)}^2 + \\
 &\quad + \frac{|h|^2}{2\nu} \int_{B_t} \xi^2 \|K_0\|_{L^\infty}^2 |Du|^2 dx. \tag{3.4}
 \end{aligned}$$

Next we estimate  $I_2$ .

$$\begin{aligned}
 |I_2| &\leq 2 \int_{B_t - B_s} |\xi| |A(x + he_i)\tau_h u| |D\tau_h u| |\nabla \xi| dx \\
 &\leq \int_{B_t - B_s} \xi^2 |D\tau_h u|^2 dx + \int_{B_t - B_s} |\nabla \xi|^2 |A(x + he_i)\tau_h u|^2 dx \\
 &\leq \int_{B_t - B_s} \xi^2 |D\tau_h(u)|^2 dx + \frac{1}{(t-s)^2} \int_{B_t - B_s} |A(x + he_i)\tau_h u|^2 dx. \tag{3.5}
 \end{aligned}$$

Now we estimate  $I_3$ . Again by Hölder’s and Young’s inequalities, we get

$$\begin{aligned}
 |I_3| &\leq 2 \int_{B_t} |\xi| |A(x + he_i) - A(x)| |Du| |\nabla \xi| |\tau_h u| dx \\
 &\leq \int_{B_t} |\xi|^2 |A(x + he_i) - A(x)|^2 |Du|^2 dx + \int_{B_t - B_s} |\nabla \xi|^2 |\tau_h u|^2 dx \\
 &\leq 2 \int_{B_t} |h|^2 |\xi|^2 |K(x) - K_0|^2 |Du|^2 dx + 2 \int_{B_t} |h|^2 |\xi|^2 \|K_0\|_{L^\infty}^2 |Du|^2 dx \\
 &\quad + \int_{B_t - B_s} |\nabla \xi|^2 |\tau_h u|^2 dx \\
 &\leq 2|h|^2 \|K(x) - K_0\|_{L^{n,\infty}}^2 \|\xi Du\|_{L^{2^*,2}}^2 + 2|h|^2 \|K_0\|_{L^\infty}^2 \int_{B_t} \xi^2 |Du|^2 dx \\
 &\quad + \frac{1}{(t-s)^2} \int_{B_t - B_s} |\tau_h u|^2 dx.
 \end{aligned}$$

Applying Theorem 2.8, then

$$\begin{aligned}
 |I_3| &\leq 4S_{2,n}^2|h|^2\|K(x) - K_0\|_{L^{n,\infty}}^2\|\xi D^2u\|_{L^2}^2 + \\
 &\quad + 2|h|^2\|K_0\|_{L^\infty}^2\int_{B_t}\xi^2|Du|^2 dx + \\
 &\quad + \frac{1}{(t-s)^2}\int_{B_t-B_s}|\tau_h u|^2 dx + \\
 &\quad + 4|h|^2S_{2,n}^2\frac{1}{(t-s)^2}\|K(x) - K_0\|_{L^{n,\infty}}^2\int_{B_t-B_s}|Du|^2 dx. \tag{3.6}
 \end{aligned}$$

Finally we estimate  $I_4$ :

$$\begin{aligned}
 |I_4| &= \left| \int_{B_t} \langle \tau_h F, D(\xi^2 \tau_h u) \rangle dx \right| \\
 &\leq \frac{\nu}{2} \int_{B_t} \xi^2 |D\tau_h u|^2 dx + \frac{1}{2\nu} \int_{B_t} \xi^2 |\tau_h F|^2 dx \\
 &\quad + \int_{B_t} \xi^2 |\tau_h F|^2 dx + \int_{B_t} |\nabla \xi|^2 |\tau_h u|^2 dx \\
 &\leq \frac{\nu}{2} \int_{B_t} \xi^2 |D\tau_h u|^2 dx + \left( \frac{1}{2\nu} + 1 \right) \int_{B_t} \xi^2 |\tau_h F|^2 dx \\
 &\quad + \frac{1}{(t-s)^2} \int_{B_t-B_s} |\tau_h u|^2 dx, \tag{3.7}
 \end{aligned}$$

and  $I_5$ :

$$\begin{aligned}
 |I_5| &= \left| h \int_{B_t} g \frac{\tau_{-h}(\xi^2 \tau_h u)}{h} dx \right| \\
 &\leq \frac{1}{2\nu} |h|^2 \int_{B_t} |g(x)|^2 dx + \frac{\nu}{2} \int_{B_t} \left| \frac{\tau_{-h}(\xi^2 \tau_h u)}{h} \right|^2 dx \\
 &\leq \frac{1}{2\nu} |h|^2 \int_{B_t} |g(x)|^2 dx + \frac{\nu}{2} \int_{B_t} \left| \frac{\tau_{-h}(\xi^2)(-\tau_{-h}u) + \xi^2 \tau_{-h}(\tau_h u)}{h} \right|^2 dx \\
 &\leq \frac{1}{2\nu} |h|^2 \int_{B_t} |g(x)|^2 dx + \nu \int_{B_t} \left| \tau_{-h}(\xi^2) \frac{\tau_{-h}u}{h} \right|^2 dx \\
 &\quad + \nu \int_{B_t} \xi^4 \left| \frac{\tau_{-h}(\tau_h u)}{h} \right|^2 dx \\
 &= \frac{1}{2\nu} |h|^2 \int_{B_t} |g(x)|^2 dx + \nu |h|^2 \int_{B_t} \left| \frac{\tau_{-h}(\xi^2)}{h} \right|^2 \left| \frac{\tau_{-h}u}{h} \right|^2 dx \\
 &\quad + \nu \int_{B_t} \xi^4 \left| \frac{\tau_{-h}(\tau_h u)}{h} \right|^2 dx. \tag{3.8}
 \end{aligned}$$

Combining estimates (3.3)–(3.8), we get

$$\begin{aligned}
 \int_{B_t} \xi^2 |D\tau_h u|^2 dx &\leq \frac{3}{2}\nu \int_{B_t} \xi^2 |D\tau_h u|^2 dx \\
 &+ \left(\frac{1}{\nu} + 4\right) |h|^2 S_{2,n}^2 \|K(x) - K_0\|_{L^{n,\infty}}^2 \int_{B_t} |\xi D^2 u|^2 dx \\
 &+ \frac{1}{(t-s)^2} \int_{B_t} |A(x + he_i)\tau_h u|^2 dx + \left(2 + \frac{1}{2\nu}\right) |h|^2 \|K_0\|_{L^\infty}^2 \int_{B_t} \xi^2 |Du|^2 dx \\
 &+ \frac{2}{(t-s)^2} \int_{B_t - B_s} |\tau_h u|^2 dx + \left(1 + \frac{1}{2\nu}\right) \int_{B_t} \xi^2 |\tau_h F|^2 dx + \int_{B_t - B_s} \xi^2 |D\tau_h u|^2 dx \\
 &+ |h|^2 \left(\frac{1}{\nu} + 4\right) \|K - K_0\|_{L^{n,\infty}}^2 S_{2,n}^2 \frac{1}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx \\
 &+ \frac{1}{2\nu} |h|^2 \int_{B_t} |g(x)|^2 dx + \nu |h|^2 \int_{B_t} \left| \frac{\tau_{-h}(\xi^2)}{h} \right|^2 \left| \frac{\tau_{-h}u}{h} \right|^2 dx \\
 &+ \nu \int_{B_t} \xi^4 \left| \frac{\tau_{-h}(\tau_h u)}{h} \right|^2 dx.
 \end{aligned} \tag{3.9}$$

Next we divide by  $|h|^2$  in (3.9) and, by Lemma 2.13, as  $h \rightarrow 0^+$ , we get

$$\begin{aligned}
 \left(1 - \frac{5}{2}\nu\right) \int_{B_t} \xi^2 |D^2 u|^2 dx &\leq \left(\frac{1}{\nu} + 4\right) S_{2,n}^2 \|K - K_0\|_{L^{n,\infty}}^2 \int_{B_t} \xi^2 |D^2 u|^2 dx \\
 &+ \int_{B_t - B_s} \xi^2 |D^2 u|^2 dx + \frac{1}{(t-s)^2} \int_{B_t} |A(x) Du|^2 dx \\
 &+ \left(\frac{1}{2\nu} + 2\right) \|K_0\|_{L^\infty}^2 \int_{B_t} \xi^2 |Du|^2 dx + \frac{2}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx \\
 &+ \frac{4\nu}{(t-s)^2} \int_{B_t} |Du|^2 dx + \left(\frac{1}{2\nu} + 1\right) \int_{B_t} \xi^2 |DF|^2 dx \\
 &+ \left(\frac{1}{\nu} + 4\right) S_{2,n}^2 \|K - K_0\|_{L^{n,\infty}}^2 \frac{1}{(t-s)^2} \int_{B_t - B_s} |Du|^2 dx + \frac{1}{2\nu} \int_{B_t} |g(x)|^2 dx.
 \end{aligned} \tag{3.10}$$

Now we put  $\nu = \nu_0 := \frac{\sqrt{65}-5}{20}$ , in order to maximize the function  $\frac{1-\frac{5}{2}\nu}{4+\frac{1}{\nu}}$ , and  $\eta := \frac{1}{S_{2,n}} \sqrt{\frac{1-\frac{5}{2}\nu_0}{4+\frac{1}{\nu_0}}} = \frac{\sqrt{9-\sqrt{65}}}{4S_{2,n}}$ . Let  $\varepsilon_0$  be a number such that  $0 < \varepsilon_0 < \eta$ . If

$$\mathcal{D}_K < \varepsilon_0, \tag{3.11}$$

then we can choose  $K_0 \in L^\infty(\Omega)$  such that  $\left(1 - \frac{5}{2}\nu_0 - \left(\frac{1}{\nu_0} + 4\right) S_{2,n}^2 \|K - K_0\|_{L^{n,\infty}}^2\right) > \left(1 - \frac{5}{2}\nu_0 - \left(\frac{1}{\nu_0} + 4\right) S_{2,n}^2 \varepsilon_0^2\right) > 0$ . Then, by reabsorbing the first term of the right hand side of (3.10) in the left hand side, since  $\xi = 1$  on  $B_s$  and  $0 \leq \xi \leq 1$ , we get

$$\begin{aligned}
 C \int_{B_s} |D^2u|^2 dx &\leq \int_{B_t - B_s} |D^2u|^2 dx + \frac{1}{(t-s)^2} \int_{B_t} |A(x) Du|^2 dx \\
 &+ \left(\frac{1}{2\nu_0} + 2\right) \|K_0\|_{L^\infty}^2 \int_{B_t} |Du|^2 dx + \left(3 + \frac{3}{2}\nu_0\right) \frac{1}{(t-s)^2} \int_{B_t} |Du|^2 dx \\
 &+ \left(\frac{1}{2\nu_0} + 1\right) \int_{B_t} |DF|^2 dx + \frac{1}{2\nu_0} \int_{B_t} |g(x)|^2 dx,
 \end{aligned}$$

where  $C = 1 - \frac{5}{2}\nu_0 - \left(\frac{1}{\nu_0} + 4\right) S_{2,n}^2 \varepsilon_0^2$ .

Now we fill the hole, having

$$\begin{aligned}
 \int_{B_s} |D^2u|^2 dx &\leq \frac{1}{C+1} \int_{B_t} |D^2u|^2 dx + \frac{1}{C+1} \left(\frac{1}{(t-s)^2} \int_{B_t} |A(x) Du|^2 dx \right. \\
 &+ \left(\frac{1}{2\nu_0} + 2\right) \|K_0\|_{L^\infty}^2 \int_{B_t} |Du|^2 dx + \left(3 + \frac{3}{2}\nu_0\right) \frac{1}{(t-s)^2} \int_{B_t} |Du|^2 dx \\
 &\left. + \left(\frac{1}{2\nu_0} + 1\right) \int_{B_t} |DF|^2 dx + \frac{1}{2\nu_0} \int_{B_t} |g(x)|^2 dx\right).
 \end{aligned} \tag{3.12}$$

Then by Lemma 2.14

$$\begin{aligned}
 \int_{B_R} |D^2u|^2 dx &\leq c \left(\frac{1}{R^2} \int_{B_{2R}} (|A|_*^2 |Du|^2 + |F|^2) dx \right. \\
 &+ \left(1 + \frac{1}{R^2}\right) \int_{B_{2R}} |Du|^2 dx \\
 &\left. + \int_{B_{2R}} |DF|^2 dx + \int_{B_{2R}} |g(x)|^2 dx\right),
 \end{aligned} \tag{3.13}$$

where  $c = c(n, \mathcal{D}_K)$ , and therefore we have the result. □

*Remark 3.2.* The bound in (3.11) could be not optimal. Anyway it is comparable with analogous bounds in [14].

*Remark 3.3.* We explicitly observe that the dependence of the constant in (3.13) on  $\mathcal{D}_K$  occurs only through the norm of  $K_0$  in  $L^\infty$ .

### 4. Regularity

Given a symmetric matrix-valued function  $A(x) \in BMO(\Omega, \mathbb{R}^{n \times n})$ ,  $\Omega$  Lipschitz domain, we assume that

$$\langle A(x)Y, Y \rangle \geq \|Y\|^2$$

for every matrix  $Y \in \mathbb{R}^{n \times n}$ . We consider the following system with Dirichlet boundary conditions :

$$\begin{cases} \operatorname{div}(A Du) = \operatorname{div} F + g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where  $u : \Omega \mapsto \mathbb{R}^n$  is a  $W_0^{1,2}$  vector-valued function,  $F$  is a field in  $W^{1,2}(\Omega, \mathbb{R}^{n \times n})$  and  $g$  is a field in  $L^2(\Omega, \mathbb{R}^n)$ . Notice that we do not require that  $A$  belongs to  $L^\infty$ , the system have to be understood in the weak sense

$$\int_{\Omega} \langle A(x)Du(x), D\varphi(x) \rangle dx = \int_{\Omega} \langle F(x), D\varphi(x) \rangle dx + \int_{\Omega} \langle g(x), \varphi(x) \rangle dx$$

$$\forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^n).$$

Because  $A \in BMO$ , we know that  $A \in L^2$  but  $A$  doesn't need to be bounded; however we know, by Theorem 1.1 in [28], that the Dirichlet problem (4.1) admits a unique solution  $u$  in  $W_0^{1,2}$ .

Now we are in position to prove Theorem 1.1.

*Proof of Theorem 1.1.* We first extend the matrix  $A$  to  $\mathbb{R}^n$ , putting zero outside of  $\Omega$ . Then we take  $\varrho \in C_0^\infty(\mathbb{R}^n)$  such that  $supp \varrho \subset \overline{B_1(0)}$ ,  $\varrho \geq 0$ ,  $\varrho \not\equiv 0$  and  $\int \varrho = 1$ , and we consider the convolution  $A_N = A \star \varrho_N$ , with  $\varrho_N = \frac{N^n \varrho(Nx)}{\int \varrho}$  and defined by:

$$A_N(x) = \int_{\mathbb{R}^n} A(y)\varrho_N(x - y) dy, \quad x \in \overline{\Omega}.$$

We notice that

1.  $A_N \in C^\infty(\overline{\Omega}, \mathbb{R}^{n \times n}) \cap L^\infty(\Omega, \mathbb{R}^{n \times n})$ ,
2.  $\langle A_N Y, Y \rangle \geq \|Y\|^2$ ,
3.  $|A_N(x + he_i) - A_N(x)| \leq K_N(x)|h|$ ,
4.  $K_N(x) = (K \star \varrho_N)(x) \in L^{n,\infty}$ ,
5.  $\|A_N\|_* \leq \|A\|_*$ ,
6.  $A_N$  converges to  $A$  in  $L^2$ .

We find solutions  $u_N \in W_0^{1,2}(\Omega, \mathbb{R}^n)$  of the Dirichlet problems :

$$\begin{cases} \operatorname{div}(A_N Du_N) = \operatorname{div} F + g & \text{in } \Omega \\ u_N = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

that converge weakly in  $W_0^{1,2}$ , and strongly in  $L^2$ , to  $u$  (see [28]) and it is well known that  $u_N \in W_{loc}^{2,2}(\Omega, \mathbb{R}^n)$  (see [18]).

Let  $\varepsilon_0 > 0$  be the number fixed in (3.11) and let us assume

$$\mathcal{D}_K < \varepsilon_0.$$

We notice that, from Lemma 2.9, we have

$$\mathcal{D}_{K_N} \leq \mathcal{D}_K < \varepsilon_0.$$

More precisely, if  $K_0 \in L^\infty(\Omega)$  is a function such that  $\|K - K_0\|_{L^{n,\infty}} < \varepsilon_0$ , we get

$$\|K_N - K_0\|_{L^{n,\infty}} \leq \|K - K_0\|_{L^{n,\infty}} + \|(K_0)_N - K_0\|_{L^{n,\infty}}.$$

Since  $K_0 \in L^p(\Omega)$  for every  $p \geq n$ , thanks to Theorem 2.10 the second term in the right hand side of the previous inequality goes to 0 as  $N \rightarrow +\infty$ . Then we can assume that

$$\|K_N - K_0\|_{L^{n,\infty}} < \varepsilon_0$$

for  $N$  sufficiently large.

Now, arguing as in Theorem 3.1, we get the following relation

$$\begin{aligned} \int_{B_s} |D^2 u_N|^2 dx &\leq \frac{1}{C+1} \int_{B_t} |D^2 u_N|^2 dx \\ &+ \frac{1}{C+1} \left( \frac{1}{(t-s)^2} \int_{B_t} |A_N(x) Du_N|^2 dx \right. \\ &+ \left( \frac{1}{2\nu_0} + 2 \right) \|K_0\|_{L^\infty}^2 \int_{B_t} |Du_N|^2 dx \\ &+ \left( 3 + \frac{3}{2}\nu_0 \right) \frac{1}{(t-s)^2} \int_{B_t} |Du_N|^2 dx + \left( \frac{1}{2\nu_0} + 1 \right) \int_{B_t} |DF|^2 dx \\ &\left. + \frac{1}{2\nu_0} \int_{B_t} |g(x)|^2 dx \right), \end{aligned}$$

where the constant  $C$  is the same of (3.12).

Applying Lemma 2.14 we deduce

$$\begin{aligned} \int_{B_R} |D^2 u_N|^2 dx &\leq c(n, \mathcal{D}_K) \left( \frac{1}{R^2} \int_{B_{2R}} ( \|A\|_*^2 |Du_N|^2 + |F|^2 ) dx \right. \\ &+ \left( 1 + \frac{1}{R^2} \right) \int_{B_{2R}} |Du_N|^2 dx \\ &\left. + \int_{B_{2R}} |DF|^2 dx + \int_{B_{2R}} |g(x)|^2 dx \right), \end{aligned}$$

and

$$\int_{B_R} |D^2 u_N|^2 dx \leq c \int_{B_{2R}} \left( \left( 1 + \frac{1}{R^2} \right) |Du_N|^2 + \frac{1}{R^2} |F|^2 + |DF|^2 + |g|^2 \right) dx,$$

with  $c = c(n, \mathcal{D}_K, \|A\|_*)$ . From the previous relation, we deduce that  $|Du_N|$  is a bounded sequence in  $W^{1,2}(B_R)$ . Then, by compactness, up to a sequence not relabeled, we deduce that  $|Du_N|$  converges to  $|Du|$  in  $L^2(B_R)$ . Finally, by the semicontinuity of the norm, we get

$$\begin{aligned} \int_{B_R} |D^2 u|^2 dx &\leq \liminf_N \int_{B_R} |D^2 u_N|^2 dx \\ &\leq c(n, \mathcal{D}_K, \|A\|_*) \int_{B_{2R}} \left( \left( 1 + \frac{1}{R^2} \right) |Du|^2 + \frac{1}{R^2} |F|^2 + |DF|^2 + |g|^2 \right) dx. \end{aligned} \quad \square$$

Now we prove a global version of Theorem 1.1.

**Proposition 4.1.** *Let  $\Omega$  be a regular domain with  $C^2$  boundary. There exists  $\varepsilon_1 > 0$ , depending on  $n$  and  $\Omega$ , such that, if*

$$\mathcal{D}_K \equiv \text{dist}_{L^{n,\infty}}(K(x), L^\infty) < \varepsilon_1,$$

then  $u \in W^{2,2}(\Omega, \mathbb{R}^n)$  and

$$\int_{\Omega} |D^2 u|^2 dx \leq c \int_{\Omega} ( |Du|^2 + |F|^2 + |DF|^2 + |g|^2 ) dx,$$

for a constant  $c$ , depending on  $n, \mathcal{D}_K, \Omega$  and the BMO-norm of  $A$ .



*Proof.* We cover  $\Omega$  by a family of open sets  $\Omega', \Omega'', U_1, \dots, U_m, V_1, \dots, V_m$  such that

1.  $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ ;
2.  $U_l, V_l$  are neighbourhoods centered in  $x_l \in \partial\Omega$ , with  $l = 1, \dots, m$ ;
3.  $V_l \subset\subset U_l$ , with  $l = 1, \dots, m$ ;
4.  $\cup_{l=1}^m V_l \supseteq \partial\Omega$ ;
5.  $\Omega \not\subset\subset \cup_{l=1}^m V_l \cup \Omega'$ .

We consider the solutions  $u_N$  of the problems (4.2). Covering  $\bar{\Omega}'$  by a finite number of balls, we have that

$$\int_{\Omega'} |D^2 u_N|^2 dx \leq c(n, \mathcal{D}_K, \Omega, \|A\|_*) \int_{\Omega} (|Du_N|^2 + |F|^2 + |DF|^2 + |g|^2) dx.$$

Now we focus on the boundary regularity of the solutions. Fixed  $l \in \{1, \dots, m\}$ , on every  $U_l$  we can consider the diffeomorphism  $\Phi(x) = (\Phi_1(x), \dots, \Phi_n(x))$  which maps  $\Omega_l \equiv U_l \cap \Omega$  to an open set of  $\mathbb{R}^n$  and defined by

$$\begin{aligned} \Phi_i(x) &= x_i, \quad i = 1, \dots, n-1; \\ \Phi_n(x) &= x_n - \psi_l(x'), \quad \text{with } x' = (x_1, \dots, x_{n-1}), \end{aligned}$$

where  $\psi_l : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  is  $C^2$  and whose graph coincides with  $\partial\Omega$  in  $U_l$ .  $\Phi(x) =: y$  is such that

$$\Phi(U_l \cap \Omega) \subset \{y \in \mathbb{R}^n : y_n > 0\}, \quad \Phi(U_l \cap \partial\Omega) \subset \{y \in \mathbb{R}^n : y_n = 0\}.$$

It can be seen that  $\Phi$  is invertible, both  $\Phi$  and  $\Phi^{-1}$  are  $C^2$  functions. Let  $\tilde{u}_N$  be such that  $u_N(x) = (\tilde{u}_N \circ \Phi)(x)$ ,  $x \in U_l \cap \bar{\Omega}$ , and we check that  $\tilde{u}_N$  solves in  $\tilde{\Omega}_l = \Phi(U_l \cap \Omega)$  the system

$$\begin{cases} \operatorname{div} \tilde{A}_N(x) D\tilde{u}_N = \operatorname{div} \tilde{F} + \tilde{g} & \text{in } \tilde{\Omega}_l \\ \tilde{u}_N = 0 & \text{on } \{y_n = 0\} \cap \partial\tilde{\Omega}_l, \end{cases}$$

where

$$\tilde{A}_N = [D\Phi \cdot A_N \cdot (D\Phi)^t] \circ \Phi^{-1}, \quad \tilde{F} = (F \cdot D\Phi) \circ \Phi^{-1}, \quad \tilde{g} = g \circ \Phi^{-1}.$$

These formulas can be derived starting from the weak formulation of the problem and applying a change of variables in order to express the different integrals in terms of the new coordinates.

For instance, for  $\varphi \in C_0^\infty(\Omega_l)$ ,

$$\int_{\Omega_l} \langle g(x), \varphi(x) \rangle dx = \int_{\tilde{\Omega}_l} \langle g \circ \Phi^{-1}(y), \varphi \circ \Phi^{-1}(y) \rangle \det(D\Phi^{-1}(y)) dy,$$

just letting  $x = \Phi^{-1}(y)$ , but then  $\det(D\Phi^{-1}) = 1$  and we can set  $\varphi = \eta \circ \Phi$  so that equivalently  $\eta = \varphi \circ \Phi^{-1}$  and

$$\int_{\Omega_l} \langle g(x), \varphi(x) \rangle dx = \int_{\tilde{\Omega}_l} \langle \tilde{g}(y), \eta(y) \rangle dy.$$

We have to prove that the conditions on  $A$  still hold true for  $\tilde{A}$ . About the uniform ellipticity, we have that

$$\begin{aligned} \langle \tilde{A}_N(y)Y, Y \rangle &= \langle A_N(\Phi^{-1}(y)) D\Phi(\Phi^{-1}(y)) Y, D\Phi(\Phi^{-1}(y)) Y \rangle \\ &\geq \langle D\Phi(\Phi^{-1}(y)) Y, D\Phi(\Phi^{-1}(y)) Y \rangle \\ &\geq c\|Y\|^2, \quad c > 0. \end{aligned}$$

Hence,  $\tilde{A}_N$  satisfies the uniform ellipticity condition; moreover  $BMO$  and  $L^{n,\infty}$  are preserved under  $C^2$  transformations. Finally, we have

$$|\tilde{A}_N(y + h e_i) - \tilde{A}_N(y)| \leq \tilde{K}_N(y)|h|, \quad y \in \tilde{\Omega}_l,$$

with  $\mathcal{D}_{\tilde{K}_N} = \mathcal{D}_{K_N}$ .

Therefore, if  $D^s \tilde{u}_N$  indicates any derivatives  $D_s \tilde{u}_N$  with  $s \neq n$ , we have that  $D^s \tilde{u}_N \in W_{loc}^{1,2}(\tilde{\Omega}_l, \mathbb{R}^n)$  and

$$\int_{B_R^+} |DD^s \tilde{u}_N|^2 dy \leq c \int_{B_{2R}^+} \left( \left(1 + \frac{1}{R^2}\right) |D\tilde{u}_N|^2 + \frac{1}{R^2} |\tilde{F}|^2 + |D\tilde{F}|^2 + |\tilde{g}|^2 \right) dy,$$

where  $B_R^+ = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : \|y\| < R, y_n > 0\}$  and  $B_{2R}^+ \subset \subset \tilde{\Omega}_l$ . In order to have the estimate for the derivatives  $D_{nn}^2 \tilde{u}_N$ , the equation readily implies that  $D_n(\tilde{A}_N^{i,n} D_n \tilde{u}_N^j) \in L^2(\tilde{\Omega}_l)$  for  $i, j \in \{1, \dots, n\}$ ; by Lemma 2.13 the difference quotients  $\Delta_h(\tilde{A}_N^{i,n} D_n \tilde{u}_N^j)$  have uniformly bounded  $L^2$  norm in  $\tilde{\Omega}_{l_{|h|}}$  and the same is true for  $\tilde{A}_N^{i,n} \Delta_h D_n \tilde{u}_N^j$ . The uniform ellipticity condition gives that  $D_{nn}^2 \tilde{u}_N \in L^2(\tilde{\Omega}_l)$  and the analogous estimate for  $D_{nn}^2 \tilde{u}_N$ .

Taking the covering introduced above with  $U_l = \Phi^{-1}(B_{2R}^+)$  and  $V_l = \Phi^{-1}(B_R^+)$  and coming back to the original variables, we get that

$$\int_{V_l} |D^2 u_N|^2 dx \leq c \int_{V_l} (|Du_N|^2 + |F|^2 + |DF|^2 + |g|^2) dx,$$

and summing

$$\int_{\Omega} |D^2 u_N|^2 dx \leq c \int_{\Omega} (|Du_N|^2 + |F|^2 + |DF|^2 + |g|^2) dx.$$

Arguing as in the proof of the Theorem 1.1, we get the desired result. □

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