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Two-player rationalizable implementation ☆

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ABSTRACT

The paper characterizes the class of two-player social choice functions implementable in rationalizable strategies under complete information.

Contents

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1. Introduction

The main objective of implementation theory is to study conditions under which we can design a mechanism within which, at every state (of the world), the outcome of players' strategic interaction coincides with the outcome prescribed by a given social choice function (SCF) for that state. Players' strategic interaction is modeled via game-theoretic solution concepts, each giving rise to a different notion of implementation.

Following Palfrey (2002), we can divide the implementation problem into two components. The first component is *incentive compatibility*. The mechanism must be devised so that players' incentives give rise to an outcome that coincides with the goal set by the SCF. The second component is *uniqueness*. The mechanism must be devised so that players' incentives never give rise to an outcome that does not coincide with the goal set by the SCF. There is tension between these two components when there are only two players (see, for instance, Hurwicz and Schmeidler (1978) and Maskin (1999)) or when information is incomplete among players (see, for instance, Jackson (1991), Oury and Tercieux (2012), and Jain and Lombardi (2022)). However, incentive compatibility is not an issue when information is complete, and there are three or more players.[1]

The idea of Nash equilibrium is fundamental to much of economic theory. An extensive literature on implementation theory assumes Nash equilibrium as the solution concept.[2] However, Nash equilibrium relies on the assumption that each player correctly predicts the strategic choices of his opponents. If players' conjectures are not correct and players' rationality is common knowledge among players, then (correlated) rationalizability (Brandenburger and Dekel (1987)) is the appropriate solution concept.

This paper fully identifies the class of SCFs that are implementable in rationalizable strategies when there are two players. Our implementing condition is stronger than the implementing condition for two-player Nash implementation. This point is made in the arbitrator selection problem discussed in Section 3. The same example shows that our two-player implementing condition is stronger than the implementing condition for rationalizable implementation when there are three or more players.

The study of two-player implementation problems has always been at the heart of implementation theory. For instance, a classical impossibility result can be found in Hurwicz and Schmeidler (1978) and Maskin (1999), which show that any Pareto-optimal two-player multi-valued SCF that is Nash implementable is dictatorial if the domain of preferences is unrestricted. Laslier et al. (2021)) provides a solution to this classical problem by considering stochastic mechanisms that are deterministic-in-equilibrium. Moreover, the understanding of two-player problems has a bearing on a wide variety of bilateral contracting and negotiating problems (Moore and Repullo (1988), Moore and Repullo (1990), and Dutta and Sen (1991)). For instance, De Clippel et al. (2014) studies the problem of selecting arbitrators from the perspective of implementation theory in a setting with complete information and no monetary transfers. Note that arbitrator selection involves only two parties.

Seminal works on two-player Nash implementation are Moore and Repullo (1990), Dutta and Sen (1991) and Sjöström (1991). Moore and Repullo (1990) and Dutta and Sen (1991) provide a full characterization of the class of Nash implementable functions, whereas Sjöström (1991) provides a constructive way of checking whether or not an SCF can be implemented in Nash equilibria. Their condition comes from recognizing that two-player implementation requires a mechanism to distinguish the true state when the two players report distinct messages. Thus, any implementable SCF needs to satisfy a two-player condition, which can be described as follows: The condition requires that a "punishment outcome" exists when players report distinct messages. Moreover, it requires that when the punishment outcome is a Nash equilibrium outcome at the true state, it must be consistent with the SCF. The incentive compatibility issues that arise for the two-player case do not have any bite in separable environments, that is, in environments with an outcome that, at every state, every player deems strictly worse than the outcome prescribed by the SCF (see, for instance, Jackson (2001) and Chen et al. (2021)). However, they have a bite in many important applications, such as in environments with no monetary transfers or in classic exchange economies where free disposal is not allowed.

The two-player condition for Nash implementation does not solve the incentive compatibility issues that arise under rationalizability. The reason is that rationalizability imposes more stringent incentive compatibility requirements than Nash equilibrium on selecting the punishment outcome when the two players report distinct messages. Let f be the goal of the planner. Suppose that player 1 reports θ as the actual state, and player 2 reports θ'. Given that the planner cannot identify the liar and the mechanism must be devised so that players' incentives give rise to an outcome that coincides with the goal set by the SCF, Nash incentive compatibility constraints force us to punish the players by selecting an outcome that lies in $L_1(f(\theta),\theta)\bigcap L_2\left(f\left(\theta'\right),\theta'\right)$–that is, that lies in the intersection between player 1's lower contour set at $(f(\theta),\theta)$ and player 2's lower contour set at $(f(\theta'),\theta')$. Let player 1's strict lower contour set at $(f(\theta),\theta)$ be denoted by $SL_1(f(\theta),\theta)$. What outcome can be selected as a punishment outcome when

$$SL_1(f(\theta),\theta)\bigcap L_2\left(f\left(\theta'\right),\theta'\right)$$

is empty? This situation may arise naturally in situations where two parties have to select an arbitrator to settle a dispute (see Section 3), where players may have diametrically opposite preferences.

The two-player condition for Nash implementation allows us to choose any outcome on the indifference curve of player 1 generated by $f(\theta)$ at θ that is also in player 2's lower contour set at $(f(\theta'),\theta')$. However, rationalizable implementation forces us to select $f(\theta)$.

[1] Information is complete when players' preferences and possible outcomes are common knowledge among all the players.

[2] Maskin (1999; circulated since 1977) shows that only Maskin monotonic SCFs are Nash implementable. He also shows that when there are three or more players, an SCF is Nash implementable if it is Maskin monotonic and satisfies the condition of no veto power. Additional characterization results for Nash implementation that do not rely on the condition of no veto power can be found, for example, in Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991), Danilov (1992), Yamato (1992), Saijo et al. (1996), Bochet (2007), Benoît and Ok (2008), Korpela (2010), Lombardi and Yoshihara (2013), and Xiong (2023a).

The reason is that any other point e of player 1's indifference curve becomes a rationalizable outcome at θ but $e \neq f(\theta)$. This choice is problematic when the two distinct messages constitute an unwanted Nash equilibrium at some other θ^* where $f(\theta) \neq f(\theta^*)$. The more stringent incentive compatibility constraints imposed by rationalizability are at the core of the contextualizing example presented in Section 3, where we construct a two-player SCF that is Nash implementable but not implementable in rationalizable strategies.

Since the two-player condition for Nash implementation cannot help solve two-player rationalizable implementation problems, we develop our solution to the two-player rationalizable implementation problems in the space of deceptions. This contrasts the solution developed by Bergemann et al. (2011) (BMT, henceforth) when there are three or more players because they develop it in the space of partitions. We connect the two solutions in Section 7.[3]

Finally, the necessary and sufficient condition for Nash implementation in abstract environments relies on the existence of certain sets. Thanks to Sjöström (1991), when we have three or more players, we have a constructive way of checking whether an SCF can be implemented in Nash equilibria. Our implementing condition also relies on the existence of a deception profile. In the spirit of Sjöström (1991), we show how to construct the deception profile underlying our two-player implementing condition. The construction is very instructive because we can use a similar algorithm to construct the partition underlying the implementing conditions proposed in the existing literature on rationalizable implementation. Indeed, the implementing conditions proposed by BMT, Xiong (2023b), and Jain (2021), rely on the existence of a partition of the set of states Θ. This existential clause, without any guidance, makes their conditions difficult to check as the number of partitions of Θ grows exponentially with the size of Θ.[4]

The rest of the paper is organized as follows. Section 2 presents the theoretical framework and outlines the basic model, with the contextualizing example presented in Section 3. Section 4 presents our characterization result, with an application discussed in Section 5. Section 6 presents the algorithm for computing the deception profile underlying our two-player implementing condition. Section 7 connects our implementing condition to the implementing condition provided by Xiong (2023b) when there are three or more players and, finally, discusses the extension of our characterization result to the case of social choice correspondences. Omitted proofs can be found in the Appendices.

2. Setup

The environment consists of two players, which we write as $\mathcal{I} = \{1,2\}$, a finite set of states Θ and a countable set of pure outcomes X. Let $Y \equiv \Delta(X)$ denote the set of lotteries over X. A generic player ℓ's preferences over lotteries are described by a utility function $u_\ell : Y \times \Theta \to \mathbb{R}$, with $u_\ell(y,\theta) = \sum_{x\in X} y_x u_\ell(x,\theta)$, where y_x is the probability of pure outcome x. For all $\theta \in \Theta$, $u_\ell(\cdot,\theta)$ satisfies the expected utility hypothesis. To save writing, for all $\ell \in \mathcal{I}$, we write $-\ell$ for player ℓ's opponent.

Given a state $\theta \in \Theta$, a player $\ell \in \mathcal{I}$, and a lottery $y \in Y$, the lower contour set of $u_\ell(\cdot,\theta)$ at y is $L_\ell(y,\theta) = \{y' \in Y | u_\ell(y,\theta) \geq u_\ell(y',\theta)\}$; the strict lower contour set of $u_\ell(\cdot,\theta)$ at y is $SL_\ell(y,\theta) = \{y' \in Y | u_\ell(y,\theta) > u_\ell(y',\theta)\}$; and the strict upper contour set of $u_\ell(\cdot,\theta)$ at y is $SU_\ell(y,\theta) = \{y' \in Y | u_\ell(y',\theta) > u_\ell(y,\theta)\}$.

A mechanism $\mathcal{M}$ is a pair $\mathcal{M} \equiv (M,g)$, where $M \equiv \prod_{\ell\in\mathcal{I}} M_i$, with each M_ℓ being a nonempty countable set, and $g : M \to Y$. As usual, we refer to M_ℓ as the (pure) strategy space of $\ell \in \mathcal{I}$, to a member of M, denoted by m, as a (pure) strategy profile, and to g as an outcome function. For all $M' \subseteq M$, let $g[M'] = \{g(m) \in Y | m \in M'\}$.

The environment, when combined with the mechanism, describes a game (of complete information) for all state $\theta \in \Theta$, which is denoted by $(\mathcal{M},\theta)$. We will use (correlated) rationalizability as a solution concept. Bernheim (1984) and Pearce (1984) provide a definition of rationalizability in which players' conjectures over their opponents' play are independent. In this paper, we follow the convention of some of the recent literature (e.g., Osborne and Rubinstein (1994) in using "rationalizability" for the correlated version of rationalizability (we refer the reader to Brandenburger and Dekel (1987)). Our definition of rationalizability coincides with the standard definition when strategy spaces are compact. However, our definition allows for infinite, non-compact strategy spaces. In this case, our definition is equivalent to one introduced by Lipman (1994).

Formally, let $\mathcal{S}$ be the set of all strategy-set profiles, defined by $\mathcal{S} \equiv \prod_{\ell\in\mathcal{I}} \mathcal{S}_\ell$, where $\mathcal{S}_\ell \equiv 2^{M_\ell}$ for all $\ell \in \mathcal{I}$, with $S = (S_\ell)_{\ell\in\mathcal{I}}$ as a typical profile of $\mathcal{S}$. The family $\mathcal{S}$ is a lattice with the natural ordering of the set inclusion: $S \leq S'$ if $S_\ell \subseteq S'_\ell$ for all $\ell \in \mathcal{I}$. The smallest element of $\mathcal{S}$ is denoted by $\underline{S} \equiv (\emptyset,\dots,\emptyset)$, whereas the largest element is denoted by $\bar{S} \equiv M$.

Fix any game $(\mathcal{M},\theta)$. The strategy $m_\ell \in M_\ell$ is player ℓ's best-response to his belief $\lambda_\ell \in \Delta(M_{-\ell})$ at θ if $m_\ell \in \arg\max_{m'_\ell\in M_\ell} \sum_{m_{-\ell}\in M_{-\ell}} \lambda_\ell(m_{-\ell}) u_\ell\left(g\left(m'_\ell, m_{-\ell}\right),\theta\right)$, where $\lambda_i(m_{-\ell})$ is player ℓ's belief that his opponent will play the strategy $m_{-\ell}$.

By following Bergemann et al. (2011), let us define an operator $b^{\mathcal{M},\theta} : \mathcal{S} \to \mathcal{S}$, where $b^{\mathcal{M},\theta} \equiv \left(b^{\mathcal{M},\theta}_\ell\right)_{\ell\in\mathcal{I}}$ and $b^{\mathcal{M},\theta}_\ell : \mathcal{S} \to \mathcal{S}_\ell$ is defined, for all $S \in \mathcal{S}$, by

[3] Xiong (2023b) provides a full characterization in the space of partitions when there are three or more players.

[4] In combinatorial mathematics, the number of partitions of a set of size n is referred to as bell number. Bell numbers can be recursively defined as follows: $B(1) = 1$ and for every $n+1$,

$$B(n) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

$$b_\ell^{\mathcal{M},\theta}(S) = \left\{ m_\ell \in M_\ell \;\middle|\; \begin{array}{l} \text{there exists } \lambda_\ell^{m_\ell,\theta} \in \Delta(M_{-\ell}) \text{ s.t.} \\ (1)\ \lambda_\ell^{m_\ell,\theta}(m_{-\ell}) > 0 \implies m_{-\ell} \in S_{-\ell}, \\ (2)\ m_\ell \text{ is a best response to } \lambda_\ell^{m_\ell,\theta} \text{ at } \theta \end{array} \right\}.$$

Note that $b^{\mathcal{M},\theta}$ is increasing (that is, $S \leq S' \implies b^{\mathcal{M},\theta}(S) \leq b^{\mathcal{M},\theta}(S')$).

By Tarski's fixed point theorem, there exists a largest fixed point of $b^{\mathcal{M},\theta}$, which is denoted by $S^{\mathcal{M},\theta}$.[5] That is, (1) $b^{\mathcal{M},\theta}(S^{\mathcal{M},\theta}) = S^{\mathcal{M},\theta}$ and (2) $b^{\mathcal{M},\theta}(S) = S \implies S \leq S^{\mathcal{M},\theta}$. We refer to $m_\ell \in S_\ell^{\mathcal{M},\theta}$ as a player ℓ's rationalizable strategy of $(\mathcal{M},\theta)$, and to a member of $S^{\mathcal{M},\theta}$ as a rationalizable strategy profile of $(\mathcal{M},\theta)$.

We say that a profile $S \in \mathcal{S}$ has the best-response property in state θ if $S \leq b^{\mathcal{M},\theta}(S)$, or equivalently, if for all $\ell \in \mathcal{I}$ and all $m_\ell \in S_\ell$, there exists $\lambda_\ell \in \Delta(M_{-\ell})$ s.t. $\lambda_\ell(m_{-\ell}) > 0 \implies m_{-\ell} \in S_{-\ell}$, and m_ℓ is a best-response to λ_ℓ at θ. It can be checked that $S \leq S^{\mathcal{M},\theta}$ when S has the best-response property in state θ.

A player ℓ's mixed-strategy σ_ℓ is a probability distribution over M_ℓ. The space of player ℓ's mixed-strategies is denoted by Σ_ℓ, where $\sigma_\ell(m_\ell)$ is the probability that σ_ℓ assigns to m_ℓ. The space of mixed-strategy profiles is denoted by $\Sigma = \prod_{\ell\in\mathcal{I}} \Sigma_\ell$, with element σ as a typical strategy profile. A mixed-strategy may assign probability one to a single strategy m_ℓ, that is, $\sigma_\ell(m_\ell) = 1$. In this case, we refer to such a mixed-strategy as a (pure) strategy and denote it by m_ℓ. The support of a mixed-strategy σ_ℓ is the set of pure strategies that are played with positive probability, that is, $supp(\sigma_\ell) = \{m_\ell \in M_\ell | \sigma_\ell(m_\ell) > 0\}$. A mixed-strategy profile σ is a Nash equilibrium of $(\mathcal{M},\theta)$ if for all $\ell \in \mathcal{I}$, $u_\ell(g(\sigma_\ell,\sigma_{-\ell}),\theta) \geq u_\ell(g(\sigma'_\ell,\sigma_{-\ell}),\theta)$, for all $\sigma'_\ell \in \Sigma_\ell$. Write $NE(\mathcal{M},\theta)$ for the set of Nash equilibrium profiles of $(\mathcal{M},\theta)$, and write $g(NE(\mathcal{M},\theta))$ for the set of Nash equilibrium outcomes of $(\mathcal{M},\theta)$. A social choice function (SCF) f is a function $f : \Theta \to Y$. For any $\theta \in \Theta$, we refer to $f(\theta)$ as the socially optimal outcome at θ.

Definition 1. A mechanism $\mathcal{M}$ implements $f : \Theta \to Y$ in rationalizable strategies if for all $\theta \in \Theta$, $S^{\mathcal{M},\theta} \neq \emptyset$ and $m \in S^{\mathcal{M},\theta} \implies g(m) = f(\theta)$. If such a mechanism exists, f is said to be rationalizably implementable.

BMT shows that Maskin monotonicity fully identifies the class of rationalizably implementable SCFs when there are three or more players and when SCFs satisfy the following two auxiliary conditions.

Definition 2. $f : \Theta \to Y$ satisfies NWA provided that for all $\theta \in \Theta$ and all $\ell \in \mathcal{I}$, $SL_\ell(f(\theta),\theta) \neq \emptyset$.

The condition requires that a player never obtains his worst outcome under the SCF.

Definition 3. $f : \Theta \to Y$ is responsive provided that for all distinct $\theta,\theta' \in \Theta$, it holds that $f(\theta) \neq f(\theta')$.

Responsiveness requires that the SCF "responds" to a change in the state with a change in the outcome selected by f.

Definition 4. $f : \Theta \to Y$ satisfies Maskin monotonicity provided that for all $\theta,\theta' \in \Theta$, if $f(\theta) \neq f(\theta')$, then there exists $\ell \in \mathcal{I}$ s.t. $L_\ell(f(\theta),\theta) \bigcap SU_\ell(f(\theta),\theta') \neq \emptyset$.

Maskin monotonicity states that in the case the socially optimal outcome differs at θ and θ', there exists a player ℓ who, if the actual state is θ' and all other players claim that it is θ, could be offered a outcome y that would give him a strict incentive to "announce" θ', where y does not give any incentive when θ is the actual state.

BMT's Proposition 2. (BMT, p. 1261) Suppose that there are more than two players. If $f : \Theta \to Y$ is responsive and it satisfies NWA and Maskin monotonicity, then f is rationalizably implementable.

In contrast to NWA and responsiveness, Maskin monotonicity is necessary for rationalizable implementation. Moreover, BMT propose ways to relax responsiveness while keeping NWA when there are more than two players. Indeed, BMT formulated a condition in the space of partitions of Θ. A fundamental difference between Nash equilibrium and Rationalizability is that the latter has a product structure. Partitions are used to capture this feature of rationalizability (for further discussion, we refer the reader to p. 1266 of BMT). A partition of Θ is a correspondence $P : \Theta \rightrightarrows \Theta$ satisfying the following requirements: (i) $\theta \in P(\theta)$ for all $\theta \in \Theta$, (ii) $\cup_{\theta\in\Theta} P(\theta) = \Theta$, and (iii) $P(\theta) \cap P(\theta') = \emptyset$ if $P(\theta) \neq P(\theta')$. Given an f, P_f is the partition of Θ induced by f, that is, $P_f = \{\Theta_y\}_{y\in f(\Theta)}$ where $\Theta_y = \{\theta \in \Theta | f(\theta) = y\}$. A partition P of Θ is at least as fine as P_f, or equivalently, P_f is coarser than P if $P(\theta) \subseteq P_f(\theta)$ for all $\theta \in \Theta$. Let $\mathcal{P}_f$ denote the set of partitions that are at least as fine as P_f.[6]

For our purposes, we will work in the space of deceptions, which includes BMT's framework as a special case (see Remark 1). Let us call any map $\beta_\ell : \Theta \to 2^\Theta \setminus \{\emptyset\}$ as player ℓ's deception. A special deception for player ℓ is the truth-telling deception, β_ℓ^t, defined by $\beta_\ell^t(\theta) = \{\theta\}$ for all $\theta \in \Theta$. It is clear that a partition is also a deception but the converse is not true.

[5] The fixed point set can indeed be empty. Our notion of implementation requires that such a fixed point set is never empty.

[6] Formally, $\mathcal{P}_f = \{P | P \text{ is a partition of } \Theta \text{ s.t. } P(\theta) \subseteq P_f(\theta) \text{ for all } \theta \in \Theta\}$.

For any β_ℓ and β'_ℓ, we write $\beta_\ell \subseteq \beta'_\ell$ if $\beta_\ell(\theta) \subseteq \beta'_\ell(\theta)$ for all $\theta \in \Theta$. Let $\mathcal{B}_\ell$ denote the set of player ℓ's deceptions containing the truth-telling deception, that is,

$$\mathcal{B}_\ell \equiv \left\{ \beta_\ell : \Theta \to 2^{\Theta} \setminus \{\emptyset\} \,\middle|\, \beta^t_\ell \subseteq \beta_\ell \right\}. \tag{1}$$

Let $\mathcal{B} \equiv \prod_{\ell \in \mathcal{I}} \mathcal{B}_\ell$, with $\beta = (\beta_\ell)_{\ell \in \mathcal{I}}$ as a typical deception profile of $\mathcal{B}$. For all $\beta, \beta' \in \mathcal{B}$, we write $\beta \subseteq \beta'$ if $\beta_\ell \subseteq \beta'_\ell$ for all $\ell \in \mathcal{I}$. The collection $\mathcal{B}$ is a complete lattice with the natural ordering set inclusion: $\beta \leq \beta'$ if $\beta \subseteq \beta'$. The largest element is $\bar{\beta} = (\Theta, ..., \Theta)$. The smallest element is β^t.

$f : \Theta \to Y$ is measurable with respect to (wrt, henceforth) $\beta \in \mathcal{B}$ if for all $\theta, \theta' \in \Theta$,

$$f(\theta) \neq f(\theta') \implies \beta(\theta) \cap \beta(\theta') = \emptyset.$$

Given any $f : \Theta \to Y$, let us define the set $\mathcal{B}_f$ by

$$\mathcal{B}_f = \{\beta \in \mathcal{B} | f : \Theta \to Y \text{ is measurable wrt } \beta\}.$$

Remark 1. Any deception $\beta \in \mathcal{B}_f$, induces a partition on Θ that is at least as fine as P_f. Indeed, let us define $P : \Theta \to 2^{\Theta} \setminus \{\emptyset\}$ by $P(\theta) = \{\theta' \in \Theta | \beta(\theta) = \beta(\theta')\}$. It is clear from definition that P is a partition of Θ.[7] Let us show that $P \in \mathcal{P}_f$. Suppose that $\bar{\theta} \in P(\theta) \cap P(\theta')$ for some $\theta, \theta', \bar{\theta} \in \Theta$. It follows from definition of P that $\beta(\theta) = \beta(\theta')$. Since $\beta \in \mathcal{B}_f$, it follows that $f(\theta) = f(\theta')$. Thus, $P \in \mathcal{P}_f$.

3. Contextualizing example

In this section, we show through an example that the incentive compatibility issues arising in the two-player rationalizable implementation problems are more severe than those in the two-player Nash implementation problems. However, these issues do not have any bite in rationalizable implementation problems with three or more players or in environments with an outcome that, at every state, every player deems strictly worse than the outcome prescribed by the SCF (Jackson (2001)).

We do it by constructing a two-player allocation rule f such that (a) it satisfies all sufficient conditions of BMT's Proposition 2 except the requirement of three or more players; (b) it is Nash implementable; and (c) it is not implementable in rationalizable strategies.[8]

Suppose that two parties have to select an arbitrator to settle a dispute (two-party mechanisms for choosing arbitrators are studied, for instance, in De Clippel et al. (2014) and Barberà and Coelho (2024)). Suppose that the parties agreed in the arbitration agreement that the arbitrator needs to be chosen from a pool of eight arbitrators $X = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ with a priori well-known abilities and that the selected arbitrator needs to be Pareto efficient and should try to minimize disadvantages as much as possible, in the sense of equalizing agents' utilities as much as possible. Let us suppose that θ, θ' and θ'' are the possible parties' preferences over X. Our setting allows that the parties may strongly disagree on their preferences over arbitrators. Indeed, their utilities over X at each state are displayed in the table below, where $\varepsilon \in \left(\frac{1}{2}, 1\right)$.

	$u_1(\cdot,\theta)$	$u_2(\cdot,\theta)$	$u_1(\cdot,\theta')$	$u_2(\cdot,\theta')$	$u_1(\cdot,\theta'')$	$u_2(\cdot,\theta'')$
a_1	1	$-(1-\varepsilon)$	1	-1	1	1
a_2	0	0	0	0	0	0
a_3	0	0	0	0	0	0
a_4	-1	1	$-(1-\varepsilon)$	1	-1	1
a_5	1	-2	-2	-1	1	-1
a_6	2	$-(2-\epsilon)$	2	-2	-2	-2
a_7	3	-3	-3	-3	3	-3
a_8	0	0	0	0	-3	-3

[7] By definition of P, it follows that $P(\theta) \neq \emptyset$ and $\cup_{\theta \in \Theta} P(\theta) = \Theta$. Fix any $\theta, \theta' \in \Theta$ and suppose that $P(\theta) \neq P(\theta')$. We show that $P(\theta) \cap P(\theta') = \emptyset$. Assume, to the contrary, that there exists $\bar{\theta} \in P(\theta) \cap P(\theta')$. By definition of P, $\beta(\theta) = \beta(\bar{\theta})$ and $\beta(\theta') = \beta(\bar{\theta})$, and so $\beta(\theta) = \beta(\theta')$. It follows from the definition of P that $P(\theta) = P(\theta')$, which is a contraction. Thus, P is a partition of Θ.

[8] This example also sheds further light on the relationship between Nash implementation and rationalizable implementation. Indeed, BMT's Proposition 2 (see Section 2) implies that in a complete information environment with three or more players, if an SCF satisfies responsiveness and the so-called no-worst alternative condition (NWA), then its Nash implementation is equivalent to its rationalizable implementation. This equivalence result is a conceptual puzzle because the two solutions concepts are very different. In a complete information environment with three or more players, Jain (2021) provides an example showing that the equivalence breaks down when f violates responsiveness, but it satisfies NWA. In the same environment, Xiong (2023b) shows that it breaks down when f satisfies responsiveness but violates NWA. Our example shows that the equivalence breaks down in a complete information environment with two players, even when f satisfies responsiveness and NWA.

Efficient arbitrators that mitigate disadvantages are a_2 at state θ, a_3 at θ' and a_1 at θ''.[9] Then, the goal of the concerned parties can be described as follows:

$$f(\theta) = \{a_2\}, f(\theta') = \{a_3\}, \text{ and } f(\theta'') = \{a_1\}.$$

This selection rule is not rationalizably implementable. The feature of the example that allows us to make the selection rule not implementable is that party 1's preference at θ and party 2's preference at θ' are diametrically opposed, which makes the intersection $SL_1(f(\theta),\theta) \bigcap L_2(f(\theta'),\theta')$ empty.[10] However, the selection rule is such that it satisfies NWA, responsiveness, and Maskin Monotonicity (see Appendix A). Thus, our selection rule satisfies all the conditions of Proposition 2 of BMT, except that $|\mathcal{I}| = 2$.[11]

When f satisfies NWA and stochastic mechanisms can be employed, it can be shown that Moore and Repullo (1990)'s necessary and sufficient condition for Nash implementation, called condition $\mu 2$, simplifies as follows.[12]

Definition 5. $f : \Theta \to Y$ satisfies condition $\mu 2$ provided that there exists $e : \Theta \times \Theta \to Y$ such that for all $\theta, \theta' \in \Theta$, (a) $e(\theta,\theta') = f(\theta)$ if $\theta = \theta'$; (b) $e(\theta,\theta') \in L_1(f(\theta'),\theta') \cap L_2(f(\theta),\theta)$ if $\theta \neq \theta'$; and (c) for all $\theta^* \in \Theta$, $f(\theta^*) = e(\theta,\theta')$ if

$$L_1(f(\theta'),\theta') \subseteq L_1(e(\theta,\theta'),\theta^*) \text{ and } L_2(f(\theta),\theta) \subseteq L_2(e(\theta,\theta'),\theta^*). \tag{2}$$

Part (b) of condition $\mu 2$ is a *self-selection* constraint due to incentive compatibility issues. It requires that when players report different states, a feasible outcome exists that can punish both players simultaneously. Part (a) of condition $\mu 2$ requires that the punishment outcome is consistent with f when players report the same state. Part (c) of condition $\mu 2$ states that if such a punishment outcome is a Nash equilibrium outcome at θ^*, then it should be selected by f at θ^*. Note that part (c) of condition $\mu 2$ implies Maskin monotonicity when $\theta = \theta'$. The challenge to satisfy condition $\mu 2$ consists in finding a feasible outcome $e : \Theta \times \Theta \to Y$ such that parts (a)–(c) are satisfied *simultaneously*.

In Appendix A, we verify that f satisfies condition $\mu 2$ and thus f is Nash implementable. Though f is Nash implementable, satisfies responsiveness, NWA and Maskin monotonicity, below we argue that f is not rationalizably implementable.

The easiest way to make this point without being distracted by details is to suppose that $\mathcal{M}$ implements f in rationalizable strategies and in (pure strategy) Nash equilibria.

For each state $\bar{\theta} \in \Theta$, let $m(\bar{\theta}) = (m_1(\bar{\theta}), m_2(\bar{\theta}))$ be a Nash equilibrium strategy profile for the game $(\mathcal{M}, \bar{\theta})$. Since $\mathcal{M}$ implements f in Nash Equilibrium, $g(m_1(\theta'), m_2(\theta)) \in L_1(f(\theta),\theta) \cap L_2(f(\theta'),\theta')$. Since $SL_1(f(\theta),\theta) \cap L_2(f(\theta'),\theta') = \emptyset$ and our assumption that $\mathcal{M}$ rationalizable implements f, it follows that $m_1(\theta') \in S_1^{\mathcal{M},\theta}$. The reason is that $m_1(\theta')$ is a best response to $m_2(\theta)$ at θ. Thus, $(m_1(\theta'), m_2(\theta)) \in S^{\mathcal{M},\theta}$.

What about $L_1(f(\theta),\theta) \cap SL_2(f(\theta'),\theta')$? Note that by construction, we have that $L_1(f(\theta),\theta) \cap SL_2(f(\theta'),\theta') = \emptyset$. Thus, it follows from our assumption that $\mathcal{M}$ rationalizable implements f that $m_2(\theta) \in S_2^{\mathcal{M},\theta'}$. The reason is that $m_2(\theta)$ is a best response to $m_1(\theta')$ at θ'. Thus, $(m_1(\theta'), m_2(\theta)) \in S^{\mathcal{M},\theta'}$. Thus, we have shown that $(m_1(\theta'), m_2(\theta)) \in S^{\mathcal{M},\theta} \bigcap S^{\mathcal{M},\theta'}$. Finally, implementation requires that

$$g(m_1(\theta'), m_2(\theta)) = g(m_1(\theta'), m_2(\theta')) = g(m_1(\theta), m_2(\theta)) = f(\theta),$$

which is a contradiction as $f(\theta) \neq f(\theta')$.

From the above discussion, it is clear that rationalizable implementation imposes more restrictions than Nash implementation in selecting the punishment outcome. Indeed, when player 1 reports θ' and player 2 reports θ, Nash implementation allowed us to select g as a punishment outcome and so to satisfy part (c) of condition $\mu 2$. However, this option is not available in the case of rationalizable implementation. The reason is that $SL_1(f(\theta),\theta) \cap L_2(f(\theta'),\theta')$ is empty and rationalizable implementation forced us to select $f(\theta)$ as a punishment outcome. Therefore, any necessary condition for two-player rationalizable implementation needs to select as a punishment outcome $f(\theta)$ when player 1 reports θ' and player 2 reports θ and $SL_1(f(\theta),\theta) \cap L_2(f(\theta'),\theta')$ is empty. One such condition can be obtained by strengthening condition $\mu 2$ of Definition 5 as follows.

Definition 6. $f : \Theta \to Y$ satisfies strong condition $\mu 2$ provided that there exists $e : \Theta \times \Theta \to Y$ such that for all $\theta \in \Theta, e(\theta,\theta) = f(\theta)$ and for all distinct $\theta, \theta' \in \Theta$:

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[9] An arbitrator x is efficient at θ if there does not exist any other feasible arbitrator y such that $u_\ell(y,\theta) \geq u_\ell(x,\theta)$ for each player ℓ, and $u_\ell(y,\theta) > u_\ell(x,\theta)$ for some party ℓ.

[10] To see, it is supposed that there exists $z \in SL_1(f(\theta),\theta) \bigcap L_2(f(\theta'),\theta')$. Then, $u_1(z,\theta) < u_1(f(\theta),\theta) = 0$ and $u_2(f(\theta'),\theta') = 0 \geq u_2(z,\theta')$, and so $u_1(z,\theta) + u_2(z,\theta') < 0$. However, since $u_1(\cdot,\theta) = -u_2(\cdot,\theta')$ by construction, it follows that $u_1(z,\theta) + u_2(z,\theta) = 0$, which is a contradiction.

[11] The indifference between a_2 and a_3 can be dropped from the example. Indeed, we can achieve the same goal by setting $f(\theta) = f(\theta') = \{a_2\}$ and dropping a_3 from the environment. However, to show that f satisfies all sufficient conditions of BMT's except the requirement of three or more players, one can check that the modified example also satisfies a condition stronger than Maskin monotonicity, called Maskin Monotonicity*. For sake of simplicity, we keep the above-mentioned indifference.

[12] It can be checked that when f satisfies NWA, it is without loss of generality to verify condition $\mu 2$ (condition β by Dutta and Sen (1991) or condition $M2$ of Sjöström (1991)) under the specifications that the set $B = \Delta(X)$ and $C_i(f(\bar{\theta}),\bar{\theta}) = L_i(f(\bar{\theta}),\bar{\theta})$ for all $i \in \mathcal{I}$.

(a) (i) If $SL_1(f(\theta),\theta)\cap L_2(f(\theta'),\theta')=\emptyset$, then $e(\theta',\theta)=e(\theta,\theta)=f(\theta)$. (ii) If $L_1(f(\theta),\theta)\cap SL_2(f(\theta'),\theta')=\emptyset$, then $e(\theta',\theta)=e(\theta',\theta')=f(\theta')$.
(b) $e(\theta',\theta)\in L_1(f(\theta),\theta)\cap L_2(f(\theta'),\theta')$.
(c) For all $\theta^*\in\Theta$, $f(\theta^*)=e(\theta',\theta)$ if

$$L_1(f(\theta),\theta)\subseteq L_1(e(\theta',\theta),\theta^*) \text{ and } L_2(f(\theta'),\theta')\subseteq L_2(e(\theta',\theta),\theta^*). \tag{3}$$

When f satisfies condition strong $\mu2$, the punishment function $e:\Theta\times\Theta\to Y$ induces the following set-valued deception $\beta:\Theta\to\Theta\times\Theta$ defined by

$$\beta(\bar\theta)\equiv e^{-1}(f(\bar\theta)) \tag{4}$$

for all $\bar\theta\in\Theta$. In other words, for each $\bar\theta\in\Theta$, β identifies profiles of states (θ,θ') that are outcome equivalent to $f(\bar\theta)$ according to the punishment function $e:\Theta\times\Theta\to Y$. Clearly, the deception identified by condition $\mu2$ can be a proper subset of that identified by strong condition $\mu2$. To see it, observe that in our example the deception induced by strong condition $\mu2$ includes pair of states (θ',θ) such that $\theta'\neq\theta$, whereas the deception induced by condition $\mu2$ includes pair of states (θ',θ) such that $\theta'=\theta$.

The language of deceptions allows us to formulate the constraints that arise due to rationalizable implementation beyond those arising from condition $\mu2$. Our two-player implementing condition builds on this insight, and it is defined on the space of deceptions. Moreover, since we are using rationalizability as a solution concept, the deception β has a product structure–i.e., $\beta(\bar\theta)=\beta_1(\bar\theta)\times\beta_2(\bar\theta)$, for every $\bar\theta\in\Theta$. The next section formalizes our implementing condition.[13]

4. A full characterization

In this section, we provide a full characterization of the class of SCFs that are two-player rationalizably implementable.

What prevented us from rationalizably implementing the two-player SCF presented in the previous section was the fact that the intersection $SL_1(f(\theta),\theta)\cap L_2(f(\theta'),\theta')$ was empty. This implies that any necessary condition for two-player implementable SCFs has to take care of these situations.

Before presenting our implementing condition, we need additional notation. Given any f, we say that player ℓ violates NWA at θ if $SL_\ell(f(\theta),\theta)=\emptyset$. Since players can violate NWA, we can rank them according to the number of times they satisfy it.[14] Let i be the player who satisfies NWA for the highest number of times. Formally, player i is s.t. $|\{\theta\in\Theta|SL_i(f(\theta),\theta)\neq\emptyset\}|\geq|\{\theta\in\Theta|SL_{-i}(f(\theta),\theta)\neq\emptyset\}|$. We follow the following convention, unless otherwise specified. For any $\theta\in\Theta$ and any $\beta\in\mathcal{B}$, $\beta(\theta)\equiv\beta_i(\theta)\times\beta_{-i}(\theta)$.

The condition can now be stated as follows.

Definition 7. $f:\Theta\to Y$ satisfies Two-Player Generalized Maskin Monotonicity** (2P-GSMM**) provided that there exists $\beta\in\mathcal{B}_f$ s.t. the following requirements are satisfied.

(a) (Intersection Property): For all $(\ell,\theta',\theta)\in\mathcal{I}\times\Theta\times\Theta$,

$$SL_\ell(f(\theta),\theta)\bigcap L_{-\ell}(f(\theta'),\theta')=\emptyset \implies \beta_\ell(\theta')\subseteq\beta_\ell(\theta)$$

(b) (Incentive Compatibility): There exists a function $e:\Theta\times\Theta\to Y$ defined, for all $(\theta',\theta)\in\Theta\times\Theta$, by

$$e(\theta',\theta)\in\begin{cases} f(\theta') & \text{if } \mathcal{I}^{\beta(\theta')}\neq\mathcal{I} \\ f(\theta')=f(\theta) & \text{if } \mathcal{I}^{\beta(\theta')}=\mathcal{I} \text{ and } (\theta',\theta)\in\beta(\theta)\cap\beta(\theta') \\ SL_i(f(\theta),\theta)\cap SL_{-i}(f(\theta'),\theta') & \text{otherwise,}\end{cases}$$

where

$$\mathcal{I}^{\beta(\theta')}\equiv\left\{\ell\in\mathcal{I}\,\middle|\,SL_\ell(f(\hat\theta),\hat\theta)\neq\emptyset \text{ for all } \hat\theta\in\Theta \text{ s.t. } \beta(\theta')=\beta(\hat\theta)\right\}.$$

(c) (Generalized Strict Maskin Monotonicity**): For all $\theta,\theta'\in\Theta$, if for all $\ell\in\mathcal{I}^{\beta(\theta')}$, there exists $\hat\theta\in\beta_{-\ell}(\theta)$ s.t.

$$SL_\ell(f(\hat\theta),\hat\theta)\subseteq L_\ell(f(\theta),\theta') \implies \beta(\theta)\subseteq\beta(\theta').$$

Let us make the following remarks.

[13] Traditionally, the idea of deceptions is used to study implementation problems with incomplete information. However, even under the assumption of complete information, formulating the condition in the space of deceptions has proved pivotal in characterizing social choice rules that are repeatedly Nash implementable and in making the connection with static Nash implementation transparent. For instance, Mezzetti and Renou (2017) show that dynamic monotonicity, a nontrivial but natural generalization of Maskin monotonicity defined on the space of deceptions, is necessary and almost sufficient for repeated Nash implementation.
[14] Breaking a tie arbitrarily.

Remark 2. Let us observe that when f satisfies 2P-GSMM**, $\beta(\theta') \subseteq \beta(\theta)$ if $\beta(\theta) \subseteq \beta(\theta')$.[15]

Remark 3. An implication of part a) of 2P-GSMM** is that $\beta_\ell(\theta) = \Theta$ if player $\ell \notin \mathcal{I}^{\beta(\theta)}$ and f satisfies 2P-GSMM**.[16]

Let us explain why 2P-GSMM** is necessary and sufficient for rationalizable implementation. For the sake of brevity, let us suppose that f satisfies NWA and that it is rationalizably implementable by a 'direct' mechanism $\mathcal{M}$, where $M_i = M_{-i} = \Theta$ and $g : M_i \times M_{-i} \to Y$.[17]

Implementation in any solution concept imposes restrictions on g. To understand the restrictions under rationalizability, let $\beta(\theta) \equiv S^{\mathcal{M},\theta}$ denote the set of rationalizable strategy profiles of $\mathcal{M}$ at θ. Rationalizable implementation implies that for all $\theta \in \Theta$ and all $m \in \beta(\theta)$, $g(m) = f(\theta)$.

Let us first connect part a) of our condition to rationalizable implementation. Suppose that $SL_\ell(f(\theta),\theta) \bigcap L_{-\ell}(f(\theta'),\theta') = \emptyset$. Since $\mathcal{M}$ rationalizably implements f, it holds that $g(\hat{\theta},\theta) \in L_\ell(f(\theta),\theta) \bigcap L_{-\ell}(f(\theta'),\theta')$ for all $\hat{\theta} \in S_\ell^{\mathcal{M},\theta'}$. Our assumption that $SL_\ell(f(\theta),\theta) \bigcap L_{-\ell}(f(\theta'),\theta') = \emptyset$ implies that $g(\hat{\theta},\theta)$ lies on the indifference curve of $f(\theta)$ at θ. However, since f is rationalizably implementable and $u_\ell(f(\theta),\theta) = u_\ell(g(\hat{\theta},\theta),\theta)$, it must be the case that $g(\hat{\theta},\theta) = f(\theta)$. Therefore, it holds that $S_\ell^{\mathcal{M},\theta'} \subseteq S_\ell^{\mathcal{M},\theta}$, and so $\beta_\ell(\theta') \subseteq \beta_\ell(\theta)$.

Now, let us connect part b) of our condition to rationalizable implementation. Suppose that θ' is a rationalizable strategy for player i at θ and θ'–that is, $\theta' \in \beta_i(\theta) \cap \beta_i(\theta')$–and that θ is a rationalizable strategy for player $-i$ at θ and θ'–that is, $\theta \in \beta_{-i}(\theta) \cap \beta_{-i}(\theta')$. Since (θ',θ) is a rationalizable strategy profile of $\mathcal{M}$ at θ and θ', implementability requires that $g(\theta',\theta) = f(\theta) = f(\theta')$, which is what part b) of our condition requires. However, when $(\theta',\theta) \notin \beta(\theta) \bigcap \beta(\theta')$, that is, when θ' is not a rationalizable strategy for player i at θ and θ', or when θ is not a rationalizable strategy for player $-i$ at θ and θ', part b) of the condition requires the existence of a punishment outcome $g(\theta',\theta)$ which lies in $SL_i(f(\theta),\theta) \bigcap SL_{-i}(f(\theta'),\theta')$.

So far, we have connected part a) and part b) of our condition to rationalizable implementation. To connect part c), suppose that (θ,θ') is such that for each player $\ell \in \mathcal{I}$, there exists a strategy $\hat{\theta} \in \beta_{-\ell}(\theta)$ s.t. $SL_\ell(f(\hat{\theta}),\hat{\theta}) \subseteq L_\ell\left(f(\theta),\theta'\right)$. Recall that $\beta(\theta)$ is the set of rationalizable strategy profiles at θ, which is a best-reply set at θ. The premise of part c) tells us that when the true state is θ' and we do not have any 'whistleblower' to rule out $\beta(\theta)$ as a best-reply set at θ', it must hold that $\beta(\theta) \subseteq \beta(\theta')$.

The role of part c) of our condition in rationalizable implementation is similar in spirit to the role played by Maskin Monotonicity in Nash implementation. The critical difference is that, for each state θ', Maskin monotonicity ensures the elimination of undesirable single-valued strategy profiles as a Nash equilibrium of $\mathcal{M}$ at θ'. In contrast, our condition eliminates undesirable set-valued strategy profiles as best-reply sets at θ'.

2P-GSMM** is not only necessary but is also sufficient for rationalizable implementation. We show this result by identifying a direct mechanism induced by the deception profile β and then by appropriately augmenting it. In what follows, suppose that f satisfies 2P-GSMM** wrt β.

Since f satisfies 2P-GSMM** wrt β, the direct mechanism induced by β, denoted by $\mathcal{M}^\beta \equiv (M, g^\beta)$, is such that $M_i = M_{-i} = \Theta$ and $g^\beta : M_i \times M_{-i} \to Y$ is such that for each $\theta \in \Theta$ and each $m \in \beta(\theta)$, $g^\beta(m) = f(\theta)$. By taking into account the restrictions imposed by part b) of our condition, we have that $g^\beta(\theta',\theta) = f(\theta) = f(\theta')$ if $(\theta',\theta) \in \beta(\theta) \bigcap \beta(\theta')$; otherwise–that is, if $(\theta',\theta) \notin \beta(\theta) \bigcap \beta(\theta')$–$g^\beta(\theta',\theta) \in SL_i(f(\theta),\theta) \bigcap SL_{-i}(f(\theta'),\theta')$.

Our implementing mechanism, denoted by $\tilde{\mathcal{M}}^\beta \equiv (\tilde{M}, \tilde{g}^\beta)$, is an augmentation of $\mathcal{M}^\beta$, where $\tilde{M}_i = \tilde{M}_{-i} = \Theta \times \mathbb{Z}_+$ and where $\tilde{g}^\beta$ satisfies the following property: for all pairs $(\theta',\theta) \in M$, it holds that $\tilde{g}^\beta((\theta',0),(\theta,0)) = g^\beta(\theta',\theta)$. A generic strategy of player i in the mechanism $\tilde{\mathcal{M}}^\beta$ is denoted by $m_i \equiv (m_i^1, m_i^2)$, where player i announces a state of world $m_i^1 \in \Theta$ and a non-negative integer $m_i^2 \in \mathbb{Z}_+$.

The augmented direct mechanism $\tilde{\mathcal{M}}^\beta$ has the following two main properties:

- Every rationalizable strategy profile m of $\tilde{\mathcal{M}}^\beta$ at θ–that is, $m \in S^{\tilde{\mathcal{M}}^\beta,\theta}$, is such that every player announces zero as the second component of his strategy–that is, $m_i^2 = m_{-i}^2 = 0$.
- Part c) of 2P-GSMM** guarantees that every rationalizable strategy profile m of $\tilde{\mathcal{M}}^\beta$ at θ is such that $(m_i^1, m_{-i}^1) \in \beta(\theta)$.

We can state our full characterization as follows.

Theorem 1. *$f : \Theta \mapsto Y$ is rationalizably implementable if and only if it satisfies 2P-GSMM**.*

Proof. See Appendix B. □

A key feature that distinguishes the two-agent case from the three or more agents case is the fact that for any pair (θ,θ'), it is possible that $SL_i(f(\theta),\theta) \cap L_{-i}\left(f\left(\theta'\right),\theta'\right)$ is empty, although $L_i(f(\theta),\theta) \cap L_{-i}\left(f\left(\theta'\right),\theta'\right)$ is non-empty. However, there are

[15] The reason is as follows. Since f is measurable wrt β and $\beta(\theta) \subseteq \beta(\theta')$, it follows that $f(\theta) = f(\theta')$. Moreover, since for all $\ell \in \mathcal{I}^{\beta(\theta)}$, it holds that $\theta \in \beta_{-\ell}(\theta')$ and $SL_\ell(f(\theta),\theta) \subseteq L_\ell(f(\theta'),\theta)$, part c) of 2P-GSMM** implies that $\beta(\theta') \subseteq \beta(\theta)$.

[16] The reason is that when player $\ell \notin \mathcal{I}^{\beta(\theta)}$, ℓ violates NWA at some $\hat{\theta}$ such that $\beta(\hat{\theta}) = \beta(\theta)$. Thus, it holds that $SL_\ell\left(f\left(\hat{\theta}\right),\hat{\theta}\right) = \emptyset$, and so $SL_\ell\left(f\left(\hat{\theta}\right),\hat{\theta}\right) \cap L_{-\ell}\left(f\left(\theta'\right),\theta'\right)$ is always empty for all $\theta' \in \Theta$. Part a) of 2P-GSMM** implies that $\cup_{\theta' \in \Theta}\beta_\ell\left(\theta'\right) \subseteq \beta_\ell\left(\hat{\theta}\right) = \beta_\ell(\theta)$, and so $\beta_\ell(\theta) = \Theta$.

[17] Recall that i is the player who satisfies NWA for the highest number of times.

important environments where $SL_i(f(\theta),\theta) \cap SL_{-i}\left(f\left(\theta'\right),\theta'\right)$ is never empty. For instance, it is non-empty in classical exchange economies where the SCF always picks an interior allocation and agents' indifference curves never touch the axes. Moreover, it is also non-empty in environments with transfer or common bad outcomes.

In these environments, part (a) of 2P-GSMM** is always vacuously satisfied for any $\beta \in \mathcal{B}_f$. Moreover, the existence requirement of part (b) of 2P-GSMM** is satisfied for any $\beta \in \mathcal{B}_f$. Thus, to check whether an SCF is implementable, we need only to check whether it satisfies part (c) of 2P-GSMM** or not. Can this condition be further simplified? The answer is yes.

To see it, suppose that f satisfies part (c) of 2P-GSMM** wrt $\beta \in \mathcal{B}_f$. Then, it follows from the definition of part (c) of 2P-GSMM** that f satisfies part (c) of 2P-GSMM** wrt $\tilde{\beta}$, where $\tilde{\beta}_\ell = P$ for each player ℓ and where P is the partition induced by β (see Remark 1). The reason is that $\tilde{\beta} \subseteq \beta$. Since in environments where for all $\theta,\theta' \in \Theta$, $SL_i(f(\theta),\theta) \bigcap SL_{-i}(f(\theta'),\theta') \neq \emptyset$, parts (a)-(b) of 2P-GSMM** are satisfied for any $\beta \in \mathcal{B}_f$, and so 2P-GSMM** can equivalently be written in the space of partitions as follows:

Proposition 1. *Suppose that for all $\theta,\theta' \in \Theta$, $SL_i(f(\theta),\theta) \bigcap SL_{-i}(f(\theta'),\theta') \neq \emptyset$. f satisfies 2P-GSMM** if and only if there exists $P \in \mathcal{P}_f$ such that for all $\theta,\theta' \in \Theta$, if for all $\ell \in \mathcal{I}$, there exists $\hat{\theta} \in P(\theta)$ s.t.*

$$SL_\ell(f(\hat{\theta}),\hat{\theta}) \subseteq L_\ell\left(f(\theta),\theta'\right) \implies P(\theta) \subseteq P(\theta').$$

The implementation of the Walrasian or competitive market allocations has always been a central theme in the implementation literature. It is well-known that under the assumption of interiority of the Walrasian allocations, these allocations are Nash implementable.[18] Under the same assumption, Walrasian allocations are rationalizable implementable when agents have Cobb-Douglas preferences. To clarify the discussion, let U^{CD} be the class of Cobb-Douglas utility functions. Let W defined over $\Theta \subseteq U^{CD} \times U^{CD}$ be the Walrasian SCF.[19] Recall that P_W is the partition of Θ induced by W. Under our preference domain assumption and the fact that there are only two agents, for any $\theta,\theta' \in \Theta$, the following properties hold: 1) $SL_1(W(\theta),\theta) \cap SL_2\left(W\left(\theta'\right),\theta'\right) \neq \varnothing$; 2) for each agent i, $SL_i(W(\theta),\theta) \cap SU_i\left(W(\theta),\theta'\right) \neq \varnothing$ if $\theta' \notin P_W(\theta)$; and 3) for each agent i, $SL_i(W(\theta),\theta) = SL_i\left(W(\theta),\theta'\right)$ if $\theta' \in P_W(\theta)$. From these properties, it is plain that the premises of Proposition 1 are satisfied, and that for each $\theta,\theta' \in \Theta$ such that $P_W(\theta) \neq P_W\left(\theta'\right)$, there exists an agent i such that $SL_i\left(W\left(\theta''\right),\theta''\right) \cap SU_i\left(W(\theta),\theta'\right) \neq \varnothing$ for all $\theta'' \in P_W(\theta)$. As a result of Proposition 1, one can see that the Walrasian SCF W defined on Θ satisfies 2P-GSMM**, and so it is rationalizably implementable by Theorem 1.

5. An application

In this section, we provide an application of Theorem 1 in the context of the selection of an arbitrator by two parties in litigation. Its main point is to show how to check our implementing condition.

The example has two parties, denoted by 1 and 2, three states, denoted by θ, θ' and θ'', and six arbitrators, denoted by a_1, a_2, a_3, a_4, a_5 and a_6. Parties' utilities from the pure outcomes are summarized in the table below, where $\varepsilon \in \left(\frac{1}{2}, 1\right)$.

	$u_1(\cdot,\theta)$	$u_2(\cdot,\theta)$	$u_1\left(\cdot,\theta'\right)$	$u_2\left(\cdot,\theta'\right)$	$u_1\left(\cdot,\theta''\right)$	$u_2\left(\cdot,\theta''\right)$
a_1	1	$-(1-\varepsilon)$	1	-1	1	$-(1-\varepsilon)$
a_2	0	0	0	0	0	0
a_3	-1	1	$-(1-\varepsilon)$	1	$-(1-\varepsilon)$	1
a_4	1	-2	-2	-1	-2	-2
a_5	2	-3	-2	-2	-2	-3
a_6	-3	-3	-3	3	-3	-3

Suppose that the goal of the parties, described by f, is to select a Pareto efficient arbitrator at each state. Specifically:

$$f(\theta) = f\left(\theta'\right) = \left\{a_2\right\} \text{ and } f\left(\theta''\right) = \left\{a_1\right\}.$$

Note that party 1's preference at θ and party 2's preference at θ' are diametrically opposed, which makes the intersection $SL_1(f(\theta),\theta) \bigcap SL_2\left(f\left(\theta'\right),\theta'\right)$ empty. Also, note that parties' preferences are such that the intersection of strict lower contour sets is never empty in all other cases. Let us finally note that party 1's preferences are the same at θ' and θ'', whereas they are the same for party 2 at θ and θ''.

To show that this f is rationalizably implementable, it is sufficient to show that f satisfies our implementing condition. This requires that we need to construct a deception profile β such that parts a)-c) of our condition are satisfied. In the claim below, we show that f satisfies our condition wrt the following deception profile:

$$\beta_1(\theta) = \beta_2\left(\theta'\right) = \left\{\theta,\theta'\right\}, \beta_1\left(\theta'\right) = \left\{\theta'\right\},$$
$$\beta_2(\theta) = \{\theta\} \text{ and } \beta_1\left(\theta''\right) = \beta_2\left(\theta''\right) = \left\{\theta''\right\}.$$

[18] Hurwicz et al. (1994) show the Walrasian allocations that are at the boundary of the feasible set are not Nash implementable.

[19] Let x be an allocation such that the total consumption of good j is equal to its total supply. For each $\theta \in \Theta$, the Walrasian SCF W is defined by $W(\theta) = \{x\}$ if there exists $p \in \Delta = \left\{p' \in \mathbb{R}^\ell_+ \mid \sum_{j=1}^{\ell} p_j = 1\right\}$ such that for each agent i, $p \cdot x_i = p \cdot \omega_i$ and $u_i\left(y_i\right) > u_i\left(x_i\right)$ implies $p \cdot y_i > p \cdot \omega_i$, where $\ell > 1$ is the number of goods.

Table I
$e:\Theta\times\Theta\to Y$.

	θ	θ'	θ''
θ	a_2	a_6	a_6
θ'	a_2	a_2	a_5
θ''	a_6	a_6	a_1

Claim 1. *f is rationalizably implementable.*

Proof. We show that parts a)-c) of our condition are satisfied wrt the deception profile β defined above. Note that $\beta\in\mathcal{B}$.

The key feature of the example is that $SL_{\ell}(f(\hat{\theta}),\hat{\theta})\bigcap SL_{-\ell}(f(\bar{\theta}),\bar{\theta})=\emptyset$ if $(\hat{\theta},\bar{\theta})=(\theta,\theta')$ and $\ell=1$, otherwise, $SL_{\ell}(f(\hat{\theta}),\hat{\theta})\bigcap SL_{-\ell}(f(\bar{\theta}),\bar{\theta})\neq\emptyset$.

To check part a), note that since, by construction, it holds that $SL_{\ell}(f(\hat{\theta}),\hat{\theta})\cap L_{-\ell}\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\neq\emptyset$ if $\ell\neq 1$ or $(\hat{\theta},\bar{\theta})\neq\left(\theta,\theta'\right)$, we need only to check the case that $\ell=1$ and $(\hat{\theta},\bar{\theta})=\left(\theta,\theta'\right)$. Part a) is satisfied because $\beta_1\left(\theta'\right)\subseteq\beta_1(\theta)$.

To check part b), note that since $SL_{\ell}\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\neq\emptyset$ for each party $\ell\in\mathcal{I}$ and each $\bar{\theta}\in\Theta$ and, moreover, $\beta\left(\bar{\theta}\right)\neq\beta\left(\hat{\theta}\right)$ for all distinct $\bar{\theta},\hat{\theta}\in\Theta$, it follows that $\mathcal{I}^{\beta(\bar{\theta})}=\mathcal{I}$ for all $\bar{\theta}\in\Theta$. Since f satisfies NWA, we set $i=1$ for checking part b) of the condition. Given the key feature of our example discussed above, it can easily be verified that the function $e:\Theta\times\Theta\to Y$ defined in Table I above satisfies part b) of our condition, where $\Theta=\{\theta,\theta',\theta''\}$ and the states of party i are listed on rows. Note that $e\left(\theta'',\theta''\right)=f\left(\theta''\right)$ and $e(\theta,\theta)=e\left(\theta',\theta'\right)=e\left(\theta',\theta\right)=f(\theta)=f\left(\theta'\right)$. In all other cases, $e(\bar{\theta},\hat{\theta})\in SL_i(f(\hat{\theta}),\hat{\theta})\cap SL_{-i}(f(\bar{\theta}),\bar{\theta})$.

We are left to show that f satisfies part c) of 2P-GSMM**. Below, we check all cases.

Suppose that the state moves from $\bar{\theta}\in\left\{\theta',\theta''\right\}$ to θ. Part c) is satisfied when $\bar{\theta}=\theta'$ because $\beta\left(\theta'\right)\nsubseteq\beta(\theta)$ and party $2\in\mathcal{I}^{\beta(\theta)}$ has a preference reversal because $\frac{1}{2}a_1+\frac{1}{2}a_3\in SL_2(f(\hat{\theta}),\hat{\theta})\cap SU_2\left(f\left(\theta'\right),\theta\right)$ for all $\hat{\theta}\in\beta_1\left(\theta'\right)=\left\{\theta'\right\}$. It is also satisfied when $\bar{\theta}=\theta''$ because $\beta\left(\theta''\right)\nsubseteq\beta(\theta)$ and party $1\in\mathcal{I}^{\beta(\theta)}$ has a preference reversal because $a_5\in SL_1(f(\hat{\theta}),\hat{\theta})\cap SU_1\left(f\left(\theta''\right),\theta\right)$ for all $\hat{\theta}\in\beta_2\left(\theta''\right)=\left\{\theta''\right\}$.

Suppose that the state moves from $\bar{\theta}\in\left\{\theta,\theta''\right\}$ to θ'. Part c) is satisfied when $\bar{\theta}=\theta$ because $\beta(\theta)\nsubseteq\beta\left(\theta'\right)$ and party $1\in\mathcal{I}^{\beta(\theta')}$ has a preference reversal because $\frac{1}{2}a_1+\frac{1}{2}a_3\in SL_1(f(\hat{\theta}),\hat{\theta})\cap SU_1\left(f(\theta),\theta'\right)$ for all $\hat{\theta}\in\beta_2(\theta)=\{\theta\}$. It is also satisfied when $\bar{\theta}=\theta''$ because $\beta\left(\theta''\right)\nsubseteq\beta\left(\theta'\right)$ and party $2\in\mathcal{I}^{\beta(\theta')}$ has a preference reversal because $a_6\in SL_2(f(\hat{\theta}),\hat{\theta})\cap SU_2\left(f\left(\theta''\right),\theta'\right)$ for all $\hat{\theta}\in\beta_1\left(\theta''\right)=\left\{\theta''\right\}$.

Finally, suppose that the state moves from $\bar{\theta}\in\left\{\theta,\theta'\right\}$ to θ''. Part c) is satisfied when $\bar{\theta}=\theta$ because $\beta(\theta)\nsubseteq\beta\left(\theta''\right)$ and party $1\in\mathcal{I}^{\beta(\theta'')}$ has a preference reversal because $\frac{1}{2}a_1+\frac{1}{2}a_3\in SL_1(f(\hat{\theta}),\hat{\theta})\cap SU_1\left(f(\theta),\theta''\right)$ for all $\hat{\theta}\in\beta_2(\theta)=\{\theta\}$. It is also satisfied when $\bar{\theta}=\theta'$ because $\beta\left(\theta'\right)\nsubseteq\beta\left(\theta''\right)$ and party $2\in\mathcal{I}^{\beta(\theta'')}$ has a preference reversal because $\frac{1}{2}a_1+\frac{1}{2}a_3\in SL_2(f(\hat{\theta}),\hat{\theta})\cap SU_2\left(f\left(\theta'\right),\theta''\right)$ for all $\hat{\theta}\in\beta_1\left(\theta'\right)=\left\{\theta'\right\}$. □

6. Endogenizing deceptions for two-player problems

This section shows how to construct the deception β in our implementing condition. Let us first connect our approach with that used by BMT. These authors discuss the role of partitions in their characterization result. In particular, they show that the required partition must be as fine as P_f and as coarse as the partition obtained in their Lemma 1, which BMT called "pairwise inclusion property" (see BMT, p. 1266, for a discussion). However, BMT also argues that this property cannot pin down the partition by stating:

We finally observe that the partition P may yet have to be coarser than is indicated by the pairwise inclusion property (BMT, p. 1266).

In what follows, we formalize the approach of BMT and we show how it can help us construct the deception profile underlying our implementing condition. To this end, we first identify an important implication of part (a) of 2P-GSMM**. This implication is central in establishing the sufficiency of our condition and in the construction of the deception profile β underlying our condition.

Part a) of 2P-GSMM** helps us to narrow down the space of deception profiles over which one has to search for the deception profile satisfying 2P-GSMM**. To show this, let us introduce a special class of deceptions as follows. For all $\ell\in\mathcal{I}$ and all $P\in\mathcal{P}_f$, let us define $\beta^P_{\ell,0}:\Theta\to 2^{\Theta}\setminus\{\emptyset\}$ by

$$\beta^P_{\ell,0}(\theta)=P(\theta)\bigcup\left(\bigcup_{\theta'\in\Theta}\left\{\quad P\left(\theta'\right)\;\middle|\;\begin{array}{l}\exists\left(\bar{\theta},\hat{\theta}\right)\in P(\theta)\times P\left(\theta'\right)\\ SL_{\ell}\left(f\left(\bar{\theta}\right),\bar{\theta}\right)\bigcap L_{-\ell}\left(f\left(\hat{\theta}\right),\hat{\theta}\right)=\emptyset\end{array}\right\}\right). \tag{5}$$

For all $\ell\in\mathcal{I}$ and all $P\in P_f$, let us define $\beta^P_{\ell}:\Theta\to 2^{\Theta}\setminus\{\emptyset\}$ by

$$\beta^P_{\ell}(\theta)\equiv P(\theta)\bigcup\left(\bigcup_{\theta'\in\Theta}\left\{\quad P\left(\theta'\right)\;\middle|\;\begin{array}{l}\text{there exist a positive integer } n \text{ and}\\ \theta_0,\theta_1,...,\theta_n\in\Theta \text{ s.t. } \theta=\theta_0 \text{ and } \theta'=\theta_n,\\ \text{and } P\left(\theta_k\right)\subseteq\beta^P_{\ell,0}\left(\theta_{k-1}\right) \text{ for } k=1,...,n.\end{array}\right\}\right) \tag{6}$$

Remark 4. Note that for all $\theta,\theta'\in\Theta$ s.t. $P(\theta)=P\left(\theta'\right)$, it follows from (6) that $\beta^P_{\ell}(\theta)=\beta^P_{\ell}\left(\theta'\right)$.

Remark 5. Note that for all $\theta \in \Theta$ and for every $\ell \in \mathcal{I}$, if $\beta^P_{\ell,0}(\theta) = P(\theta)$, then $\beta^P_\ell(\theta) = \beta^P_{\ell,0} = P(\theta)$.

Based on (6), we can define a subset of $\mathcal{B}_f$ as follows.

$$\mathcal{B}_f(\mathcal{P}_f) = \left\{ \beta \in \mathcal{B}_f \middle| \beta \equiv \beta^P \text{ for some } P \in \mathcal{P}_f \right\}$$

The following lemma states that for any SCF satisfying 2P-GSMM**, we can restrict our focus on the space $\mathcal{B}_f(\mathcal{P}_f)$ for the existence of deception profiles satisfying parts a)-c) of 2P-GSMM**.

Lemma 1. *If f satisfies 2P-GSMM** wrt $\beta \in \mathcal{B}_f$, then f satisfies 2P-GSMM** wrt $\beta \in \mathcal{B}_f(\mathcal{P}_f)$.*

Proof. See Appendix C. □

We can now define our algorithm over $\mathcal{B}_f(\mathcal{P}_f)$ which identifies the deception profile meeting the requirements of our implementing condition. For all $\theta \in \Theta$, let $P^t(\theta) = \{\theta\}$. Clearly, $P^t \in \mathcal{P}_f$. Moreover, for all $\theta' \in \Theta$, let the sequence $\left\{P^\ell\right\}_{\ell\geq 0}$ be defined iteratively as follows. For $\ell = 0$, let

$$P^0\left(\theta'\right) = P^t\left(\theta'\right), \tag{7}$$

and, for all $\ell - 1 \geq 0$ s.t. $P^{\ell-1} \in \mathcal{P}_f$, let us define $P^\ell(\theta')$ as follows.

1. For all odd positive integer $\ell > 0$,

$$P^\ell(\theta) = \left\{ \theta' \in \Theta \middle| \beta^{P^{\ell-1}}(\theta) = \beta^{P^{\ell-1}}\left(\theta'\right) \right\}. \tag{8}$$

2. For all even positive integer $\ell > 0$,

$$P^\ell\left(\theta'\right) = \bigcup_{\theta\in\Theta} \left\{ P^{\ell-1}(\theta) \;\middle|\; \begin{array}{l} \text{for all } j \in \mathcal{I}^{P^{\ell-1}(\theta')}, \exists \hat{\theta} \in \beta^{P^{\ell-2}}_{-j}(\theta) \text{ s.t.} \\ SL_j\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_j\left(f(\theta), \theta'\right). \end{array} \right\} \tag{9}$$

Suppose that the sequence $\left\{P^\ell\right\}_{\ell\geq 0}$ is s.t. $P^\ell \in \mathcal{P}_f$ for all $\ell \geq 0$. Then, $P^\ell \subseteq P^{\ell+1}$ for all $\ell \geq 0$. Since the sequence is increasing and Θ is finite, the limit of the sequence exists–that is, there exists ℓ^* s.t. $P^\ell = P^{\ell^*}$ for all $\ell \geq \ell^*$. Let us denote the limit of $\left\{P^\ell\right\}_{\ell\geq 0}$ by P^* when the sequence $\left\{P^\ell\right\}_{\ell\geq 0}$ is s.t. $P^\ell \in \mathcal{P}_f$ for all $\ell \geq 0$.

Theorem 2. *$f : \Theta \mapsto Y$ satisfies 2P-GSMM** if and only if it satisfies 2P-GSMM** wrt β^{P^*}.*

Proof. It is obvious that $f : \Theta \mapsto Y$ satisfies 2P-GSMM** provided that it satisfies 2P-GSMM** wrt β^{P^*}.

For the converse, suppose that f satisfies 2P-GSMM** wrt β. Lemma 1 implies that f satisfies 2P-GSMM** wrt $\beta^P \in \mathcal{B}_f(\mathcal{P}_f)$. We show that $\beta^{P^*} \subseteq \beta^P$. We show this by showing that $P^\ell \subseteq P$ and $P^\ell \in \mathcal{P}_f$ for all $\ell \geq 0$. Let us proceed by induction. Clearly, $P^0 = P^t \subseteq P$ and $P^0 = P^t \in \mathcal{P}_f$.

Then, suppose that there exists $\ell \geq 0$ s.t. $P^{\ell'} \subseteq P$ and $P^{\ell'} \in \mathcal{P}_f$ for all $\ell \geq \ell' \geq 0$. Let us show that $P^{\ell+1} \subseteq P$ and $P^{\ell+1} \in \mathcal{P}_f$.

Since $P^{\ell'} \subseteq P$ for all $\ell \geq \ell' \geq 0$ and $P \in \mathcal{P}_f$, (6) implies that $\beta^{P^{\ell'}} \subseteq \beta^P$ for all $\ell \geq \ell' \geq 0$. Let us proceed according to whether $\ell + 1$ is odd or even. Suppose that $\ell + 1$ is odd. It follows from (8) that $P^{\ell+1} \subseteq P$. Suppose that $\ell + 1$ is even. It follows from (9) that $P^{\ell+1} \subseteq P$.

By the principle of mathematical induction, we have that $P^\ell \subseteq P$ for all $\ell \geq 0$. It follows that $P^* \subseteq P$, and so (6) implies that $\beta^{P^*} \subseteq \beta^P$. Since f satisfies 2P-GSMM** wrt β^P and f is measurable wrt β^P, and since, moreover, $\beta^{P^*} \subseteq \beta^P$, we have that f is measurable wrt β^{P^*}, and so $\beta^{P^*} \in \mathcal{B}_f$. Thus, f satisfies 2P-GSMM** wrt β^{P^*}. □

7. Concluding remarks

7.1. Connecting 2P-GSMM** with strict event monotonicity**

Xiong (2023b) shows that rationalizable implementation of an SCF is equivalent to the Strict Event monotonicity** (SEM**, henceforth) when there are three or more players. The condition contains two axioms, SEM and Dictator Monotonicity (DM), that are based on a common partition $P \in \mathcal{P}_f$. His condition can be stated as follows for the two-player case.

Definition 8. $f : \Theta \mapsto Y$ satisfies Two-Player SEM** (2P-SEM**, henceforth) if there exists $P \in \mathcal{P}_f$ s.t. for all $\theta, \theta' \in \Theta$,

$$\underbrace{\left[\begin{array}{l}\forall i \in \mathcal{I}^{P(\theta')}, \exists \hat{\theta} \in P(\theta) \text{ s.t.} \\ SL_i\left(f(\theta), \hat{\theta}\right) \subseteq L_i\left(f(\theta), \theta'\right)\end{array}\right]}_{\text{PART A: SEM}} \vee \underbrace{\left[\begin{array}{l}\mathcal{I}^{P(\theta)} = \{i\}, \exists \hat{\theta} \in \Theta \text{ s.t.} \\ L_i\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \subseteq L_i\left(f(\theta), \theta'\right)\end{array}\right]}_{\text{PART B: DM}} \Longrightarrow P\left(\theta'\right) = P(\theta),$$

where $\mathcal{I}^{P(\theta')}$ is defined by:

$$\mathcal{I}^{P(\theta')} \equiv \left\{\ell \in \mathcal{I} \middle| SL_\ell\left(f\left(\hat{\theta}\right), \hat{\theta}\right) \neq \emptyset \text{ for all } \hat{\theta} \in P(\theta')\right\}.$$

Part A is the premises of SEM, while Part B is the premises of DM.

SEM requires that for any θ and θ', with $P\left(\theta'\right) \neq P(\theta)$, there exists a whistle-blower i who, at true state θ', can find for each $\hat{\theta} \in P(\theta)$, a state-contingent blocking plan $y^{\hat{\theta}}$ that works for this $\hat{\theta}$–that is, $y^{\hat{\theta}} \in SL_i(f(\theta), \hat{\theta}) \cap SU_i(f(\theta), \theta')$. Note that SEM requires the whistle-blower i to be an active player in $\mathcal{I}^{P(\theta')}$–i.e., $i \in \mathcal{I}^{P(\theta')}$. In contrast to SMM**, SEM requires that player i must be an active player in $\mathcal{I}^{P(\theta')}$.

DM requires that at the true state θ' if player i reports θ when he is a dictator–that is, he is the only one active player in $\mathcal{I}^{P(\theta)} = \{i\}$, while the opponent reports $\hat{\theta}$, and $P(\theta) \neq P(\theta')$, then player i can be the only whistle-blower who must have a blocking plan y that must be credible, i.e., $y \in L_i(f(\hat{\theta}), \hat{\theta})$, and strictly profitable, i.e., $y \in SU_i\left(f(\theta), \theta'\right)$, when the state moves from $\hat{\theta}$ to θ'.

A priori, 2P-GSMM** and 2P-SEM** may seem very different conditions. The reason is that 2P-GSMM** is written in the space of deceptions, whereas 2P-SEM** is written in the space of partitions. However, Remark 1 allows us to form a bridge between these two conditions.

To see this, suppose that f satisfies 2P-GSMM** wrt $\beta \in \mathcal{B}_f$, then f satisfies 2P–SEM** wrt the partition induced by β. Indeed, since $P(\theta) \subseteq \beta_\ell(\theta)$ for all $\ell \in \mathcal{I}$ and, moreover, $\mathcal{I}^{P(\theta')} = \mathcal{I}^{\beta(\theta')}$ for all $\theta' \in \Theta$, one can see that 2P-GSMM** implies SEM. To see that 2P-GSMM** implies DM wrt P, let the premises of DM be satisfied. Then, suppose that $\mathcal{I}^{P(\theta)} = \{i\}$ and that there exists $\hat{\theta} \in \Theta$ s.t. $L_i(f(\hat{\theta}), \hat{\theta}) \subseteq L_i(f(\theta), \theta')$. Since $-i \notin \mathcal{I}^{\beta(\theta)} = \mathcal{I}^{P(\theta)}$, Remark 3 implies that $\beta_{-i}(\theta) = \Theta$. Thus, it follows that there exists $\hat{\theta} \in \beta_{-i}(\theta)$ s.t. $L_i(f(\hat{\theta}), \hat{\theta}) \subseteq L_i(f(\theta), \theta')$. Moreover, since $-i \notin \mathcal{I}^{\beta(\theta)}$, there exists $\tilde{\theta} \in \beta_i(\theta)$ s.t. $SL_{-i}(f(\tilde{\theta}), \tilde{\theta}) = \emptyset$, and so there exists $\tilde{\theta} \in \beta_i(\theta)$ s.t. $SL_{-i}(f(\tilde{\theta}), \tilde{\theta}) \subseteq L_{-i}(f(\theta), \theta')$. By applying part (c) of 2P-GSMM** when the state moves from θ to θ', we conclude that $\beta(\theta) \subseteq \beta(\theta')$. Moreover, since, in light of Remark 2, it holds that $\beta(\theta) \subseteq \beta(\theta')$, it follows that $\beta(\theta) = \beta(\theta')$, and so $P(\theta) = P(\theta')$.

Given the discussion provided above and the example discussed in Section 3, we state without proving that our two-player implementation condition is strictly stronger than 2P-SEM**.

Proposition 2. *If $f : \Theta \mapsto Y$ satisfies 2P-GSMM**, then it satisfies 2P-SEM**. The converse implication is false.*

7.2. Social choice correspondences

We restricted our attention to the study of SCFs. Let us briefly discuss the extension of the results to social choice correspondences (SCC). An SCC defines a set of outcomes for each state, and rationalizability is a set-based solution concept. Jain (2021) and Kunimoto and Serrano (2019) derive partial characterizations under different notions of rationalizable implementation for SCCs in environments with complete information and three or more players.

Using the characterization obtained for SCFs by Bergemann et al. (2011), Jain (2021), in his online appendix, formulates a condition termed r-monotonicity**. r-monotonicity** reduces to Maskin monotonicity** when we focus on SCFs. Under NWA and in an environment with more than three players, r-monotonicity** can be shown to be sufficient for rationalizable implementation of an SCC under Jain (2021)'s notion of implementation. Following Jain (2021)'s approach, it is straightforward to formulate a two-player sufficient condition for rationalizable implementation of SCCs, which will reduce to 2P-GSMM** for SCFs.

Although a complete characterization of the rationalizable implementation of SCCs is beyond the scope of this paper, SEM** of Xiong (2023b) and our 2P-GSMM** must be a theoretical benchmark for any work focusing on rationalizable implementation of SCCs. We conjecture that our framework of deceptions could be useful in providing a complete characterization of the rationalizable implementation of SCCs.

CRediT authorship contribution statement

Ritesh Jain: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Ville Korpela:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Michele Lombardi:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no relevant material or financial interests that relate to the research described in this paper

Appendix. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jet.2025.106031.

Data availability

No data was used for the research described in the article.

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