Singular Surfaces

▶ Kinematics of Singular Surfaces and Jump Equations of Balance
▶ Propagation of Wavefronts in Thermoelastic Media
▶ Singular Surfaces in Thermoviscoelastic Materials with Voids

Singular Surfaces in Thermoviscoelastic Materials with Voids

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Synonyms

Materials with voids; Singular surfaces; Thermoviscoelasticity; Wave propagation

Overview

The theory of elastic materials with voids is a generalization of the classical theory of elasticity. The basic premise underlying this theory is that the volume fraction corresponding to the void volume may be taken as an independent kinematical variable. The bulk density is written as the product of two fields: the matrix material field and the volume fraction field. This representation introduces an additional degree of kinematic freedom.

An early attempt to describe the change in void volume, induced by the deformation of a porous body, by means of an independent kinematical variable was made by Goodman and Cowin [1]. They introduced the notion of *volume distribution function* as the basis of a theory of granular materials with interstitial voids which, in contrast to the previous conventional treatments of the topic, was formulated from the formal arguments of continuum mechanics. An account of the historical development of the porous media theories as well as reference to various contributions may be found in the monographs by De Boer [2] and Rajagopal and Tao [3].

The nonlinear theory of elastic material with voids was introduced by Nunziato and Cowin [4]. They subsequently established a linear theory of elastic material with voids [5]. Extensions of the theory to thermoelastic materials were formulated by Iesan [6] and by De Cicco and Diaco [7]. The viscoelastic behavior of materials with voids has been discussed in [8] and [9]. In recent years, the mechanics of porous media has been a subject of intensive study, and many works have been devoted to the applications of the theory (see, e.g., [10]). These studies seem to be an adequate mathematical tool to describe the behavior of porous materials like rock and soils, biological tissue, and manufactured porous materials.

Some problems in the theory of viscoelastic anisotropic materials can be studied only under the condition that the relaxation functions are symmetric. In the classical theory of linear viscoelasticity, Day [11] first drew attention to a necessary and sufficient condition for the symmetry of the relaxation function. If $G(t)$ is a relaxation function, he proved that $G(t)$ is symmetric for every t in $0 \leq t \leq \infty$ if and only if the work done on every closed path starting from the virgin state is invariant under time reversal. Time-reversal symmetry was discussed by Gurtin [12] in the context of thermodynamics of materials with memory. In [13], De Cicco and Nappa presented a theory of thermoviscoelastic materials with voids in which the heat flux is independent on the present temperature gradient, but depends upon the past history of this gradient. In the framework of this theory, they derived an extension of Gurtin's results.

The propagation of mechanical disturbances in solids is of interest in many branches of the physical sciences and engineering. A propagating singular surface is roughly speaking a surface moving with time, across which some kinematic quantity suffers a jump discontinuity. The basic methods for the analysis of propagating surface of discontinuity in continuum mechanics were established toward the end of the nineteenth century. The definition of singular surface of order $n \geq 0$ is due to Duhem [14] and Hadamard [15]. For elastic media, a brief exposition can be found in the book by Love [16]. In more recent years, the theory was discussed in detail by Thomas [17] and was extended in the context of linear thermoelasticity by Chadwick and Powdrill [18]. The propagation of singular surfaces in viscoelastic medium has been the subject of various papers (see, e.g., [19–21]). In the isothermal case, the wave propagation in materials with voids has been extensively studied (see, e.g., [22–24]). Following the results established in [13], we study the propagation conditions and growth equations which govern the propagation of singular surfaces of order 1 in the case of linear homogeneous isotropic thermoviscoelastic materials with voids.

Preliminaries

We consider a body which at time t^0 occupies the region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The configuration of the body at time t^0 is taken as reference configuration. The motion of the body is referred to the reference configuration and a fixed system of rectangular Cartesian axes. We use vector and Cartesian tensor notation with Latin indices having the value $1, 2, 3$. Greek indices are confined to the range (1, 2). Letters in boldface stand for tensors of an order $p \geq 1$, and if v has the order p, we write $v_{ij...k}$ (p subscripts) for the components of v in the underlying rectangular Cartesian coordinate system. In all that follows, subscripts preceded by a comma denote partial differentiation with respect to corresponding material coordinate. Also, we use a superposed dot to denote differentiation with respect to t holding the material coordinates fixed. The position of a typical particle of the body at time t is x. The motion of continuum is described by the mappings

$$x_i = x_i(X_j, t), \quad (X_j, t) \in B \times I \tag{1}$$

where I is a time interval. The above functions are assumed to be sufficiently smooth for the ensuing analysis to be valid. The concept of a distributed body asserts that the mass density at time t has the decomposition

$$\rho = \nu\gamma \tag{2}$$

where γ is the density of the matrix material and ν is the volume fraction field. The relation (2) also holds for the reference configuration, $\rho_0 = \nu_0\gamma_0$ where ρ_0 is the density at time t^0 and ν_0 is the volume fraction field for the reference configuration.

We introduce the displacement vector field $\boldsymbol{u}$ defined by $u_i = x_i - X_i$; the temperature change ϑ by $\vartheta = T - T_0$, where T is the absolute temperature and T_0 the absolute temperature in the reference configuration; and the volume fraction change φ by $\varphi = \nu - \nu_0$. The infinitesimal strain tensor is given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

In the following, $\boldsymbol{t}$ denotes the stress: tensor, $\boldsymbol{b}$ the body force per unit mass, $\boldsymbol{h}$ the equilibrated stress, g the intrinsic equilibrated body force per unit mass, l the extrinsic equilibrated body force per unit mass, and κ the equilibrated inertia.

We consider an arbitrary open region ω in the continuum, bounded by a surface $\partial\omega$. In the framework of the linear theory, the materials with voids behaves according to the global balance law of linear momentum

$$\frac{d}{dt}\int_\omega \rho_0 v_i dv = \int_\omega \rho_0 b_i dv + \int_{\partial\omega} t_{ji} n_j da \tag{4}$$

and the law of balance of equilibrated force

$$\frac{d}{dt}\int_\omega \rho_0 \kappa \dot{\varphi} dv = \int_\omega (\rho_0 l + g) dv + \int_{\partial\omega} h_i n_i da \tag{5}$$

Here $\boldsymbol{n}$ is the outward unit normal to $\partial\omega$ and $v_i = \dot{u}_i$. The law of balance energy is given by

$$\frac{d}{dt}\int_\omega \rho_0 e^* dv = \int_\omega \rho_0 s dv + \int_{\partial\omega} q_i n_i da \tag{6}$$

where $e^* = T_0\eta$, η is the entropy per unit mass, s is the external heat supply per unit mass per unit time, and $\boldsymbol{q}$ is the heat flux vector.

In what follows, we consider materials that are centrosymmetric. In this case, the constitutive equations consist of (see [13])

$$\begin{aligned}
&t_{ij}(t) = \int_{-\infty}^{t} [G_{ijmn}(t-s)\,\dot{e}_{mn}(s)+\\
&B_{ij}(t-s)\dot{\varphi}(s) - b_{ij}(t-s)\dot{\vartheta}(s)]ds\\
&h_i(t) = \int_{-\infty}^{t} [A_{ij}(t-s)\,\dot{\varphi}_{,j}(s)-\\
&N_{ij}(t-s)\vartheta_{,j}(s)]ds\\
&g(t) = -\int_{-\infty}^{t} [B_{ij}(t-s)\,\dot{e}_{ij}(s)+\\
&M(t-s)\dot{\varphi}(s) - m(t-s)\dot{\vartheta}(s)]ds\\
&\rho_0 e^*(t) = T_0\int_{-\infty}^{t} [b_{ij}(t-s)\,\dot{e}_{ij}(s)+\\
&m(t-s)\dot{\varphi}(s) + a(t-s)\dot{\vartheta}(s)]ds\\
&q_i(t) = \int_{-\infty}^{t} [T_0 N_{ji}(t-s)\,\dot{\varphi}_{,j}(s)+\\
&K_{ij}(t-s)\vartheta_{,j}(s)]ds
\end{aligned} \tag{7}$$

where $\boldsymbol{G}$, $\boldsymbol{B}$, $\boldsymbol{b}$, $\boldsymbol{A}$, $\boldsymbol{N}$, M, m, $\boldsymbol{K}$, and α are the constitutive moduli of the material. We assume that the infinitesimal entropy production is invariant under time reversal. Then, the following relations hold for every $s \geq 0$ [13]:

$$\begin{aligned}
G_{ijmn}(s) = G_{jimn}(s) = G_{mnij}(s)\\
b_{ij}(s) = b_{ji}(s)\ A_{ij}(s) = A_{ji}(s)\\
K_{ij}(s) = K_{ji}(s)
\end{aligned} \tag{8}$$

In what follows, we consider the initial history conditions:

$$\begin{aligned}
u(X,t) = 0 \qquad \varphi(X,t) = 0\\
\vartheta(X,t) = 0 \qquad (X,t) \in B \times (-\infty, 0)
\end{aligned} \tag{9}$$

In the relations (7), the functions t_{ij}, h_i, g, and e^* are independent of the history of the temperature gradient if and only if the heat flux is independent of the histories of e_{ij}, φ, $\varphi_{,j}$, and ϑ. Coleman and Gurtin [25] have shown that, under the assumption of fading memory, the equilibrium heat flux vanishes when the temperature gradient vanishes. In what follows, we consider constitutive equations which are consistent with this result and restrict our attention to homogeneous and isotropic materials. Thus, we have

$$\begin{aligned}
G_{ijrs}(X,t) = \lambda(t)\delta_{ij}\delta_{rs} + \mu(t)\big(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\big)\\
B_{ij}(X,t) = b(t)\delta_{ij}, \qquad b_{ij}(X,t) = \beta(t)\delta_{ij}\\
A_{ij}(X,t) = \alpha(t)\delta_{ij}, \qquad K_{ij}(X,t) = k(t)\delta_{ij}\\
(X,t) \in B \times (-\infty,\infty)
\end{aligned} \tag{10}$$

where δ_{ij} is the Kronecker delta. We assume that the relaxation functions λ, μ, β, b, α, and k are of class C^2 on $(-\infty,\infty)$. With the help of (9), the constitutive equations become

$$\begin{aligned}
&t_{ij}(X,t) = G_{ijmn}(0)e_{mn}(X,t) + B_{ij}(0)\varphi(X,t)\\
&- b_{ij}(0)\vartheta(X,t) + \int_0^t \{\dot{G}_{ijmn}(t-s)e_{mn}(X,s)\\
&+ \dot{B}_{ij}(t-s)\varphi(X,s) - \dot{b}_{ij}(t-s)\vartheta(X,s)\}ds\\
&h_i(X,t) = A_{ij}(0)\varphi_{,j}(X,t) + \int_0^t \dot{A}_{ij}(t-s)\varphi_{,j}(X,s)ds\\
&g(X,t) = -B_{ij}(0)e_{ij}(X,t) - M(0)\varphi(X,t)\\
&+ m(0)\vartheta(X,t) + \int_0^t \{\dot{m}(t-s)\vartheta(X,s)\\
&- \dot{B}_{ij}(t-s)e_{ij}(X,s) - \dot{M}(t-s)\varphi(X,s)\}ds\\
&\rho_0 e^*(X,t) = \rho_0 T_0\eta(X,t) = T_0\{b_{ij}(0)e_{ij}(X,t)\\
&+ m(0)\varphi(X,t) + a(0)\vartheta(X,t)\\
&+ \int_0^t [\dot{b}_{ij}(t-s)e_{ij}(X,s) + \dot{m}(t-s)\varphi(X,s)\\
&+ \dot{a}(t-s)\vartheta(X,s)]ds\}\\
&q_i(X,t) = \int_0^t K_{ij}(t-s)\vartheta_{,j}(X,s)ds
\end{aligned} \tag{11}$$

where the constitutive moduli have the form (10), and we have used the notation $G_{ijrs}(0) = G_{ijrs}(X,0)$, etc.

We assume that

$$\begin{aligned}
&\lambda(0) + 2\mu(0) > 0 \quad \mu(0) > 0 \qquad \alpha(0) > 0\\
&k(0) > 0 \qquad a(0) > 0 \qquad \dot{\lambda}(0) + 2\dot{\mu}(0) < 0\\
&\dot{\mu}(0) < 0 \qquad \dot{\alpha}(0) < 0
\end{aligned} \tag{12}$$

The restrictions concerning λ, μ, $\dot{\lambda}$, $\dot{\mu}$, a, and k have been extensively studied in the classical thermoviscoelasticity [19–21] and [26]. Let us show that the restrictions on α and $\dot{\alpha}$ are compatible with the second law of thermodynamics. In the linear theory, the second law of thermodynamics may be written in the form

$$\int_0^t (\rho_0 \dot{\eta}\vartheta + t_{ij}\dot{e}_{ij} + h_i \dot{\varphi}_{,i} - g\dot{\varphi} + \frac{1}{T_0} q_i \theta_{,i}) d\tau \geq 0 \tag{13}$$

for every $t \geq 0$. We restrict our attention only to porosity effect. In this case, the relation (13) reduces to the following dissipation inequality:

$$U(X,t) \geq 0, \quad (X,t) \in B \times [0,\infty) \tag{14}$$

where the function U is defined by

$$U(X,t) = \int_0^t [h_i(X,\tau)\,\dot{\varphi}_{,i}(X,\tau) - g(X,\tau)\dot{\varphi}(X,\tau)]d\tau \quad (X,t) \in B \times [0,\infty) \tag{15}$$

Clearly, we have

$$U(t) = h_i(t)\varphi_{,i}(t) - g(t)\varphi(t) + \int_0^t [\dot{g}(\tau)\varphi(\tau) - \dot{h}_i(\tau)\varphi_{,i}(\tau)]d\tau \tag{16}$$
$$t \geq 0$$

where, for convenience, we have suppressed the argument X. In view of the constitutive equations (11) and the initial conditions, we find that

$$\begin{aligned} \dot{h}_i(t) &= A_{ij}(0)\,\dot{\varphi}_{,j}(t) + \dot{A}_{ij}(0)\varphi_{,j}(t) + \int_0^t \ddot{A}_{ij}(t-s)\varphi_{,j}(s)ds \\ \dot{g}(t) &= -M(0)\dot{\varphi}(t) - \dot{M}(0)\varphi(t) - \int_0^t \ddot{M}(t-s)\varphi(s)ds \end{aligned} \tag{17}$$

Thus, from (11), (17), and (16), we obtain

$$\begin{aligned} U(t) &= \frac{1}{2}\left[A_{ij}(0)\varphi_{,j}(t)\varphi_{,i}(t) + M(0)\varphi^2(t)\right] \\ &- \int_0^t [\dot{A}_{ij}(0)\varphi_{,j}(s)\varphi_{,i}(s) + \dot{M}(0)\varphi^2(s)]ds \\ &+ \varphi_{,i}(t)\int_0^t \dot{A}_{ij}(t-s)\varphi_{,j}(s)ds \\ &+ \varphi(t)\int_0^t \dot{M}(t-s)\varphi(s)ds \\ &- \int_0^t \left\{\varphi_{,i}(s)\int_0^s \ddot{A}_{ij}(s-\tau)\varphi_{,j}(\tau)d\tau + \varphi(s)\int_0^s \ddot{M}(s-\tau)\varphi(\tau)d\tau\right\}ds \quad t \geq 0 \end{aligned} \tag{18}$$

Now we use the following identities [27]:

$$2\varphi(t)\int_0^t \dot{M}(t-s)\varphi(s)ds = \int_0^t \dot{M}(t-s)\varphi^2(s)ds - \int_0^t \dot{M}(t-s)[\varphi(t) - \varphi(s)]^2 ds + [M(t) - M(0)]\varphi^2(t)$$

$$2\varphi_{,i}(t)\int_0^t \dot{A}_{ij}(t-s)\varphi_{,j}(s)ds = \int_0^t \dot{A}_{ij}(t-s)\varphi_{,i}(s)\varphi_{,j}(s)ds - \int_0^t \dot{A}_{ij}(t-s)\left[\varphi_{,j}(t) - \varphi_{,j}(s)\right][\varphi_{,i}(t) - \varphi_{,i}(s)]ds + [A_{ij}(t) - A_{ij}(0)]\varphi_{,j}(t)\varphi_{,i}(t)$$

$$2\int_0^t\int_0^s \ddot{A}_{ij}(s-\tau)\varphi_{,i}(s)\varphi_{,j}(\tau)dsd\tau = \int_0^t\int_0^t \ddot{A}_{ij}(|s-\tau|)\varphi_{,i}(s)\varphi_{,j}(\tau)dsd\tau = \int_0^t\int_0^t \ddot{A}_{ij}(|s-\tau|)\varphi_{,i}(s)\varphi_{,j}(s)dsd\tau - \frac{1}{2}\int_0^t\int_0^t \ddot{A}_{ij}(|s-\tau|)[\varphi_{,i}(s) - \varphi_{,i}(\tau)]\left[\varphi_{,j}(s) - \varphi_{,j}(\tau)\right]dsd\tau$$

$$2\int_0^t\int_0^s \ddot{M}(s-\tau)\varphi(s)\varphi(\tau)dsd\tau = \int_0^t\int_0^t \ddot{M}(|s-\tau|)\varphi(s)\varphi(\tau)dsd\tau = \int_0^t\int_0^t \ddot{M}(|s-\tau|)\varphi^2(s)dsd\tau -$$

$$\frac{1}{2}\int_0^t\int_0^t \ddot{M}(|s-\tau|)\,[\varphi(s)-\varphi(\tau)]^2\,dsd\tau$$
$$\int_0^t \ddot{M}(|s-\tau|)d\tau = \dot{M}(s)+\dot{M}(t-s) - 2\dot{M}(0) \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned}U(t) &= \frac{1}{2}[A_{ij}(t)\varphi_{,j}(t)\varphi_{,i}(t)+M(t)\varphi^2(t)]\\&-\frac{1}{2}\int_0^t [\dot{A}_{ij}(s)\varphi_{,j}(s)\varphi_{,i}(s)+\dot{M}(s)\varphi^2(s)]ds\\&-\frac{1}{2}\int_0^t \{\dot{A}_{ij}(t-s)[\varphi_{,i}(t)-\varphi_{,i}(s)][\varphi_{,j}(t)\\&-\varphi_{,j}(s)]+\dot{M}(t-s)[\varphi(t)-\varphi(s)]^2\}ds\\&\frac{1}{4}\int_0^t\int_0^t \{\ddot{A}_{ij}(|s-\tau)[\varphi_{,i}(s)-\varphi_{,i}(\tau)][\varphi_{,j}(s)\\&-\varphi_{,j}(\tau)]+\ddot{M}(|s-\tau|)[\varphi(s)-\varphi(\tau)]^2\}dsd\tau\\&t\geq 0\end{aligned} \tag{20}$$

In the case of isotropic bodies, the relation (20) reduces to

$$\begin{aligned}U(t) &= \frac{1}{2}\left[\alpha(t)\varphi_{,i}(t)\varphi_{,i}(t)+M(t)\varphi^2(t)\right]\\&-\frac{1}{2}\int_0^t \left[\dot{\alpha}(s)\varphi_{,i}(s)\varphi_{,i}(s)+\dot{M}(s)\varphi^2(s)\right]ds\\&-\frac{1}{2}\int_0^t \{\dot{\alpha}(t-s)\left[\varphi_{,i}(t)-\varphi_{,i}(s)\right][\varphi_{,i}(t)\\&-\varphi_{,i}(s)]+\dot{M}(t-s)\left[\varphi(t)-\varphi(s)\right]^2\Big\}dsd\tau\\&+\frac{1}{4}\int_0^t\int_0^t \{\ddot{\alpha}(|s-\tau|)\left[\varphi_{,i}(s)-\varphi_{,i}(\tau)\right][\varphi_{,i}(s)\\&-\varphi_{,i}(\tau)]+\ddot{M}(|s-\tau|)[\varphi(s)-\varphi(\tau)]^2\Big\}dsd\tau\end{aligned} \tag{21}$$

From (21) we conclude that the dissipation inequality (14) is satisfied if $\alpha \geq 0$, $M \geq 0$, $\dot{\alpha} \leq 0$, $\dot{M} \leq 0$, $\ddot{\alpha} \geq 0$, and $\ddot{M} \geq 0$. Thus, our assumptions concerning the relaxation function α are compatible with the second law of thermodynamics. For convenience, in what follows, we shall denote the material coordinates by (x_1, x_2, x_3).

Singular Surfaces

Let S be a moving surface defined by the equations

$$x_i = x_i(\theta^1, \theta^2, t)$$

where θ^1, θ^2 are curvilinear coordinates on the surface. We suppose that above functions are continuously differentiable with respect to their arguments and that S is smooth in the sense that the matrix $(\partial x_i/\partial\theta^\alpha)$ has rank two. The metric tensor of the surface is denoted by $a_{\alpha\beta}$. In what follows, we denote by n_i the unit normal to S. We note that [28]

$$\begin{aligned}&n_in_i = 1 \quad n_ix_{i;\alpha} = 0 \quad x_{i;\alpha\beta} = b_{\alpha\beta}n_i\\&n_{i;\alpha} = -a^{\lambda\rho}b_{\rho\alpha}x_{i;\lambda}\end{aligned} \tag{22}$$

where indices followed by a semicolon represent covariant partial differentiation based on the metric of S, $b_{\alpha\beta}$ is the second fundamental form of the surface, and $a^{\alpha\beta}$ are the elements of the inverse of matrix $(a_{\alpha\beta})$. We have

$$\begin{aligned}a^{\alpha\beta}x_{i;\alpha}x_{j;\beta} &= \delta_{ij} - n_in_j\\H &= \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta}\end{aligned} \tag{23}$$

where H is the mean curvature of the surface.

Let f be a function on $B \times (-\infty, \infty)$. We assume that f is a continuously differentiable function on each side of the moving surface S. We denote by $[f]$ the jump of the function f across S. The discontinuities in the first and second derivative of f satisfy the relations [17]

$$\begin{aligned}&[f_{,i}] = a^{\alpha\beta}A_{;\alpha}x_{i;\beta} + Bn_i\,[\dot{f}] = \frac{\delta A}{\delta t} - VB\\&[f_{,ij}] = a^{\alpha\beta}\left(B_{;\alpha} + a^{\lambda\rho}b_{\alpha\lambda}A_{;\rho}\right)\left(n_ix_{j;\beta} + n_jx_{i;\beta}\right)\\&\quad + a^{\alpha\beta}a^{\nu\rho}\left(A_{;\alpha\nu} - b_{\alpha\nu}B\right)x_{i;\beta}x_{j;\rho} + Cn_in_j\\&[\dot{f}_{,i}] = a^{\alpha\beta}\left(\frac{\delta A}{\delta t} - VB\right)_{;\alpha}x_{j;\beta}\\&\quad + \left(\frac{\delta B}{\delta t} + a^{\alpha\beta}A_{;\alpha}V_{;\beta} - CV\right)n_i\\&[\ddot{f}] = \frac{\delta}{\delta t}\left(\frac{\delta A}{\delta t} - VB\right)\\&\quad - V\left(\frac{\delta B}{\delta t} + a^{\alpha\beta}A_{;\alpha}V_{;\beta} - CV\right)\end{aligned} \tag{24}$$

where

$$\frac{\delta}{\delta t}=\frac{\partial}{\partial t}+Vn_i\frac{\partial}{\partial x_i}$$

is the convected derivative for an observer moving with the surface, V is the speed of propagation of the surface, and

$$A=[f] \qquad B=[f_{,i}n_i] \qquad C=[f_{,ij}n_in_j] \quad (25)$$

In what follows, we assume that the body loads b_i, l, and s are continuous on $B\times(-\infty,\infty)$. Following [18], by a wave of order 1, we mean a solution (u_i,φ,θ) of (3), (4)–(6), and (11), with the properties: (a) the functions u_i, φ, and θ are continuous on $B\times(-\infty,\infty)$; (b) the first-order derivative of the five-dimensional vector (u_i,φ,θ) has jump discontinuities across S, but are continuous elsewhere. We say also that S is a singular surface of order 1, and we shall refer to it as a wave surface of order 1. The balance laws (4)–(6) have the form

$$\frac{d}{dt}\int_{\omega}\rho_0 F dv=\int_{\omega}Q dv+\int_{\partial\omega}G_in_i da$$

which, at singular S, is equivalent to the condition [17]

$$-V[\rho_0F]=[G_i]n_i \qquad (26)$$

If we apply (26) to the global balance laws (4)–(6), then we obtain

$$\begin{aligned}\rho_0V[\dot{u}_i]+[t_{ji}]n_j&=0\\ \rho_0\kappa V[\dot{\varphi}]+[h_i]n_i&=0\\ \rho_0V[e^*]+[q_i]n_i&=0\end{aligned} \qquad (27)$$

It follows from (24) that

$$\begin{aligned}&[u_{i,j}]=\xi_in_j \qquad [\varphi_{,i}]=\zeta n_i \qquad [\vartheta_{,i}]=\gamma n_i\\ &[\dot{u}_i]=-V\xi_i \qquad [\dot{\varphi}]=-V\zeta \qquad [\dot{\vartheta}]=-V\gamma\end{aligned} \qquad (28)$$

where

$$\xi_i=\left[\frac{\partial u_i}{\partial n}\right] \qquad \zeta=\left[\frac{\partial\varphi}{\partial n}\right] \qquad \gamma=\left[\frac{\partial\vartheta}{\partial n}\right] \quad (29)$$

We denote by n the distance measured along the normal to the wave surface. In what follows, we shall use the following result established by Fisher and Gurtin [20].

Lemma 1. *Let u and v be functions on $B\times(-\infty,\infty)$ with the following properties: (1) u is continuous, (2) v is continuous everywhere except for a possible jump discontinuity across S, (3) v is bounded on every compact subset of $B\times(-\infty,\infty)$. Then, the function*

$$w(x,t)=\int_0^t u(x,t-s)v(x,s)ds$$

is continuous on $B\times(-\infty,\infty)$.

In view of Lemma 1, from (11), (3), and (28), we obtain

$$\begin{aligned}&[t_{ij}]=G_{ijrs}(0)\xi_rn_s\\ &[h_i]=A_{ij}(0)\zeta n_j\\ &[g]=-B_{ij}(0)\xi_in_j\\ &[\rho_0e^*]=T_0b_{ij}(0)\xi_in_j\\ &[q_i]=0\end{aligned} \qquad (30)$$

With the help of (28) and (30), the conditions (27) become

$$\left(G_{ijrs}(0)n_sn_j-\rho_0V^2\delta_{ir}\right)\xi_r=0 \qquad (31)$$

$$\left(A_{ij}(0)n_in_j-\rho_0\kappa V^2\right)\zeta=0 \qquad (32)$$

$$\beta(0)\xi_in_i=0 \qquad (33)$$

The jumps ξ_i, ζ, and γ cannot all be zero. If $\beta(0)=0$, then (33) is identically satisfied. The equations (31) admit a nontrivial solution for ξ_i if and only if

$$\det\left(G_{ijrs}(0)n_jn_s-\rho_0V^2\delta_{ir}\right)=0$$

Taking into account (10), this equation reduces to

$$\left(c_1^2 - V^2\right)\left(c_2^2 - V^2\right)^2 = 0$$

where

$$c_1 = \{[\lambda(0) + 2\mu(0)]\rho_0\}^{\frac{1}{2}} \qquad c_2 = [\mu(0)/\rho_0]^{\frac{1}{2}} \tag{34}$$

If $V = c_1$, the wave is longitudinal ($\xi_i = \xi n_i$). When $V = c_2$, we obtain transverse waves ($\xi_i n_i = 0$).

If $\zeta \neq 0$, the wave is a wave of compaction (or distension). The possible speed of propagation of this wave is $V = c_3$ where

$$c_3 = [\alpha(0)/(\rho_0\kappa)]^{\frac{1}{2}} \tag{35}$$

We now assume that $\beta(0) \neq 0$. Then, from (33), we obtain $\xi_i n_i = 0$ so that in this case there are two type of singular surfaces of order 1: transverse waves and waves of compaction. We remark that the transverse mechanical waves are not coupled with compaction waves or thermal waves.

The Growth of Waves

The local forms of the balance laws (4)–(6) are

$$t_{ji,j} + \rho_0 b_i = \rho \ddot{u}_i \tag{36}$$

$$h_{i,i} + g + \rho_0 l = \rho_0 \kappa \ddot{\varphi} \tag{37}$$

$$q_{i,i} + \rho_0 s = \rho_0 \dot{e}^* \tag{38}$$

respectively. Using the fact that V is constant for all waves, from (24), we have

$$\begin{aligned}
\left[u_{s,ij}\right] &= a^{\alpha\beta}\xi_{s;\alpha}\left(n_i x_{j;\beta} + n_j x_{i;\beta}\right) \\
&\quad - a^{\alpha\beta}a^{\nu\rho}b_{\alpha\nu}\xi_s x_{i;\beta}x_{j;\rho} + \mu_s n_i n_j \\
\left[\dot{u}_{i,j}\right] &= \left(-V\mu_i + \frac{\delta\xi_i}{\delta t}\right)n_j - Va^{\alpha\beta}\xi_{i;\alpha}x_{j;\beta} \\
\left[\ddot{u}_i\right] &= V^2\mu_i - 2V\frac{\delta\xi_i}{\delta t}
\end{aligned} \tag{39}$$

where $\mu_i = \left[u_{i,rs}n_r n_s\right]$. It follows from (36) that

$$\left[t_{ji,j}\right] = \rho_0[\ddot{u}_i] \tag{40}$$

From (24), we get

$$\begin{aligned}
V\left[t_{ji,j}\right] &= V\left[t_{ri,j}n_j\right]n_r + Va^{\alpha\beta}\left[t_{ji}\right]_{;\alpha}x_{j;\beta} \\
&= -\left[\dot{t}_{ji}\right]n_j + n_j\frac{\delta}{\delta t}\left[t_{ji}\right] + Va^{\alpha\beta}\left[t_{ji}\right]_{;\alpha}x_{j;\beta}
\end{aligned} \tag{41}$$

When the constitutive relations for t_{ij} is differentiated with respect to t and jumps are taken across S,we obtain

$$\begin{aligned}
\left[\dot{t}_{ji}\right] &= -VG_{ijrs}(0)\mu_r n_s - VB_{ij}(0)\zeta \\
&\quad + V\gamma b_{ij}(0) + G_{ijrs}(0)n_s\frac{\delta\xi_r}{\delta t} \\
&\quad - VG_{ijrs}(0)a^{\alpha\beta}\xi_{r;\alpha}x_{s;\beta} + G^{(1)}_{ijrs}(0)\xi_r n_s
\end{aligned} \tag{42}$$

Here we have used the notation $\boldsymbol{G}^{(1)} = \dot{\boldsymbol{G}}$.

With the aid of equations (39), (41), and (42), the equation (40) may be written as

$$\begin{aligned}
&V\left\{G_{ijrs}(0)n_j n_r - \rho_0 V^2\delta_{is}\right\}\mu_s + \\
&VB_{ij}(0)n_j\zeta - V\gamma b_{ij}(0)n_j - \\
&G_{ijrs}(0)n_s n_j\frac{\delta\xi_r}{\delta t} + G_{ijrs}(0)n_j\frac{\delta}{\delta t}(\xi_r n_s) \\
&+ 2\rho_0 V^2\frac{\delta\xi_i}{\delta t} + VG_{ijrs}(0)a^{\alpha\beta}\left(\xi_r n_s\right)_{;\alpha}x_{j;\beta} \\
&+ VG_{ijrs}(0)n_j a^{\alpha\beta}\xi_{r;\alpha}x_{s;\beta} - G^{(1)}_{ijrs}(0)\xi_r n_s n_j = 0
\end{aligned} \tag{43}$$

We note that

$$\begin{aligned}
&a^{\alpha\beta}\xi_{i;\alpha}x_{i;\beta} = a^{\alpha\beta}\left(x_{i;\beta}\xi_i\right)_{;\alpha} - 2Hn_i\xi_i \\
&\frac{\delta n_i}{\delta t} = 0
\end{aligned} \tag{44}$$

With the help of (10), (22), (23), and (44), the equations (43) reduce to

$$\begin{aligned}
&V\left\{[\lambda(0) + \mu(0)]n_i n_r + \mu(0)\delta_{ir} - \rho_0 V^2\delta_{ir}\right\}\mu_r \\
&+ 2\rho_0 V^2\frac{\delta\xi_i}{\delta t} + Vb(0)n_i\zeta - V\beta(0)n_i\gamma \\
&+ V[\lambda(0) + \mu(0)]\left\{a^{\alpha\beta}n_i\left(x_{r;\beta}\xi_r\right)_{;\alpha}\right. \\
&\left. + a^{\alpha\beta}\left(\xi_r n_r\right)_{;\alpha}x_{i;\beta} - 2H\xi_r n_r n_i\right\} - 2HV\mu(0)\xi_i \\
&- \left[\lambda^{(1)}(0) + \mu^{(1)}(0)\right]\xi_r n_r n_i - \mu^{(1)}(0)\xi_i = 0
\end{aligned} \tag{45}$$

We denote $\xi = \xi_i n_i$. If we multiply (45) by n_i and sum on i, we obtain

$$V\{\lambda(0) + 2\mu(0) - \rho_0 V^2\}\mu_s n_s + V\zeta b(0) + V\gamma\beta(0) + 2\rho_0 V^2 \frac{\delta\xi}{\delta t} + V[\lambda(0) + \mu(0)]a^{\alpha\beta}\left(x_{r;\beta}\xi_r\right)_{;\alpha} - 2V[\lambda(0) + 2\mu(0)]H\xi - \left[\lambda^{(1)}(0) + 2\mu^{(1)}(0)\right]\xi = 0 \quad (46)$$

In the case of longitudinal waves, (46) yields the growth equation

$$\frac{1}{c_1}\frac{\delta\xi}{\delta t} = \frac{d\xi}{dn} = \xi(H - J_1) \quad (47)$$

where we have used the notation

$$J_1 = -\left[\lambda^{(1)}(0) + 2\mu^{(1)}(0)\right]\left(2\rho_0 c_1^3\right)^{-1} \quad (48)$$

If, at some instant $t = t_0$, the mean and Gaussian curvatures of surfaces are H_0 and K_0, respectively, then at a subsequent time t,

$$H = \frac{H_0 - nK_0}{1 - 2nH_0 + n^2K_0} \quad (49)$$

and (47) may be integrated to give

$$\xi = \xi_0\left(1 - 2nH_0 + n^2K_0\right)^{-\frac{1}{2}}\exp(-nJ_1) \quad (50)$$

where ξ_0 is the strength of the wave at time $t = t_0$.

The speed of propagation of irrotational waves is c_2, and for these waves, we have $\xi_i n_i = 0$. In this case, assuming that $c_2 \neq c_3$, the equation (46) reduces to

$$[\lambda(0) + \mu(0)]\left[a^{\alpha\beta}\left(x_{r;\beta}\xi_r\right)_{;\alpha} + \mu_s n_s\right] + \beta(0)\gamma = 0$$

Thus, from (45), we obtain

$$\frac{1}{c_2}\frac{\delta\xi_i}{\delta t} = \frac{\delta\xi_i}{dn} = (H - J_2)\xi_i \quad (51)$$

where

$$J_2 = -\mu^{(1)}(0)\left(2\rho_0 c_2^3\right)^{-1} \quad (52)$$

As before, we obtain

$$\xi_i = \xi_i^0\left(1 - 2nH_0 + n^2K_0\right)^{-\frac{1}{2}}\exp](-\mathrm{nJ}_2) \quad (53)$$

where $\xi_i^0 = \xi_i(t_0)$

In the case of a wave of compaction, we have $V = c_3$. If we assume that $c_3 \neq c_1$ also $c_3 \neq c_2$, then from (45) we obtain

$$\{[\lambda(0) + \mu(0)]n_i n_s + \mu(0)\delta_{is} - \rho_0 c_3^2\delta_{is}\}\mu_s = -b(0)n_i\zeta + \beta(0)n_i\gamma \quad (54)$$

This relation implies that

$$\rho_0\left(c_1^2 - c_3^2\right)\mu_s n_s = -b(0)\zeta + \beta(0)\gamma \quad (55)$$

By using (55) in (54), we obtain

$$\mu_i = -[b(0)\zeta - \beta(0)\gamma]n_i\left[\rho_0\left(c_1^2 - c_3^2\right)\right]^{-1} \quad (56)$$

We see that a thermal wave or a wave of compaction induces a longitudinal mechanical acceleration discontinuity.

It follows from (37) that

$$[h_{i,i}] + [g] = \rho_0\kappa\left(V^2\tau - 2V\frac{\delta\zeta}{\delta t}\right) \quad (57)$$

where $\tau = \left[\varphi_{,ij}n_i n_j\right]$. With the help of relations

$$\begin{aligned} V\left[h_{i,i}\right] &= -\left[\dot{h}_i\right]n_i + n_i\frac{\delta}{\delta t}[h_i] + Va^{\alpha\beta}\left[h_j\right]_{;\alpha}x_{j;\beta} \\ \left[\dot{h}_i\right] &= \alpha(0)\{(\frac{\delta\zeta}{\delta t} - V\tau)n_i - Va^{\alpha\beta}\zeta_{;\alpha}x_{i;\beta}\} + \alpha^{(1)}(0)\zeta n_i \\ [h_i] &= \alpha(0)\zeta n_i \\ [g] &= -b(0)\xi_i n_i \end{aligned} \quad (58)$$

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from (57), we find

$$\left[\alpha(0)-\rho_0\kappa V^2\right]V\tau-2HV\alpha(0)\zeta- \alpha^{(1)}(0)\zeta-Vb(0)\xi+2\rho_0\kappa V^2\frac{\delta\zeta}{\delta t}=0 \tag{59}$$

We assume now that $V=c_3$ with $c_3\neq c_1$ and $c_3\neq c_2$. Then, (59) reduces to

$$\frac{1}{c_3}\frac{\delta\zeta}{\delta t}=(H-J_3)\zeta \tag{60}$$

where

$$J_3=-\alpha^{(1)}(0)\left(2\rho_0\kappa c_3^3\right)^{-1} \tag{61}$$

It follows from (60) that

$$\zeta=\zeta^0\left(1-2nH_0+n^2K_0\right)^{-\frac{1}{2}}\exp(-nJ_3) \tag{62}$$

where $\zeta^0=\zeta(t_0)$. If $V=c_1\neq c_3$, then (59) reduces to

$$\left[\alpha(0)-\rho_0\kappa c_1^2\right]\tau=b(0)\xi \tag{63}$$

so that a longitudinal wave induces an acceleration discontinuity in the waves of compaction.

Let us consider now the equation (38). This equation implies that

$$\left[q_{i,i}\right]=\left[\rho_0\,\dot{e}^*\right] \tag{64}$$

In the view of relations

$$\begin{aligned}&[q_i]=0\quad V\left[q_{i,i}\right]=-[\dot{q}_i]n_i\\&[\dot{q}_i]=k(0)\gamma n_i\\&[\rho_0\,\dot{e}^*]=T_0\{\beta(0)(n_i\frac{\delta\xi_i}{\delta t}-V\mu_i n_i\\&-Va^{\alpha\beta}\xi_{i;\alpha}x_{i;\beta})-V\zeta m(0)-V\gamma a(0)\\&+\beta^{(1)}(0)\xi_r n_r\}\end{aligned} \tag{65}$$

from (64), we get

$$\begin{aligned}&\left[k(0)-T_0a(0)V^2\right]\gamma=T_0V\{\beta(0)[V\mu_i n_i\\&+Va^{\alpha\beta}\xi_{i;\alpha}x_{i;\beta}-n_i\frac{\delta\xi_i}{\delta t}\Big]+V\zeta m(0)\\&-\beta^{(1)}(0)\xi_j n_j\}\end{aligned} \tag{66}$$

Let us assume that $V\neq c_1$, $V\neq c_2$, and $V\neq c_3$. Then, from (46), we obtain

$$\rho_0\left(c_1^2-V^2\right)\mu_i n_i=-\beta(0)\gamma$$

If $\beta(0)=0$, then we find that $\mu_i n_i=0$ and (66) reduces to

$$\left(c_4^2-V^2\right)\gamma=0 \tag{67}$$

where

$$c_4=\left[k(0)/T_0a(0)\right]^{\frac{1}{2}}$$

In this case, we see that the possible speed of propagation of thermal waves is $V=c_4$. If $\beta(0)\neq 0$, then from (33), we obtain $\xi_i n_i=0$ so that $V=c_2$ or $V=c_3$. For $V=c_2\neq c_3$, the equation (46) reduces to

$$\rho_0\left(c_1^2-c_2^2\right)\left(a^{\alpha\beta}x_{r;\beta}\xi_{r;\alpha}+\mu_s n_s\right)=-\beta(0)\gamma$$

Thus, the equation (66) can be written in the form

$$R\gamma=0$$

where

$$R=a(0)\rho_0\left(c_1^2-c_2^2\right)\left(c_4^2-c_2^2\right)+\beta^2(0)c_2^2$$

If $R\neq 0$, then $\gamma=0$ so that the wave is purely mechanical.

For $V=c_3(\neq c_2)$, the equations (50) and (66) imply that

$$\Lambda\gamma=c_3^2\left[\rho_0 m(0)\left(c_1^2-c_3^2\right)-b(0)\beta(0)\right]$$

where

$$\Lambda=\rho_0 a(0)\left(c_1^2-c_3^2\right)\left(c_4^2-c_3^2\right)+c_3^2\beta^2(0)$$

Thus, in general, a compaction wave induces a thermal wave. We note that the eigenvalue problems from Section 4 govern also the propagation of thermoelastic waves studied in [6]. However, in the present paper, the amplitudes of waves contain new attenuation factors related to the relaxation functions (see (48), (50), (52), (53), (61), and (62)).

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Singularities of the Thermo-Magneto-Electro-Elastic Fields

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Overview

This entry investigates the influence of a mixed mode crack on the coupled response of a functionally graded magnetoelectroelastic material (FGMEEM) subjected to thermal loading. The crack is embedded at the center of a 2D infinite medium, and the material is graded in the direction orthogonal to the crack plane and is modeled as a nonhomogeneous medium with anisotropic constitutive laws. The heat equation is first solved using Fourier transform to yield the