

Tamsin Meaney · Christiane Benz ·
Antonella Montone · Benedetto Di Paola ·
Michele G. Fiorentino *Editors*

Engaging with Mathematics in the Early Years

Results from the POEM6 Conference

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Introduction



**Tamsin Meaney, Christiane Benz, Michele G. Fiorentino, Antonella Montone,
and Benedetto Di Paola**

The POEM Conference

The sixth Perspective on Early Mathematics learning between the poles of instruction and construction (POEM) conference was held in Bari, Italy 20–21 May 2024. The first conference at Frankfurt am Main, Germany, in 2012. POEM was initiated by Götz Krummheuer to provide a space to continue discussions amongst predominantly European early childhood mathematics researchers, about the role of construction and instruction in how young children engage with and learn mathematics. This distinction can be connected to the two approaches to early childhood recognized by Bennett (2005), 20 years ago. The first approach focuses on preparing children for school, while the second approach focuses on children developing through play and exploration. Since then the discussions have continued at conferences in Malmö, Sweden, in 2014, Karlsruhe, Germany, in 2016, Kristiansand, Norway, in 2018 and in Gothenburg, Sweden, in 2022. Therefore, every other year, POEM brings together researchers within the field of early childhood mathematics education to exchange current research findings and to collaborate and discuss new ideas. The latest iteration of the conference was held for the first time in Italy, at the Department of Education, Psychology and Communication Sciences of the

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University of Bari Aldo Moro. The conference gathered more than 50 researchers and included 22 presentations.

The POEM conferences remain small, with participants being those who had previously participated or those who asked to be invited. The smallness of the conference provides adequate time for discussions of each presentation. Presenters are asked to prepare a conference paper which is distributed to all the participants with an expectation that it will be read before the conference. After each conference a book has been published by Springer, in which the presenters expand on their presentations, based on the conversations which were held at the conference. The subsequent book chapters were all peer reviewed.

The conference also had two keynote speeches presented by Professors Anna Baccaglioni-Frank (University of Pisa, Italy) and Associate Professor Iliada Elia (University of Cyprus). Anna Baccaglioni-Frank's lecture focused on a topic gaining considerable attention in early childhood mathematics education, that of how multi-touch technology can support the development of number sense in preschool-aged children. As can be seen in her chapter in this book, she described how different apps, each of them designed to develop a specific aspect of number sense through dynamic, embodied interactions. Her lecture was based on two research studies that explored the connection between the development of number sense and the use of the fingers. In both studies, this relationship is closely tied to specific digital apps (Fingu, Ladybug Count, and TouchCounts), and how the teacher or the researcher as an educator mediator, supported the construction of number sense. Thus, the approach taken in this chapter was one of highlighting the instructional approach to young children's learning of mathematical understandings.

Iliada Elia discussed early geometrical thinking and learning, from different research perspectives. In her keynote, she reflected on the significance of culturally responsive pedagogy and the importance of contextualizing mathematical experiences within children's everyday lives. Elia advocated for empowering teachers to listen deeply to children's mathematical expressions, positioning children not merely as learners but as co-constructors of mathematical knowledge. Her lecture can be considered as promoting understandings about the importance of young children's construction of mathematical knowledge.

The call for papers included several themes, which could be responded to from either the construction or instruction approach which was the underlying aim for having early childhood mathematics researchers discuss differences and similarities in their research. The themes were:

- Mathematics learning, development, teaching, instruction and education in the early years
- Early Childhood Education and Care practices and contexts for mathematics learning
- Early childhood teachers' professional development
- Children's perspective and the role of play in early mathematics education
- Theoretical perspectives on early mathematics education
- Methodological views in research on early mathematics education

The chapters included in this volume, followed a structured review process. Both an internal reviewer and an external reviewer to the book were asked to review each chapter. The external reviewer was a researcher with recognized expertise in the field of early childhood mathematics education. Reviewers were asked to provide detailed feedback to the authors, who were then given the opportunity to revise their manuscripts accordingly. After the revisions, when needed, a further round of revision was requested. This process ensured the scientific quality and coherence of the contributions, while also respecting the individual perspectives and research approaches of the authors.

The Chapters in the Book

From the latest POEM conference, the kind of papers which were accepted from presentation was expanded from being purely empirical-based to also include developmental papers. This was in part a recognition of the collaboration work being done particularly in Italy, but also in other places between researchers and teachers to engage young children in mathematical thinking and doing.

The chapters represent a varied approach to mathematics education in early childhood, which includes the first year of schools as well as Early Childhood Education and Care (ECEC) institutions, such as kindergartens and preschools. The majority of the studies were qualitative, but with a few studies including statistical analyses of quantitative data, such as Gasteiger and Benz and Vee et al.

In this book, many of the chapters discuss the planning and implementation of interventions, in which specially-developed activities or games were used with children, generally with the aim of understanding their mathematical sense making. For example, in Italy, Casi et al. had developed a version of the game Memory to evaluate children's use of spatial language. The rules of the game proved quite difficult for the children to follow as it required them to logically work out what cards to remove based on the answers they received for different questions. In Germany, Reuter and Benz used a well-known book as a starting point for having children reason about how much food was eaten by a caterpillar during the weekdays compared with the weekend, with a focus on comparisons of amounts. Activities were also designed and implemented to determine children's reasoning, such as by Markken et al. who investigated children's reasoning about fairness by designing an activity in which a number of biscuits had to be shared with some stuffed animals. In a similar study in Germany, Wiggelinghoff and Peter-Koop investigated understandings about division with remainders of students in their first year of school. The children were interviewed about their solutions to different division problems, with remainders to see how the concrete materials used to illustrate the problems affected the kinds of responses the children gave. These examples, illustrate the different mathematical foci that were the focus of the investigations.

Also in Italy, DiPaola et al. developed a game that could contribute to kindergarten children's understandings about natural numbers. This game had several parts

and supported children to develop an understanding of how the ordering of quantities and the use of digits to represent those quantities could be used to solve certain problems. In Sweden in a study by van Brommel et al., 6-year-old children's representations of their solutions to a combinatorics problem were analysed. The responses showed that the majority of the children were able to adopt a systematic approach to finding all the solutions, although a few children misunderstood the problem.

Although the majority of the work was about how children learnt or used specific mathematics, there were also investigations that focused on how children interacted with others. For example, Acar Bayrakta and Brandt investigated how a multilingual child watched and made use of the interaction between his grandmother and his older brother in regard to engaging in block play. Similarly in Switzerland, Ott et al. describe a study with a multilingual kindergarten child and how they used different languages and other linguistic resources to communicate in the kindergarten and with a parent about the placement of eggs in an egg carton. Both studies highlight how children engage with mathematics in multiple contexts and can learn and choose the ways that they want to communicate their understandings.

As was the case with Reuter and Benz, Mellone et al. used a story as a basis for an activity with 5–6-year olds. In Mellone et al.'s case the aim of the activity was to stimulate children's interactions in relationship to proportional amounts. This task was more open-ended than that of Reuter and Benz, as it prompted a more inquiring interaction, rather than one which focused on how children reasoned towards finding a specific response. These two studies show how similar materials, in this case stories, can be used to support different kinds of interactions.

In other chapters, the focus was on how mathematical understandings developed across time or relied on children being introduced to specific critical aspects of a concept so they increased their understanding of that concept. For example, as a result of work with teachers, Robotti and Boscolo developed a learning trajectory for children from kindergarten to Grade 2. They looked at how children gained understandings about particular aspects connected to the processes and products of measuring. In a Norwegian study, Bjørnebye focused on how one child engaged with a set of activities that required them to use different representations, including with their body, to identify and gain the key aspects of visual patterns. In contrast, Lüken wrote in her chapter about the non-patterns that German children produced when playing with different concrete materials and everyday objects. The identification of the features of the non-patterns is discussed in relationship to how they could inform understandings about children's development of pattern understandings. In a study of 7-year old children posing and solving two step arithmetic problems, Fosse et al. examined differences in the representations of the problems in relationship to mathematical modelling. Although a very limited number of examples were analysed, the child who was able to conflate two potential rounds of the modelling cycle into one problem solution, did so at the loss of the everyday context in which the problems were set. This has insights into developing children's understanding of two-step problems in everyday contexts.

The materials or assessment materials that are provided to young children were also investigated in some chapters. For example, Manolino et al. analysed the value of Cubarithm for blind students learning how to do two column addition problems. Although the material provides some possibilities for manipulation that can support doing two column addition, the authors raise issues about how the material can also hinder children's learning. In another paper by Marx et al., the process was described for developing a digital resource to support young children's understandings about part/whole. The focus was on what could be learnt about how digital resources could be collaboratively designed by people with different professional backgrounds, in particular mathematics educators and software developers. Furness et al. identified the mathematics in the game design processes of preservice teachers. Although the games mostly focused on understandings about counting and problem solving, other mathematical understanding could potentially be developed by the children playing the games, highlighting the support that teacher educators might need to develop in their ECEC teacher education courses.

Evaluations of assessment materials that could be used with young children in the normal course of ECEC were also discussed in two chapters. Gasteiger and Benz looked at whether two everyday activities in ECEC, reading a story and playing a game, about placing eggs in egg cartons, could be used to assess young children's number knowledge. They found that the children's understanding identified in the assessments based on these everyday activities gave similar results to those of more formal tests. Assessment activities that were based on everyday activities seemed to fulfil the needs expressed by ECEC educators to find out about the mathematics that the children in their care knew. Ringvold and Skorpen also evaluated three different sets of assessments of young children's knowledge of cardinality that were based on observations. They evaluated observation assessments from different perspectives to determine the different aspects that they could fulfill. Two of the assessment had originated in Norway while the third one was from Germany. As they did not examine the assessment being used in ECEC, they considered that cultural differences may have limited their evaluations.

Apart from children, there was also some studies that focused on the knowledge of preservice teachers. Along with the study of Furnes et al.'s study about preservice teachers' games, in Germany, Sprenger investigated school students and preservice teachers' knowledge of quadrilaterals to see if they shared the same misunderstandings. The results showed that understandings of quadrilaterals increased with age but there were still some preservice teachers who were not able to identify appropriate quadrilaterals from a set of shapes. Vee et al. undertook a mixed method study on the views of preservice teachers in Norway and Sweden about how to use digital tools to support children's engagement with mathematics. The differences in the survey results across countries were minimal, suggesting that even though there were different public discussions about young children's use of digital devices, these seemed to have had a limited effect on how preservice teachers considered they should be used.

As can be seen from this brief synopsis of the chapters in the book, early childhood mathematics education in Europe is flourishing. It also shows that there are a

number of overlaps in approaches and foci, regardless of the country in which studies are undertaken. This provides rich possibilities for further research discussions.

What Is Missing for the Next POEM Conference?

The studies described in this book do not include much focus on very young children, as there had been in earlier POEM conferences. Lüken's study is the exception as it included three 1-year olds and 17 2-year olds, as well as older children. Although there are now more studies about mathematics education in the first few years of school, at the next conference it would be interesting to also consider how very young children relate to mathematics, given that in many countries curricula for ECEC do not exempt these children from engaging with mathematical ideas.

The other possibility that could be considered to be missing is that of cross-country or cross-cultural studies. As noted in the previous section, there are overlaps in interests and foci but there was only one study, Vee et al., that involved a cross-country analysis. Perhaps one of the possibilities that could be promoted for the next conference is that of doing research across contexts.

When POEM was first established the main aim was for researchers to engage in discussions about instruction and construction in relationship to young children engaging with mathematical ideas. Most of the chapters that were originally presented at the sixth POEM conference were based on how children engaged with activities designed and implemented by teachers and could be considered as related to how instruction connected to children's learning and using mathematics. Lüken's chapter is the only one which investigated what children did when playing without clear instructions, although the researcher had chosen what materials should be made available to the children. It may, thus, be the only study which investigated what mathematics children show when they explore in their play and, thus, construct their own understandings.

It seems that with the turn to focus on the teacher's work and the focus on intervention studies, there has been a turn away from exploring what children can do by themselves based on their own interests. Having more understandings about children's own explorations is likely to better inform the work of teachers and the design of intervention studies. It is only when the two poles of instruction and construction are brought together that young children's learning can be improved in relationship to early childhood approaches based on Bennett's (2005) social policy pedagogy approach as well as the preparation for school.

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Reference

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Number Sense, Fingers and Multi-Touch Apps in Preschool



Anna Baccaglini-Frank

Number Sense and the Fundamental Role of Fingers

The integration of technology in educational contexts has revolutionized the way mathematics can be taught and learned, particularly in early childhood education. With the proliferation of touch-based devices such as tablets and smartboards, researchers have been exploring the potential of interactive technologies in the classroom. Among these, in the context of early childhood mathematics education, multi-touch technologies—those that respond to multiple simultaneous touches on a screen—are especially promising for fostering key aspects of number sense. Before exploring the potential of multi-touch technologies for developing young children’s number sense, I first examine the concept of number sense and the critical role of fingers in its development.

Number Sense

Number sense refers to a child’s intuitive understanding of numbers, their relationships, and their manipulation; it is a critical component of early mathematical learning and includes abilities such as subitizing, counting, understanding quantities, recognizing number patterns, and grasping basic relationships between numbers (Butterworth, 2005; Coles & Sinclair, 2018; Elia et al., 2018). These abilities form the foundation upon which more complex mathematical concepts are built, such as arithmetic, place value, and algebraic reasoning. Developing strong number sense at an early age has been found to be predictive of future mathematical achievement

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(Lyons et al., 2014), and it is considered a necessary condition for learning formal arithmetic at the early elementary level (Griffin et al., 1994; Mulligan et al., 2018). Moreover, a recent literature review by Elia and colleagues found that:

Number sense development is globally recognized as the fundamental foundational knowledge for children's mathematical growth. Findings in the literature suggest that stimulating children's number sense at an early toddler stage leads to the children's future benefit. (Elia et al., 2023, p. 81)

Number sense is believed to rely on a number of cognitive skills, processes and mechanisms, such as the following: children's spontaneous focusing on numerosity, the frequency with which children focus their attention on the exact number of objects in a set (Hannula & Lehtinen, 2005); (perceptual) subitizing, the rapid, accurate and confident judgment of the number of items in small collections 'at a glance', without counting; the approximate number system, a cognitive mechanism that generates experiences of approximate number (Halberda & Odic, 2015); recognizing parts of a whole that allows a child to recognize the complementarity of two numbers with respect to a given one; one-to-one correspondence, that is establishing a one-to-one correspondence between two numerosities in analogical form (e.g., dots on a screen); estimation, a (complex) ability that consists in guessing a discrete or continuous quantity without counting; the counting principles (Gelman & Gallistel, 1978).

Many of the basic processes and mechanisms listed above are believed to be "innate" and even common to both humans and many animals (Butterworth, 1999, 2005; Dehaene, 1997), so, supposedly, they are not dependent on an individual's culture or education. Recent behavioral genetic research indicates that while the heritability of number sense is limited, its connection to mathematical ability is significantly mediated by genetic factors. However, the developmental course, the direction of causal links between number sense and mathematics, the effects of education on these links remain unclear (Davydova et al., 2014), and socio-cultural factors seem to play a significant role in the development of "number sense", as intended by much of the mathematics education community (Elia et al., 2023).

Moreover, while most educational studies on number sense have focused on children's managing cardinality, neuroscientists, psychologists, and some educational researchers have brought attention to the importance of managing ordinality, as well. Managing ordinality consists of skills, such as: associating number symbols to number words, knowing the sequence of number symbols, knowing the sequence of number words, and knowing which number precedes or follows a given one. Educational research suggests promoting ordinal awareness earlier (Coles & Sinclair, 2018; Sinclair & Coles, 2017) and to a greater extent than what is done traditionally in Western schools—note that there are different traditions in which a greater focus is put on measurement, ordinality and a continuous, as opposed to discrete, approach to number (e.g., Mellone et al., 2020). Specifically, a hypothesis is that a great source of children's difficulties with numbers is related to linking symbols to other symbols (as opposed to symbols to sets of objects); also, ordinality is extremely relevant to place value (Coles & Sinclair, 2018; Lyons & Beilock, 2013). In the rest of this chapter, I will be referring to studies and research conducted from within a Western cultural perspective—not because I consider such a view in any way "superior", but because it is the one in which I am immersed.

Fingers and Number Sense

Research across neuroscience, developmental psychology, and mathematics education highlights the critical role of fingers in developing number sense. According to Butterworth (1999),

Without the ability to attach number representations to the neural representations of fingers and hands in their normal locations, the numbers themselves will never have a normal representation in the brain. (Butterworth, 1999, pp. 249–250)

Abilities such as subitizing, fine motor skills (e.g., finger tapping), and finger gnosis (mentally representing one's fingers) all support numerical representations. Butterworth's hypothesis is further supported by later studies. Gracia-Bafalluy and Noël's research (2008), showed that finger gnosis training positively impacts number sense and calculation skills.

Other research has suggested the importance of finger-based counting in establishing numerical practices (Andres et al., 2007). Finger use in counting also facilitates one-to-one correspondences, essential for understanding number relationships, even when numbers are not explicitly used. Margolinas and Vosniak (2012) emphasize the concept of "finger symbol sets" (Brissiaud, 1992), where finger gestures represent numbers and operations. Working with finger symbol sets can help children grasp part-whole relationships, particularly the decomposition of numbers up to 10, which is foundational for addition and subtraction. Developing such decompositions has been found to support a deeper understanding of the complementarity of numbers, contributing to the "part-whole concept" (Resnick et al., 1991).

Affordances of Multi-Touch Technology for Number Sense Development

Unlike traditional single-point input devices, such as a mouse or keyboard, multi-touch screens allow for more natural interactions, such as tapping, dragging, and pinching, that mirror physical actions. This is particularly relevant in the context of developing number sense in early childhood because of the aspects presented in the previous section (Kortenkamp et al., 2024; Ladel & Kortenkamp, 2014; Lembrér & Meaney, 2016). Some multi-touch apps, like Fungu, have been designed explicitly to support young children's development of aspects of number sense: the designers state that the app is

designed to support young children's learning and development of number concepts and flexible arithmetic competence [...] [targeting] the understanding and mastering of the basic numbers 1–10 as part-whole relations, which according to the literature on early mathematics learning is critical for this development. (Holgersson et al., 2016, p. 123).

In this game (Fig. 1a), floating fruits appear briefly before disappearing. Players must place the same number of fingers on the screen as the number of fruits shown. The answer is given in the form of a multi-touch gesture, and it needs to occur within a limited timeframe.

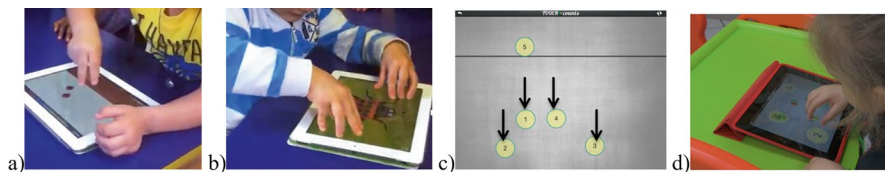


Fig. 1 (a) a child interacting with Fingu; (b) a child interacting with Ladybug Count; (c) TouchCounts “Numbers World”; (d) child interacting with the TouchCounts “Operations World”

Another example is Ladybug Count (Fig. 1b); here the player is shown a ladybug depicted on a leaf, with a varying number of dots on its back. The player needs to place simultaneously on the screen the corresponding number of fingers, in order to make the ladybug move.

A third example is the app TouchCounts (Sinclair & Jackiw, 2011; Sinclair & Pimm, 2015), that offers two open environments (Fig. 1c and d), in which children can create and manipulate numbers through finger gestures, while receiving immediate, dynamic feedback.

In the “Numbers World” (Fig. 1c) each touch creates a disc with a number that either falls out of the screen if it is below the shelf, or it stays fixed on the shelf if it is created above. Sequential touches create discs numbered sequentially (it is impossible to generate two discs with the same number). In the “Operations World” (Fig. 1d), each touch creates a herd with the number of discs corresponding to the number of screen touches in a multi-touch gesture. The herds stay on the screen, and they can be “pinched” together, or separated using multi-touch gestures.

Studies have advanced and explored the hypothesis that apps like those in the examples above may uniquely influence children’s mathematical understandings and strategy development concerning number sense (Baccaglini-Frank & Maracci, 2015; Tucker & Johnson, 2020; Baccaglini-Frank et al., 2020; Tucker & Johnson, 2020). One of the perspectives used to study such learning is that of embodiment in research and practice related to development of number sense (e.g., Holgersson et al., 2016; Segal et al., 2014; Tucker, 2018). Indeed, the role attributed to the use of fingers in the development of number sense is highly resonant with the frame of *embodied cognition* (Gallese & Lakoff, 2005). Within this perspective doing, touching, moving and seeing are essential components of mathematical thinking processes. Hypotheses like that of Butterworth (1999), highlighting the necessity of linking the representation of numbers to the neural representations of fingers, seem to be completely in line with the embodied cognition approach (Baccaglini-Frank & Maracci, 2015; Tucker & Johnson, 2020).

Based on the available literature, in 2016, with Nathalie Sinclair, in our chapter in the *Handbook of International Research in Mathematics Education*, I presented a summary of the basic features of multi-touch technology that can be used to foster the development of number sense skills and component abilities (Sinclair & Baccaglini-Frank, 2015, p. 673). Shortly after, for a chapter in the volume *Contemporary Research and Perspectives in Early Childhood Mathematics Education* (Baccaglini-Frank, 2018), I revised such a summary in light of recent

research (see Baccaglioni-Frank & Maracci, 2015; Kortenkamp et al., 2024; Lembrér & Meaney, 2016), introducing Table 1.

Many of the affordances presented in Table 1 are exploited by apps such as those described above.

Specifically, Fingu and Ladybug Count exploit the following affordances: detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers; showing numerosities on the screen for very brief

Table 1 Affordances of multi-touch technology to foster specific number sense skills or component abilities

Aspect of number sense skills or component abilities	Affordances of multi-touch technology with respect to the potential of fostering development of the specific aspect
Finger tapping	<ul style="list-style-type: none"> • Detecting and differentiating rapid sequences of inputs from different areas of the screen • Accepting input in the form of sequences of rapid taps to identify a target numerosity • Detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers • Recording different gestures as separate inputs (swipe with 1 finger, swipe with two fingers, lasso, pinch, un-pinch/enlarge...) • Manipulating virtual hands by user or by computer • Simulating pianos or string instruments
Subitizing	<ul style="list-style-type: none"> • Showing numerosities on the screen for very brief amounts of time (possibly even fractions of a second) • Returning immediate feedback in response to the input given by the user • The objects to be considered may appear still and placed randomly on the screen or in given arrangements (e.g., dots on dice) or they can move all together, or in relationship to one another • Input may be given not only as typed numbers (in Arabic code or letters) but also in terms of a number of fingers placed simultaneously on the screen, as a number of sequential taps
Recognizing parts of a whole	<ul style="list-style-type: none"> • Detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers • Manipulating virtual objects
Establishing one-to-one correspondence (with fingers)	<ul style="list-style-type: none"> • Detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers • Accurate timing of the user’s performance
Estimation	<ul style="list-style-type: none"> • Providing stimuli with different (large) numerosities which may remain on the screen or disappear after a given time • Providing immediate feedback on the input received as a product of the estimation process
Managing counting principles	<ul style="list-style-type: none"> • Adding verbal feedback in the form of verbal symbolic number representations to sets of fingers placed on the screen, or to numbers presented in analogical form • Detecting gestures such as simultaneous taps and sequential, or their combination, and providing different feedback in response to each of them • Arranging objects on the screen through dragging

amounts of time (possibly even fractions of a second); returning immediate feedback in response to the input given by the user; the objects to be considered may appear still and placed randomly on the screen or in given arrangements (e.g., dots on dice) or they can move all together, or in relationship to one another; input may be given in terms of a number of fingers placed simultaneously on the screen; detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers; accurate timing of the child's performance; providing immediate feedback on the input received as a product of the estimation process.

Also the two “worlds” available in TouchCounts (Fig. 1c and d) exploit many of the affordances in Table 1, for example: adding verbal feedback in the form of verbal symbolic number representations to sets of fingers placed on the screen, or to numbers presented in analogical form; detecting gestures such as simultaneous taps and sequential, or their combination, and providing different feedback in response to each different gesture; arranging objects on the screen through dragging (especially in the “Operations World”); manipulating virtual objects; input may be given in terms of a number of fingers placed simultaneously on the screen, as a number of sequential taps; returning immediate feedback in response to the input given by the child.

Various studies have investigated whether and how these apps not only exploit multi-touch affordances but also elicit certain aspects of number sense (Baccaglioni-Frank, 2018; Baccaglioni-Frank & Maracci, 2015; Barendregt et al., 2012). To exemplify and discuss some of these results, I will briefly present two studies that I conducted, respectively, with Mirko Maracci, and with Nathalie Sinclair and Gemma Carotenuto. A more general problem that we try to solve in these studies is how to infer what and how young children “learn” concerning number sense, when there are very few spoken words upon which to base such inferences (Baccaglioni-Frank et al., 2024). After introducing these examples, I will discuss their limitations and share some of my concerns about how multi-touch apps can and should be used in educational settings.

Study 1

The first study (Baccaglioni-Frank & Maracci, 2015) focuses on the potential of two of the multi-touch apps introduced earlier, Ladybug Counts and Fingu, to elicit aspects of number sense (see introduction and Table 1) in preschool-aged children (ages 4–5). We did this by investigating the *schemes* that children developed in their interactions with the apps, and, in particular, how they used their fingers; and by attempting to relate the schemes enacted to the development of number sense.

We operationalized Vergnaud's construct of *scheme* (Vergnaud, 2009), which has the function “both to describe ordinary ways of doing, for situations already mastered, and give hints on how to tackle new situations” (Vergnaud, 2009, p. 88). A







scheme is made up of several components, of which, for our purposes, we recall the operational invariants, that is, implicit knowledge which structures the whole scheme. These drive the identification of the situation and of its relevant aspects, and they allow selecting suitable goals and inferring the rules for generating appropriate sequences of actions for achieving those goals.

In this study, by analyzing videos of children playing with the apps, we identified and described the situations which the children faced when interacting with the apps, and the stable recurring strategies which children enacted in those situations, that is “regularities” in the children’s behavior. So, stable recurring strategies are what they relate to the operational invariants, belonging to schemes.

This study utilized data previously collected by Giorgia Giacomini, a pre-service teacher, as part of her master’s thesis. The data involved 25 children ages 4–5 (6 did not speak Italian) in groups of 5 children, who interacted with the apps. Before starting the sessions in which the children interacted with the apps on a tablet, the pre-service teacher met the children and asked them to “draw your hand”: the children were asked to put the hand that they did not use to write under the table, so that they would not trace it out. Some of the drawings produced are in the pre-intervention drawing column in Table 2, below. A final session, after the end of the intervention with the apps was dedicated to the same task, which gave rise to drawings such as those in the post-intervention drawing column in Table 2. These pre- and post-drawings were collected to gain insight into children’s finger gnosis, one of the hypothesized component abilities of number sense (Butterworth, 1999).

As for the intervention sessions, at first, the apps were introduced in two free-play sessions. Then for 2 weeks, every day each group of 5 children, under the guidance of the pre-service teacher, interacted with the apps for 25 min (turns of 5 min per child; the others watched); in total each child during their turns interacted actively for 50 min. When not interacting with the apps the children watched their classmates work, and, if prompted by the pre-service teacher, they were allowed to show finger configurations and to express verbal utterances. The pre-service teacher managed all the group sessions, without ever showing any child what to do: she intervened only to call each child within the group when it was their turn to interact

Table 2 Three children’s pre- and post-drawings of their hands

Child	Pre-intervention drawing	Post-intervention drawing
Child 1		
Child 2		
Child 3		

with the tablet; to draw the children's attention to their classmate's work with the tablet, or to ask them to help their classmate. All sessions were video-recorded; for example, Fig. 1a and b come from such footage.

I now summarize the findings. In the Ladybug Count app, children seemed to use strategies based on explicit counting, or on a first approximate estimation of quantities followed by successive adjustments. On the other hand, Fingu, being timed and requiring simultaneous finger touches, seemed to inhibit explicit counting and successive adjustments; it seemed to contribute more to the development of strategies triggering the representation of numbers with fingers and the ability to subitize (possibly two small quantities simultaneously). In general, the children tended to rely on very few (possibly just one) strategies and using a variety of different strategies did not appear to be a possibility that they drew upon spontaneously.

An unexpected finding was that in many cases, when playing Fingu, children tried to place their fingers as close as possible to the floating objects, so as to "catch" them, or to reproduce with their finger the same spatial arrangement of the floating objects. A few children quickly counted the floating objects and then placed their fingers on the screen simultaneously; but pressed by the time constraints some children failed to count correctly or to place their fingers properly on the screen.

An interesting hypothesis emerged concerning a potential of Fingu to inhibit the spontaneous urge to "capture" the floating fruits. Indeed, it might be the case that an inhibitory control needs to be exercised over the more intuitive catching strategies in order to carry out one of the more successful strategies that are based on other aspects of number sense. Such inhibitory control in various children seemed to dissolve when too many conflicting and/or time-pressing stimuli concur.

Finally, in Giacomini's unpublished dissertation (2014) the children's drawing of their hands pre- and post-intervention were compared. Table 2 shows three of the children's drawings.

Notice how children 2 and 3 initially draw their hands as a disproportionately large palm with unclear numbers of fingers: child 2 seems to differentiate a larger finger, possibly their thumb; while the fingers drawn by child 3 are many more than five, and they almost look like "hair" surrounding the palm. Child 1's pre-intervention drawing displays a proper proportion between the palm and the fingers, but only 4 fingers. In their post-intervention drawings, all three children draw exactly 5 fingers: child 1's drawing is extremely precise. Children 2 and 3 seem to have shifted their focus from the palm to the fingers, that are now represented in the correct amount, while the palms are no longer represented. Similar results were obtained for more than half of the children in the study.

While I would like to claim that this result suggests strengthening of the children's finger gnosis over the 2-week intervention, I suggest caution in generalizing, because of the small sample size and of the fact that other activities conducted at school or at home during the same time period could have produced the same sorts of changes in the drawings.

Study 2

The study that I conducted with Gemma Carotenuto, and Nathalie Sinclair (2020) also investigated how multi-touch apps can foster the development of number sense in preschool children (ages 3–4); it focuses, in particular, on the open-ended “worlds” in TouchCounts (Fig. 1c and d). The study builds on the following set of findings in the literature: different children attend to different affordances, and scaffolding is key (Bullock et al., 2017); while representing the majority of apps, instructive apps are less beneficial than manipulative and constructive ones (e.g., Goodwin & Highfield, 2013; Sinclair & Zazkis, 2017); certain multi touch apps can support one-to-one correspondence between fingers and virtual objects, finger gnosis, and counting—both of virtual objects and of fingers (Baccaglioni-Frank, 2018; Baccaglioni-Frank & Maracci, 2015; Broda et al., 2018; Holgersson et al., 2016); instruction often overemphasizes counting (Clements et al., 2019) and cardinality (Sinclair & de Freitas, 2016).

The research objectives in this study were twofold: (1) to design tasks in TouchCounts that could elicit various number sense abilities in children; (2) to analyze preschoolers’ interactions with the two “worlds” in the app to understand the range of number sense abilities mobilized, particularly focusing on cardinality and ordinality (for the definitions used see Table 3). In considering possible number sense skills elicited during the children’s interactions with the app, the authors updated their review of the literature on basic abilities and skills¹ related to human cognitive development of managing (small) Natural Numbers, and in particular

Table 3 Skills considered as characteristic of number sense (Baccaglioni-Frank et al., 2024)

Characteristic Number Sense skills	Definition
Finger knowledge and control	Knowing the position of your fingers in your hand and making precise gestures with your fingers
Subitizing	Detecting small numbers (up to *****) without triggering counting processes
Finger-to-finger number representation	Knowing the number-word (e.g., “four”) or the symbol corresponding to a certain finger configuration, and vice versa. Configurations can be achieved with counting (constructive) support or on the fly (immediate)
Ordinality skills	Associating number symbols with number-words, and vice versa; knowing the sequence of number symbols; knowing the sequence of number-words; knowing the previous and next of a certain number
Cardinality skills	Establishing one-to-one correspondences between elements of two different sets (of number-words, of material or virtual objects, of fingers); associating a numerosity to a set of objects (this could occur through subitizing, or by counting and associating such a numerosity to the last number-word spoken); assigning the correct numeric symbol or the correct number-word to a set, depending on its numerosity

¹We choose to speak broadly of “skills” to avoid discussions of innateness, which was beyond the scope of the study.

ordinality and cardinality. Such a review (Baccaglioni-Frank et al., 2020; Baccaglioni-Frank & Maracci, 2015; Elia et al., 2023) and a critical analysis of the literature from related studies (Ladel & Kortenkamp, 2014; Mulligan et al., 2018; Tucker & Johnson, 2020) led to the five characteristic number sense skills² in Table 3.

The theoretical framework is based on the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), which draws on Vygotskian socio-constructivism, and connects the visible parts of children's interactions with digital environments to their development of the skills in Table 3. Mathematics is seen as a cultural product, the shared knowledge of a society, expressed by systems of signs (Mariotti, 2009). The Theory of Semiotic Mediation interprets both Mathematics and artifacts as historical-cultural products, recognizing the way an artifact works as a product of the relationship between tasks it is used to solve and specific mathematical notions. A central focus concerns the relationship between *artifacts* and *signs* within the solution of a *task*. The artifact has the potential to stimulate the solvers' production of signs (in this study: words and gestures). As they solve the task with the artifact, solvers will develop personal meanings for such signs; an expert will recognize, in such signs, mathematical meanings that can be expressed with mathematical signs. The teacher plays a crucial role both in designing the tasks and in leading the transition from artifact signs to mathematical signs. Developing mathematical signs concerning the "intended" mathematics (we chose this to be the characteristic number sense skills in Table 3) is how the theory conceptualizes learning. However, in the context of our research with very young children, we needed to clarify what signs we could consider, and how to interpret them: we needed to interpret the (few) words spoken and the gestures made by the children in their interactions with the app and with the researcher/teacher. For this, we elaborated on the previously used (Baccaglioni-Frank & Maracci, 2015) notion of scheme by Vergnaud (2009), focusing specifically on two components of scheme (Baccaglioni-Frank et al., 2020; Baccaglioni-Frank et al., 2024): the intentional component, consisting of one or more goals, with any sub-goals, and anticipations; the generative component, consisting of rules of action, i.e. sequences of actions that are carried out generating the activity.

We conducted the study in a preschool in southern Italy, involving 19 children aged 3.5–4.5. After initial interviews to assess the children's existing number sense skills, eight children were selected based on their ability to answer at least four out of 13 interview questions correctly. The children participated in two individual sessions, during which they interacted with the two worlds in TouchCounts. Two of the tasks among those designed and implemented are:

- Task 1: "Can you put only this on the shelf?" (in the Numbers World)—The child was expected to tap the screen and create numbered discs, placing only the specified disc (e.g., "4") on the shelf.
- Task 2: "Make Nadia count" (in the Operations World)—The child was expected to tap the screen with different numbers of fingers in sequence so that the voice in the app (that we referred to as "Nadia's") would say them in order (i.e., count).

²Note that these are not assumed to be a partition.

The sessions were video recorded and the children's gestures, utterances, and interactions with the app were transcribed and analyzed. The qualitative analysis of the children's interactions with the two tasks revealed different strategies and number sense abilities.

In the case of one of the children, Fred, a careful analysis of his gestures as well as the length of pauses between gestures, suggests that he was on the verge of developing a new understanding of the previous and the next of each number up to four (ordinality). During Fred's brief interaction with TouchCounts and the researcher, the child was able to experiment with different gestures and rapidly developed a successful scheme for Task 1. He seemed to have discovered that the disc with the number four (as the spoken sound "four" or as the symbol "4") comes after the disc with the number three, which is an important aspect of ordinality. Such learning seemed to be very much supported also by the characteristics of TouchCounts, and, in particular, its being an *open* environment, allowing free exploration and naturally inviting interaction between the child and the researcher/teacher. Specifically, not having tasks automatically proposed by design, the app affords the teacher's active engagement in proposing tasks and offering personalized support to the child.

I would also like to share insights gained from closely studying the interaction of another child, Maria, in the Operations World with the researcher/teacher working on Task 2. Initially, for numbers 1–4, she demonstrated immediate finger-to-finger number representation skills, but for the numbers 5–7, she needed to count and prepare her fingers more deliberately. By the end of the session, Maria developed an intuitive sense for finger configurations (for details see pp. 787–789 of Baccaglini-Frank et al., 2020), even demonstrating finger-to-finger number representation for the number 7—where she raised all seven fingers simultaneously without needing to count.

Overall, the study suggested that an open multi-touch environment like TouchCounts can effectively elicit a range of number sense abilities in young children, especially when combined with appropriate task design and mediation of the researcher/teacher. The open-ended nature of the app allowed children to experiment with different strategies, enabling the development of both ordinal and cardinal understanding, as well as finger control and subitizing skills. In particular, the study emphasized the possibility of promoting ordinality in early childhood mathematics through appropriately-designed environments and tasks proposed and scaffolded by the researcher/teacher.

Discussion and Conclusions

Overall, the literature presented and, in particular the studies I focused on in this chapter, highlight the importance of fingers in children's development of number sense, and the potentially positive role that certain multi-touch apps can have in fostering such a development. These findings align with embodied cognition theories, which propose that physical or digital interactions enhance conceptual

understanding (e.g., Shvarts et al., 2021; Sinclair & de Freitas, 2016; Tucker & Johnson, 2020). However, I would like to discuss some important differences between the kinds of interactions promoted by the apps and the role of the researcher/teacher interacting with the children in each learning situation.

On the one hand, closed-interaction learning environments provided by apps like Ladybug Count and Fingu in which tasks are proposed automatically can enhance early number sense development, especially certain aspects of number sense that are directly related with the strategies that children tend to use repeatedly to “win” (Baccaglini-Frank & Maracci, 2015; Ladel & Kortenkamp, 2014). However, recent studies are starting to suggest that not always are the strategies used by children to make progress in a game like Fingu elicit the knowledge expected by its designers (e.g., Kortenkamp et al., 2024). Moreover, the fact that the tasks (often very repetitive) are set and instantiated automatically ensures immediate usability of these environments, but limits their educational benefits (Elia et al., 2023; Sinclair & Baccaglini-Frank, 2015). Finally, although I have not introduced this aspect in the paper, a current study that I am carrying out using re-enactment along the lines of the recent study by Sinclair (2024), suggests that when children are left alone to play with an app like Fingu by themselves, or in an environment in which many children are each playing separately while next to each other in “training sessions”, many stressful factors seem to emerge, as well as incorrect ideas of what mathematics is about (answering fast, without thinking, and not making mistakes).

On the other hand, the open-ended nature of the environments in an app like TouchCounts encourages children to engage in various manipulations, discovering new relationships, guided both by the immediate dynamic feedback and by the nearby researcher/teacher, who needs to be involved in designing and proposing tasks that exploit the affordances of the environment (Baccaglini-Frank et al., 2020, 2024; Sinclair & Baccaglini-Frank, 2015). These environments foster not only the child’s development of the number sense characteristics discussed, but they also offer other possibilities, such as: pursuing educational objectives adapted to children’s development levels; encouraging diverse strategies for solving a same task; discussing such strategies for gathering deeper insight into children’s learning, as well as enriching children’s learning; encouraging free exploration and fostering children’s problem-posing skills (Baccaglini-Frank et al., 2020; Baccaglini-Frank et al., 2024). For example, research has shown that children who work together in open environments by setting their own tasks can also work with a range of mathematical ideas. It is important that open digital environments can support children to make their own problems and then try to solve them (Fosse et al., 2023).

Looking ahead, the potential for multi-touch technology in early childhood education is vast. As multi-touch devices continue to evolve, so too will the educational apps that take advantage of these technologies. The findings from the studies we discussed suggest that multi-touch technology can serve as a powerful tool for developing number sense, offering children a highly engaging and interactive way to learn. However, the effectiveness of these tools ultimately depends on thoughtful design and skilled implementation by educators. Greater emphasis should be placed on the educator’s role, as their interventions may lack productivity without specific

guidance (e.g., Baccaglioni-Frank, 2016; Elia et al., 2017). A final caveat: I firmly believe that digital experiences should always be well-integrated with multiple physical experiences and direct interactions with the other people in the children's learning environment.

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References

- Andres, M., Seron, X., & Olivier, E. (2007). Contribution of hand motor circuits to counting. *Journal of Cognitive Neuroscience*, 19, 563–576. <https://doi.org/10.1162/jocn.2007.19.4.563>
- Baccaglioni-Frank, A. E. (2016, July). Educational multi-touch applications, number sense, and the homogenizing role of the educator. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany. <https://doi.org/10.48550/arXiv.1605.02580>.
- Baccaglioni-Frank, A. (2018). What schemes do preschoolers develop when using multi-touch applications to foster number sense (and why)? In I. Elia, J. Mulligan, A. Anderson, A. Baccaglioni-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education, ICME-13 Monographs* (pp. 223–243). Springer. https://doi.org/10.1007/978-3-319-73432-3_12
- Baccaglioni-Frank, A., Carotenuto, G., Funghi, S., Lisarelli, G., & Miragliotta, E. (2024). Digital artifacts in mathematics education: How can we study the learning processes they promote? *Bollettino dell'Unione Matematica Italiana*. <https://doi.org/10.1007/s40574-024-00439-2>
- Baccaglioni-Frank, A., Carotenuto, G., & Sinclair, N. (2020). Eliciting preschoolers' number abilities using open, multi-touch environments. *ZDM Mathematics Education*, 52(4), 779–791. <https://doi.org/10.1007/s11858-020-01144-y>
- Baccaglioni-Frank, A., & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education*, 1(1), 7–27. <https://doi.org/10.1007/s40751-015-0002-4>
- Barendregt, W., Lindstrom, B., Rietz-Leppanen, E., Holgersson, I., & Ottosson, T. (2012). Development and evaluation of Fingu: A mathematics iPad game using multi-touch interaction. In *IDC 2012*, June 12–15, 2012, Bremen, Germany (pp. 1–4). <https://doi.org/10.1145/2307096.2307126>.
- Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Routledge.
- Brissiaud, R. (1992). A toll for number construction: Finger symbol sets. In J. Bidaud, C. Meljac, & J.-P. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities*. Lawrence, Erlbaum Associates.
- Broda, M., Tucker, S., Ekholm, E., Johnson, T. N., & Liang, Q. (2018). Small fingers, big data: Preschoolers' subitizing speed and accuracy during interactions with multitouch technology. *The Journal of Educational Research*, 112(2), 211–222. <https://doi.org/10.1080/00220671.2018.1486281>
- Bullock, E. P., Shumway, J. F., Watts, C. M., & Moyer-Packenham, P. S. (2017). Affordance access matters: Preschool children's learning progressions while interacting with touch-screen math-

- ematics apps. *Technology Knowledge and Learning*, 22, 485–511. <https://doi.org/10.1007/s10758-017-9312-5>
- Butterworth, B. (1999). *The mathematical brain*. Macmillan.
- Butterworth, B. (2005). The development of arithmetical abilities. *Journal of Child Psychology and Psychiatry*, 46, 3–18. <https://doi.org/10.1111/j.1469-7610.2005.00374.x>
- Clements, D. H., Sarama, J., & MacDonald, B. L. (2019). Subitizing: The neglected quantifier. In A. Norton & M. W. Alibali (Eds.), *Constructing number: Merging perspectives from psychology and mathematics education* (pp. 13–45). Springer International Publishing.
- Coles, A., & Sinclair, N. (2018). Re-thinking ‘normal’ development in the early learning of number. *Journal of Numerical Cognition*, 4(1), 136–158. <https://doi.org/10.5964/jnc.v4i1.101>
- Davydova, Y., Sabirova, Y., Lysenkova, I., & Tikhomirova, T. (2014). Number Sense and mathematics across development and cultures. *Personality and Individual Differences*, 60, S9. <https://doi.org/10.1016/j.paid.2013.07.179>
- Dehaene, S. (1997). *The number sense*. Oxford University Press.
- Elia, I., Baccaglioni-Frank, A., Levenson, E., Matsuo, N., Feza, N., & Lisarelli, G. (2023). Early childhood mathematics education research: Overview of latest developments and looking ahead. *Annales de Didactique et de Sciences Cognitives*, 28, 75–129. <https://journals.openedition.org/adsc/3113#text>
- Elia, I., Mulligan, J., Anderson, A., Baccaglioni-Frank, A., & Benz, C. (2018). Contemporary research and perspectives on early childhood mathematics education. In *ICME-13 monographs*. Springer. <https://doi.org/10.1007/978-3-319-73432-3>
- Elia, I., Mulligan, J., Anderson, A., et al. (2017). Topic Study Group no. 1: Early childhood mathematics education (up to age 7). In G. Kaiser (Ed.), *Proceedings of the 13th international congress on mathematical education, ICME-13 monographs* (pp. 375–379). Springer. https://doi.org/10.1007/978-3-319-62597-3_28
- Fosse, T., Lange, T., & Meaney, T. (2023). Lek med problemer. *Nordisk Barnehageforskning*, 20(2), 89–109. <https://doi.org/10.23865/nbf.v20.280>
- Gallese, V., & Lakoff, G. (2005). The brain’s concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, 22(3–4), 455–479.
- Gelman, R., & Gallistel, C. R. (1978). *The child’s understanding of number*. Harvard University Press.
- Giacopini, G. (2014). *iPad per Contare a Quattro Anni*. Unpublished thesis. Dipartimento di Educazione e Scienze Umane, Università di Modena e Reggio Emilia.
- Goodwin, K., & Highfield, K. (2013). A framework for examining technologies and early mathematics learning. In L. D. English and J. T. Mulligan (Eds.), *Reconceptualising early mathematics learning* (pp. 205–226). New York: Springer.
- Gracia-Bafalluy, M. G., & Noël, M. P. (2008). Does finger training increase young children’s numerical performance? *Cortex*, 44, 368–375. <https://doi.org/10.1016/j.cortex.2007.08.020>
- Griffin, S. A., Case, R., & Siegler, R. S. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), *Classroom lessons: Integrating cognitive theory and classroom practice* (pp. 25–49). The MIT Press.
- Halberda, J., & Odic, D. (2015). The precision and internal confidence of our approximate number thoughts. In D. C. Geary, D. B. Berch, & K. M. Koepke (Eds.), *Evolutionary origins and early development of number processing* (pp. 305–333). Elsevier Academic Press. <https://doi.org/10.1016/B978-0-12-420133-0.00012-0>
- Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction*, 15(3), 237–256. <https://doi.org/10.1016/j.learninstruc.2005.04.005>
- Holgersson, I., Barendregt, W., Emanuelsson, J., Ottosson, T., Rietz, E., & Lindström, B. (2016). Fingu: A game to support children’s development of arithmetic competence—Theory, design and empirical research. In P. S. Moyer-Packenham (Ed.), *International perspectives on teach-*

- ing and learning mathematics with virtual manipulatives. *Mathematics education in the digital era* (pp. 123–146). Springer. https://doi.org/10.1007/978-3-319-32718-1_6
- Kortenkamp, U., Ladel, S., & Larkin, K. (2024). Understanding children's learning of part-whole relations using Fingu. *Digital Experiences in Mathematics Education*. <https://doi.org/10.1007/s40751-024-00160-x>
- Ladel, S., & Kortenkamp, U. (2014). Number concepts—Processes of internalization and externalization by the use of multi-touch technology. In U. Kortenkamp, B. Brandt, C. Benz, G. Krummheuer, S. Ladel, & R. Vogel (Eds.), *Early mathematics learning: Selected papers of the POEM 2012 Conference* (pp. 237–253). Springer. https://doi.org/10.1007/978-1-4614-4678-1_15
- Lembré, D., & Meaney, T. (2016). Preschool children learning mathematical thinking on interactive tables. In T. Meaney, O. Helenius, M. L. Johansson, T. Lange, & A. Wernberg (Eds.), *Mathematics education in the early years - Results from the POEM2 conference, 2014* (pp. 235–254). Springer. https://doi.org/10.1007/978-3-319-23935-4_13
- Lyons, I., & Beilock, S. (2013). Ordinality and the nature of symbolic numbers. *The Journal of Neuroscience*, 33(43), 17052–17061. <https://doi.org/10.1523/JNEUROSCI.1775-13.2013>
- Lyons, I., Price, G., Vaessen, A., Blomert, L., & Ansari, D. (2014). Numerical predictors of arithmetic success in grades 1–6. *Developmental Science*, 17(5), 714–726. <https://doi.org/10.1111/desc.12152>
- Margolinas, C., & Wosniak, F. (2012). *Le nombre à l'école maternelle. Une approche didactique*. De Boeck.
- Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: the role of the teacher. *ZDM*, 41, 427–440. <https://doi.org/10.1007/s11858-009-0199-z>
- Mellone, M., Baccaglioni-Frank, A., & Martignone, F. (2020). Measuring rice in early-childhood education activities: A bridge across discrete and continuous magnitudes. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years* (pp. 161–176). Springer. https://doi.org/10.1007/978-3-030-34776-5_10
- Mulligan, J., Verschaffel, L., Baccaglioni-Frank, A., et al. (2018). Whole number thinking, learning and development: Neuro-cognitive, cognitive and developmental approaches. In M. G. Bartolini Bussi & X. H. Sun (Eds.), *Building the foundation: Whole numbers in the primary grades* (New ICMI study series) (pp. 137–167). Springer. https://doi.org/10.1007/978-3-319-63555-2_7
- Resnick, L. B., Bill, V., Lesgold, S., & Leer, M. (1991). Thinking in arithmetic class. In B. Means, C. Chelemer, & M. S. Knapp (Eds.), *Teaching advanced skills to at-risk students: Views from research and practice* (pp. 27–53). San Francisco: Jossey-Bass.
- Segal, A., Tversky, B., & Black, J. (2014). Conceptually congruent actions can promote thought. *Journal of Applied Research in Memory and Cognition*, 4(3), 124–130. <https://doi.org/10.1016/j.jarmac.2014.06.004>
- Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics*, 107, 447–469. <https://doi.org/10.1007/s10649-021-10053-0>
- Sinclair, N. (2024). Knowing as remembering: Methodological experiments in embodied experiences of number. *Digital Experiences in Mathematics Education*, 10, 29–46. <https://doi.org/10.1007/s40751-023-00132-7>
- Sinclair, N., & Baccaglioni-Frank, A. (2015). Digital technologies in the early primary school classroom. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 662–686). Routledge. <https://doi.org/10.4324/9780203448946>
- Sinclair, N., & Coles, A. (2017). Returning to ordinality in early number sense: Neurological, technological and pedagogical considerations. In F. Ferrara, E. Faggiano, & A. Montone (Eds.), *Innovation and technologies in mathematics education* (Mathematics education in the digital era) (Vol. 9, pp. 39–58). Springer. https://doi.org/10.1007/978-3-319-61488-5_3
- Sinclair, N., & de Freitas, E. (2016). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture*, 14(3), 351–374. <https://doi.org/10.1075/gest.14.3.04sin>
- Sinclair, N. & Jackiw, N. (2011). *TouchCounts* [software application for the iPad].

- Sinclair, N., & Pimm, D. (2015). Mathematics using multiple sense: Developing finger gnosis with three- and four-year-olds in an era of multi-touch technologies. *Asia-Pacific Journal of Research in Early Childhood Education*, 9(3), 99–109. <https://doi.org/10.17206/apjrece.2015.9.3.99>
- Sinclair, N., & Zazkis, R. (2017). Everybody counts: Designing tasks for TouchCounts. In A. Leung & A. Baccaglioni-Frank (Eds.), *Digital technologies in designing mathematics education tasks* (Mathematics education in the digital era) (Vol. 8, pp. 175–191). Springer. https://doi.org/10.1007/978-3-319-43423-0_9
- Tucker, S. I. (2018). Applying the modification of attributes, affordances, abilities, and distance for learning framework to a child's multi-touch interactions with an idealized number line. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach, & C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education* (pp. 35–57). Springer. https://doi.org/10.1007/978-3-319-76575-4_3
- Tucker, S. I., & Johnson, T. N. (2020). Developing number sense with Fingu: A preschooler's embodied mathematics during interactions with a multi-touch digital game. *Mathematics Education Research Journal*, 34, 393–417. <https://doi.org/10.1007/s13394-020-00349-4>
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52(2), 83–94. <https://doi.org/10.1159/000202727>

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Negotiation Types and Participation Forms in a Family Interaction During Block Play



Ergi Acar Bayraktar and Birgit Brandt

Introduction

It is undisputed that early mathematical learning does not only take place institutionally in kindergarten and school. Rather, the family is a place where learning can take place and be shaped in a special way. Bronfenbrenner (1979) describes development—and thus also learning processes in terms of content—in his ecosystemic developmental model as a reciprocal, interactionist process between the child as an individual and his or her social environment. In this model, the family is a system at the micro level. In addition to parents or mother and father, siblings and grandparents may also play a decisive role as sub-systems, each with their own routines and dynamics. The child is thus integrated into a variety of interaction processes with individual family members, which offer different developmental opportunities, including mathematical learning opportunities. However, there have been few systematic studies in which the shaping of interaction processes in a family context have been realized.

The chapter represents a contribution to the research field, from an interactionist perspective (Blumer, 1969). Blumer (1969) formulates “three simple premises” (p. 2) of symbolic interactionism: According to the first premise, “human beings act toward things on the basis of the meanings that the things have for them” (Blumer, 1969, p. 2). According to the second premise, “the meaning of such things derived from, or arises out of, the social interaction that one has with one’s fellows” (Blumer, 1969, p. 2). And the third premise states, “that these meanings are handled in, and modified through, an interpretative process used by the person in dealing with the things he encounters” (Blumer, 1969, p. 2). On the basis of these three premises, Krummheuer (1992) states that symbolic interactionism contributes to learning theory.

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Mathematical Learning as a Social Process: Learning-as-Participation

Mathematics learning is fundamentally a social process, situated within and shaped by social relationships and contexts (e.g., Cobb & Bauersfeld, 1995). The interactionist perspective emphasizes that learning processes arise from negotiation of meaning and, in particular, through collective argumentation (Krummheuer, 1995; Krummheuer & Brandt, 2001). Thus, individuals' cognitive development is closely linked to their participation in social interactions, where they establish interpretations, expectations, and obligations (Blumer, 1969). Learning occurs through increasing autonomy within stable interactional structures, particularly through participation in collective argumentation processes. Thus, participation in social interactions is closely linked to cognitive development. Participants coordinate goals and intentions, adjusting actions and negotiating understandings of the situation. *Learning-as-participation* extends the understanding of participation in learning processes (Krummheuer, 2011; Sfard, 2001, 2008).

The theory of situated learning (Lave & Wenger, 1991) is also based on the idea that learning takes place through participation in social events. Challenging traditional views of learning as the acquisition of abstract knowledge, this theory emphasizes that learning is deeply rooted in social interactions and authentic engagement within communities of practice. A central concept within this framework, *legitimate peripheral participation* (Lave & Wenger, 1991), describes how individuals, such as children or newcomers, begin as peripheral participants in a community or practice and gradually "move toward full participation" (Lave & Wenger, 1991, p. 29) through meaningful interaction and engagement with more experienced members. This process highlights that learning is not merely about mastering abstract concepts, but involves becoming an active participant in the sociocultural practices of a community, "A person's intentions to learn are engaged and the meaning of learning is configured through the process of becoming a full participant in a sociocultural practice" (Lave & Wenger, 1991, p. 29).

In the context of mathematics education, the implications of this theory are important. Lave and Wenger's perspective reframes learning as a dynamic, socially embedded process, where competence is developed through active participation in meaningful and socially situated activities. Similarly, Krummheuer and Brandt (2001) argue that students' learning processes are shaped by their modes of participation, which can range from passive observation to active collaboration in problem-solving. This *learning-as-participation* approach shifts the focus from individual cognitive processes to the interactional and discursive practices within the classroom, where students co-construct knowledge through engagement in classroom discourse (Cobb & Bauersfeld, 1995). Beyond classroom settings, this framework is equally relevant in family contexts, particularly during mathematical play, where legitimate peripheral participation sheds light on how children learn mathematics through everyday interactions with family members. These learning processes are inherently social and contextual, demonstrating that mathematics learning, like all

forms of situated learning, is shaped by participation in meaningful, real-world activities.

Talk Types and Negotiation Processes

In his work, Mercer (1995, 2000) examines the essential function of language as a tool for fostering collaborative thought and negotiation within diverse social contexts, including school lessons, interactions among children as well as interactions in the family environment. Based on these ideas, Mercer and his colleagues (Fernández et al., 2001) have examined the scaffolding concept of asymmetrical interactions between teachers and students and considered the extent to which scaffolding processes are also integrated into more symmetrical interactions between peers. They found that in symmetrical talk, children follow implicit ground rules, leading to scaffolding effects without conscious intent (Fernández et al., 2001, p. 68). Despite children not consciously scaffolding each other's learning, effective communicative strategies facilitate problem-solving. Three different types of talk can be distinguished, which "can be considered as social ways of thinking" (Fernández et al., 2001, p. 57): disputational, cumulative, and exploratory talk. While disputational talk in symmetrical groups can restrict children's development, exploratory talk can expand it. Exploratory talk among peers has the greatest possibilities for intellectual advancement. These talk types represent distinct communication styles, integrated with the concept of flexible and dynamic scaffolding. Thus, it is not so much the conscious intention (of a teacher) that is decisive for the effectiveness of scaffolding, but rather the nature of the conversation and the dynamic process between the participants (Fernández et al., 2001).

These theoretical perspectives can be used in understanding family interactions, including possible integrated negotiation and scaffolding processes that could initially appear in asymmetric interaction processes. Incorporated into an interactionist learning theory, Acar Bayraktar (2017) describes these familial interaction processes as "negotiating taken-as-shared meanings" (p. 110). In this context, she emphasizes that the negotiation process goes beyond mere verbal exchange and includes both verbal and non-verbal elements, with or without embodied expressions, thus emphasizing shared responsibility for the structure and meaning of contributions (Acar Bayraktar, 2017). With reference to the work of Fernández et al. (2001), she elaborates on these styles as follows:

- Negotiating taken-as-shared meanings in a disputational way: This kind of negotiation consists of disagreements, individualized decision-making, counter-assertions through verbal, nonverbal and motional outputs. In this type of negotiating each individual seems to perceive others as competitive or incompetent.
- Negotiating taken-as-shared meanings in a cumulative way: This kind of negotiation consists of agreements, positive and uncritical assertions through verbal,

nonverbal and motional outputs such as repetitions, confirmations and elaborations. In this type of negotiating each individual seems to achieve agreement without any critiques.

- Negotiating taken-as-shared meanings in an exploratory way: This kind of negotiation consists of alternative hypotheses, offered justifications, rationalized practices and common pursuit of the best solutions. Its orientation is being constructive and cooperative so that all relevant features about the thematized issue can be shared. In this type of negotiating individuals seem to achieve agreement through contribution by verbal, nonverbal and motional outputs. (Acar Bayraktar, 2017, p. 438)

In the following, the participation of a child in a family interaction process with the grandmother and the elder brother is to be examined more closely, and in doing so, these forms of negotiating taken-as-shared meanings are to be considered in more detail for the individual participants. To this end, the next section presents the method of participation analysis (Krummheuer, 2011; Krummheuer & Brandt, 2001), which takes into account the different forms of integration, responsibility and modes of articulation.

Analysis of Interaction and Participation

Krummheuer and Brandt (2001) described learning as the product of participation in interaction processes, especially in collective argumentation. Following the work of Goffman (1981), they classified each utterance in the collective argumentation into two groups, according to “direct participation of the recipient in the utterance” and “non-direct participation of the recipient in the utterance” (Krummheuer & Brandt, 2001, p. 51). Each of these groups is separated into two subgroups according to their equal and unequal involvement in the developing conversation, depending on the direct participation or indirect participation (Krummheuer, 2011, p. 84). Considering the interactional theory of mathematics learning, Krummheuer and Brandt call this categorization “recipient design” (Krummheuer & Brandt, 2001, p. 51), which is shown in Table 1.

In the “direct participation of the recipient in the utterance” the utterance of the speaker addresses the recipient directly (see Table 1; Acar Bayraktar, 2017, p. 123). In such participation the speaker has a conversation partner, “to whom the speaker

Table 1 Recipient design (Krummheuer, 2011, p. 84)

Accessibility of an utterance			
Direct participation of the recipient in the utterance		Non-direct participation of the recipient in the utterance	
Addressed by the speaker	Not addressed by the speaker	Tolerated by the speaker	Excluded by the speaker
Conversation partner	Co-hearer	Over-hearer	Eavesdropper

allocates the right to take the next turn” (Krummheuer, 2011, p. 83). When the recipient “is directly addressed, but not assumed to be the next speaker but may be later involved” (Krummheuer, 2011, p. 83). In the current situation, then such recipient is called a ‘co-hearer’; “this recipient status is associated with the obligation of a high level of attentiveness” (Krummheuer, 2011, p. 83). In the example below, the child is initially the grandmother’s *conversation partner* (1–10) and using body language presents himself as an attentive recipient. In “not direct participation of the recipient in the utterance” the speaker does not address the recipient directly (see Table 1; see also Acar Bayraktar, 2017, p. 123). The recipient is “tolerated by the speaker, but not considered in the same manner to take part in the conversation” and so the recipient is an “over-hearer” (Krummheuer, 2011, p. 84). Lastly, the “eavesdropper” can be defined as a listener who is deliberately excluded from the utterance (Krummheuer, 2011, p. 84).

Krummheuer and Brandt (2001) also distinguished between four different forms of ‘speaking persons’ in the interaction process, based on the constitution of participants. This categorization is grounded in the speaker’s responsibility for three components of a contribution: realisation (gestural or acoustic appearance—sound box), formulation (syntactic construction using specific words and expressions), and content (the function/thematic and semantic meaning of an utterance). They referred to this categorization as “production design” and identified four corresponding forms of participation: author, spokesman, ghostee, and relayer (Krummheuer & Brandt, 2001, p. 41).

Brandt (2017) extended this model by incorporating the modality through which a contribution is realized during interaction: (1) verbally—expressed through spoken language; (2) gesturally—expressed through hand movements that can be interpreted as gestures; and (3) concretely—realized through material actions involving physical objects.

The extended production design—or, in other words, the extended analytical model based on the degree of participation and activities in interaction through their modes of representation—is shown in Table 2 (Acar Bayraktar & Brandt, 2023, p. 2023) and each term is defined in the following:

- Author: The speaker constructs the contribution in full, taking responsibility for its performance, how it is expressed (formulation), and the underlying idea (content). Expression can occur through any modality—verbal, gestural, or concrete

Table 2 Extended production design (Acar Bayraktar & Brandt, 2023, p. 27)

Speaking person	Realisation	Formulation			Content
		Verbally	Gestural	Concrete	
Author	+	+	+	+	+
Spokesmann	+	+/-	+/-	+/-	-
Ghostee	+	-/+	-/+	-/+	+
Relayer	+	-	-	-	-

(Brandt, 2017). In the table, this role is marked with “+” across all dimensions, indicating full responsibility.

- **Spokesman:** The speaker conveys an idea taken from someone else but formulates it in their own modality—verbal, gestural, or concrete. The formulation may show traces of previous contributions but also includes elements of individual structuring. The “+/-” notation in the formulation column indicates that the speaker may rephrase existing material or combine their own form with inherited structure, while “-” under content shows that they do not contribute new meaning.
- **Ghostee:** The speaker adopts a previous formulation but uses it to express a new idea. While the content is original (“+”), the formulation—whether verbal, gestural, or concrete—is borrowed. The “-/+” notation reflects that the speaker may initially rely on the existing form but gradually adapt it into their own expression.
- **Relayer:** The speaker neither contributes a new idea (content) nor rephrases the existing formulation. Their role is limited to repeating a contribution exactly as it was presented—in verbal, gestural, or concrete form. This role is marked with “-” across all components, indicating that the speaker takes no responsibility for either the form or the meaning of the utterance.

This theoretical framework provides a systematic basis for analyzing different forms of participation by distinguishing speakers’ responsibilities for formulation and content, and by incorporating various modalities of expression (verbal, gestural, concrete), thereby enhancing the transparency, depth, and interpretive rigor of the interaction analysis.

In particular, in play activities, many activities are carried out with concrete material, such as building blocks. Following the Actor-Network-Theory (Latour, 2005), the building blocks can be seen as actors. Fetzer and Tiedemann (2018) have integrated the idea of object actors into learning theory from an interactionist perspective: “objects can participate in the social process of negotiating a common context and can, in this way, influence processes of empirical abstraction” (p. 97).

The concepts of recipient design and production design illustrate how learners contribute to and receive knowledge during collaborative interactions, transitioning between roles such as peripheral participants and active contributors—depending on the extent of their responsibility for the contributions made, as outlined in Table 2. This nuanced understanding aligns with Lave and Wenger’s (1991) notion of participation, demonstrating how learners negotiate their roles within the community of practice through engagement and interaction. Such insights emphasize the importance of studying the micro-dynamics of interactions to better understand how mathematical understanding emerges as a product of collaborative and participatory practices. Building on this theory of Lave and Wenger (1991), Krummheuer (2011) elaborates on how the *learning-as-participation* perspective can be represented in everyday classroom practices, particularly in mathematics education. While this perspective has gained significant theoretical traction (Cobb & Yackel, 1996), Krummheuer underscores the need for empirical exploration to substantiate its practical application.

By integrating the notion of “legitimate peripheral participation” (Lave & Wenger, 1991) with the analytical frameworks of production design and recipient design, Krummheuer (2011, p. 88) emphasizes the critical role of participation in learning processes—both in classroom and home environments. This conceptual synthesis highlights how learners’ roles and degrees of engagement shift dynamically through interaction, enabling both individual and collective meaning-making. Building on distinctions between recipient roles and speaking person participation, Krummheuer illustrates how empirical participation categories (Tables 1 and 2) can be interpreted through broader theoretical lenses. While roles like co-hearer or relayer align with the concept of legitimate peripheral participation, others like conversation partner or author correspond to full participation. This interplay between theoretically grounded and empirically observed roles allows for a more nuanced understanding of how participation unfolds in situated learning contexts.

Building on this background, this paper aims to delve into the functions and modalities through which family members engage in the negotiation process. The analysis focuses on the negotiating taken-as-shared meanings in the interaction and emphasizes the application of recipient design in the negotiation of meanings. Under this focus, the production design is used to reconstruct the individual contributions and responsibilities for the content of the interaction.

The Block Play Activity of the Gül Family


Family Gül is a German-Turkish family who lives in a major German city. Berk is the focus child who is aged 7 years and 1 month old and has an elder brother, who is about 13 years old. The elder brother goes to a secondary school in Germany. Both are fluent in German and Turkish. The child’s grandmother, the mother of the child’s father, has a close relationship with Family Gül and cares for Berk and his elder brother when they return from school. Her education consisted of 5 years in the Turkish school system. She speaks Turkish and rudimentary German.

Transcription of the Selected Scene

The transcript comes from a play situation, which focuses on geometry and spatial thinking. The aim of the task is for the family to build a three-dimensional version from a picture of a design built with wooden blocks, which all had the same size and weight. It was anticipated that the family would relate the two-dimensional representations to the three-dimensional representation that they were building. One player chooses a card from the deck, placed face down on the table, and then reproduces the building on the card.

In the transcript, Berk is playing with his elder brother and grandmother, referred to as granny. They take turns playing across ten rounds. Throughout the interaction,

they speak both German and Turkish, frequently code-switching between the two languages. In the transcript below, speech originally spoken in German is written in normal font, while speech originally spoken in Turkish is underlined>. In this situation, wooden blocks of identical size and color are used. In the transcript¹ extract, the sides of the blocks are labeled as X, Y, and Z (see Fig. 1). The extract comes from the first round, specifically granny's turn. The blocks used by granny are labeled as K1, K2, K3, and K4, while the blocks handled by the elder brother are labeled as BK1, BK2, BK3, and BK4.

S1	S2	S3	S4
01		Granny	<i>picks up a card from the deck, looks at it about 1 sec. by holding it with both</i>
02			<i>hands, sets it in front of her on the table, takes first block K1 from the pile</i>
03	>		<i>of blocks and places it on its Y side on the table</i> 
04	>	Brother	<i>pushes all blocks from the pile on the table toward his Granny with both hands, then</i>
05			<i>places his arms on the table resting them with his elbows bent while facing the table</i>
06		Granny	<i>takes K2 from the pile of blocks and sets it in a same way parallel to the right</i>
07	<		<i>side of K1. takes K3, places it on its X side horizontally on K1 and K2</i> 
08	<	Berk	<i>places both hands on the table, slightly pushes himself up, looks around, then</i>
09	>		<i>turns to the side facing Granny and his brother, lightly leans forward facing</i>
10			<i>Granny and his brother.</i>
11	>	Granny	<i>takes K4 and then sets it on its Z side vertically on K3</i> 
12			<i>looks at Berk's brother <u>like this/</u></i>
13		Brother	<i>shakes his head from left to right while facing the table</i>

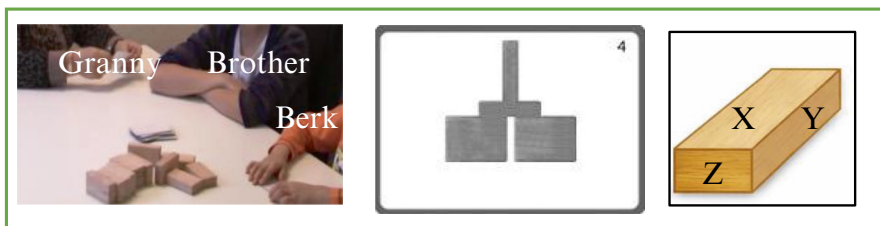
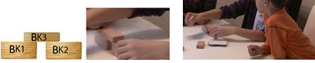




Fig. 1 Recording position, the chosen card, and sizes of a block

¹Column 1 (S1) contains continuous line numbers in two-digit format. Column 2 (S2) indicates simultaneity or overlap between utterances using the symbols ">" and "<". Column 3 (S3) identifies the speaker or actor (agent) responsible for the action or utterance. Column 4 (S4) includes all verbal (regular) and non-verbal (italic) contributions as well as paralinguistic features.

S1	S2	S3	S4
14		Granny	No/looks down at the card
15		Brother	shakes his head from left to right while facing the table and grinning
16		Granny	how/ (make it-) turns the card slides it in front of Berk's brother with her right hand while leaning forward
17			
18	#	Berk	leans forward with his arms resting on the table, positioned slightly closer to his brother, facing Granny and his brother
19			
20	#	Brother	dismantles Granny's construction, randomly takes two blocks(BK1, BK2)from it, places them next to each other on their Y sides, places the third block(BK3) on top of the Y sides of BK1 and BK2 while holding K1 with his left hand on the table, BK3
21			
22			
23			falls down, smirks, picks it up and places it back on them again, still holding same
			
24	<		takes BK4 and sets it on its Z side vertically on K3
			
25	<	Granny	wow\ while facing the table
26			oh\ looks at it, slightly turns the card in front of her with the fingertips of her right hand while keeping her eyes on it
27			
28	>	Brother	slides it in front of him with his left hand on the table
29	>	Berk	keeps both hands around the tower his brother built, cupping all the blocks with his hands, removes them to the pile on the table
30	<		
31	<	Granny	Now is Berk
32		Berk	says <u>Brother</u> -casting a brief glance at Granny while continuing to move the blocks to the pile
33			

Participation Analysis: Recipient and Production² Design of Block Play Interaction

The extract began with granny picking up a card from the deck and starting to build immediately (see Fig. 1). While the elder Brother pushes all the blocks from the pile toward granny, she sequentially places Block K1 and then Block K2 on their Y

² For a detailed Production Design of the selected scene, see also Acar Bayraktar and Brandt (2024).

sides, next to and parallel to each other (see Fig. 2) <01–06>.³ Then she puts one block (K3) horizontally on the initial two blocks. After this, she places the final block (K4) vertically upon GK3 <06–07> (see Fig. 2). From a production design perspective, she looks like an *author*⁴ in that she performs her own idea about reproducing the building. She uses wooden blocks as “object actors” to present her idea through concrete actions (see also Fetzer & Tiedemann, 2018).

Meanwhile, Berk places his hands on the table, pushes himself up slightly, and looks around. Immediately afterward, he leans forward slightly while facing granny and his brother <08–09>. His reaction suggests that he is becoming increasingly interested in granny’s building activity and he presents himself as an attentive recipient through his posture.

Afterwards granny asks whether the building construction is like this <12>. Most probably she was asking if the construction was correct. A possibility for her reaction is that she tried to initiate discussion about the building and implicitly attempted to open up opportunities for Berk or his brother to contribute their own ideas within the ongoing interaction. In doing so, she may have been gently prompting both children to think again about the incorrectness of her construction, which might expand their possibilities for participation. If this is the case, then she regarded Berk and his brother as her *conversation partners*, from a recipient design perspective.

In the figure on the card, there are two block-parts (a, b) which can be seen in Fig. 2. Furthermore, the transitions between the blocks in the image on the chosen card have been eliminated, causing the individual blocks to appear visually merged, which makes it difficult to determine how many blocks are required and how they should be arranged to replicate the tower depicted on the card. In granny’s building there are two blocks (K1, K2) (see Fig. 2), those parts seem smaller than block-parts of a and b. Thus, the building resembles but is not identical to the figure on the card. To construct a nonidentical but similar building to that on the card could be because of either a ‘static balance’ problem (1) in which granny focused only on ensuring

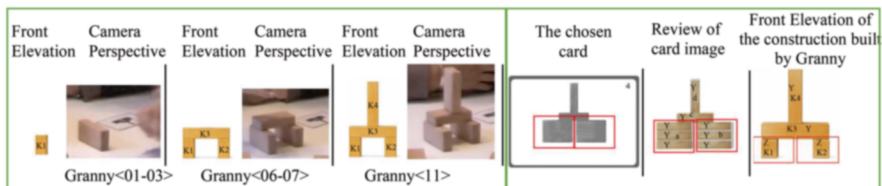


Fig. 2 Building steps of granny and review of building constructions

³In the analysis of the transcript extract, notations such as <01> are used to indicate the corresponding line numbers.

⁴Key concepts discussed in the theoretical section above are *italicized* during the conduct of the participation analysis to emphasize their relevance throughout the analysis.

that the construction was stable; or a ‘gestalt’ problem (2) in which she saw only the whole and not the individual parts (see Fig. 2). Topologically she might have had either an adaptation problem by decomposing and then composing shapes or a spatial problem by regulating spatial relations of the objects and figures. Maybe she realized the difference between the card and the construction, but she focuses only on stability of the construction. Maybe she tried to offer Berk and his brother the possibility to recognize the essential fault in the construction and solve it without any assistance.

Berk’s brother shakes his head from left to right while facing the table <13>. His reaction can be interpreted as a way of showing his disagreement of either granny’s construction or to the granny’s geometric understandings. His spatial understanding is shown through gestures. Without any verbal comments, his reaction provided insights about his idea of the incorrectness of the construction. In this sense, the brother took the role of an autonomous person, who expressed his ideas and had the freedom to act independently (Krummheuer & Brandt, 2001). From a recipient and production design perspective, he acted as an *author*, while reserving the role *conversation partner* for the granny.

Granny asked whether it was a “no” and looked down at the card again <14>. She seemed to want to get clear feedback from the brother. It is possible that she intended to ask whether he meant “no”, implying that her construction might not have been correct. Another possibility is that she was asking whether he was saying “no” in response to her question. By asking for disapproval, she signaled that she was cooperative and wanted to have input from Berk and his brother and maintain the negotiation with them. Moreover, by asking for disapproval, she reinforced her suggestion, that Berk or his brother could offer their ideas and take the role of *author*. The interaction process provided an opportunity for expanding the possibility for participation to all participants.

Again, Berk’s brother shakes his head from left to right while grinning <15>, repeating his disagreement with the correctness of the construction by using gestures instead of verbal statements. He expressed the same idea in a similar way. In this sense he acts as a *relayer*.

This time, granny asks for a clarification, while sliding the chosen card in front of Berk’s brother <16–17>. It may be that she hoped that the brother would give her detailed information about the correct construction. From a recipient and production design perspective, Granny’s gesture of sliding the card toward Berk’s brother can be interpreted as a concrete form of interactional framing: she—acting as an author—not only initiates a shift in the participation structure, but also designs the interaction in a way that allocates specific roles and responsibilities to the participants. She appears to assign the brother the role of advisor. At the same time, Berk is positioned more peripherally, perhaps as a *ghostee* or even observer, with the possibility to appropriate the knowledge implicitly. She seemed to hope that the brother can solve the task. In this way, she offered Berk the opportunity to learn about how to build an identical structure to the one on the card. The subsequent actions of both children support this interactional arrangement, validating Granny’s recipient design.

Berk leant forward with his arms resting on the table, positioned slightly closer to his brother, and facing both granny and his brother <18–19>. His posture suggested that he was becoming increasingly interested in granny’s question and his brother’s response. Meanwhile, without offering any verbal explanation, Berk’s brother knocks over the completed construction and immediately begins rebuilding with the same blocks. He sets two blocks next to each other on their Y sides and puts on their Y sides another block (BK3) <20–23> (see Fig. 3). He takes one more block (BK4) from the pile of blocks and sets it (BK4) vertically upon BK3 <24> (see Fig. 3). In this sense, he offered his hypothesis and demonstrated his own idea through concrete actions, without any accompanying verbal statements. Thereby, he expressed his assertions through spatial and structural arrangements of the building blocks. From a production design perspective, he acted as an author who constructed the contribution in full, took responsibility for his action—including how it was expressed (formulation) and the underlying idea (content)—and was free to choose and combine different modes of expression. At the same time, the wooden blocks became *object actors*. They in connection with his actions helped Berk’s brother to clarify his idea through shared meanings (see also Fetzer & Tiedemann, 2018).

When the first construction that granny built and the second construction that the Berk’s brother built (Fig. 3) are compared from the front view, their differences show that they both represent rectangles, but the rectangle of the granny’s construction is smaller than the older brother’s construction (in Fig. 4 outlined both with red). Furthermore, the chosen card and the built construction are reviewed by naming and analyzing blocks and their sides (X; Y; Z) from the front view. It is clear that the Z side of the blocks is four times smaller than the X side and two times smaller than the Y sides, whereas the X side is two times broader than the Y sides (see Fig. 4). The image on the card shows that there are two rectangles (a, b) and in each there are exactly three blocks, which are set on their X sides and the front view comprises of three times the Y sides (see Fig. 4).

Keeping in mind that the number of blocks in the built tower does not matter, Berk’s brother appeared to construct a tower identical to the image on the card.

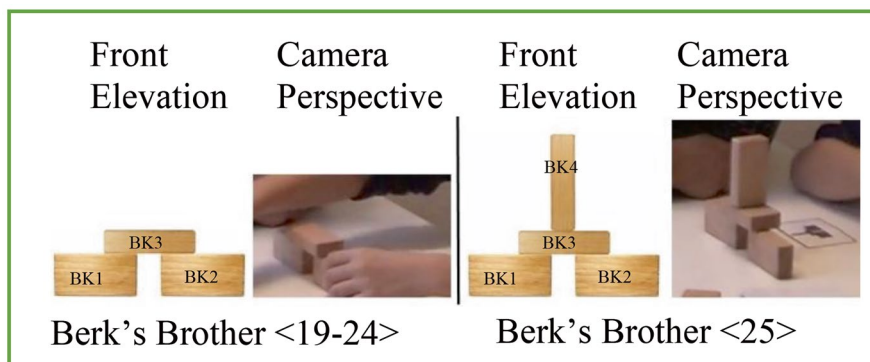


Fig. 3 Building steps of the elder brother

Although the transitions between the blocks in the image on the chosen card had been eliminated—causing the individual blocks to appear visually merged—the front view of the tower did match the image.

“Wow” granny exclaims <25>. She seems impressed. “Oh” she murmured, turning the card slightly with her fingertips while keeping her eyes fixed on the blocks <26–27>. Saying “wow” and “oh” indicated admiration and approval for the construction built by Berk’s brother. In so doing, she confirmed his hypothesis and agreed that he had reproduced the image correctly.

While Berk’s brother slid the card in front of him with his left hand on the table, Berk placed his hands around the tower, cupping all the blocks protectively <28–29>. He carefully removed them and placed them back in the pile on the table, resetting the scene <30>. In doing so, he appeared to signal agreement with his brother’s construction being correct. From a recipient and production design perspective, Berk actively engages by dismantling his brother’s construction. He became an active participator in the content-related negotiation process. Beyond agreeing with his brother, his action marked the end of granny’s turn so that a new turn could begin. Berk made this content-related contribution gesturally, by taking up the role of an *author*. Up to this point, Berk could be considered a *legitimate peripheral participant*, primarily observing and witnessing the situated activities of his elder brother and grandmother. However, by dismantling and reorganizing the blocks, Berk transitioned into a more active role within the interaction. The negotiation process between his grandmother and brother seemed to guide Berk to accept that his brother’s construction was correct. This shift highlighted Berk’s progression through different participation profiles. Initially, Berk participated as a *co-hearer*, absorbing the *disputational* and *cumulative* negotiation processes between his grandmother and brother as an attentive recipient, which could be seen in his body language. The verbal and nonverbal exchanges between granny, the older brother and the blocks, with or without embodied expressions create a co-constructive space, enabling Berk to join the interaction smoothly. By physically reorganizing the blocks, Berk took on the role of *full participant*, contributing meaningfully and actively to the process. In this regard, the different negotiation styles of the grandmother and the elder brother contributed to Berk’s participation, fostering his gradual shift from peripheral observation to active engagement.

Granny said “now is Berk” <31>. Most probably she means, “now it’s Berk’s turn”. Granny’s verbal reaction not only signified the conclusion of her turn but facilitated Berk’s seamless integration into the game. Following Berk’s active

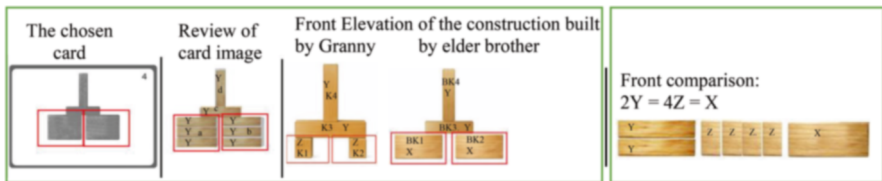


Fig. 4 Reviews of built constructions, wooden blocks, and the chosen card

engagement in removing the blocks, granny explicitly acknowledged his involvement by verbally recognising Berk's action, which ended her turn, as a *ghostee*. By addressing Berk directly, granny not only signaled a change of role, but also reinforced his transition from a *legitimate peripheral participant* to a *full participant*.

Berk responded, glancing briefly at granny and saying, "Brother," while continuing to move the blocks to the pile <32–33>. Although this response reinforced the end of granny's turn, it is multifaceted. While acknowledging the conclusion of granny's role in the interaction, Berk simultaneously reaffirms his brother's authority over the construction process. His verbal and non-verbal contributions demonstrated an understanding of the negotiation dynamics and a willingness to engage within the boundaries of the roles previously established by his grandmother and brother. By continuing to move the blocks, Berk consolidated his position as a *full participant* while still aligning himself with the framework set by his elder brother. Berk's response, "Brother," might reflect his recognition of the turn-taking rules and that it was his elder brother's turn rather than his own. Consequently, Berk demonstrated an awareness of and adherence to the superordinate play rules. This attention to structure and turn-taking indicates that Berk was not merely observing but was actively engaged in understanding and maintaining the order of the activity. Berk's adherence to these implicit rules reveals a nuanced awareness of the social and procedural dynamics within the interaction, further solidifying his role as an active participant.

Summary of the Interpretation with Regard to Types of Negotiation Processes

The analysis of the block-play interaction illustrates how negotiation and participation foster fluid shifts in roles and responsibilities within family dynamics. Verbal, non-verbal, and material actions—such as confirmations, elaborations, and spatial arrangements—emerge as key tools for managing interaction, negotiating meaning, and maintaining relational harmony.

Different negotiation styles emerged between family members. While the brother expressed disagreement and decision-making through the structural use of blocks, Granny guided the process in a cumulative and affirming manner. Through confirmations, elaborations, and physical contributions, she fostered agreement without critique. Together, they created an *exploratory* negotiation process for Berk, shaped by their spatial arrangements and construction strategies, enabling his learning through observation and interaction. Their collaboration culminated in a jointly constructed working consensus on how to build the correct structure, exemplifying the co-construction of taken-as-shared meanings. This dynamic is further illustrated in Fig. 5, which provides an overview of the negotiation process.

Throughout the process, Granny and the brother formed an asymmetrical but complementary partnership—balancing *disputational* and *cumulative* negotiation

styles leading the negotiation process in an *exploratory* direction for Berk. Granny also implicitly created space for Berk's engagement by assigning responsibility to him and his brother to contribute their own ideas. Their cooperative negotiation allowed for verbal and non-verbal coordination, adjustment of intentions, material-based reasoning and co-constructing meanings, all of which supported Berk's emerging spatial understanding and enabled opportunities for individual learning.

Initially participating as a *co-hearer* and an *attentive recipient*, Berk was given the opportunity to observe both actors and gradually transition toward more active involvement. By attending to his brother's construction activities, material actions, and use of blocks as *object actors*, he was able to develop emerging spatial awareness and geometric insight. His shift from *co-hearer* to active contributor—taking apart and reconfiguring the blocks—reflected a movement from *legitimate peripheral participation* toward *full participation*. Though minimal, this contribution aligns with the notion of *author*, as he assumed responsibility for both formulation and content. This progression reflects Lave and Wenger's (1991) theory of learning through legitimate peripheral participation, in which learners gradually increase their engagement in social practices. Ultimately, the negotiation process unfolded through distinct yet complementary interactional styles, enabling shared meaning-making and fostering opportunities for situated learning. Figure 6 visually summarizes the co-constructive negotiation process, illustrating how complementary negotiation styles—*disputational* and *cumulative*—converged to support Berk's participation in an exploratory manner.

Conclusion

This study highlights how collaborative construction tasks in family contexts are shaped by diverse forms of participation, as conceptualized through recipient and production design. Participants engaged in multimodal communication—relying not only on verbal interaction, but also on gestures and material actions—to express geometric reasoning and coordinate joint activity. In this interactional setting, wooden blocks functioned as *object actors* (Fetzer & Tiedemann, 2018), mediating shared meaning-making through concrete material-related actions.

The dynamic negotiation of interactional roles was central to this process. Although explicit verbal dialogue was limited, the participants mutually positioned one another as *conversation partners*. At the same time, Berk—initially not directly involved as a conversation partner—occupied a peripheral role as a *co-hearer* and *attentive recipient*, thus entering the process through legitimate peripheral participation (Lave & Wenger, 1991), in which learning occurs through observation and increasing levels of engagement within practice.

The dynamics of family relationships shaped the interaction and lead to different negotiation processes depending on the perspective of the participants, so that a *cumulative* and a *disputational* mode of negotiation emerge simultaneously. These styles do not function in isolation, but rather complement each other, creating a rich

Granny builds a construction <1-11>.		
Granny	concrete material actions	→ author
Berk + Brother		→ her conversation partners
Berk turns to the side facing Granny and his brother and leans forward with his arms resting on the table <08-10>.		
Berk		→ attentive recipient
Granny asks whether the building construction is like this <12>.		
Granny	concrete material actions	→ author
Berk + Brother		→ her conversation partners
Berk's brother shakes his head from left to right while facing the table <13>.		
Brother	gestural	→ author
Granny		→ his conversation partner
Berk		→ co-hearer
Berk		→ attentive recipient
Granny asks him if it is no while looking down at the card <14>.		
Granny	verbal	→ ghostee
Brother		→ her conversation partner
Berk		→ co-hearer
Berk		→ attentive recipient
Berk's brother shakes his head from left to right while facing the table and grinning <15>.		
Brother	gestural	→ relayer
Brother		→ her conversation partner
Berk		→ co-hearer
Berk		→ attentive recipient
Granny responds with "How" and tells him to make it, leaning forward, slides the card toward Berk's brother with her right hand <16-17>.		
Granny	verbal + gestural	→ author
Brother		→ her conversation partner
Berk		→ co-hearer
Berk		→ attentive recipient
Berk leans forward with his arms resting on the table, positioned slightly closer to his brother, facing Granny and his brother <18-19>.		
Berk		→ attentive recipient
Without offering any verbal explanation, Berk's brother knocks over the completed construction and immediately begins rebuilding with the same blocks <19-24>.		
Brother	concrete material actions	→ author
Granny		→ his conversation partner
Berk		→ co-hearer
Berk		→ attentive recipient
Granny reacts "Wow", then says "Oh" looks at the card, slightly turns the card in front of her with the fingertips of her right hand while keeping her eyes on it <25-27>.		
Granny	verbal + gestural	→ author
Brother		→ her conversation partner
Berk		→ co-hearer
Berk		→ attentive recipient
Berk keeps both hands around the tower his brother built, cupping all the blocks with his hands, removes them and places them back in the pile on the table <29-30>.		
Berk	gestural + concrete material actions	→ author
Brother + Granny		→ his conversation partners
Granny says "Now is Berk" <31>.		
Granny	verbal + gestural	→ author
Brother + Berk		→ her conversation partners
Berk says "Brother-" casting a brief glance at Granny while continuing to move the blocks to the pile on the table <32-33>.		
Berk	gestural + concrete material actions	→ author
Brother + Granny		→ his conversation partners

Fig. 5 An overview of the collective familial negotiation process

learning environment that other participants—as attentive recipients—can perceive as a negotiation process in an *exploratory* way and gradually transition from peripheral observation to more autonomous participation.

The analysis demonstrates that negotiation processes are not monolithic but composed of layered, interwoven modes of engagement. It is important to note that collective negotiation processes reflect a dynamic interplay between *explorative*, *cumulative* and *disputative* elements in the construction of meanings that are

Brother & Granny → conversation partners				
Berk	<i>legitimate peripheral participant</i> co-hearer / attentive recipient		→	<i>full participant</i> conversation partner
Brother	disagreement →	disputationally	→	Berk agreement → exploratorily
Granny	agreement →	cumulatively		

Fig. 6 The co-constructive negotiation process

assumed to be shared. In negotiations in a *disputational* way, family members often present opposing arguments, revise intentions and adjust interpretations in order to convince each other. At the same time, *cumulative* processes foster trust and enable a critical discussion of the topic and the exchange of theses and antitheses—which occurs in a *diputational* way—without disturbing the harmony in familial interaction. Importantly, these negotiation forms coexisted within a harmonious collaborative framework, enabling both shared understanding and the emergence of individual learning trajectories through observation, experimentation, and progressive participation.

Future Research and Limitation

Different negotiation styles jointly construct shared meanings that not only support collective sense-making but also stimulate individual learning. By fostering participation in such processes, family environments contribute to intellectual growth and illustrate how the interplay of negotiation can generate meaningful learning opportunities. While this study provides insight into the dynamics of participation and negotiation within one specific family interaction, its context-bound nature limits generalizability. Future research should explore more diverse cultural and social contexts to further examine how such processes unfold across settings.

References

Acar Bayraktar, E. (2017). *Interactional niche in the development of geometrical and spatial thinking in the familial context* [Doctoral dissertation, Johann Wolfgang Goethe University, Frankfurt am Main]. Published online in 2023. Goethe University Publication Server. <https://doi.org/10.21248/gups.74492>.

Acar Bayraktar, E., & Brandt, B. (2023). How do we make an E? Participation theory and theoretical consideration of supportive activities on block play in the family. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 2022–2029). Alfréd Rényi Institute of Mathematics and ERME. hal-04422693.

- Acar Bayraktar, E., & Brandt, B. (2024). Production design of an interaction process between grandmother and grandchildren in a block play. In *Proceedings of the 15th International Congress on Mathematics Education (ICME-15)*. Sydney, Australia, 7–14 July 2024.
- Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. Prentice Hall.
- Brandt, B. (2017). Spiel-Räume der Partizipation. Zur situationalen, inhaltspezifischen Ausgestaltung mathematischer Spiel- und Erkundungssituationen. In R. Vogel & M. Beck (Eds.), *Geometrische Aktivitäten und Gespräche von Kindern im Blick qualitativen Forschens–Mehrperspektivische Ergebnisse aus den Projekten erStMaL und MaKreKi* (pp. 107–131). Waxmann.
- Bronfenbrenner, U. (1979). *The ecology of human development: Experiments by nature and design*. Harvard University Press. <https://doi.org/10.2307/j.ctv26071r6>
- Cobb, P., & Bauersfeld, H. (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Lawrence Erlbaum.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3–4), 175–190.
- Fernández, M., Wegerif, R., Mercer, N., & Rojas-Drummond, S. (2001). Re-conceptualizing “scaffolding” and the zone of proximal development in the context of symmetrical collaborative learning. *Journal of Classroom Interaction*, 36(2), 40–54.
- Fetzer, M., & Tiedemann, K. (2018). The interplay of language and objects in the process of abstracting. In J. Moschkovich, D. Wagner, A. Bose, J. Rodrigues, & M. Schütte (Eds.), *Language and communication in mathematics education: International perspectives* (pp. 91–104). Springer.
- Goffman, E. (1981). *Forms of talk*. University of Philadelphia Press.
- Krummheuer, G. (1992). *Lernen mit “Format”: Elemente einer interaktionistischen Lerntheorie*. Deutscher Studien Verlag.
- Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning (Interaction in classroom cultures)* (pp. 229–269). Lawrence Erlbaum.
- Krummheuer, G. (2011). Representation of the notion “learning-as-participation” in everyday situations of mathematics classes. *ZDM Mathematics Education*, 43, 81–90.
- Krummheuer, G., & Brandt, B. (2001). *Paraphrase und Traduktion. Partizipationstheoretische Elemente einer Interaktionstheorie des Mathematiklernens in der Grundschule*. Beltz Wissenschaft Deutsche Studien Verlag.
- Latour, B. (2005). *Reassembling the social. An introduction to actor-network theory*. Oxford University Press.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.
- Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners*. Multilingual Matters.
- Mercer, N. (2000). *Words and minds: How we use language to think together*. Routledge. <https://doi.org/10.4324/9780203464984>
- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46, 13–57. <https://doi.org/10.1023/A:1014097416157>
- Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

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The Construction of the Natural Number's Concept in Kindergarten: The “Go!” Game



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Introduction

Natural number is present in daily activities and children's lives. From the early age, they memorably and repetitively count small quantities and express various situations (age, date of birth, family members, measurements, orderings, etc.). Numbers represent one of the fundamental pillars of mathematics, as well as one of its main topics, as highlighted in the Italian National Guidelines for kindergarten and the first cycle of school (MIUR, 2012). The concept of natural numbers is highlighted as being a very sophisticated and complex concept (Mann, 2014; NCTM, 2000). In the Italian National Guidelines, it is suggested that it should be addressed and built up gradually in a vertical perspective from the early years of school and preschool to grade 13 (MIUR, 2012, p. 49).

It often happens that the need to count and use natural numbers by memorizing the numerical sequence is not accompanied by equally in-depth work on the understanding of numbers. This may be a more frequent situation for children who experience difficulties, as they may find comfort and security in reproductive and schematic learning, rather than more complex learning. The consequence of these learning choices means that they risk exclusion from more meaningful learning experiences. Our didactic assumption is that the process of formalizing leading so that it characterizes natural number, through understanding three aspects, cardinal, ordinal and recursive, can be achieved through the mediation of suitable artefacts.

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This can be done in accordance with a specific teaching and learning model, based on the framework of the Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008). The TSM provides a reference for the design and the implementation of a teaching sequence, as well as the analysis of the data coming from the experiment. The main aim of the experimental educational path described in this chapter is to support children's development of their sense of natural numbers, using games and artefacts, as outlined in the TSM.

In particular, games are selected as a means to approach the discovery and construction of new mathematical concepts, while artefacts are chosen based on the problem situations presented to the children, in accordance with the principles of semiotic mediation theory. The analysis of the transcriptions of children's talking during the game reveals the evolution of the concept of one-to-one correspondence, the equipotency between sets (thus an initial idea of cardinal numbers), the representation of numbers, and their ordering. Additionally, in the final phase of the game, comparing the cardinality of plastic bags and ordering them based on their cardinality evokes the idea of a number's successor through the +1 operator. These ideas lay the foundation for building the ordinal, cardinal, and recursive aspects of natural numbers.

The cardinal aspect of Natural number refers to the idea of biunivocal correspondence between elements belonging to two equipotent sets (Piaget, 1997). In mathematical literature (Kunen, 2011), the term equipotent refers to sets that have the same cardinality, meaning there exists a bijective function between them. Two sets A and B are equipotent if there is a one-to-one correspondence (bijection) between their elements. The ordinal aspect of Natural number concerns the comparison of cardinal numbers establishing which set contains the greatest number of elements. This aspect focuses on sorting, gradually leading to the construction of the number line (Piaget, 1997). The recursive aspect of the Natural number, stated in Peano's axioms, fixes a starting point by assuming zero as the first Natural number, and constructs the next number of n , through adding 1 to any number.

In summary, the cardinal aspect allows the elements of a set to be counted; the ordinal aspect allows them to be reordered, according to an ascending or descending order; the recursive aspect avoids "jumps" in the Natural number's sequence (Butterworth, 1999). This idea is linked to the construct "number sense" that is seen as essential for learning arithmetic. Many researchers also believe that the development of number sense is a fundamental skill for acquiring awareness of the structural aspects of number and for promoting pre-algebraic thinking (Fuson, 2012; Griffin et al., 1994).

The educational path we have designed, identified different phases related to each aspect of Natural numbers (Usmanov, 2020). The design, implementation and analysis of results in the educational path are based on the theoretical framework of TSM. This framework provides an important reference for studying the relationship between artefacts, the actions they enable, and the construction of mathematical concepts by the learners who use them. Finally, the analysis focuses on the semiotic potential of the artefact used.

Theoretical Framework

From a Vygotskian perspective, the Theory of Semiotic Mediation (TSM) considers the complex system of semiotic relations between the elements involved in the construction of mathematical meanings using artefacts: the artefact, the task in which the artefact is used, the mathematical knowledge that is the objective of the didactic intervention, and the teaching/learning processes taking place in the classroom.

The notion of artefact, utilized in this chapter, is consistent with the distinction between artefact and instrument, introduced by Rabardel (Bartolini Bussi & Mariotti, 2008), and identifies an artefact with any device conceived and realized by human beings for any purpose. The distinction between artefact and instrument aims at distinguishing between the device and the use that any person can conceive of for such a device to fulfil a specific purpose. The combination of different schema that might be conceived for the use of the artefact will constitute a new entity, namely an instrument. The notion of artefact and that the schema of utilization (Rabardel, 1995) is at the core of the TSM's construct of semiotic potential, which is the twofold relationship that the artefact has with the personal meanings emerging from its use and the mathematical meanings that might be evoked by such use.

on the one hand, personal meanings are related to the use of the artefact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artefact and its use. This double semiotic relationship will be named the semiotic potential of an artefact. (Bartolini Bussi & Mariotti, 2008, p. 754)

The analysis of the semiotic potential will describe what is expected to emerge in the classroom, both actions accomplished, and signs produced by the children and their relationship with the mathematical meanings in focus. For this reason, TSM is at the core of the design of any teaching sequence, and it is the basis for the analysis of any experimental realization of a didactic intervention. Meanings related to the use of the artefact can be referred to the schema of utilization that are displayed by the children with respect to a specific task. Therefore, the design of the tasks develops on the base of a fine grain a priori analysis of the solution processes and specifically on the identification of the schema of utilization that are expected.

According to the TSM, personal meanings emerging from the activities carried out with an artefact may evolve into mathematical meanings that constitute the objective of the teaching intervention. Such an evolution can occur, fostered by specific semiotic activities in the pair interaction during the accomplishment of the task and in the collective discussions, orchestrated by the expert guide of the teacher. The collective construction of shared mathematical meanings is a complex and long process, where it is possible to identify evolution paths (semiotic chains) described by the appearance and connection of different types of signs: artefact signs, mathematical signs and pivot signs. The artefact signs are characterized by being highly subjective and clearly related to the specific experience with the artefact, while the mathematical signs constitute the goal of the teaching /learning process and are related to the mathematical meanings as shared in the institution. Pivot signs with their hybrid nature, both referring to the use of the artefact and to the mathematical

domain, are characterized by their function in the evolution process, fostering through their intrinsic ambiguity the move from artefact to mathematical signs (Bartolini Bussi & Mariotti, 2008; Faggiano et al., 2018; Montone et al., 2019).

Finally, the notion of didactic cycle constitutes the unit of design. The didactic cycle organizes the coordination between activities with the artefact and semiotic activities so that the expected evolution of signs is achieved. According to this structure, the description of the sequence will be framed by an iteration of didactic cycles. These educational assumptions align with the use of laboratory-based methodologies in teaching, as the National Guidelines (Miur, 2012) emphasize the importance of developing refined and complex concepts starting from preschool. They also provide methodological suggestions for gradually and cyclically approaching abstract concepts.

In particular, the document recommends (Miur, 2012, p. 27):

Implementing educational activities in a laboratory format to encourage hands-on experiences while also fostering dialogue and reflection on what is being done. A well-organized laboratory is the most effective approach to encourage inquiry and planning, engage students in thinking, creating, and evaluating shared activities, and can be implemented both within different spaces and opportunities inside the school as well as by leveraging the surrounding environment as a learning resource.

Furthermore, it states, “Experimentation, manipulation, play, and storytelling are privileged opportunities for learning through practice, which later becomes the foundation for more advanced theoretical and experimental knowledge” (Miur, 2012, p. 7).

The entire approach is structured as a problem-solving/game-based activity that fosters experimentation, creativity, discussion, analysis, and interpretation of situations within a meaningful context related to the task and the knowledge to be constructed. Problem-solving, when embedded in real-world scenarios, mirrors the typical procedures used by experts in the field. This enables students to develop awareness and build disciplinary concepts while adopting the role of experts (Dunlap, 1996). The game was the primary tool through which children constructed mathematical concepts using the artefact. At this age, play is “serious business”, it represents a “problem” to be tackled and solved for the sake of enjoyment. Moreover, play plays a crucial role in motivation and inclusion, as it is an inherently cooperative activity (Butterworth, 1999). During play, dialectical confrontation is encouraged, making it easier for children to accept failure (as they play for the sake of playing, not to win) and to engage with others’ ideas. While playing, children do not feel judged by the teacher, but feel free to justify their choices, express their opinions, and compare their ideas with those of their peers (D’Amore & Sbaragli, 2011). Thus, play regulates self-discipline, which is essential for the success of the activity. It fosters social intelligence and serves as a catalyst for learning, facilitating devolution—the process through which children become active protagonists in their own learning, taking conscious responsibility for their actions and decisions (D’Amore & Marazzani, 2011). Play has a highly positive social value (Di Paola et al., 2020) while simultaneously fostering a positive attitude toward mathematics, as prescribed by the National Guidelines. Since play is the primary methodology underlying the

proposed teaching approach, the entire design of this experimental research is based on the Theory of Semiotic Mediation (TSM) for the planning, implementation, and analysis of the activities presented in this article (Fiorentino et al., 2023).

With these assumptions, our research questions are:

- Does the artefact, related to a specific given problem concerning natural numbers, involving a game, support children's understanding of the concept of one-to-one correspondence, the equipotency between sets, the representation of numbers, and their ordering emerge, also in kindergarten?
- Is it possible to highlight, only with the children's verbal contributions and gestures in kindergarten, the evolution of personal meanings about elementary mathematical concepts?

Research Methodology

In accordance with TSM, a teaching sequence was designed following the general scheme of successive didactic cycles. Specifically, the structure of the iteration of the Didactic Cycle is constituted by working on the previously designed task with an artefact, by producing individual signs evoked by using the artefacts and finally in a mathematical discussion collective signs are produced.

The experimental activity follows the design phase, with the overall educational path being carried out with a group composed of 12 children. In this study we have focused on the observation and analysis of the development of the semiotic potential of the three artefacts and in this chapter, we present the results related to the use of one of them, specifically the "Go!" game.

The collected data from the teaching experiments, focusing on the unfolding of the semiotic potential of the artefact involved in the "Go!" game, included videotapes and transcriptions. Activities were videotaped using two cameras, a fixed one facing the students and a second one focused on the artefact in use; conversations were transcribed, which also identified the specific actions taken with the artefacts. The choice of artefacts, the activity design and analysis of results has been undertaken by all the authors. The experimental activity in the class was conducted by one of the authors (Ricciardiello), while the other three authors were observers in the class.

The Experimental Activity

To validate the effectiveness of the designed educational path a research study was conducted in three different kindergarten schools with a total of 45 children coming from different Italian cities. In this paper, we analyse some excerpts, which provide

evidence of the evolution of signs, related to this experimental research. The steps for the analysis were:

First step: The “Go!” game is introduced through proposing a situation problem. It is first suggested to create spare pencil cases to keep on hand in the classroom, to help students that forget their pencil cases at home. The teacher takes the cardboard case containing the colours lost by the students and simulates an accident. She accidentally drops the briefcase and the colours spill on the floor. A transparent plastic bag is distributed to each student and the first task is suggested:

TASK 1: Collect and group all the same coloured pencils in a transparent plastic bag. The randomness of the situation means that not all colours are present (e.g., red and some other colours are missing). Furthermore, not all colours are present in the same amount (e.g., there are 5 yellow pencils, 4 green, 8 brown, and so on). After collecting colours, each student has a plastic bag of the same colour; there are 15 empty boxes in the back of the classroom, which will later be used to collect equipotent bags. A game leader is identified, and the game begins. At this moment the teacher gives two tasks, one for the game leader and another one for the whole class:

TASK 2a (for the game leader): Remove one coloured pencil at a time from your plastic bag and say “Go!” each time. Repeat the action until your plastic bag is empty. When this happens, exclaim “Stop!”

TASK 2b (for the whole class): Each time the game leader says “Go!” remove a coloured pencil from your plastic bag and put it on the desk, until your plastic bag is empty. When this happens, exclaim “Stop!”

The leader starts the game. They say, “Go!” and takes out a coloured pencil and places it on the desk. At the same time, all the other students perform the same action. In relation to Tasks 2a and 2b, the semiotic potential of the artefact emerges. The action of simultaneously taking out a coloured pencil from each plastic bag at the command “go!” evokes one-to-one correspondence between the coloured pencils in the bags; through the simultaneous call of “Stop!” by several players and the putting the coloured pencils back into their own plastic bag, which created bags with the same numerosity. These actions evoke the idea of correspondence between the bags, that is mathematically the correspondence between the sets, namely, the idea of equipotency. After a few calls of “Go!”, it may happen that, due to the way the game is structured, someone will have emptied their plastic bag and will say “Stop!” to indicate that they have run out of coloured pencils. When this happens, the student or students will be asked to collect their coloured pencils and place them back into their empty bags. The children will then place those bags in one of the boxes on the floor, as part of the next task:

TASK 3: If there are no more colours in your bag and you have said “Stop!”, put them back in your empty plastic bag and place it in one of the boxes.

Difficulties are expected to emerge about the choice of the box in which the students, who said “Stop!” at the same time, are to place the plastic bags of coloured pencils, that is, the plastic bags which have the same numerosity. At this point, the teacher leads a discussion (D1) in which he/she aims to bring out the concept of equipotency related to the number of coloured pencils that are in the boxes, when two or more students said “Stop!” at the same time. As a result of Task 3, the

semiotic potential of the artefact emerges. The action of putting all the plastic bags of students who said "Stop!" simultaneously into the same box evokes the idea of the equipotential relation between sets.

Second step: The following day, the teacher brings more coloured pencils and other school objects (rubbers, pens, protractors, etc) (see Fig. 1). She repeats the game, in order for the students to create new plastic bags to compare with the ones used in the previous day. Compared with the day before, randomly arranged boxes were placed on the floor. The main aim of this second step is to label the boxes, that is, to bring up the need to denote the equivalence classes with a symbolic representative, which specifically will be the cardinal number. To create the problematic situation connected with this aim, the "Go!" game is played with the new materials, repeating the tasks from the first phase of the game. When some players say "Stop!", a difficulty arises, as it is not easy to find the appropriate box for placing the new plastic bag full of objects, among the boxes, created the previous day. To decide which box to put the new bags in, students should take out a bag already placed in any of the boxes and so they can match the objects in their own plastic bag and the objects in the plastic bag they just took out of the box. If the two bags are in a one-to-one correspondence, between each of the objects in the first plastic bag and each in the second bag, then both bags belong in the same box. If this is not the case, students should take a plastic bag from another box and proceed to check the match by comparing one-to-one. The search for the appropriate box may be long, arduous, and inconclusive. However, this offers an opportunity for the children to reflect on the need to place an identifying symbol on the outside of the boxes. A second discussion is also triggered (D2) in this part of the task. This discussions is about the

Fig. 1 New objects and bags



need and decision to affix a recognition symbol to each box. The action of labelling each box with as many circles as there are objects in the bags evokes the idea of a one-to-one correspondence between the objects and the signs.

When the circles become too many to add to the outside of the box and to do the matching, a need arises to use a unique symbol to denote each numerosity, which provides the opportunity for the Arabic numbers (0, 1, 2, 3, 4...) to be introduced. The action of labelling each box with numbers evokes the idea of using a representative symbol to denote the equivalent class of the characteristic property, common to all the objects in the box, namely equipotency. Specifically, the common characteristic of the bags is that they are in one-to-one correspondence with each other and thus have the same cardinality. Such a representative element will be the cardinal number.

Third step: The boxes have labels, but are not placed in an order. Therefore, an additional difficulty that emerges during the game is to quickly find the “right” box at the “Stop!” moment. In other words, it may happen that students have to compare their plastic bag with the numerosity defined by the cardinal numbers on the boxes before finding its equipotential class. The students can respond to this difficulty by “putting the boxes in ascending order” so they do not have to search randomly to locate the appropriate box. Therefore, the goal of this phase is to order the boxes, and that is to bring about the need to use an ordering relationship. The teacher will then lead a third discussion (D3) aimed at sorting the boxes according to their numerosity, to identify the correct position of each box, through a two-by-two comparison. In relation to this discussion, the semiotic potential of the artefact used emerges indeed, the action of sorting the boxes, according to their numerosity, naturally evokes an ordering relationship.

Fourth step: During the previous step, it could happen that a box is missing. For this reason, a new discussion (D4) will be led to identify how to obtain the “missing” numbers in the sequence of the first 15 Natural numbers. In addition, if there is no box containing plastic bags that are biunivocal matched to the new plastic bag of objects, that represent the missing numbers, students will fill in the “gaps”, which will result in creating a new box or boxes. The missing numbers will be added through the addition or subtraction of an element, respectively, from the previous or the next box. Such procedures evoke the idea of recursiveness and the construction of the number line for Natural numbers.

Fifth step: In the last step, after labelling the boxes with number symbols, the number line is constructed by drawing a long line on the wall and placing a mark with the number at each box.

The “Go!” Game Artefacts and Its Semiotic Potential

The artefact was chosen for its semiotic potential, in terms of the underlying mathematical meanings that can be evoked when performing tasks connected to the game. The main aim of this game/artefact is to construct the cardinal number aspect

as representative of the equipotential classes. Moreover, the game/artefact allows the construction of the ordinal and recursive aspects through the solution of problem situations during the different tasks associated with the game. To construct the meaning of a Natural number and its complexity, we outline the specific aims: 1. Constructing sets to play with; 2. Matching equipotent sets with a biunivocal correspondence; 3. Construct the concept of cardinal number through the realization of equivalence classes related to the equipotential relationship between sets (represented by boxes that will in turn contain plastic bags with the same number of school objects); 4. Identify a way to recognize boxes (because externally they are equal) by labelling them with the cardinal number symbol as representing the equivalence class; 5. Introducing Arabic numeral symbols; 6. Sort the boxes, comparing the plastic bags contained in them (two by two) and identifying which box comes first and which comes next (Sorting principle: $5 > 3$ because it does exist the number 2 that, if added to 3, is 5); 7. Complete the number sequence, "filling in" the "jump" between two non-consecutive Natural numbers, by using the principle of recursiveness (+1), with the missing number; 8. Creating the number line based on the previous number sequence. The game/artefact evokes the idea of equivalence class because the transparent resealable plastic bags model a set, the objects contained in the bags model the elements of the set, and the white boxes naturally simulate the equivalence class. The small objects allow for the construction of the one-to-one correspondence when being extracted from the plastic bags simultaneously when the game leader says: "Go!". Two or more transparent resealable plastic bags of the players, who finished at the same time, provide the possibility to evoke the idea of equipotency among the sets. The action of placing them together into a white empty box, promotes the idea of equivalence class.

Results and Discussion

Below we respond to the research question by discussing some crucial episodes in the mathematical discussions that arose during the different steps of the activity.

Some Crucial Episodes in Students' Classroom Discussion

We analyse three specific episodes which appeared during the didactical path developed across the activity connected to the game. These episodes were considered significant because they point out the role played by the chosen three artefacts in contributing to the students' cognitive process in the construction of three fundamental aspects, cardinal, ordinal and recursive, of the concept of Natural numbers.

Episode 1: Once all the colours were collected the students started to play. Mik started the game by taking the first colour out of his plastic bag and said the first "Go!". His classmates did the same thing. Mik took another turn and said "Go!"

three times, until four of his classmates said “Stop!” simultaneously. These four students should understand the need to place their four plastic bags in the same box. The teacher starts the discussion (D1):

- Teacher: What happens now? Any suggestions?
 Federica: They finished their colours together and now they want to put their bags in a box.
 Teacher: Do they want to store them in the same box? Why in the same box?
 Federica: Because ... because they said “Stop!” ... at the same time.
 Giovanni: Who finishes at the same time, goes in the same box.
 Teacher: Ok, but why? Why in the same box?
 Francesca: Because they are equal.
 Teacher: What do you mean? Why equal?
 Federica: The quantity...is equal.
 Matteo: I got it!! The amount is the same! They said stop after 4 “Go!” turns... Since you have the equal number, you have to put it in the same box.
 Teacher: Equal number...! Mhmm... I don’t see any numbers here!
 Laura: Yes, that’s right! They said “Stop!” together at the same time, and so this means that their plastic bags have the same quantity of coloured pencils, that is an equal number of coloured pencils
 Matteo: They all have four colours; the colours are correspondent.

During this discussion, signs were produced that are gradually evolving into a fundamental mathematical idea, that the plastic bags of colour pencils have the same numerosity because each child said “Stop!” at the same time. Saying “Stop” together represented, through bi-univocal correspondence, the concept of equipotency. This contributed to the idea that plastic bags of the same quantity should be placed in the same box, representing an early idea of the concept of “equivalence class” and cardinality. The change in how the ideas were expressed showed the evolution of signs from the personal meaning of “stop at the same time” to the collective sign “same quantity of coloured pencils, to “the colour pencils are correspondent”. During the next lesson, the teacher summarised what happened in the first phase and asked the students to tell a classmate, who was not at school, what happened. Federica tells the main rules of the game and Giulia explains how to choose the box in which to place their plastic bag at the “Stop!” moment:

- Giulia: When someone was left with the empty bag and said “Stop!”, he had to put them back in the bag and put the bag in a box.
 Federica: The children who said “Stop!” would get up and put their plastic bag in a box ...
 Antonio: That is, they would put the coloured pencils back in their plastic bag and put it in one of the boxes and if students said “Stop!” together, ... that is, the colour pencils of the bags matched each other, ... the bags had to be put in the same box.
 Maria: Yes, but I don’t understand why they put the bags in the same box!

- Federico: Yes, we put our bags in the same box because we said "Stop!" together and finished at the same time... our bags were equal.
- Antonio: Not equal in the coloured pencils but in the numbers of the coloured pencils.
- Teacher: Well, as Federico and Antonio say, if you said "Stop!" together, you would put the bags in the same box because your bags were equal in the number of the coloured pencils. Do you all agree?
- All together: Yes!

In this excerpt, students showed an evolution of the signs, related to the one-to-one correspondence between the plastic bags and again the link between the action of saying "Stop!" simultaneously and the equipotency between the bags whose elements are in one-to-one correspondence.

Episode 2: In this episode we analyse an excerpt of the discussion (D2) in which we point out the need to use a number as a representative of the equivalence class to identify each box. The children started the "Go!" game and must place a "Stop!" bag in a box, but there are many boxes and they do not know where to put it.

- Teacher: So, we have a problem! How do we do...?
- Andrea: We can remember the box.
- Teacher: How can we remember the box?
- Federica: Like, if here is the three [points to the first box], we remember that in the first one is the three.
- Marco: But the three could be here [with finger points to first box] or here [with finger points to third box] or right here [with finger points to last box], we don't know which is the... we can't remember... also because yesterday we put them away in disorder...
- Maria: Or we can put marks on it, for example a drawing of a pencil like this [and she sketches a pencil in the air, with a metaphoric gesture, moving her finger downwards] ...
- Mattia: Yes it's true we can do as Maria says, every time we take a colour we put a little pencil drawing, so if we have two coloured pencils we put a drawing of two little pencils, if we have three coloured pencils we put a drawing of three little pencils... so we remember...
- Teacher: Interesting proposal from Mattia and Maria, what do you think? Shall we put the drawing of pencils outside the boxes?
- Marco and Federica: Yes, let's put them on!
- Teacher: Who wants to tell me how we do it?
- Maria: (takes the box) For example, this is the box and here there are the bags with two coloured pencils, we put a drawing of two pencils to see which bags are in it.
- Laura: Here there are the plastic bags with five coloured pencils... on the box I draw five pencils, here there are the plastic bags with four coloured pencils and on the box, I draw four pencils...

- Luca: Yes, but if we have a thousand coloured pencils, will we draw a thousand pencils? there is not enough space
- Teacher: Good problem! What do you think? Can we find a solution?
- Maria: Let's use numbers... for example here [points to the box with four pencils] we can write four [Maria draws with her finger the symbol for the number 4 in the air].
- Teacher: Do you agree with Maria?
- All: Yes, let's use numbers!

This excerpt highlights the need for young students to identify a representative of the equivalence class that to indicate the numerosity of the objects in the plastic bags, in a specific box. In this way it is easier to find the right box in which to store the "Stop!" bags. To solve the problem, students propose to represent each object in the bags with a symbol, a circle, creating a correspondence between the material object (coloured pencils) and a sign representing it. Finally, one student, Luca, poses a very interesting problem that will lead toward the choice of using conventional number symbols (see Fig. 2). In this excerpt, the need to easily find the box, in which to put the bags, suggests the need to draw the number of pencils on the box before substituting the drawings with symbols for the natural numbers. In this way students firstly use the one-to-one correspondence between the coloured pencils and their drawings, then the drawings with the cardinality of the bags using the symbols for Natural numbers.

This excerpt shows that the choice to use equal bags for gathered objects, led to the idea of using a symbolic representation showing the amount of objects inside the bags. In this way, the equipotency concept emerged with the necessity to use a representative symbol for each equivalence class.

Episode 3: After putting the numbers on the boxes and ordering them, in discussion D4, the students noticed that some boxes were missing, after the box with the sign "4" there was a jump to the box with the sign 7. The students used several procedures to recursively identify the missing numbers, based on the numbers which were already present.

- Teacher: Instead of skipping them, is there a way to add them?
- Anna: Yes, we can create two new bags one with 5 and the other with 6 objects, and so we can put them into two different new boxes.



Fig. 2 Bags with symbols

Mary: If I add a coloured pencil to a bag in the number four box, I get five.

Teacher: So four...

Several students: plus one makes five.

Teacher: and what does plus one mean?

Aurora: that you add one to four, so five is formed....

Luca: and then to five I add a coloured pencil and it makes six, five plus one makes six.

Mary: but we can also have six from seven, I can remove a coloured pencil and make six.

In this brief excerpt, it seems that the concept of recursiveness is emerging. Indeed, students easily created the boxes that were missing, being able to operate with the previous number and/or the next number (see Fig. 3). The students slowly move from counting objects using the manipulative artefact "Go!" game (with bags, coloured pencils, boxes...) to the abstract idea of numbers when they construct the missing numbers by adding or subtracting one.

Fig. 3 Children sorting the bags



Conclusions

Throughout the activity, the complexity of the construction of the concept of Natural numbers concept was evident as it emerged in relationship to the various aspects. The choice of using a game to construct the three aspects of natural numbers, highlighted how students refer implicitly or explicitly to the artefact. The spontaneity and fluency with which the students intervened during the discussions should be noted. This highlights how this concept, in an intuitive and unstructured mode, was built into the kindergarten, facilitating the achievement of the goal to have the students come to understand and use the three aspects of the concept of natural numbers. The intervention was developed from a rigorously scientific proposal built on TSM. The implementation of the activity with the “Go” game showed how effective it was with its choice of appropriate artefacts which supported students, in a non-random way, to gradually construct understandings about Natural numbers, such as one-to-one correspondence, the equipotency between sets, the representation of numbers, and their ordering. Furthermore, analysing verbal signs and gestures of children in kindergarten made it possible to observe the evolution of personal meanings of these elementary mathematical concepts.

References

- Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 746–783). Routledge.
- Butterworth, B. (1999). *What counts: How every brain is hardwired for math*. Free Press.
- D’Amore, B., & Marazzani, I. (2011). *Problemi e laboratori. Metodologie per l’apprendimento della matematica [Methodologies for learning mathematics]* (p. 19). Pitagora editrice.
- D’Amore, B., & Sbaragli, S. (2011). *Principi di base della Didattica della Matematica [Basic principles of mathematics education]* (pp. 47–51). Pitagora editrice.
- Di Paola, B., Montone, A., & Ricciardiello, G. (2020). Drawings, gestures and discourses: A case study with kindergarten students discovering Lego bricks. In I. Erfjord, M. Carlsen, & P. S. Hunderland (Eds.), *Mathematics education in the early years: Results from the POEM4 conference, 2018* (pp. 199–212). Springer.
- Dunlap, J. C. (1996). Rich environments for active learning in the higher education classroom. In B. G. Wilson (Ed.), *Constructivist learning environments: Case studies in instructional design* (pp. 65–82). Educational Technology Publications.
- Faggiano, E., Montone, A., & Mariotti, M. A. (2018). Synergy between manipulative and digital artefacts: A teaching experiment on axial symmetry at primary school. *International Journal of Mathematical Education in Science and Technology*, 49(5), 733–751. <https://doi.org/10.1080/0020739X.2018.1449908>
- Fiorentino, M. G., Montone, A., Rossi, P. G., & Telsoni, A. I. (2023). A digital educational path with an interdisciplinary perspective for pre-service mathematics primary teachers’ professional development. In G. Fulantelli, D. Burgos, G. Casalino, M. Cimitile, G. Lo Bosco, & D. Taibi (Eds.), *Higher education learning methodologies and technologies online: HELMeTO 2022* (Vol. 1779). Springer. https://doi.org/10.1007/978-3-031-29800-4_50

- Fuson, K. C. (2012). *Children's counting and concepts of number*. Springer.
- Griffin, S. A., Case, R., & Siegler, R. S. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), *Classroom lessons: Integrating cognitive theory and classroom practice* (pp. 24–49). MIT Press.
- Kunen, K. (2011). *Set theory*. College Publications.
- Mann, E. L. (2014). Review of *Principles to actions: Ensuring mathematical success for all*. *Mathematics Teacher*, 108(4), 319.
- Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR). (2012). *Indicazioni nazionali per il curricolo della scuola dell'infanzia e del primo ciclo di istruzione (National Guidelines for the curriculum for Kindergarten and the first cycle of education)* (Annali della Pubblica Istruzione, Numero speciale, LXXXVIII). Le Monnier.
- Montone, A., Fiorentino, M. G., & Mariotti, M. A. (2019). Learning translation in geometric transformations through digital and manipulative artefacts in synergy. In P. Zaphiris & A. Ioannou (Eds.), *Learning and collaboration technologies: Designing learning experiences* (Vol. 25, pp. 191–205). Springer. https://doi.org/10.1007/978-3-030-21814-0_
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. NCTM.
- Piaget, J. (1997). *The child's conception of number* (Vol. 4). Psychology Press.
- Rabardel, P. (1995). *Les hommes et les technologies: Approche cognitive des instruments contemporains*. Armand Colin.
- Usmanov, M. (2020). The concept of number: The establishment of the concept of natural number and zero. *International Journal of Academic Information Systems Research (IJASIR)*, 4(12), 7–9.

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From Concept to Implementation: Lessons Learnt from Developing a Mathematical Digital Learning Resource



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and Christiane Benz

Introduction

Understanding numbers in terms of part-whole relations is widely recognized as one of the most important milestones in early mathematical development (Kullberg & Björklund, 2020; Resnick, 1983; Young-Loveridge, 2002). According to Resnick (1992), developing symbolical/numerical part-whole understanding, children proceed through four steps. Building on their *protoquantitative knowledge*, children are thought to develop *the mathematics of quantities* including part-whole understanding closely linked to increase/decrease as well as composition/decomposition operations performed on concrete sets of objects. Subsequently, in *the mathematics of numbers* step, children then start thinking about numbers/non-physical quantities as compositions/decompositions of other numbers, and, thus, understand that all numbers can be composed/decomposed into larger/smaller numbers. This enables them to understand, *the mathematics of operators* reflecting relations between triples of numbers, such as 2–4–6, in which 6 (as a whole) can be decomposed into 2 and 4 (i.e., its parts). Building on this, children develop an understanding of the

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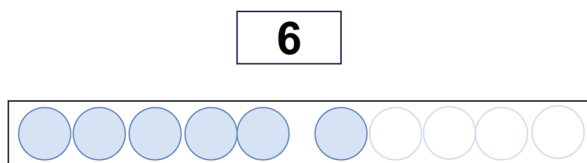
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conceptual relation between, for example, $6 = 2 + ?$ or $6 - 2 = ?$. Thereby, they adopt approaches to mathematical problem solving and interpretation (Resnick, 1983). As such, understanding additive part-whole relations is central to learning arithmetic (Kilpatrick et al., 2001), because it serves as a building block for more advanced mathematical concepts such as the commutative or the complement principle (Björklund et al., 2021; Ekdahl, 2019) as well as additive compositions (Sarama & Clements, 2009).

Consequently, learning environments facilitating part-whole understanding need to promote children's development from protoquantitative to symbolic part-whole understanding by systematically integrating enactive, iconic, and symbolic-digital representations (Bruner et al., 1971). One successful approach to facilitate this development is the use of structured visualizations, such as the use of a ten-strip (see Fig. 1). Interestingly, however, recent research indicated that current analogue (see Lenz & Wittmann, 2023) as well as Digital Learning Resources (DLRs)¹ mainly focus on automating rather than on systematically developing part-whole understanding and, thus, hardly provide a systematic approach to promote part-whole relations (Marx et al., 2025). The latter seems unexpected as DLRs such as, for example, educational apps are typically seen as an attractive, affordable, and successful tools for fostering foundational mathematical skills (Kim et al., 2021; Outhwait et al., 2023a; Outhwaite et al., 2023b). In particular, the use of DLRs in mathematics education offers several advantages especially regarding the acquisition of conceptual knowledge (Larkin et al., 2019), such as part-whole understanding (Ladel & Kortenkamp, 2011). For example, Hillmayr et al. (2020) stated that DLRs offer new and motivating ways for visualizing mathematical relationships that might be acquired by students in different ways, such as in collaborative settings or individually supported by differentiated, targeted feedback. As such, implementing DLRs in mathematics education may help support both, learning and teaching processes (Cirneanu & Moldoveanu, 2024) as they might help reduce obstacles teachers face when facilitating adaptive learning in heterogeneous learner groups (Bang et al., 2023).

Taken together, the importance of developing DLRs that not only promote part-whole understanding in a systematic way by facilitating development from protoquantitative to symbolic part-whole understanding as described by Resnick (1992), but also augment typically-used analogue learning materials is obvious. Accordingly, our aim was to develop a DLR that offers a systematic way to facilitate part-whole understanding and allow teachers to individually support their students' learning

Fig. 1 Example of the ten-strip



¹The term digital learning resource (DLRs) refers to and comprises different digital technologies such as, e.g., applications, software, web-based programs or websites (Bauer et al., 2021).

processes. In particular, we are developing a DLR that considers content-related findings and recommendations from mathematics education [e.g., the use of structured visualizations and introducing systematic decomposition of numbers, e.g., Björklund et al., 2021], but is technically feasible in terms of programming demands and resources needed at the same time.

In this article, we aim to share and discuss experiences and decisions which were made during the development of our DLR. In particular, we will elaborate on the process of balancing out aspects that would be desirable from a mathematics education point of view with constraints due to programming demands and resources. As a framework to discuss the iterative development process, we chose the *Design Thinking* approach. This approach is not only common in the field of education, but has also been recommended as useful for DLR development (Schmidberger et al., 2022).

In the next sections, we briefly introduce the rationale of the *Design Thinking* approach and describe the development process of our DLR applying the steps of this approach. We then reflect on possible lessons learned from a mathematics education perspective that might be useful for future DLR development.

The Design Thinking Approach

The *Design Thinking* approach has its origins in business and economics with the idea to support multidisciplinary teams in the development of new products through an iterative proceeding (Meinel et al., 2011). To achieve this, the *Design Thinking* approach comprises (1) *Empathize*, (2) *Define and Ideate*, (3) *Prototype* and (4) *Test* stages (Schmidberger et al., 2022). The goal of the *Empathize* stage is to clarify the guiding question, the so-called *Design Challenge*, of the respective project and to gain a deeper understanding of the project itself by gathering existing knowledge and understanding different facets of the topic (Schmidberger et al., 2022). The goal of the *Define and Ideate* stage is to bring together all the people who are involved to establish a common knowledge base, to define a common goal and to generate first ideas for the implementation (Samadhiya & Agrawal, 2022). In the following *Prototype* stage, a first draft of the intended object is designed and generated through iterative processes (Samadhiya & Agrawal, 2022), for example by creating sketch notes or mock ups (Schmidberger et al., 2022). In the *Test* stage, future users test the respective prototype and provide an initial evaluation. This includes collecting systematic feedback to further improve the quality of the prototype (Samadhiya & Agrawal, 2022). Afterwards, the prototype will be further developed based on the collected feedback (Schmidberger et al., 2022). How we applied these stages within the development process of our DLR will be described in the next section.

DLR Development Following the Design Thinking Approach

For the present project, the *Design Challenge* was identified as: *How can additive part-whole understanding be facilitated systematically in a DLR?* However, to our knowledge, there is no comprehensive framework that specifies how systematic instruction on additive part-whole understanding in either analogue or digital learning resources should look like. Therefore, the aim of the *Empathize stage* was to identify, based on existing literature, the important aspects to consider in promoting part-whole understanding that are relevant to both, analogue as well as digital learning environments.

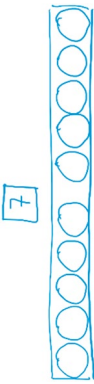
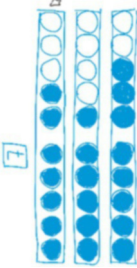
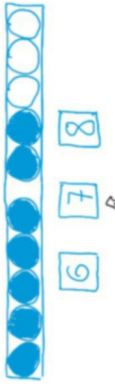
Empathize Stage

Considering aspects for promoting part-whole understanding, recent evidence suggests an association between children's visual structuring ability, which is their ability to mentally structure a given quantity of elements (i.e., to subdivide, based on known facts, the given amount of ten dots in Fig. 1 in *six blue* and *four white dots*) and their part-whole understanding (Lüken, 2012; Young-Loveridge, 2002). Accordingly, to encourage children to apply visual structuring, analogue and digital learning environments should provide tasks using structured visual representations, such as, for instance, the ten-strip (i.e., see Fig. 1). This way, children's experience of quantities/numbers being composed of/decomposed into other quantities/numbers is facilitated. Consequently, conceptual subitizing, which refers to the process of perceiving sets as a combination of subsets and determining cardinality based on known facts (i.e., to quickly determine the cardinality of the blue dots in Fig. 1 as *six*), may replace enumerating single units as the main quantification strategy. Overall, this supports the formation of structured mental representations of part-whole relations between numbers (Lüken, 2012).

Additionally, it is important to provide children with tasks that enable them to systematically work out decompositions of sets on their own, following a constructivist approach (Marx et al., 2025), for example by combining activities allowing for enactive, iconic, and symbolic experiences (Bruner et al., 1971). In digital learning environments, this might be offered through using different visual and dynamic representations (Highfield & Goodwin, 2013), for instance, by linking enactive, iconic, and symbolic representations and user actions in an interactive way (see Table 1, task 3 and 4) or by implementing the possibility to dynamically move/rotate visual representations to foster children's understanding of the commutative principle.

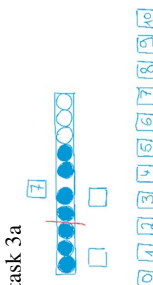
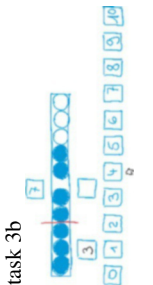
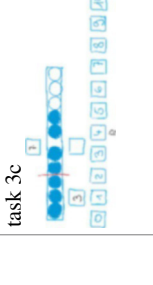
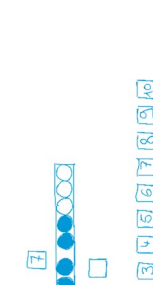
Moreover, it seems necessary to provide tasks that offer opportunities to systematically decompose all numbers in the number range up to ten to help children acquire, memorize, and apply number decompositions (Wartha et al., 2019). This should also include *zero* (e.g., the number triple $0-5-5$), to facilitate children's understanding of *zero* as a natural number (Hartmann et al., 2022). Additionally, the

Table 1 The four tasks as sketched in the Define and Ideate stage

<p>Task 1: Introduction of the ten-strip structure</p>	
<p>task 1a</p> 	<p>task 1b</p>  <p>In task 1a, an empty ten-strip and a digit (e.g., 7) is presented. Children have to click on the respective number of white dots (e.g., seven dots), which subsequently turn blue.</p> <p>In task 1b, a digit (e.g., 7) and three different ten-strips are shown. Children have to identify and click on the corresponding ten-strip (e.g., the first one showing seven blue dots).</p>
<p>Task 2: Brief presentation of the ten-strip structure</p>	
<p>task 2</p> 	<p>In task 2, a ten-strip (e.g., showing seven blue dots), which is presented only for 0.75 s and then disappears, as well as three digits are presented. Children have to identify the matching digit (e.g., 7) by clicking on it.</p>

(continued)

Table 1 (continued)

Task 3: Decomposition of quantities (enactive and iconic level):		
<p>task 3a</p>  <p>In task 3a, a digit (e.g., 7), representing the whole, and a ten-strip showing the respective quantity (e.g., seven blue dots) are depicted. Children have to divide the whole into two parts by clicking between the respective subsets of blue dots in the ten-strip (see red line). The corresponding digits then will be displayed automatically in the two boxes below (e.g., 3 and 4).</p>	<p>task 3b</p>  <p>In task 3b, a digit (e.g., 7), representing the whole, and a ten-strip showing the respective quantity (e.g., seven blue dots) are depicted. Children have to divide the whole into different parts by clicking between the respective subsets of blue dots in the ten-strip (see red line), and, they have to indicate the correct number corresponding with the respective parts by clicking on the corresponding digits presented below the ten-strip (e.g., 3 and 4).</p>	<p>task 3c</p>  <p>In task 3c, a ten-strip, in which two subsets are already separated by the red line, and three empty boxes are shown. Children have to work out the respective number triple by clicking on the matching digits representing the whole and its parts according to the ten-strip given (e.g., 7, 3, and 4).</p>
<p>task 3d</p>  <p>In task 3d, a ten-strip showing only one subset (e.g., three blue dots) and a digit, representing the whole (e.g., 7) are depicted. Children have to determine the hidden subset by clicking on the respective digit, from a series of digits presented below the ten-strip (e.g., 4).</p>		
Task 4: Decomposition of numbers (symbolic level)		
<p>task 4</p>  <p>In task 4, two digits, representing the whole (e.g., 7) and one part (e.g., 3), are depicted. Children have to complete the number triple by selecting the missing part (e.g., 4) from a series of digits presented below the ten-strip.</p>		

structured visual representations might be used in tasks in which they are presented only very briefly (i.e., less than 1 s). This discourages any counting based strategies and, thus, further encourages conceptual subitizing and the formation of respective mental representations of the respective part-whole relations (Lüken, 2012).

Furthermore, considering digital learning environments in particular, recent findings indicate that DLRs support children's understanding best when they offer the possibility to individually select learning paths (e.g., Outhwait et al., 2023a; Outhwaite et al., 2023b) and provide explanatory (e.g., by explaining the correct solution) or motivational (e.g., motivational messages: "Good Job!" or "Try it again!") feedback (e.g., Hirsh-Pasek et al., 2015).

Taken together, the following aspects for promoting part-whole relations in both, analogue and digital learning environments, can be deduced:

- The use of *structured visualizations* (e.g., the ten-strip) throughout allowing children to use conceptual subitizing to determine the respective cardinality and to facilitate mental representations of the whole and its parts.
- Implementation of tasks that combine *different levels of actions* (enactive, iconic, symbolic) and, thus, offer the possibility for children to work out number decompositions on their own (enactive level), to match them with visual representations (iconic level) and to write them down systematically (symbolic level).
- *Systematic decomposition* of all numbers in the number range up to ten by encouraging children to work out, for example, all six different number decompositions for the quantity five (i.e., 5–0, 4–1, 3–2, 2–3, 1–4, 0–5).
- Consideration of number triples with *Zero* e.g., by enabling children to select triples representing 5/0–5 or 5/5–0.

In addition, the following aspects can be derived specifically for digital learning environments that aim to promote part-whole understanding:

- The use of structured visualizations that might be *dynamically moved/rotated* in all directions to allow children understand the commutative principle of cardinalities (e.g., $4 + 2 = 6$ and $2 + 4 = 6$).
- *Automatic navigation*: To offer systematic decomposition, the DLR should allow to automatically present all decomposition tasks for the selected quantity in a sequence from enactive to iconic to symbolic tasks.
- *Individual selection* of tasks/quantities to allow personal adaptation to each child and, thus, to allow gradual access to more advanced decomposition tasks.
- Implementation of different forms of *feedback* such as, explanatory or motivational feedback, so that children receive immediate feedback.
- *Brief presentation* of visualizations to discourage counting based strategies and encourage the use of conceptual subitizing to support the formation of structured mental representations.

To conclude, we expect our DLR to close the gap of missing possibilities for learning part-whole relations by considering and implementing these aspects, which are important for both analogue and digital learning environments and, thus, by exploiting advantages offered by DLRs. Accordingly, we reflect upon these aspects during the development process.

Define and Ideate Stage

Getting together with all persons involved in the project, we established a common knowledge base by introducing not only the *Design Challenge*, but also what was desirable from a mathematics education point of view as clarified in the *Empathize* stage. In addition, necessary programming demands and resources were discussed. Considering desirable aspects and (resource) limitations on both sides, we generated and roughly sketched four tasks that systematically build on each other, shown in Table 1.

Additionally, to develop a first draft of these tasks in the upcoming *Prototype* stage, the following aspects were discussed, which we considered important from the software development perspective:

We decided to develop a browser-based DLR in pure JavaScript instead of a native mobile Android or iOS app to ensure maximal accessibility independent of the end user devices. To make sure that our DLR might be used on different hardware, we put emphasis on implementing a mobile-first design by using Bootstrap² to obtain a user interface that is scalable across different devices. Considering the question how to administrate data, we used an existing software archetype as a starting point that offers a Jetty server, a Java-based RESTful API and a PostgreSQL database. It provides basic database connection for user administration and can be expanded on a modular basis due to its implementation in docker. The archetype has been used for a range of different projects and, thus, provides a tried-and-tested starting point (Chen et al., 2022). Additionally, to reduce programming workload, it was necessary to draft tasks as consistent and as precisely as possible, for instance, by mainly using the same input types (e.g., click³ action) or by designing the user-screens as similar as possible for all tasks (e.g., by placing the ten-strip always at the same position or by using the same buttons for all tasks, i.e., see Fig. 2). Therefore, it was decided to sketch and define every single wireframe (in other words: every single screen that will be visible to the user on the front-end) step-by-step as precisely and detailed as possible in an online drawing tool (i.e., see Fig. 2).

Finally, to allow iterative circles in the subsequent *Prototype* stage, we agreed to use an online communication tool that provided access for all persons involved.

Prototype Stage

To generate a first draft of our DLR, the *Prototype* stage comprised mainly two phases: i) the design of each and every single screen in an online drawing tool (i.e., see Fig. 2), and ii) programming of tasks 3 and 4 as a browser-based DLR.

²<https://getbootstrap.com/>

³Throughout this paper, the term *click actions* refers to and includes click, tip or tap actions that may occur depending on whether the DLR is used on a computer or smartphone device.

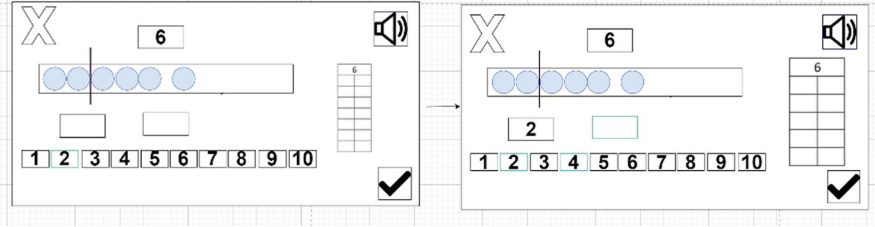


Fig. 2 Screenshot of two wireframes in the drawing tool draw.io

Between and within both phases, all persons involved repeatedly discussed and reflected above introduced aspects relevant for analogue and digital learning environments to promote part-whole understanding and added further details to the tasks as follows:

- To implement *structured visualizations*, we discussed different layouts typically used in analogue learning materials such as the use of a ten-strip or a tens block. As previous findings indicated that linear layouts of quantities had a beneficial effect on children's numerical understanding (Siegler & Ramani, 2009), we decided to use a ten-strip. Additionally, to draw on the power of five (Flexer, 1986), we drafted a ten-strip that separated the fifth and the sixth dot visually (see Fig. 1). Finally, this draft was presented to a small group of children, aged 6–7 years. All of them were able to draw on the power of five by applying conceptual subitizing to determine given quantities.
- *Tasks that combine different levels of actions*: In task 3, children are encouraged to construct the respective decompositions by clicking between the respective parts (enactive and iconic level). All correct decompositions found will be visualized on the symbolic level in a structured way in boxes, and, additionally, in a table (see Fig. 3).
- *Systematic decomposition*: To reduce programming demands in task 3 and 4, we decided to implement quantities from three to ten only.
- *Zero*: To select zero as one part of the number triple, we defined a click movement left of the first (blue) dot of the ten-strip and right of the last blue dot shown as a *true click*.

In addition, considering aspects for digital learning environments, we reflected on the following:

- *Dynamically rotating representations*: We decided not to implement visualisations that can be dynamically moved/rotated mainly because of the following reasons: i) Our DLR will probably be used by younger primary school children, for whom both, swiping as well as drag and drop actions, might be difficult. ii) Combining different input types (such as click or drag and drop actions) would be more time- and resource-consuming to be programmed, at least at this point of the project. Thus, we mainly implemented click actions in all tasks.
- *Automatic navigation*: The user will be automatically navigated through different levels of actions (enactive, iconic, and symbolic) for one quantity. For example,

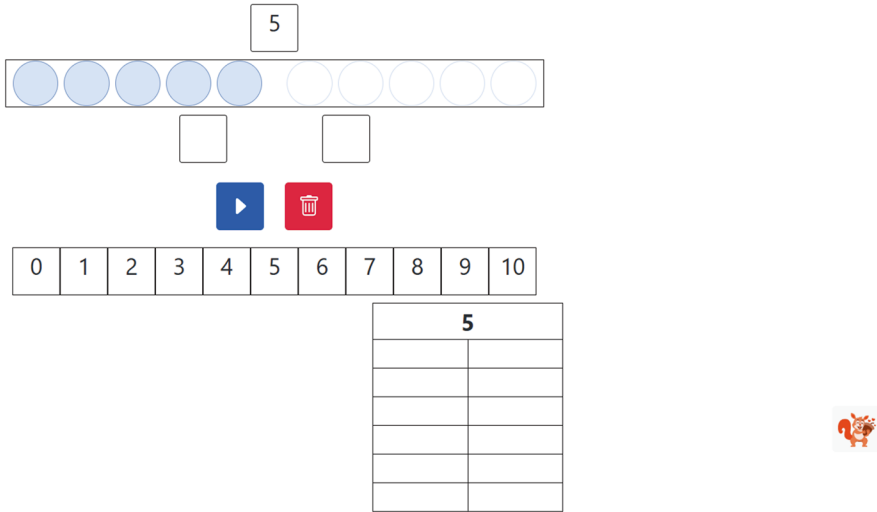


Fig. 3 Example for task 3b, quantity 5

having selected the quantity four, the DLR automatically navigates through all tasks from 3a up to 4 for that respective quantity.

- *Individual selection:* Before and after each task, it is possible to individually select either another task or another quantity to work on next. This means, taken together with an *automatic navigation*, the DLR now offers the possibility to personally adapt the tasks to each child and, thus, allows gradual progress to more advanced decomposition tasks.
- *Feedback:* All tasks are embedded within a narrative of a squirrel that aims to collect nuts for the winter and children can help it by solving the part-whole tasks. Additionally, the squirrel gives all instructions and provides motivational feedback. After completion of a series of tasks for one quantity (e.g., after having completed all six decomposition tasks for the quantity five in task 3b, see Fig. 3), children will get an overview of the total number of points awarded in the form of collected nuts. A maximum of three nuts will be awarded in case over 90% of the answers have been correct. When at least 70% were solved correctly, the player gets two nuts and in case at least 50% were solved correctly, one nut is collected. Additionally, all task instructions given by the squirrel can be repeated by clicking on the respective button. To implement the squirrel narrative, suitable images were selected and the rights for the use of these images within the tasks were clarified. Considering the aim to reduce programming workload, it was, at least until now, not possible to implement instructional feedback. Nevertheless, by implementing the squirrel narrative in the described way, not only motivational, but also meaningful feedback is provided and children's engagement to engage with the tasks may be maintained (Hirsh-Pasek et al., 2015).
- *Brief presentation:* Brief presentations will be implemented in task 2, which, for now, were not yet programmed.

To exemplify how the front-end screens look like at that time of the development process, task 3b is explained in more detail for the quantity five (see Fig. 3): In task 3b a blue ten-strip (e.g., showing five blue dots), the digit 5 (e.g., representing the whole), two empty boxes, an empty table (e.g., consisting of six rows), an arrow button, a trash button and a check mark button (showing the squirrel holding a nut) are displayed. In this example, children have to find all the correct decompositions of 5 by clicking between the blue dots in the ten-strip (e.g., between the second and third). To check whether their decomposition is correct and to receive immediate feedback, they can press the arrow button. To cancel their last input, they can press the trash-can button. Additionally, to check whether their input is correct, they are allowed to press the check mark button any time and will get an overview of the number of nuts collected. In the table, all decompositions found will be automatically noted in structured order.

Subsequently, a first prototype of our DLR comprising tasks 3 and 4 was programmed. Referring to the DLR-definition from Bauer et al. (2021), our DLR can be considered to be a web-based learning program which we trialled in some first grade classrooms in an initial test phase.

Test Stage

To test and evaluate the prototype, we are currently running a first evaluation study involving 111 first graders, aged between 6 and 8 years. As a pre- post-test design, we focused on part-whole understanding (using a self-developed test) and basic mathematical skills. In addition, general cognitive abilities and language skills were assessed as control variables. In the intervention phase, 67 children worked with the prototype three times a week for about 15 minutes over a period of 4 weeks, while 44 children in a control group received business-as-usual mathematics instruction, but had the opportunity to work with the prototype after the study ended. As the study is still ongoing, we cannot report any results on learning gains. However, we also asked participating teachers to evaluate the prototype and report their observations during the intervention.

Based on first reports, we have already started to reflect what worked well and what needs to be improved regarding the design of our DLR. Importantly, all participating teachers reported that students were highly motivated to work with our app, that the students quickly grasped the squirrel narrative or how the different tasks were structured and that they liked the motivational feedback. Additionally, it was reported that the ten strip as a *structured visualization* seemed to facilitate conceptual subitizing. Furthermore, teachers noted that the *automatic navigation* through all tasks seemed to allow for *systematic decomposition* of all quantities from three to ten. Moreover, it was stated that the implementation of *tasks that combine different levels of actions*, in particular the implementation of the enactive and the iconic level, seemed helpful, especially for children who struggled with part-whole relations on the symbolic level. Considering what needs to be improved, it

was observed that the layout of some screens needed to be reorganized (e.g., more distance between buttons, larger buttons, implementing more icons to reduce text, shorter time delay between input and feedback). Finally, it was reported that tasks on the symbolic level seemed to be challenging. This underlines the need to programme tasks 1 and 2 to ensure a deeper understanding of quantities and their compositions/decompositions on the enactive and the iconic level as a basis for solving tasks automatically on the symbolic level.

We are aware that referring to an ongoing study without being able to report findings on the effectiveness of using the developed DLR may be a limitation. Nevertheless, the explicit focus of this article is to report on and discuss experiences and decisions made during the development of our DLR across the different stages of an established model of DLR development from the *Empathize*, over the *Define and Ideate* as well as *Prototype* to the *Test* stage. Explicit consideration of these stages provides mathematics education insights at the different levels of the development process from which future development projects may benefit. Notwithstanding this, we acknowledge that future empirical evidence on the effectiveness of the DLR is needed to substantiate the successful development of our DLR (Outhwait et al., 2023a; Outhwaite et al., 2023b).

Lessons Learned from a Mathematics Education Perspective

Reflecting on the process so far, the following is intended to summarize what was helpful for the development process at the different stages, before discussing some lessons learned from a mathematics education perspective. During the *Empathize Stage*, it was important for both disciplines involved, mathematics education and software development, to reflect and summarize what was relevant from their respective perspective. In particular, from the perspective of mathematics education, it was essential to pursue the *Design Challenge* by collecting and elaborating on the topic-specific, scientific background, and summarizing it in a way understandable to project partners.

This was specifically useful for the *Define and Ideate* stage to help create a common knowledge base. An in-person workshop day was effective as a joint kick-off for the project. During that day, relevant aspects and boundary conditions on both sides, mathematics education and software development, were discussed directly and to the point, accomplishing the *Design Challenge* with an as-specific-as-possible shared understanding of the to-be-developed DLR. Furthermore, it was crucial to sketch potential tasks to allow for a balancing of desirable instructional features and their necessary, but also available software developing demands and resources. Additionally, we made initial decisions on relevant technical aspects of the DLR (such as going for a browser-based DLR, defining data management or user roles).

Moreover, it was essential to clearly set out and define the next steps for the subsequent *Prototype stage*, including an agreement on which specific tools to be used

to keep joint reflection and discussion on the development processes going (e.g., *Webex* as a communication tool and *draw.io* as a tool to draft every single screen).

Reflecting on the *Prototype stage*, it became clear that the most important aspect was to take the time needed to develop and specify the individual screens as the foundation of any programming and implementation of the respective tasks. At this point, initial inconsistencies between tasks (e.g., different positions of the same buttons) were identified before the actual implementation started. This helped to avoid considerable reprogramming work. In addition, it was crucial to have a shared communication tool to allow for spontaneous queries during these development processes, and, thus, to allow for several iterative circles.

Moreover, it can be recapitulated that, at least until now, not all aspects, that were considered desirable from a mathematics education point of view, could be realized when implementing the prototype (e.g., to implement *brief presentation* of visualizations or to program all four tasks) due to related programming demands and limitations in programming resources. In retrospect, it turned out that from the mathematics education perspective, we often underestimated the amount of programming work that some of our desirable aspects require. As such, realizing the first prototype was a continuous balancing out of aspects considered important from our mathematics education perspective with resource limitations on the software development side.

Nevertheless, regarding the *Test stage*, it can be summarized that our prototype was well received by children and teachers alike. For example, children's motivation when testing the prototype, the opportunity to proceed systematically from enactive and iconic to symbolic tasks (*combination of different levels of actions*), the *systematic decomposition* of all numbers up to ten, the ten-strip as a *structured visualization* as well as the squirrel narrative that provides feedback were evaluated very positively by the teachers. However, testing the prototype in class also revealed what needs to be developed further, such as some layout features (e.g., the position/size of buttons, time delay between student input and feedback). Furthermore, it became clear that part-whole understanding on the symbolic level remained difficult, which emphasizes the need to implement tasks 1 and 2 to facilitate the acquisition of a basic of part-whole understanding instead of automating too early.

Finally, against this background our lessons learned from the mathematics education perspective are:

1. *Be aware of software development demands.* For our project, we had to learn that the programming side of DLR development comprises a lot more than what is visible to the user in the end. In fact, preparation of the entire infrastructure needed (including back-end, user management, etc.) was a lot more effort than we naively expected—even though we were lucky to be able to build on an existing archetype. Programming all back-end related matters from scratch would have further increased programming demands immensely. To limit the programming effort, the most important thing in the *Prototype Stage* was to specify all individual screens, using the same layout and structure (e.g., the same buttons, input types, visual elements) as often as possible, as the foundation of any programming and implementation of the respective tasks.

2. *Be open to the perspectives, needs, and demands of the other disciplines involved.* For the present project, it became very clear already in the initial workshop of the *Define and Ideate Stage* that some aspects desirable from a mathematics education perspective (e.g., to use *dynamically moving/rotating representations*, to provide *automatic navigation* or to program all four tasks), as they were elaborated in the *Empathize Stage*, may not be possible to implement considering the given software development resources. Nevertheless, continuous communication and discussion of what may be difficult why and potential alternatives allowed to develop tasks that, on the one hand, serve their instructional purpose well. On the other hand, these tasks can be implemented economically because they largely rely on the same elements (e.g., the ten-strips, the same buttons, etc.), which are displayed at the same locations and use of the same answer mechanics (e.g., clicking on buttons).
3. *Ensure low-threshold communication at all stages of the project to fully exploit the potential of iterative development circles.* Continuous communication and discussions were key to making progress throughout all stages of the project. As such, it is of critical importance for comparable interdisciplinary projects to make use of technical tools and structures that allow for and stipulate low-threshold communication and exchange. In our case this was (in addition to crucial in-person workshops), the use of *Webex*, as a platform for communication through different channels from chats to online meetings, also allowing the sharing of documents. Furthermore, we used the open drawing tool *draw.io*, upon which all members of the project team could follow, contribute to, and comment on the progression of creating individual screens.

Importantly, these experiences are consistent with other findings that DLR development is complex and requires cooperation from multiple perspectives (Dilling et al., 2022). Furthermore, it is considered resource consuming (Thomas & Devi, 2021), in particular so when pitfalls, such as developing educational DLRs that contain *empty calories* rather than rich educational content should be avoided (Hirsh-Pasek et al., 2015). Therefore, future development projects should consider all different perspectives as early as possible to fully exploit the potential of interdisciplinary work, which any DLR development process will require to successfully produce the envisaged product (Outhwaite et al., 2017).

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References

- Bang, H. J., Li, L., & Flynn, K. (2023). Efficacy of an adaptive game-based math learning app to support personalized learning and improve early elementary school students' learning. *Early Childhood Education Journal*, 51(4), 717–732. <https://doi.org/10.1007/s10643-022-01332-3>
- Bauer, A., Lamag, A., Eckerson, A., Diaz, A., Costas, A., Abel, A., Rojas, C., Conrick, C., Keane, C., Brunson, D., Rudzinski, D., Tucker, E., Weeks, E., Meyer, G., McLaughlin, H., Harris, J., Grossmann, J., Ringo, K., Loneragan, K., ... Kaplan, T. (2021). Technology: Where it started and where it's going. <https://opentextbooks.clemson.edu/sts1010fidlerfall2021/>
- Björklund, C., Ekdahl, A.-L., & Runesson Kempe, U. (2021). Implementing a structural approach in preschool number activities. Principles of an intervention program reflected in learning. *Mathematical Thinking and Learning*, 23(1), 72–94. <https://doi.org/10.1080/10986065.2020.1756027>
- Bruner, J., Olver, R., & Greenfield, P. (1971). *Studien zur kognitiven Entwicklung* (Studies on cognitive development). Klett. <http://d-nb.info/456209816/04>
- Chen, X., Bear, E., Hui, B., Santhi-Ponnusamy, H., & Meurers, D. (2022). Education theories and AI affordances: Design and implementation of an intelligent computer assisted language learning system. In M. M. Rodrigo, N. Matsuda, A. I. Cristea, & V. Dimitrova (Eds.), *Artificial intelligence in education. Posters and late breaking results, workshops and tutorials, industry and innovation tracks, practitioners' and doctoral consortium* (pp. 582–585). Springer International Publishing. https://doi.org/10.1007/978-3-031-11647-6_120
- Cirneanu, A.-L., & Moldoveanu, C.-E. (2024). Use of digital technology in integrated mathematics education. *Applied System Innovation*, 7(4), Article 4. <https://doi.org/10.3390/asi7040066>
- Dilling, F., Pielsticker, F., & Witzke, I. (Eds.). (2022). *Neue Perspektiven auf mathematische Lehr-Lernprozesse mit digitalen Medien: Eine Auswahl grundlagenorientierter und praxisorientierter Beiträge* (New perspectives on mathematical teaching-learning processes using digital media: A review of articles). Springer Fachmedien Wiesbaden. <https://doi.org/10.1007/978-3-658-36764-0>
- Ekdahl, A.-L. (2019). *Teaching for the learning of additive part-whole relations: The power of variation and connections*. Jönköping University, School of Education and Communication.
- Flexer, R. J. (1986). The power of five: The step before the power of ten. *The Arithmetic Teacher*, 34(3), 5–9. <http://www.jstor.org/stable/41193003>
- Hartmann, J., Herzog, M., & Fritz, A. (2022). Zero – An uncommon number: Preschoolers' conceptual understanding of zero. *International Electronic Journal of Elementary Education*, 14(3), 353–361. <https://www.iejee.com/index.php/IEJEE/article/view/1740>
- Highfield, K., & Goodwin, K. (2013). Apps for mathematics learning: A review of 'educational' apps from the iTunes app store. In *Annual conference of the Mathematics Education Research Group of Australasia*. Mathematics Education Research Group of Australasia. <https://eric.ed.gov/?id=ED572919>
- Hillmayr, D., Zierwald, L., Reinhold, F., Hofer, S. I., & Reiss, K. M. (2020). The potential of digital tools to enhance mathematics and science learning in secondary schools: A context-specific meta-analysis. *Computers & Education*, 153, 103897. <https://doi.org/10.1016/j.compedu.2020.103897>
- Hirsh-Pasek, K., Zosh, J. M., Golinkoff, R. M., Gray, J. H., Robb, M. B., & Kaufman, J. (2015). Putting education in "educational" apps: Lessons from the science of learning. *Psychological Science in the Public Interest*, 16(1), 3–34. <https://doi.org/10.1177/1529100615569721>
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academic Press.
- Kim, J., Gilbert, J., Yu, Q., & Gale, C. (2021). Measures matter: A meta-analysis of the effects of educational apps on preschool to grade 3 children's literacy and math skills. *AERA Open*, 7, 233285842110041. <https://doi.org/10.1177/23328584211004183>
- Kullberg, A., & Björklund, C. (2020). Preschoolers' different ways of structuring part-part-whole relations with finger patterns when solving an arithmetic task. *ZDM*, 52, 767–778. <https://doi.org/10.1007/s11858-019-01119-8>

- Ladel, S., & Kortenkamp, U. (2011). An activity-theoretic approach to multi-touch tools in early maths learning. *International Journal for Technology in Mathematics Education*, 20. https://www.researchgate.net/publication/261823263_An_activity-theoretic_approach_to_multi-touch_tools_in_early_maths_learning#fullTextFileContent
- Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT framework to evaluate the design of two geometry apps: An exploratory study. *Digital Experiences in Mathematics Education*, 5(1), 59–92. <https://doi.org/10.1007/s40751-018-0045-4>
- Lenz, K., & Wittmann, G. (2023). Zur Erarbeitung des Teile-Ganzes-Konzepts im mathematischen Anfangsunterricht: Welche Lerngelegenheiten bieten Schulbücher für die erste Klasse (Developing part-whole understanding in early mathematical instruction: What learning opportunities do first grade textbooks offer)? *Journal für Mathematik-Didaktik*. <https://doi.org/10.1007/s13138-023-00218-0>
- Lüken, M. M. (2012). Young children's structure sense. *Journal für Mathematik-Didaktik*, 33(2), 263–285. <https://doi.org/10.1007/s13138-012-0036-8>
- Marx, C., Roesch, S., Moeller, K., & Benz, C. (2025). From the whole to its parts – A systematic analysis of affordances for learning part-whole-relations in digital apps. *International Electronic Journal of Mathematics Education*, 20(1), em0802. <https://doi.org/10.29333/iejme/15677>
- Meinel, C., Leifer, L., & Plattner, H. (Eds.). (2011). *Design thinking*. Springer. <https://doi.org/10.1007/978-3-642-13757-0>
- Outhwaite, L. A., Early, E., Herodotou, C., & Van Herwegen, J. (2023a). Understanding how educational maths apps can enhance learning: A content analysis and qualitative comparative analysis. *British Journal of Educational Technology*, 54(5), 1292–1313. <https://doi.org/10.1111/bjet.13339>
- Outhwaite, L. A., Early, E., Herodotou, C., & Herwegen, J. V. (2023b). *Can Maths apps add value to learning? A systematic review*. CEPEO Working Paper(23–02). <https://EconPapers.repec.org/RePEc:ucl:cepeow:23-02>
- Outhwaite, L. A., Gulliford, A., & Pitchford, N. J. (2017). Closing the gap: Efficacy of a tablet intervention to support the development of early mathematical skills in UK primary school children. *Computers & Education*, 108, 43–58. <https://doi.org/10.1016/j.compedu.2017.01.011>
- Resnick, L. B. (1983). A developmental theory of number understanding. In *The development of mathematical thinking* (pp. 109–151). Academic.
- Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a Foundation of Everyday Knowledge. In *Analysis of arithmetic for mathematics teaching* (1st ed., pp. 373–429). Routledge. <https://doi.org/10.4324/9781315044606>
- Samadhiya, A., & Agrawal, R. (2022). Developing a handloom through d.schools design thinking approach. *Technology in Society*, 71, 102134. <https://doi.org/10.1016/j.techsoc.2022.102134>
- Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.
- Schmidberger, I., Wippermann, S., Stricker, T., & Müller, U. (Eds.). (2022). *Design Thinking im Bildungsmanagement: Innovationen in Bildungskontexten erfolgreich entwickeln und umsetzen*. Springer VS. <https://doi.org/10.1007/978-3-658-36951-4>
- Siegler, R. S., & Ramani, G. B. (2009). Playing linear number board games—But not circular ones—Improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology*, 101(3), 545–560. <https://doi.org/10.1037/a0014239>
- Thomas, C. G., & Devi, J. (2021). A study and overview of the mobile app development industry. *International Journal of Applied Engineering and Management Letters (IJAEML)*, 5(1), 1. <https://supublication.com/index.php/ijaeml/article/view/424>
- Wartha, S., Hörhold, J., Kaltenbach, M., & Schu, S. (2019). *Grundvorstellungen aufbauen Rechenprobleme überwinden—Zahlen, Addition und Subtraktion bis 100 (Building basic concepts and overcoming math difficulties – numbers, addition and subtraction up to 100)*. Westermann.
- Young-Loveridge, J. (2002). Early childhood numeracy: Building an understanding of part-whole relationships. *Australasian Journal of Early Childhood*, 27(4), 36–40. <https://doi.org/10.1177/183693910202700408>

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Storytelling Activity for Stimulating Proportional Reasoning in Young Children



Maria Mellone, Marina Spadea, Rosalia Maria Lo Sapio,
and Giancarlo Artiano

Introduction

“Can you tell me a story?” The relationship between adults and children is often characterised by storytelling. Whenever you read or listen to a story, you embark on a journey into the imagination! Through the stories we are told, we learn something about the world. A state of correspondence with reality is created in which everything is possible and, above all, in which our ego enters into a close relationship with the ego of the protagonists of the stories and shares their experiences (Winnicott, 1971). In mathematics, storytelling is also a powerful pedagogical tool that allows children to appreciate and use mathematics as a possible lens to give meaning to certain aspects of stories (see e.g. Liljedhal & Zazkis, 2009), such as how to resolve a problem.

This paper presents a case study of a storytelling problem-solving activity designed for a kindergarten class. The activity introduces measurement contexts involving discrete and continuous quantities, introducing students to understanding direct proportionality. The students are encouraged to explore these proportional relationships by engaging with various materials and representations, allowing them to connect with the mathematical quantities depicted in the story physically. The research team comprised the classroom teacher (one of the authors), an experienced educator, and two researchers. According to the teacher, the researchers played a

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crucial role in designing and implementing the interventions. Their consistent presence over an extended period facilitated the seamless integration of these interventions into the students' daily school experiences. Children worked both individually and in small groups, and each activity concluded with sufficient time for whole-class discussions. During these discussions, the children shared their ideas and representations, fostering collective sensemaking (Odden & Russ, 2019) and co-construction through a sustained shared thinking approach (Siraj-Blatchford et al., 2002).

State of the Art

One of the most significant differences between real-life and school problems is that the latter are heteroposted. Indeed, in the school context the individual who presents the problem (usually a teacher or a textbook) is not the same as the person required to solve it (Zan, 2011). Educational mediation should, therefore, try to work on the gaps and find ways to involve young pupils, especially when they are still very young so that they can perceive the problem as theirs: they need to feel like protagonists of the proposed situation and have the freedom to use all their skills to find a solution. It transforms a simple word problem into a problem-solving problem. In this direction, storytelling offers a fascinating area of work to immerse children in mathematical problem-solving activities in a meaningful and engaging way (e.g., Liljedhal & Zazkis, 2009).

From a young age, being involved in a problem situation within a measurement context is fundamental for developing proportional reasoning (Piaget, 1974; Vygotsky, 1978). In the Western tradition, we tend to introduce natural numbers in the context of counting discrete sets before exploring them in the context of measurement. Historically and pedagogically, (natural) numbers are seen primarily as tools for counting and, in general, we can see that most approaches to mathematics education, with minor differences, give a primary role to natural numbers and counting discrete quantities (see, e.g. Gallistel & Gelman, 1978). For this reason, a different approach is based on the studies of the Russian psychologist Davydov (1982). According to this approach, pupils can begin their mathematical experiences by comparing continuous quantities (liquids, spatial extensions, ...) before encountering numbers (Iannece et al., 2009). Davydov suggests that students should experience comparing and dealing with continuous substances at the beginning of their mathematical education before encountering natural numbers (Mellone et al., 2020). It considers all the experimentation with measuring liquid volumes using graduated and non-graduated cylinders that students can do. Several factors make it worthwhile: -the linearity of the volume as a function of the level (proportional reasoning), -the freedom in determining the unit of measurement, -the "indefinite" repeatability of decanting with the possibility of perfecting and quickly correcting "errors", -the evaluation of the approximation in evaluating the volume, -"playing"

with water which is always very engaging for children, for example also discovering the effects of refraction in seeing “through”.

Teaching proportional reasoning is one of primary school mathematics’ most significant pedagogical challenges (see e.g. Simon & Placa, 2012). Simon and Placa (2012) particularly highlight the central role of intensive quantities in addressing this challenge. These quantities represent relationships between two values, often expressed as a ratio that conveys a measure of density or intensity (e.g., speed, color, price). Neuroscience interestingly suggests that the brain has an innate ability to recognize rates and densities, such as the density of discrete objects in continuous space or the concentration of sounds within specific time intervals (Dehaene, 2001). In this direction, the design of the experimentation of this study was built around the idea of mobilising this innate ability of children in a problem-solving activity and developing it through the appropriate use of different types of representations in a more conscious management of proportional reasoning, such an intense quantity (the price coins/quantity of cheese).

Theoretical Framework

In this study, we adopt a cultural-historical perspective on learning processes, recognising that material objects (in our study drawings, abacus, numerical rods) are not conceptually neutral but suggest a particular way of looking at the world and that this interaction is deeply dependent on the specific historical and cultural context in which the material activity takes place. In line with the theory of semiotic mediation, the creation and manipulation of some artefacts support the interiorisation of mathematical knowledge (Bartolini Bussi & Mariotti, 2008). The teacher uses semiotic mediation tools with the didactic intention of developing mathematical meanings and helping the student to perform a task or solve a problem. We will see that the teacher suggests students work with different artefacts to create different representations of the problem.

In the problem-solving activity, a crucial part is the problem-solving process, where pupils can express themselves through all the channels they know and master: the body, the hands, the drawing, and the language. In this space, the dimensions of the body and movement are therefore of particular importance. Creating compelling experiences for developing mathematical skills through an embodied approach is possible. In line with the work of Lakoff and Núñez on the embodied mind and its relationship to abstract mathematics (Lakoff & Núñez, 2000), the role of the body and the material world occupy a central role in knowledge and human development. Several researchers studying language, communication, and interaction in mathematics thinking, teaching, and learning have recently focused on the role of gestures (Robutti et al., 2022). Simulations, for example, are mental events that are conducted “offline,” away from the place of their original intent. They are an aspect of our imagination that enables us to extend our cognitive processes for present and familiar things to think about possible and unfamiliar ideas (Nemirovsky & Ferrara,

2009). Through embodied simulation, learners may mentally simulate the abstract constructs using perception and emotion (Johnson-Glenberg et al., 2016). This process occurs especially when we are involved in a storytelling activity.

An essential component of our theoretical framework is the exploration of the ethical themes inherent in mathematics. The intricate relationship between equality and equity is brought to light through storytelling, as highlighted in this work. While equality involves providing everyone with the same resources, equity recognises that not everyone starts from an equal position and emphasises the need to identify and address disparities. We are interested in exploring new approaches, such as Ethics in Mathematics and Mathematics for Social Justice, which are increasingly acknowledged within educational discourse, as recent literature underscores (Ernest, 2024).

Methodologies

The methodology is inspired by Design-based Research (Cobb et al., 2003; diSessa & Cobb, 2004) and aims to develop studies that combine both the learning process and the design that supports that process (Cobb et al., 2003; diSessa & Cobb, 2004). That approach implied that researchers designed and implemented interventions in authentic classroom contexts, whereas their regular presence over an extended time frame allowed for a seamless integration of the intervention in the students' everyday school experience. The iterative nature of the design approach involved a process of cyclic design, evaluation and revision in which interventions are planned according to provisional assessment of learning outcomes. This implies that learning and design can be studied in their mutual interdependence.

The didactical experiment involved a kindergarten class composed of twelve 5–6-year-old pupils in a problem-solving activity related to numbers as measurement and direct proportional relationships. The research team consisted of the classroom teacher (one of the authors), an experienced teacher and two researchers. The experimentation was developed in the second part of the school year (January to May 2022) for 30 hours. Children were involved in hands-on inquiries and encouraged to produce enactive, iconic, and symbolic representations of their ideas in interactions with their peers, researchers, and teachers. During their investigations, children worked both individually and in small groups. Each activity was concluded with ample time devoted to a whole class discussion in which the children shared their ideas and representations in the process of collective sensemaking (Odden & Russ, 2019) and co-construction based on a sustained shared thinking approach (Siraj-Blatchford et al., 2002). Data collection included video and audio recordings of the interventions and field notes. Also, the inscriptions produced by the students were partly photographed and partly collected and scanned. Selected data (focusing on activities related to modelling) were analysed qualitatively.

Case Study: The Mouthfuls Story

Synopsis

The case study presented in this paper is part of a broader project on using storytelling to teach mathematics problem-solving to young pupils (see e.g. Mellone et al., 2012). The objectives are: (1) to improve teaching practices by proposing an approach that uses storytelling to suggest working with direct proportional relationships with kindergarten pupils; (2) to investigate how storytelling, representations and manipulation of specific material can stimulate learning processes in this context.

The activity begins with the reading of the following storytelling:

An ant, a mouse, and a lion were friends. All three were very hungry. Finally, they found some cheese that they all liked. The ant ate ten large mouthfuls of cheese and said, "I'm full!". The mouse ate ten large mouthfuls, then said, "I'm full!". The lion ate ten large mouthfuls and said, "I'm full!". Then, the owner of the cheese came and wanted to be paid. The ant said: "We ate ten mouthfuls for one, so that's how we ate: I'll give you a coin, and the mouse and the lion will give you a penny, too." The mouse and the lion agreed. The host disagreed, and so they began to argue. Then the ant said: from now on, I will no longer trust numbers because numbers are cheating.

The problem-solving presented in the storytelling is based on contextual measurement of a continuous quantity (see Davydov, 1982): cheese. In particular, the storytelling proposes to work with the ambiguity arising from using an anthropomorphic unit of measurement to measure cheese, the mouthfuls, to create the need to seek a more objective unit of measurement. Another mathematical knot that this narrative proposes is to establish a correspondence between the quantity of cheese and the coins. This node refers to establishing an intensive quantity (Simon & Placa, 2012): the relationship between coins and amounts of cheese.

In the following, we present some excerpts (drawings, pieces of the discussion transcript) from the activity to show some essential steps in developing pupils' mathematical thinking. To facilitate reading, we divide the path into three phases: drawing and discussing, reasoning and modelling, and manipulating.

Results

Phase One: Drawing and Discussing

The first activity focused on reading and interpreting the problem through a collaborative discussion. The pupils discuss the events in a circle under the teacher's guidance and identify the story's protagonists and antagonists. Together, the story is staged to allow students to experience what they have read in the first person and freely express their feelings and emotions. This initial phase aims to determine the relationships between what has happened and its consequences. Spontaneously, the

need arises to make a series of drawings that capture the most critical moments of the story told, accompanied by short sentences that reproduce the story in the pupils' own words. Thus, the children participate in a natural collective writing laboratory in which the teacher puts the pupil's words on paper. In Fig. 1, we can see some drawings.

After this first phase, the teacher tries to stimulate the discussion and focus on mathematical aspects by asking some questions: "What does the host look at?" "What does the ant refer to?" "When I count, what do I count?". The pupils respond with different observations:

Pietro: The ant is right, they ate the same amount and paid the same amount.

Ciro: But they are not the same, they are small, medium and large.

Francesco: The lion has to pay more.

To steer the discussion, the teacher turns the situation around, puts three small plates on the table pours a spoonful of popcorn into each plate with a spoon and says: "one for the ant, one for the mouse, one for the lion". Figure 2 shows a picture of the scene.

Teacher: In this case, how much does each have to pay?

Andrea: Let's say ten euros per spoon. They must pay the same because they ate the same amount of popcorn.

Pietro: So the lion eats less.

Riccardo: The spoon is the same for all three.

Teacher: When the mum goes to buy cheese, she says, "Give me lots of cheese -spreading her arms wide - for my son who has a mouth so big", showing a picture of the mouths of all the people having lunch that day at her house.



Fig. 1 Pupils' drawings with the teacher's transcription of their words

Fig. 2 The three spoonsful of popcorn for the three animals

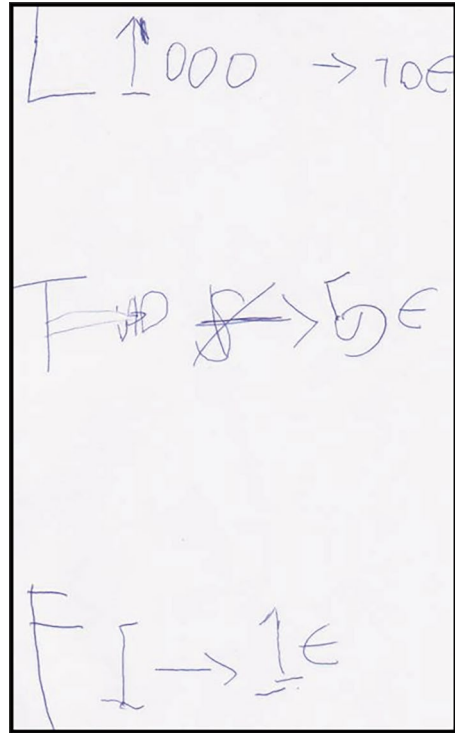


- Children: NOOO!!! [answering with a resounding laugh].
- Simona: My mum says, “Give me ten grams of cheese”.
- Francesco: Then the grocer puts it on the scales and tells her how much it costs.
- Teacher: Look at these. [She shows the children the box of 30 grams of cheese they had received for lunch that day.]
- Chiara: So the mouse eats ten grams of cheese.
- Andrea: The ant eats one gram, and the lion eats a thousand grams.
- Teacher: ...and how much does each one pay?
- Chiara: The ant pays one.
- Francesco: Then the mouse pays two and the lion ten.
- Teacher: How could we remember all these things we said without doing too many drawings? We could do something like the adults do. When Mum goes shopping, to remember what to buy, she doesn't bring drawings or pictures of fruit, salami, etc.!!!
- Martina: We have to write but don't know how to write.
- Chiara: I can do the T for “topo”-mouse.
- Martina: Let's try!
- Chiara: ok [Chiara starts to write a T in the middle of the paper].
- Giulia: Let's put an arrow to show how much cheese and money there is.
- Chiara: ok [Chiara, with the teacher's help, wrote the letters and the number by tracing the proposal that emerged in the previous expert of discussion (Fig. 1)].

Phase Two: Reasoning and Modeling

The discussion is quite productive. The teacher begins by introducing the unit of measurement of the spoon and using three plates for the three animals to make an even distribution of a spoonful of popcorn on each plate. This prompts Andrea (line 2) to suggest an exact number of coins for the payment, ten coins, an exciting number that deviates from the ability to subitize but is central in our number system of representation and linked to the number of fingers. However, the question of the size of the animals (mouths) soon arises, as Pietro (line 3) emphasizes that the lion eats less this way. The teacher decides to steer the discussion back in the right direction by suggesting that the pupils think about a scene that some have probably experienced before and get them to think about how adults communicate about quantities and units of measurement used (line 5). Following this prompt, Simona (line 7) introduces the word ‘gram’, and the teacher uses this prompt to draw the pupils’ attention to the box of cheese they received for lunch by asking them to look at the label 30 grams. This prompts Chiara (line 10) to suggest an exact amount of cheese the mouse has eaten by using the word ‘gram’, exactly ten grams. The “ten” is chosen again to show its central role in the pupils’ mathematical reasoning. Andrea, starting from Chiara’s proposal, proposes to quantify with 1000 grams and 1 gram, respectively, the lion and the ant quantity of cheese, showing Andrea’s taste for the

Fig. 3 A different payment is then associated with the amount of cheese eaten by each animal. The letter “L” is used for lion, the letter “T” is used for mouse and the letter F is used for hunt



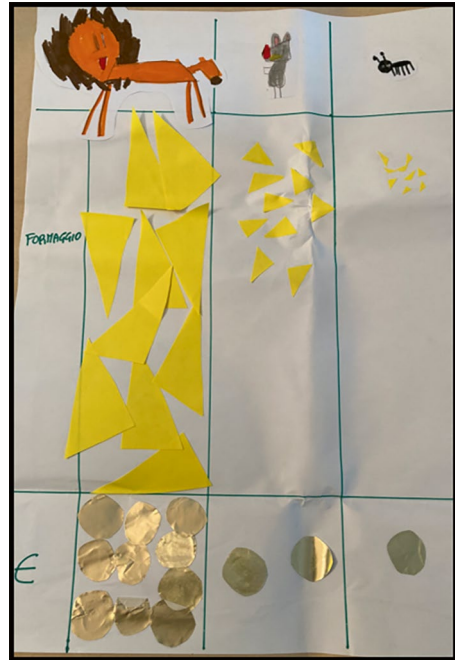
multiplies of ten. The other children propose a correspondent number of coins (13 and 14 lines). In this way, as we can see in Fig. 3, a naive idea of direct proportionality between the quantity of cheese and coins emerges and appears, even if the numbers chosen are not in proportion.

Phase Three: Manipulating

The activity continues in several following classes, after some of which the teacher proposes to create a double-entry table. Figure 4 represents the mouthfuls of cheese eaten by the animals, with yellow triangles of different sizes for each animal and a corresponding number of coins. In this phase, the teacher proposes to assign the cost of two coins to the mouse cheese. Indeed, the teacher wants to propose a second part of the activity to improve children’s mathematical skills regarding the order of number words and the related action of counting in this context. To this purpose, the problem proposed is developed further in the narrative:

The three animals found that the cheese was excellent quality, and since the price was also low, they decided to invite more of their friends. So it came about that our three protagonists invited some of their friends on the condition that the size of the width of their mouths compensated for the values between 2 (mouse) and 10 (lion).

Fig. 4 The double entry table. On the first row cheese. On the second row the coins and the pupils chose to use the symbol of euros



After an initial exchange of opinions, the pupils searched for new animals. With the help of the interactive whiteboard, they discussed and selected seven more animals with matching mouth openings that correspond to the mouth openings of the ant, mouse, and lion. This part was rich in considerations and discussions by the pupils, which we cannot report on for space reasons.

The teacher introduces another representation, namely skewers (a mouthful) with cheese balls for stacking the cheese balls and links this representation to using the abacus. The balls represent the new unit of measurement of the cheese, and each ball corresponds precisely to the amount of cheese eaten by the ant, which was chosen as the unit of measurement for the cheese. In this way, as we can see in Fig. 5, the pupils manipulate the abacus to represent the different amounts of cheese eaten by each animal. In addition, the teacher suggested displaying the amount of cheese on the board as a plate with the correct number of balls and as a skewer/abacus to provide a further experience of multiple representations. Figure 6 represents this scene. Also noteworthy is the use of the tens bar to represent the ten balls that the lion has eaten, as shown in Fig. 8. The teacher wants the pupils to begin to trust the abacus, which, together with the positional representation of the number (see its use in Fig. 3), performs its function as a semiotic mediator (Bartolini Bussi & Mariotti, 2008). After the joint discussion, the teacher helps the pupils to organize the ideas that have emerged on a board. For each animal, a plate with cheese balls and a skewer/abacus with the same number of pieces of cheese were drawn, as shown in Fig. 7.

Fig. 5 A pupil using abacus for representing the quantity of cheese



Fig. 6 Two pupils at the blackboard representing the quantities of cheese on a plate alongside the balls on a skewer abacus



The teacher shifts the focus to the cheese each animal has eaten and the coins each animal must pay to the host, as each bite of cheese costs one euro. The teacher presents the children with a box with coloured number rods as a money box, from which they take the coin cube to pay for the cheese. In this phase, each coin is represented by a unit of the number rod. The pupils explore the box with the number sticks and realise there are rods of different lengths. They also begin to recognise the relationships between the number of rods of different lengths. In this case, the teacher also wants the pupils to start to trust the coloured number rods, which, together with the positional representation of numbers and the abacus, act as semi-otic mediators (Bartolini Bussi & Mariotti, 2008).

Fig. 7 For each animal the quantity of cheese is represented as balls on a plate and on the skewer/ abacus



Simultaneously with the manipulation and discussion activities, the pupils decided that each cheese ball paid for one coin. The teacher helps the pupils to place on the board the corresponding amount of cheese for each animal, represented as a circle with the corresponding number of cheese balls inside, then the same number of balls stacked on the skewer/abacus, and then the corresponding number of coins, represented as the corresponding coloured number rod shown in Fig. 8. With this board arrangement and then on the wall chart, the teacher wants to emphasise the precise correspondence between the cheese balls (on the plate or the skewer/abacus) and the number rods representing the coins.

In the last phase, the teacher proposed a histogram to summarize the work done: The animals were drawn in ascending order according to the size of their mouth opening, and for each animal, the corresponding amount of cheese balls and coins (represented as colored number rods) was indicated. Fig. 9 shows a histogram made by a pupil. Throughout the trial, the pupils' drawings and the representations created with the teacher's help are collected on a wall chart to express the participants' contribution, skills and creativity. Figure 10 represents the wall chart of the entire path. This provides a summary picture at the end of the activities of the richness of what has been achieved, but it also seems to be a valuable tool throughout the pathway to keep track of work in progress.

Fig. 8 The completed panel

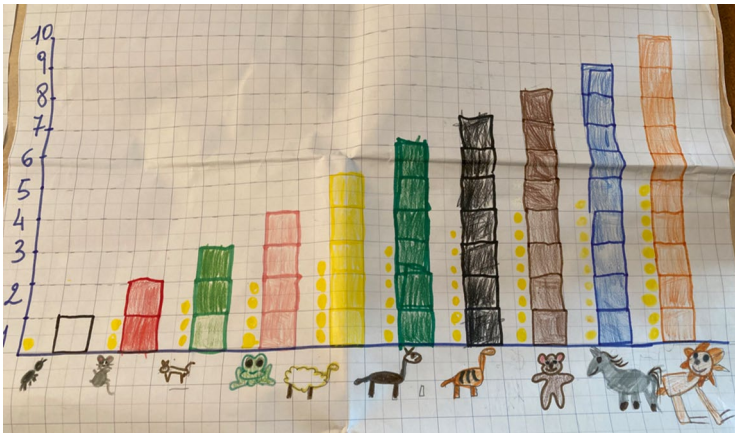
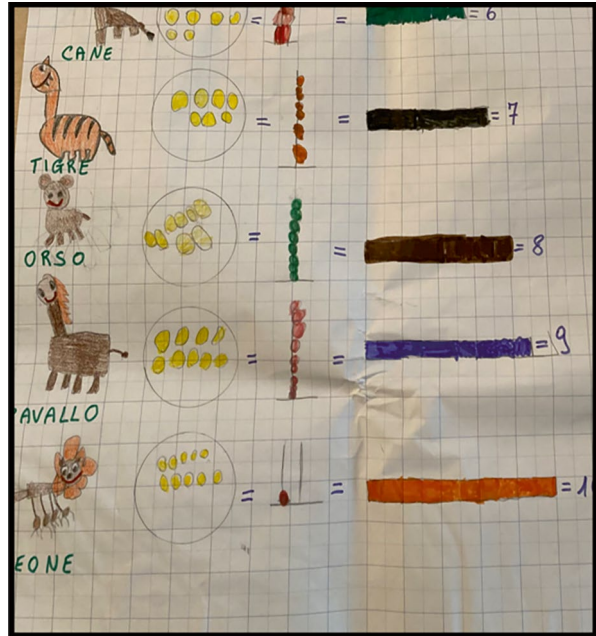


Fig. 9 A histogram where animals are in ascending order drawing by a pupil

Conclusions

The wall chart at the end summarizes all the considerations and experiences made during the path, lasting many days and characterized by various work phases. According to the theoretical framework presented at the beginning, storytelling



Fig. 10 The wall chart of the story of the ant, the mouse and the lion

fully involved the pupils—through discussions, comparisons and the production of different artefacts—and made them protagonists of the proposed situation. In particular, the initial storytelling became a “problem solving” that consisted of moving from an anthropomorphic unit of measurement to a more objective unit of measurement. As the blackboard recounts, pupils first tackled the problem of discretizing a continuum (the cheese) to make it countable. Drawings, representations and the manipulation of different artefacts allowed the kindergarten pupils to face the problem-solving raised from the storytelling, which led the pupils to identify a new unit of measurement (the amount of cheese eaten by the ant). The excerpts show the presence of a naïve idea involving direct proportionality relationships between quantities (cheese and coins), as shown in Fig. 3. The artefacts proposed by the teachers go toward discretizing a continuum (in the case of cheese) or making a discrete quantity continuous (representation of coins by colored rods). They first related the mouthful of cheese (discretized continuum) to the opening of the

animals' mouths and then to the balls of cheese (mouthful of ants) and gained a sense of numbers through a series of manipulations. Finally, by activating proportional reasoning, they were guided to link the amount of cheese to the number of coins.

The idea of organizing such rich and complex paths was born from the desire to accompany the pupils on their way to elementary school with the richest possible experiences related to numbers, taking advantage of their innate number sense and also strengthening it by introducing them to the use of some artefacts that they are likely to encounter in their future school experiences.

Acknowledgments We want to dedicate this paper to our mentor Paolo Guidoni, creator of Mouthfuls story, who we miss very much and who however continues to strongly inspire all our work as mathematics educators. Finally, we wish to thank the foundation "I Lincei per l'Infanzia" who supports us to systematize this design study for teacher education purpose.

References

- Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In *Handbook of international research in mathematics education* (2nd revised ed., pp. 746–783). New York.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9–13.
- Davydov, V. V. (1982). The psychological characteristics of the formation of elementary mathematical operations in children. In T. P. Carpenter et al. (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 224–238). Lawrence Erlbaum.
- Dehaene, S. (2001). Précis of "The number sense". *Mind and Language*, 16, 16–36.
- diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences*, 13(1), 77–103. <https://doi.org/10.1207/s15327809jls1301/4>
- Ernest, P. (2024). *Ethics and mathematics education. The good, the bad and the ugly*. Springer.
- Gallistel, C. R., & Gelman, R. (1978). *The child's understanding of number*. Harvard university Press.
- Iannece, D., Mellone, M., & Tortora, R. (2009). Counting vs. measuring: Reflections on number roots between epistemology and neuroscience. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidi (Eds.), *Proceedings of PME33* (Vol. 3, pp. 209–216). PME.
- Johnson-Glenberg, M. C., Megowan-Romanowicz, C., Birchfield, D. A., & Savio-Ramos, C. (2016). Effects of embodied learning and digital platform on the retention of physics content: Centripetal force. *Frontiers in Psychology*, 7, 213387.
- Lakoff, G., & Núñez, R. (2000). *Where mathematics come from? How the embodied mind brings mathematics into being*. Basic Books.
- Liljedhal, P., & Zazkis, R. (2009). *Teaching mathematics as storytelling*. Sense Publishers.
- Mellone, M., Baccaglioni-Frank, A., & Martignone, F. (2020). Measuring rice in early-childhood education activities: A bridge across discrete and continuous magnitudes. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years*. Springer.
- Mellone, M., Spadea, M., & Tortora, R. (2012). Designing tales for introducing the multiplicative structure at kindergarten. In B. Maj & K. Tatsis (Eds.), *Generalization in mathematics at all educational levels* (pp. 103–112). Wydawnictwo Uniwersytetu Rzeszowskiego.

- Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, 70, 159–174.
- Odden, T. O. B., & Russ, R. S. (2019). Defining sensemaking: Bringing clarity to a fragmented theoretical construct. *Science Education*, 103(1), 187–205.
- Piaget, J. (1974). *Understanding causality* (D. Miles & M. Miles, trans). W.W. Norton.
- Robutti, O., Sabena, C., Krause, C., Soldano, C., & Arzarello, F. (2022). Gestures in mathematics thinking and learning. In *Handbook of cognitive mathematics* (pp. 1–42). Springer.
- Simon, M. A., & Placa, N. (2012). Reasoning about intensive quantities in whole-number multiplication? A possible basis for ratio understanding. *For the Learning of Mathematics*, 32(2), 35–41.
- Siraj-Blatchford, I., Muttock, S., Sylva, K., Gilden, R., & Bell, D. (2002). *Researching effective pedagogy in the early years* (Vol. 356). Department for Education and Skills.
- Vygotsky, L. S. (1978). *Mind in society: Development of higher psychological processes*. Harvard University Press.
- Winnicott, D. W. (1971). *Playing and reality*. Routledge. (Italian edition “Gioco e Realtà”, 2020 Armando Editore).
- Zan, R. (2011). The crucial role of narrative thought in understanding story problems. *Current State of Research on Mathematical Beliefs*, XVI, 287–305.

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Enhancing an Early Inclusive Mathematical Learning? Investigating Cubarithm’s Semiotic Potential for Children with Visual Impairments



Carola Manolino, Cristian Bernareggi, and Tiziana Armano

Introduction

Mathematics stands as a foundational pillar of education, fostering critical thinking, problem-solving, and cognitive development (OECD, 2023). However, for children with visual impairments, the path to mathematical learning—or often just its accessibility—presents unique and daunting challenges. The reliance on visual representations in traditional mathematical education has created barriers that can be insurmountable without innovative solutions (Gerino et al., 2014). In response to this pressing need, our research undertakes an exploratory investigation into the semiotic potential (Bartolini Bussi & Mariotti, 2008) of a teaching tool widely recognized across different educational contexts throughout the world (e.g., Adelakun et al., 2025; Akbayrak & Douglas, 2021; Reddy & Sujathamalini, 2006): the Cubarithm. We have chosen it because it is paradigmatic for the initial learning of mathematics, that is emblematic of numeracy introduction in primary school for children with visual impairments. It is a tactile mathematical artifact, explicitly designed to enhance mathematical learning for children with visual impairments. In this educational landscape, the challenges extend beyond children with visual impairments, significantly impacting their educators (e.g., Rule et al., 2011). Teachers find themselves navigating complex terrain, often grappling with limited resources and a lack of professional education in using mathematical instruments

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(in the sense of Drijvers, 2019), like Cubarithm. These challenges highlight the dual nature of our research objective:

1. To uncover if and how the Cubarithm, when effectively guided by knowledgeable teachers, contributes to creating an inclusive mathematics education environment, within the classroom, for children with visual impairments.
2. To shed light on the intricate semiotic relationships within the Cubarithm that might facilitate mathematical understanding.

The purpose of this study is to provide insights about the Cubarithm as a semiotic tool, which can then inform the professional development to be provided to teachers who have to decide about the use of this resource.

Inclusive Education in Mathematics—The Case of the Italian Primary School

Inclusive education is a fundamental right for all students. Since the 1990s, the Italian primary school system has been recognised as an exemplary reference model for inclusive education policies in Europe, due to its long-standing legislative framework and its commitment to ensuring the active participation of all students in general educational settings. This development began with the enactment of Law No. 104/1992 and was subsequently reinforced by key legislative measures, including the *Guidelines for the Inclusive Education of Students with Disabilities* in 2009, Law No. 170/2010, the Directive on *Special Educational Needs* in 2012, and the *Inclusion Decree* for the years 2017–2019. The core principles underpinning this paradigm shift within primary education include the acceptance and appreciation of diversity as a source of enrichment, and the educational significance of catering to the individual needs of each student, not limited to those with specific impairments. Furthermore, the personalization and individualization of educational approaches, along with the development of inclusive pedagogical strategies, have played a crucial role in shaping a more inclusive primary education system. These initiatives extend beyond disability-related challenges, encompassing students facing socio-environmental, cultural, or familial difficulties, thereby fostering a broader culture of inclusion (cf. Italian Ministry of Education's institutional webpage on this issue: <https://www.miur.gov.it/web/guest/inclusione-e-interculturale>). Despite this long tradition which remains a crucial foundation and the main rationale for contextualizing our study, as is often the case, there is a significant gap between policy and practice: particularly in mathematics-related subjects. What happens is that, despite a strong and well-founded pedagogical background, and the existence of technologies (digital or analogical), educators often lack confidence in their mathematical content knowledge (Piroi et al., 2023). The special education tradition is well-established in both university instruction of educators and research, but it rarely engages in

dialogue with the mathematics education community, and vice versa. Things can be arguably trickier when narrowing the field to visual impairments:

There are no [at least Italian] text[book]s that adequately address the topic of visual impairment in a way that is accessible and usable by teachers and educators working with these students. [...] The need to investigate the developmental processes of these individuals arises from the fact that they are unique and specific to the disability, and cannot be compared to normal trends in terms of the presence or absence of certain abilities or skills (Bonfigliuoli & Pinelli, 2010, p. 7, our translation).

The Unique Challenges in Mathematics Education for Pupils with Visual Impairments

Blindness, in itself, does not seem to be an impediment to learning mathematics. Indeed, history shows that there have been a number of very successful blind mathematicians [...] the lack of access to the visual field does not diminish a person's ability to visualize—but modifies it, since spatial imagination amongst those who do not see with their eyes relies on tactile and auditory activity (Healy & Fernandes, 2014, pp. 61–62).

This statement offers at least two significant starting points. Firstly, it underscores how sight can be effectively replaced by other senses to access mathematical content, and the hands emerge as the most prominent surrogate for the eyes when engaging with mathematics, mainly because such content inherently entails manipulation of spatial representations and information (Healy, 2015; Marichal et al., 2022). Similarly, Vygotsky (1993) had previously argued that replacing sight with tactile exploration through the hands can lead to the emergence of unique perspectives. These perspectives differ from those of sighted students because of the different sensory tools used to access mathematics and construct mathematical understanding. Secondly, the enduring interconnectedness of visualisation and vision within the field of mathematics education is still emphasised. Indeed, Radford (2010) notes that the learning process in mathematics, particularly within the context of algebra, is linked to a gradual “domestication of the eye” (p. 10, our emphasis). This involves a lengthy evolution in which individuals learn to perceive and recognise—especially with their eyes—mathematical phenomena through culturally ingrained and effective means and methods. In this regard, according to Sfard (2008), the mathematical discourse is defined by specific *visual mediators*, i.e., “visible objects that are operated upon as a part of the process of communication” (p. 133). Certainly, these quotes do not suggest that the scholarly community limits mathematics to the visual sense. Rather, the intention is to emphasise that the existing literature portrays mathematics as primarily *visual* (Arcavi, 1999).

Sight is known for its synthetic and global nature, allowing individuals to perceive the whole. This means that vision enables the simultaneous perception of multiple elements within a mathematical representation, facilitating an immediate grasp of overall structures, patterns, and spatial relationships. In contrast, touch, which is predominantly used by learners with visual impairments, relies on haptic

exploration, where the whole emerges from the relationships between the parts. Learners with blindness tend to describe properties and relations using dynamic means that correspond to their physical interactions with objects (Healy & Fernandes, 2011). In particular, Braille codes, mainly characterised by a linear nature, serve as tactile substitutes for conventional written materials: the mathematical and thus spatial objects are converted into sequential representations within the Braille notation. This requires the incorporation of additional symbols (e.g., prefixes and suffixes),¹ which introduces an additional cognitive load on students and it naturally slows down their work (Edwards et al., 2006; Archambault et al., 2007). For example, a “number mark” must be placed before any number. However, this is just the simplest case: in practice, Braille mathematical notation requires numerous additional symbols, all arranged sequentially within the same line of text. Compounding this issue is the fact that Braille readers can only perceive one (and contextualised) part of the notation at a time, limiting their ability to gain a comprehensive overview of representations. To address the unique challenges that students with visual impairments face in accessing and learning mathematical content, educators have turned to multimodal, and in particular tactile, kinaesthetic and even speech-based approaches to learning (e.g., Healy & Fernandes, 2011). These methods allow students to physically interact with mathematical concepts; and tactile tools, including physical artifacts, are essential for creating a multisensory learning environment for pupils with visual impairments. However, it has only been a few years (e.g., Ahmetovic et al., 2017; Stylianidou & Nardi, 2019) since such unique perspectives due to different sensory access have been considered as an opportunity for the mathematics education of all students.

Mathematical manipulatives play a crucial role in mathematics education, serving as tangible tools that enable students to interact with mathematical concepts. Bartolini Bussi and Martignone (2014) distinguish between two primary categories (concrete and virtual) and define mathematical manipulatives as “artifacts used in mathematics education [...] handled by students in order to explore, acquire, or investigate mathematical concepts or processes and to perform problem-solving activities drawing on perceptual (visual, tactile, or, more generally, sensory) evidence” (p. 365). In summary, concrete manipulatives are physical objects that

¹This additional amount of Braille-specific symbols contributes to increase the cognitive load in reading mathematical notation in Braille. An attempt to mitigate this problem has been made in Braille codes design for the use of Braille displays. Braille codes on paper utilise six-dot cells, allowing for the representation of up to 64 symbols. In contrast, Braille displays employ cells with two additional dots, resulting in 8-dot Braille, which can encode up to 256 symbols per cell. These extra symbols help reduce the need for prefixes and suffixes, thereby minimising ambiguities, particularly in mathematical notation. As a result, most 8-dot Braille codes eliminate the use of signs for numbers, lowercase, and uppercase letters. This reduction enhances the tactile readability of mathematical expressions. Furthermore, there is not a universally accepted Braille code for mathematics (Marcone & Penteadó, 2013). Efforts to standardize at least 8-dot Braille codes across various European countries have been made in the LAMBDA project (Edwards et al., 2006; Schweikhardt et al., 2006). Despite these efforts, nowadays still remain differences among Braille codes for mathematics.

students can touch, perceive, and manipulate. They provide students with a rich sensory hands-on experience and allow them to explore mathematical concepts in a tangible way. Manipulatives are frequently employed in preschool and primary education, as well as in special education settings, where they serve as essential tools for developing students' foundational mathematical understanding (e.g., Kamii et al., 2001; Simon, 2022). However, it is paramount to emphasise that their educational value extends to secondary education and broader learning contexts. The National Council of Teachers of Mathematics (NCTM) has consistently encouraged the use of manipulatives at all grade levels, dating back to 1940. This advocacy reflects the belief that manipulatives can foster essential mathematical processes such as defining, conjecturing, arguing, and proving. However, regardless of context, the autonomy of students in using manipulatives remains a subject of scrutiny. Research indicates that providing teacher guidance significantly enhances learning outcomes compared to unassisted discovery, as indicated by a synthesis of the literature on instructional guidance (Alfieri et al., 2011). The introduction of an artifact into the classroom does not inherently predetermine how students will use, engage with, perceive or interpret it. Instead, it can give rise to a range of interpretations and viewpoints among students. In essence, manipulatives, like other educational tools, carry multiple layers of meaning, with the potential to generate diverse interpretations. This characteristic aligns with the concept of *polyphony* (Bartolini Bussi et al., 2005, p. 79), which refers to the multiplicity of voices or meanings that emerge during the use of these tools in classroom interactions. Each participant to the discussion may interpret the situation differently, based on their prior knowledge, cognitive frameworks, and interaction with the manipulative. This perspective is consistent with the notion of *theoretical ambiguity* discussed by Nührenböcker and Steinbring (2008).

This very ambiguity makes manipulatives suitable to all school levels, up to university [...].

This requires a very strong and deep analysis of manipulatives, from theoretical and epistemological points of view, and a study of the consequence of this analysis in teachers' design of tasks and interventions in the mathematics classroom (Bartolini Bussi & Martignone, 2014, p. 490).

The teacher assumes a crucial role in mediating mathematical meanings through the strategic use of manipulatives as tools for *semiotic mediation* (Bartolini Bussi & Mariotti, 2008). In the absence of teacher guidance, there is a risk of disconnection between students' hands-on interaction with manipulatives and the broader mathematical content entangled in the educational intentionalities of the teacher, potentially hindering learners' construction of mathematical knowledge. Within this theoretical framework, embedded within the Theory of Semiotic Mediation (Bartolini Bussi et al., 2005), the concept of the *semiotic potential* of an artifact takes centre stage. It encompasses the dual semiotic connection that an artifact can establish. On one hand, it relates to the personal meanings that emerge from the use of the artifact to solve a specific task. At the same time, it involves the mathematical meanings evoked by the use of the artifact, which are recognisable as mathematics

by an expert. This study is framed within this context and a more detailed discussion of the [theoretical framework](#) is available in the dedicated section.

The Cubarithm

Cubarithm is a specialised tactile concrete mathematical manipulative designed to facilitate the learning of basic arithmetic for individuals with visual impairments. A Cubarithm setup is shown in Fig. 1. On the left, a white box contains the digit-cubes; just below, six black digit-cubes display all the possible faces of each cube (all cubes are identical); on the right, a columnar-layout addition with an intentional mistake is represented on a white grid-slate.

There are also virtual versions of it, which can speed up the slow performance of the student using the material pieces of the artefact (Mikułowski et al., 2016), but in this study the focus is on the physical artefact.

Cubarithm consists of a set of (100) small plastic or metal cubes with an edge of 1 cm, designed to fit into a tactile slate featuring square holes. A slate is typically provided with 15 cells vertically and 20 horizontally, totalling 300 cells, with dimensions measuring approximately $1 \times 19 \times 25.3$ cm (Aparecida & Rodrigues, 2014; Turella & Conti, 2012). However, variations in size and shape are possible, such as square slates measuring about $1 \times 30 \times 30$ cm, or those comprising 15×20 cells (Fig. 1), depending on the manufacturer. These dimensions precisely match those of the cubes, ensuring easy placement and rearrangement within the grid structure, clearly demarcated for user convenience. The six-faces cubes are uniform in size and shape, featuring rounded edges and corners. The faces serve as the canvas for Braille representations of numerical values from 0 to 9. Specifically, on five

Fig. 1 A cubarithm

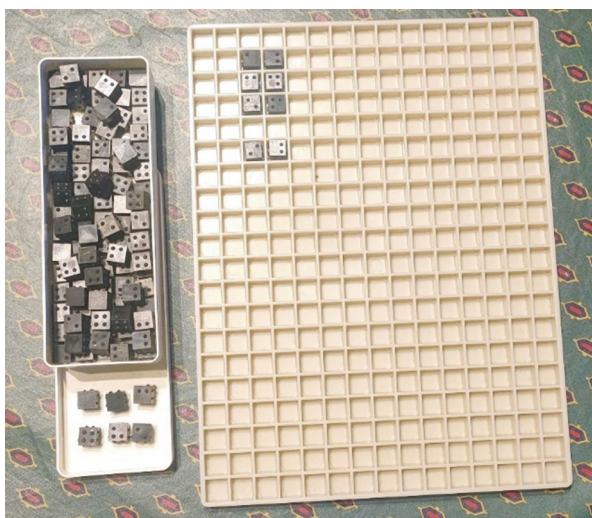


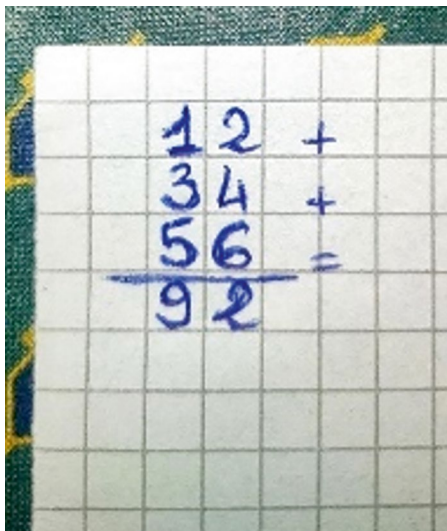
Fig. 2 An example of some Braille-labeled cubes
(Retrieved from: <https://www.braillenews.cloud/cubaritmo>)



of the cube's face a single Braille sign representing digit(s) is displayed (the sixth face remains empty) The Braille numerals are created with raised dots or tiny pyramids, enabling tactile differentiation and identification by users (Fig. 2).

Cubarithm is explicitly designed to perform the four “traditional” columnar-layout operations, sometimes referred to as “operations under the line” (Mikułowski & Terlikowski, 2020, p. 290). The Cubarithm was introduced to address the limitations of using Braille on paper for performing arithmetic operations arranged in a columnar layout. There are two primary methods for writing Braille on paper: the slate and stylus, and the Braille typewriter. Writing with a slate and stylus involves using a manual tool to emboss raised dots on paper. The slate consists of two hinged plates: the top plate has small openings for each Braille cell, while the bottom has indentations to guide the stylus. A sheet of paper is placed between the plates and the stylus puns dots into the paper from right to left. Once the writing is completed, the paper is flipped over to read the Braille from left to right. This method requires performing arithmetic operations in a columnar layout from left to right while writing so that they can be read from right to left, thus creating a significant cognitive burden that is impractical for pupils learning arithmetic. This issue is overcome by using a Braille typewriter, as characters are written from left to right by pressing key combinations corresponding to the raised dots in each Braille cell. However, difficulties arise in achieving precise vertical alignment of digits. Maintaining proper column alignment requires manually repositioning the typewriter's cursor to find the correct column and adjusting the paper around the roller to ensure correct row placement. Additionally, with both the slate and stylus and the Braille typewriter, erasing individual characters and replacing them with correct ones without damaging the paper is extremely difficult. With the Cubarithm, the digit-cubes are positioned and removed from the various horizontal and vertical cells. In Fig. 1, the slate displays an example of an incorrect columnar-layout addition ($12 + 34 + 56 = 92$). The top row and the first two columns from the left are empty. The representation of the addition “under the line” with three addends then begins. The first row contains the digit-cube 2 on the right and the digit-cube 1 on the left. The second row has the digit-cube 4 on the right and the digit-cube 3 on the left. The third row features the digit-cube 6 on the right and the digit-cube 5 on the left. An empty row marks the position of “the line” separating the addends from the result, which is placed below. Here, the digit-cube 2 is on the right and the digit-cube 9 on the left. For reference, Fig. 3 presents the analogous columnar-layout addition on paper, demonstrating how this notation is transposed onto the Cubarithm slate in Fig. 1.

Fig. 3 The incorrect
 $12 + 34 + 56 = 92$
 columnar-layout addition
 on grid paper



Theoretical Framework

As stated in the Introduction and to address the research aim, the necessary theoretical framework comprises the theoretical concept of semiotic potential, contextualised within the framework of Inclusive Mathematics as proposed by Moura (2020). This combination of frameworks allows for the analysis of whether Cubarithm can be, and how it can be, an inclusive “mediator” of educational interaction patterns and a meaningful producer of mathematical signs.

The Semiotic Potential

The Theory of Semiotic Mediation is grounded in a Vygotskian perspective on the teaching-learning of mathematics. Here, Bartolini Bussi and Mariotti (2008) intend learning as a collective process of the class group, concerning the evolution of signs used by students to discuss the mathematical meanings. In particular, the two researchers propose to structure teaching activities according to repeated iterations of the so-called didactic cycle: students are first asked to solve a task using a specific artifact; then, on the basis of artifact signs (or situated signs) emerged during that experience (i.e., informal signs, such as words and gestures, used by the students to talk about what they have experienced with the artifact), the teacher can orchestrate a phase of mathematical discussion (Bartolini Bussi, 1998). This phase is aimed at making these artifact signs evolve towards targeted mathematical signs (i.e., the signs of formal mathematics referring to the relevant mathematical meanings, that the teacher aims for students to co-construct).

For the design of a didactic cycle, a particularly important role is played by the teacher's choice of the artifact and the task for students to accomplish, in relation to the mathematical knowledge to be taught. Those choices, in fact, affect the emergence of situated signs suitable to unfold significant mathematical meanings. The artifact is conceived as a historical-cultural product, created to simplify or to solve a particularly frequent problem in a certain historical and cultural period. The artifacts of interest in mathematics usually work on the basis of culturally and historically determined parts of mathematical knowledge. For example, the compass incorporates the meaning of a circle as the locus of points equidistant from a center; the abacus incorporates different ways of writing, composing or decomposing numbers depending on their cultural origin (see Bartolini Bussi et al., 2018; Maschietto & Bartolini Bussi, 2011). The mathematical knowledge "incorporated" into the artifact's functioning, according to Bartolini Bussi and Mariotti (2008), is what makes it possible for situated signs to evolve towards targeted mathematical ones. As Baccaglini-Frank et al. (2023) explain

artifacts afford a *semiotic potential* (Bartolini Bussi & Mariotti, 2008): the use of an artifact for accomplishing a task fosters the emergence and transformation of signs (that is verbal utterances, written inscriptions, gestures, ...) related to the activity at stake (called situated signs); in turn, an expert can relate such situated signs to the appropriate mathematical knowledge. The *semiotic potential of an artifact with respect to a task and to certain mathematical meanings* is precisely its potential to foster the emergence of situated signs that are, on the one hand, related to the accomplishment of the task, and, on the other hand, relatable to mathematical meanings targeted by the teacher (p. 4).

Thus, the capacity to discern the semiotic potential of an artifact (with respect to a task and to certain mathematical meanings) is what allows the teacher to predict and manage the evolution of situated signs towards the targeted mathematical ones:

any artefact will be referred to as tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention. [...] The key point according to our hypothesis is that the twofold role played by the artefact, both as a means in accomplishing a task, and as a tool of semiotic mediation to accomplish a didactical objective, can be fully exploited. The role of the teacher is then crucial and not incidental and the teaching sequence has to present certain peculiarities [referring to the structure of the didactical cycle] (Bartolini Bussi & Mariotti, 2008, p.754).

The Inclusive Interaction Pattern

This study is in alignment with Moura's (2020) and Skovsmose's proposals of engaging all the learners in the existing curriculum (in Figueiras et al., 2016, pp. 19–20), "learning in mixed-ability groups provides richer opportunities for learning, as learning is related to processes of negotiating, explaining, and noticing". However, the use of assistive technologies does not guarantee the inclusion of students with visual impairments in teaching-learning practices (Ahmed & Chao, 2018). In our vision, "the notion of inclusion goes beyond simple, though

preliminary, accessibility. [...] [T]he presence of these technologies in classrooms, where students with and without visual impairments are together, is the first step in promoting a new classroom relationship and communication” (Piroi et al., 2023, p. 60).

In mathematics classrooms, interactions between students, and even between the class teacher and the student with visual impairments, are frequently indirect. Communication and engagement typically occur through the intermediary role of a support teacher specializing in special educational needs. Here, prevailing pedagogical approaches often substitute visual cues with alternative sensory modalities, thereby expecting students with visual impairments to conform to the behaviours of their sighted peers (Ahmed & Chao, 2018). This trend is illustrated in Fig. 4.

In an inclusive classroom, with the presence of assistive technologies as inclusive instruments, this pattern evolves into the one represented in Fig. 5, as demonstrated by Piroi et al. (2023). For instance, consider a mathematics activity in which students explore numeric patterns and sequences. In a traditional setting (Fig. 4), a student with visual impairments might rely on a support teacher to describe what is being drawn or written by their peer, creating an indirect mode of interaction. However, with swell paper or a digital board equipped with screen-reading software and a tactile display, the student can directly access and manipulate the different representations. For example, if the class is diagrammatically identifying the rule behind a sequence, the student can explore the patterns using audio feedback and tactile overlays, actively participating in the discussion rather than receiving second-hand explanations. This transformation allows all students to engage directly with mathematical structures, as shown in Fig. 5, fostering more direct and inclusive interactions. Tactile tools are essential for creating a multisensory learning environment with inclusive interactions for students with visual impairments. However, this connection between tactile perception and visual impairment often results in accommodating students with visual impairments by modifying visual materials typically designed for sighted students. While some accommodations have been successful, such as creating tactile shapes for students with visual impairments while the rest of the class uses visual shapes from the standard textbook (Argyropoulos & Stamouli, 2006), other accommodations face various limitations. These limitations can be

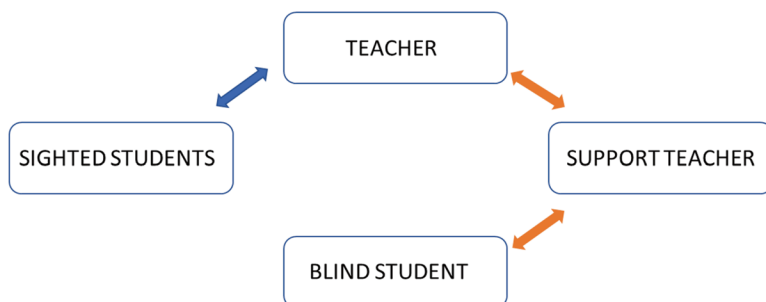


Fig. 4 Interaction patterns in classes with blind (sic.) students and a support teacher (Piroi et al., 2023, p. 61)

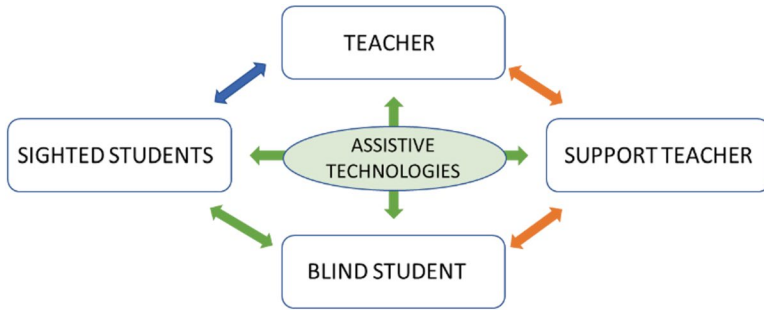


Fig. 5 Inclusive interaction patterns, mediated by assistive technologies (Piroi et al., 2023, p. 72)

categorised as technical, affective, or social. In this study, an analysis of the semiotic potential of the Cubarithm is conducted to examine how these three limitations could be related to semiotic mediation processes and the emergence of inclusive interaction patterns. By exploring these relationships, this study aims to provide deeper insights into how the Cubarithm can shape both individual meaning-making and classroom dynamics.

Cubarithm: An In-Depth Semiotic Analysis

This section presents an a priori semiotic analysis of the Cubarithm, aimed at investigating its semiotic potential to mediate mathematical meaning through haptic interaction. While this analysis is theoretically grounded, it emerges from a structured didactic observations (Manolino, 2024) of four young users (grades 1 and 2, with different visual impairments) engaged in simple tasks on columnar-layout calculations designed by their classroom teachers, as well as an interview with an adult blind user, expert in STEM, who had extensive experience interacting with the Cubarithm during his primary education. These empirical insights have informed our understanding of the challenges and affordances associated with this artifact. However, to our knowledge, no task-based studies explicitly informed by the framework of Semiotic Mediation have yet been conducted or documented in the literature. The absence of such research underscores the need for future empirical investigations that systematically explore how learners construct mathematical meaning through engagement with the Cubarithm in structured instructional settings. This study, therefore, serves as a preliminary step in theorizing the semiotic potential of the Cubarithm, providing a foundation for future research that integrates classroom-based interventions and controlled experimental designs. The unique design of the Braille systems used in the Cubarithm allows for a single cube to represent different digits based on its spatial orientation and for the same positioned cube to represent some other digits based on its orientation within the slate. For instance, the Braille character corresponding to the letter *b* (Braille dots 1 and 2), which represents the digit 2, can be rotated by 90 degrees to become the letter *c*

(Braille dots 1 and 4), which represents the digit 3 (see Fig. 6). Through this mechanism, each cube can be oriented and placed on the slate to assume any desired digit from 0 to 9. Figure 7 illustrates the symmetrical pairs of digits that emerge from this system.

Students can arrange cubes in various configurations within the slate, either placing them side by side vertically or horizontally to create different combinations of Braille digits.

This manipulation demands a significant amount of time and imposes a high cognitive effort, particularly for younger students in primary or nursery education. Several cognitive challenges arise:

- **Symbol Recognition**—At this stage of schooling, students are still in the process of learning Braille, so symbol recognition is not yet automatic.
- **Orientation Awareness**—Identifying the correct digit requires a careful consideration of the cube's orientation.
- **Hand-Cube-Slate Coordination**—The student must precisely place the cube in the correct cell while ensuring the proper orientation.
- **Memory Load**—Since students with visual impairments lack a synthetic and global visual perception of the slate, they must rely on memory to track which cubes have already been placed. Otherwise, they would need to frequently re-explore the slate through touch.

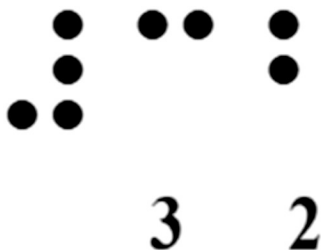
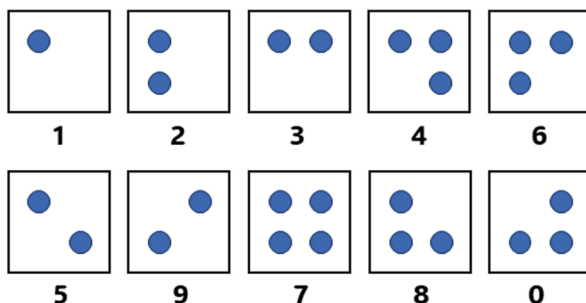


Fig. 6 A braille representation of the number 32. [Note: The first symbol is known as the number mark, i.e., a sign placed in front of digits to distinguish it from the analogous group of letters. When the number mark is omitted, this notation represents the word ‘bc’.]

Fig. 7 The digits in the Braille-labeled cubes system



The time required for this process and the associated cognitive load are non-negligible. By contrast, in its paper-based “traditional” columnar-layout operations counterpart, columnar diagrams are designed specifically to relieve the student from this effort and to alleviate this cognitive burden. The structure of these diagrams is designed to enable a quicker and more intuitive digit placement process, allowing students to focus their cognitive resources on the calculation itself rather than on managing the spatial organization of digits.

Across various bibliographic references studied and presented so far, as well as in teacher education courses conducted (at least) in Italy to date by academic institutions and associations for visual impairments, the Cubarithm has been consistently presented as a tool for mathematics education (Del Campo, 2000). Frequently cited as either the sole or primary mathematical artifact for students with visual impairments in primary education, its use appears indispensable when addressing mathematics education in this context. However, it is typically introduced solely for computational purposes, being described as “a tool for learning traditional columnar layout calculations by visually impaired students” (Mikułowski et al., 2016, p. 17). The Cubarithm is thus regarded as a faithful physical counterpart of what is commonly written on a paper sheet in mathematics lessons with squares in its entirety. Unlike the Braille slate and Braille typewriter, it enables the precise spatial placement and removal of digits within a defined grid structure, making them easy to locate through touch. Consequently, a student with visual impairments can perform traditional columnar layout calculations in a way that is considered analogous to how sighted peers work with pen and paper, although the situated signs will be predominantly haptic rather than graphic-script type. However, it is crucial to recognize that the teaching of “operations under the line”, if not strongly anchored in their meaning, leads (all) students to rely on more or less automated procedures for spatial displacements of digits. As described by Orton-Flynn and Richards (2000) in their discussion of dyslexics—a perspective that can be extended to learners more broadly:

[students] can be confused about the direction in which they should work on a particular calculation. Western languages are read and written from the top left to the bottom right of the page. For multiplication, however, pupils are taught to work out from the bottom right, diagonally across to the top left. Addition is done by calculating from the top of the furthest right-hand column, and working towards any left-hand columns. Division is, again, laid out quite differently. Is it so surprising that many pupils become confused? (p. 206).

It is even more complicated for a student with visual impairments who cannot grasp the whole view. Figure 8 provides four representations of operations on the cubarithm. It is evident how close the transposition from the paper is. But how to keep track of the intermediate steps? In a faithful transposition, there should also be traceability of intermediate signs, such as remainder and decomposition signs. How can this be done in the cubarithm without confusion? Some educators place such signs above the first row of the operation, but in this way a re-reading in case of forgetfulness, how does it take into account what was already there and what was added afterwards during the calculation?

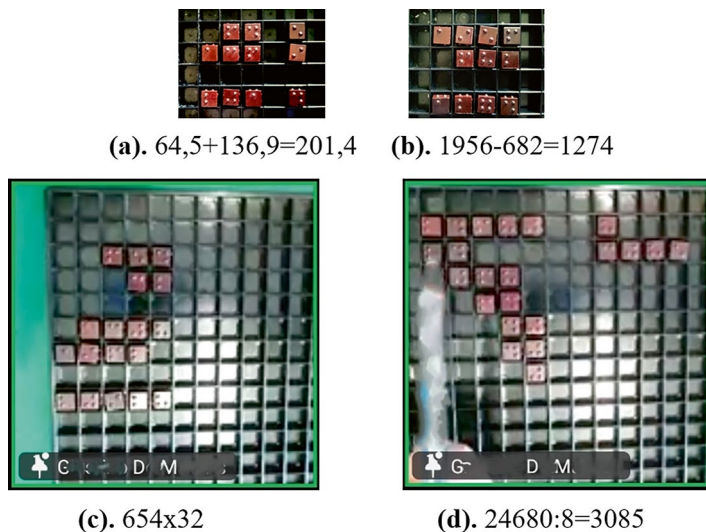


Fig. 8 Four “operations under the line” displayed during a teacher education course

Despite this, in teacher education courses, Cubarithm is almost always taken to the extent of providing rules for carrying out column operations, both with and without decimal numbers. A commonly found description is that, using the Cubarithm, arithmetic operation signs (+, −, ×, ÷, =) are generally omitted. These symbols are expected to be inferred from the context, as their inclusion is thought to lead to confusion. This arises from the fact that the Braille representations of digits within the Cubarithm are encoded using four dots, making some operation symbols visually and tactilely similar to certain numbers (e.g., ‘+’ resembling the digit 6, ‘.’ resembling 4, ‘×’ resembling 8, and ‘=’ resembling 7). Cubarithm, like Braille, is thus heavily context-dependent, relying on implicit conventions for mathematical expressions.

Another characteristic aspect of this artifact is its ephemeral nature. A frequently cited analogy in teacher education materials describes it as “a ‘rough draft’ paper notebook with squares: once the exercise is completed, all cubes are removed and stored back in the designated container”—as literally quoted from a recent Italian teacher education course handout. This contrasts with traditional paper-based work, where the permanence of written steps serves an essential educational function. Despite the Cubarithm intention to replicate the use of a sheet of paper, it does not retain all its aspects. In fact, at the educational level, one of the most essential aspects is lost: the permanence of errors and processes. The absence of a record of previous steps poses a challenge: the learning process benefits from the ability to review past work, identify errors, and reflect on the sequence of calculations. The Cubarithm, in its standard use, does not support this type of reflective engagement. In addition to the effort and slowness of manipulating and positioning the cubes on the slate, the transcription of what happens on the Cubarithm cannot occur. And if

writing on paper or digital devices is possible, in any case for students with visual impairments it is linear: the spatial dimension of the columns is lost.

Despite these limitations, the Cubarithm can serve valuable preparatory functions in early mathematics education. For instance, its interactional properties differentiate it significantly from standard paper-based mathematical notation. It can be used in activities designed to stimulate cognitive skills such as spatial alignment, laterality, and orientation tasks. Additionally, it can be employed to reinforce mathematical language and Braille notation through structured exercises (e.g., RoboBraille.org, 2016). The design of the slate features a smooth, flat surface that offers stability during mathematical interactions, providing ample space for arranging and organising the Braille-labeled cubes in the space. The grid structure provides a tactile framework for arranging haptic elements, allowing users to create modelizations and mathematical structures, and explore mathematical relationships through touch. From a semiotic perspective, the slate serves as a physical medium for a tactile sign system. Its grid structure provides a haptic framework for arranging mathematical signs, allowing the user to construct and explore through touch. This physical arrangement introduces a structured spatial representation and reasoning that can support early algebraic reasoning. The process of manipulating the cubes involves a continuous interplay between perception, action, and cognition, reinforcing the semiotic mediation between the physical manipulative and the abstract mathematical concepts it represents. The need to physically rotate and place the cubes correctly could mirror essential mathematical processes such as rearranging terms, aligning place values, and structuring relations, making it a tool that may extend beyond mere computation. At the same time, the Cubarithm requires careful instructional strategies to mitigate cognitive overload, ensuring that students can focus on conceptual understanding rather than on the mechanics of manipulation alone. Its use in mathematics education should be carefully contextualized. Rather than serving solely as a computation aid, it may be more effectively integrated into broader pedagogical strategies that incorporate multiple modalities, including digital and paper-based tools, to support a more comprehensive mathematical learning experience.

Discussion and Conclusions

The semiotic analysis conducted in this study has revealed how the Cubarithm encapsulates the technical, affective, and social limitations commonly encountered in adapting tactile mathematical tools originally designed for sighted learners. These limitations, which influence both the usability and educational effectiveness of the tool, must be considered in relation to how the mathematical meaning is constructed through interaction with the artifact. As established in the theoretical framework, an artifact's semiotic potential is crucial in determining how it can foster the emergence of mathematical signs and contribute to semiotic mediation in the classroom. The Cubarithm, while historically recognized as an essential computational

tool for students with visual impairments, presents challenges that can directly impact both the evolution of mathematical meaning and the inclusivity of interaction patterns in the learning environment.

The technical limitations of the Cubarithm concern the potential motor and spatial-perceptual difficulties that students may encounter when manipulating the small cubes (for instance, while correctly positioning them on the slate). These tasks require fine motor skills and spatial reasoning, which can be especially demanding for young learners or those still developing proficiency in Braille, and which Cubarithm does not support, but rather tires them out. Additionally, the lack of a permanent external representation of their work, particularly concerning errors, prevents students from easily reviewing and reflecting on their processes. These technical limitations may, in turn, lead to educative and affective constraints. The cognitive burden associated with handling the Cubarithm may shift students' focus away from conceptual engagement and towards the mere mechanics of digit placement. This misalignment between effort and learning outcomes can diminish students' perceived self-efficacy, potentially reducing their motivation to engage with mathematical tasks. Specifically, the slowness and challenges in placing and identifying the cubes on the Cubarithm impose a significant workload, diverting attention from the primary objective, which is to fulfil the specific mathematical task. Consequently, there is a risk that students with visual impairments may perceive a decrease in autonomy and thus self-efficacy, resulting in decreased interest in mathematics (Bernabè, 2024; Mason & McCall, 2013). Moreover, the social limitations of the Cubarithm arise from its reliance on Braille-labeled cubes, which are inaccessible to peers and teachers unfamiliar with Braille. This characteristic may result in reduced interaction between the student with visual impairments and their peers, often resulting in the isolation of the student with visual impairments rather than fostering inclusion and multimodality. As demonstrated in the analysis, the Cubarithm's role as a mediator of inclusive interaction patterns is compromised by its reliance on a not shared socio-cultural language and sign production. It does not fully facilitate shared engagement in mathematical discourse. Here the Cubarithm fails in its assistive and inclusive role (see Fig. 5). Currently, alternatives to the Cubarithm based on digital devices could render this tool more inclusive. However, the skills required to use such tools may not align with those typically expected in primary education. Young students may lack proficiency in 10-finger typing, which is necessary for efficient input on digital devices, and may also require extensive training to develop a comprehensive understanding of Braille on refreshable Braille displays. Additionally, the transition from a tangible, spatially structured artifact like the Cubarithm to a linear, text-based digital format may introduce further cognitive challenges, particularly in early mathematical learning, where direct manipulation of objects plays a crucial role in conceptual development.

Despite these limitations, the Cubarithm remains a particularly useful tool in teaching students with blindness, provided that its use is expanded beyond simple traditional use for teaching column-based computational tasks. The Cubarithm offers numerous opportunities for educational applications. Firstly, the ability to directly manipulate Braille-numbered cubes in a tangible two-dimensional space

allows for the design of educational activities aimed at developing visuospatial and visuoconstructive skills, such as Euclidean plane isometry (see the issue of cubes rotation and slate spatialisation) or visual pattern proposals. More generally, the Cubarithm can serve as an effective semiotic mediator if integrated into activities designed to enhance spatial reasoning, lateralization, and proprioception. The development of spatial reasoning and visuospatial skills is a fundamental aspect of early education, as these abilities support a wide range of cognitive and literacy-related processes. In sighted learners, the coordination between visual perception and motor actions—commonly referred to as oculomotor coordination—plays a crucial role in integrating spatial information (Pento, 2020), facilitating the discrimination of similar signs, and supporting the structured organization of written language and representation (Colina, 2015). Moreover, successful encoding relies not only on semantic acquisition but also on the ability to analyze and process spatial control and visuospatial characteristics, skills that progressively develop throughout early childhood (Cornoldi et al., 1997). For students with visual impairments, the absence of direct visual input necessitates alternative pathways for acquiring these essential competencies. The Cubarithm, by enabling direct manipulation of signs within a structured two-dimensional space, provides a haptic analogous to the role that written text and diagrammatic representations play in sighted learners' development (Olive & Passerault, 2012; Tseng & Cermak, 1993). Through targeted educational activities, this tool can support the refinement of spatial reasoning, promote intermodal integration, and foster a deeper understanding of mathematical structures. In this sense, the Cubarithm slate can serve as a physical counterpart to the sheet of paper, a conventional artifact essential for spatial control, organization, and structuring in early education. Additionally, it can be employed to develop the skills necessary for transitioning from a tangible space to a virtual one, accessible through tools like keyboards, voice feedback, or Braille displays. Individuals with blindness face difficulties in mastering virtual spaces without adequate experience in physical space through object manipulation activities (Del Campo, 2000), for which the Cubarithm may be particularly well-suited. The Cubarithm can facilitate the development of primary school-level mathematical skills, such as completing numerical sequences, and secondary school-level skills, such as introducing coordinate systems, the concept of matrices, or algebra structure sense (Bernabè, 2024; Maffia et al., 2025).

Indeed, the Cubarithm can regain its role as a mediator for inclusive interaction patterns in the classroom if it is used for educational purposes that extend beyond calculating in columns, and rather work on spatiality, lateralisation, sheet space usage and proprioception. In this case, not only does the Cubarithm become an essential tool for the student with visual impairments, but it acts as a pivotal tool for the whole class to develop these target competencies for the early years of primary school. In the context of elementary education, learning mathematics involves more than performing arithmetic calculations. It requires understanding numerical structures, recognizing patterns, and developing problem-solving skills. The Cubarithm, when used solely as a computational tool, does not inherently support these broader learning objectives. Furthermore, educational research suggests that multi-sensory

learning environments enhance concept retention and comprehension. Combining the Cubarithm with auditory feedback systems or tactile-graphical representations could mitigate some of its limitations and expand its role beyond arithmetic operations, fostering a deeper engagement with mathematical concepts.

Often mathematics education, as challenging for teachers caring for students with special needs, is conveyed through a “replication” of tools and procedures typical of traditional teaching-learning, instead of providing an opportunity to realise an inclusive mathematics teaching-learning. We look at Inclusive Mathematics as a framework to renew the conceptualisation of mathematical contents through a multimodal and sensorial interaction pattern.

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References

- Adelakun, S., Abdusalaam, A., Evangeline, C., Michael, K., & Musibau, S. (2025). Empowering BVI students in mathematics: The impact of STEM KIT tiles and board. *American Journal of Science Education Research*, 4(1), e230. <https://www.cmjpublishers.com/article-view-ajser-e229/>
- Ahmed, I., & Chao, T. (2018). Assistive learning technologies for students with visual impairments: A critical rehumanizing review. *Investigations in Mathematics Learning*, 10(3), 173–185.
- Ahmetovic, D., Alampi, V., Bernareggi, C., Gerino, A., & Mascetti, S. (2017). Math melodies: Supporting visually impaired primary school students in learning math. In *W4A '17: Proceedings of the 14th international web for all conference* (Vol. 26, pp. 1–2). Association for Computing Machinery. <https://doi.org/10.1145/3058555.3058583>
- Akbayrak, K., & Douglas, G. (2021). Examining specialist teachers’ conceptualisations of their roles in supporting learners with vision impairment: A comparative analysis of Turkey and England using Bronfenbrenner’s ecological systems theory. *European Journal of Special Needs Education*, 37(6), 979–993. <https://doi.org/10.1080/08856257.2021.1989129>
- Alfieri, L., Brooks, P. J., Aldrich, N. J., & Tenenbaum, H. R. (2011). Does discovery-based instruction enhance learning? *Journal of Educational Psychology*, 103(1), 1–18.
- Aparecida, R., & Rodrigues, C. K. (2014). Educação matemática inclusiva: Uma análise quantitativa de investigações acadêmicas [Inclusive mathematics education: A quantitative analysis of academic research]. *Revista de Educação, Ciências e Mathematics*, 4(1), 69–84.
- Arcavi, A. (1999). The role of visual representations in the learning of mathematics. In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 55–80).
- Archambault, D., Stoger, B., Fitzpatrick, D., & Miesenberger, K. (2007). Access to scientific content by visually impaired people. *Upgrade-cepis*, 8(2), 29–42.
- Argyropoulos, V., & Stamouli, M. (2006). A collaborative action research project in an inclusive setting: Assisting a blind student. *British Journal of Visual Impairment*, 24(3), 128–134. <https://doi.org/10.1177/0264619606066187>
- Bartolini Bussi, M. G. (1998). Verbal interaction in mathematics classroom: A Vygotskian analysis. In H. Steinbring, M. G. Bartolini Bussi, & A. Sierpinkska (Eds.), *Language and communication in mathematics classroom* (pp. 65–84). *NCTM*.

- Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. D. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Routledge-Taylor & Francis Group. <https://doi.org/10.4324/9780203930236>
- Bartolini Bussi, M. G., Mariotti, M. A., & Ferri, F. (2005). Semiotic mediation in the primary school: Dürer's glass. In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign-grounding mathematics educations. Festschrift for Michael Otte* (pp. 77–90). Springer.
- Bartolini Bussi, M. G., & Martignone, F. (2014). Manipulatives in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 365–372). Springer.
- Bartolini Bussi, M. G., Inprasitha, M., Arzarello, F., Bass, H., Kortenkamp, U., Ladel, S., Lajoie, C., Ni, Y., Rottmann, T., Sarungi, V., Soury-Lavergne, S., & Young-Loveridge, J. (2018). Aspects that Affect Whole Number Learning: Cultural Artefacts and Mathematical Tasks. In M. G. Bartolini Bussi & X. Sun (Eds.), *Building the Foundation: Whole Numbers in the Primary Grades* (pp. 181–226). Springer. https://doi.org/10.1007/978-3-319-63555-2_9
- Baccaglioni-Frank, A. E., Funghi, S., Maracci, M., & Ramploud, A. (2023). Learning about multiplication by comparing algorithms: “One times one, but actually they are ten times ten.” *The Journal of Mathematical Behavior*, 70, 101024. <https://doi.org/10.1016/j.jmathb.2022.101024>.
- Bernabè, V. (2024). Genesi strumentale nello studio dell'algebra lineare: un'analisi sperimentale tra soggetti con disabilità visive [Instrumental genesis in the study of linear algebra: an experimental analysis among subjects with visual impairments]. *Master dissertation*, University of Bologna.
- Bonfigliuoli, C., & Pinelli, M. (2010). *Disabilità visiva: Teoria e pratica nell'educazione per alunni non vedenti e ipovedenti [Visual impairment: Theory and practice in education for blind and visually impaired pupils]*. Erikson.
- Colina, D. (2015). *150 giochi logico-percettivi. Sviluppare i prerequisiti visuopercettivi e visuo-spaziali per l'apprendimento [150 logic-perceptive games. Developing visuoperceptive and visuospatial prerequisites for learning]*. Erikson.
- Cornoldi, C., Friso, G., Giordano, L., Molin, A., Poli, S., Rigoni, F., et al. (1997). *Abilità visuo-spaziali. Intervento sulle difficoltà non verbali di apprendimento [Visual-spatial skills. Intervention on non-verbal learning difficulties]*. Erikson.
- Del Campo, J. E. F. (2000). *La enseñanza de la matemática a los ciegos [Teaching mathematics to the blind]* (A. García Martín, Trans.). ONCE and Biblioteca italiana per ciechi “Regina Margherita” ONLUS.
- Drijvers, P. (2019). Embodied instrumentation: Combining different views on using digital technology in mathematics education. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in mathematics education* (pp. 8–28). Utrecht and ERME. <https://hal.science/hal-02436279>
- Edwards, A. D. N., McCartney, H., & Fogarolo, F. (2006). Lambda: A multimodal approach to making mathematics accessible to blind students. In *Assets '06: Proceedings of the 8th international ACM SIGACCESS conference on computers and accessibility* (pp. 48–54). Association for Computing Machinery. <https://doi.org/10.1145/1168987.1168997>
- Figueiras, L., Healy, L., & Skovsmose, O. (2016). Difference, inclusion, and mathematics education. *Journal Studies in MathEdu*, 9(3), 15–35. <https://doi.org/10.17921/2176-5634.2016v9n3p15-35>
- Gerino, A., Alabastro, N., et al. (2014). MathMelodies: Inclusive design of a didactic game to practice mathematics. In K. Miesenberger et al. (Eds.), *Computers helping people with special needs* (pp. 564–571). Springer. https://doi.org/10.1007/978-3-319-08596-8_88
- Healy, L. (2015). Hands that see, hands that speak: Investigating relationships between sensory activity, forms of communicating and mathematical cognition. In S. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education*. Springer.
- Healy, L., & Fernandes, S. H. A. A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics*, 77(2–3), 157–174.

- Healy, L., & Fernandes, S. H. A. A. (2014). Blind students, special needs, and mathematics learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 61–63). Springer.
- Kamii, C., Lewis, B., & Kirkland, L. (2001). Manipulatives: When are they useful? *The Journal of Mathematical Behavior*, 20(1), 21–31. [https://doi.org/10.1016/s0732-3123\(01\)00059-1](https://doi.org/10.1016/s0732-3123(01)00059-1)
- Maffia, A., Manolino, C., & Miragliotta, E. (2025). There is more to algebra than meets the eye: The case of blindness. *Educational Studies in Mathematics*, 118(3). <https://doi.org/10.1007/s10649-025-10394-0>
- Manolino, C. (2024). L'osservazione in Matematica [observation in mathematics]. *La Vita Scolastica*, 38(3), 6–7. <https://www.giuntiscuola.it/articoli/losservazione-in-matematica>
- Marcone, R., & Penteadó, M. G. (2013). Teaching mathematics for blind students: A challenge at the university. *Revista Internacional de Pesquisa em Educação Matemática*, 3(1), 23–35.
- Marichal, S., Rosales, A., Perilli, F. G., Pires, A. C., & Blat, J. (2022). Auditory and haptic feedback to train basic mathematical skills of children with visual impairments. *Behaviour & Information Technology*, 42(8), 1081–1109. <https://doi.org/10.1080/0144929x.2022.2060860>
- Maschietto, M., & Bartolini Bussi, M. G. (2011). Mathematical machines: From history to mathematics classroom. In O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics. Mathematics teacher education (MTEN)* (Vol. 6, pp. 227–245). Springer. https://doi.org/10.1007/978-0-387-09812-8_14
- Mason, H., & McCall, S. (2013). *Visual impairment: Access to education for children and young people*. Routledge.
- Mikułowski, D., & Terlikowski, G. (2020). Towards supporting skills of making columnar layout operations by visually impaired students. In *2020 IEEE 20th international conference on advanced learning technologies (ICALT)* (pp. 290–292). <https://doi.org/10.1109/icalt49669.2020.00094>
- Mikułowski, D., Terlikowski, G., & Brzostek-Pawłowska, J. (2016). Virtual cubarithms: Innovative assistive technology for teaching the blind and visually impaired students traditional columnar layout operations. *Studia Informatica: Systems and Information Technology*, 1–2(20), 17–25.
- Moura, A. Q. (2020). O encontro entre surdos e ouvintes em cenários para investigação: das incertezas às possibilidades nas aulas de matemática [The encounter between deaf and hearing people in scenarios for research: from uncertainties to possibilities in mathematics classes]. *Doctoral dissertation*, UNESP-Rio Claro.
- Nührenböcker, M., & Steinbring, H. (2008). Manipulatives as tools in teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education* (pp. 157–181). Sense.
- OECD. (2023). PISA 2022 mathematics framework. In *PISA 2022 assessment and analytical framework* (pp. 18–98). OECD Publishing. <https://doi.org/10.1787/7ea9ee19-en>
- Olive, T., & Passerault, J.-M. (2012). The visuospatial dimension of writing. *Written Communication*, 29(3), 326–344. <https://doi.org/10.1177/0741088312451111>
- Orton-Flynn, S., & Richards, C. C. J. (2000). The design and evaluation of an interactive calculator for children. *Digital Creativity*, 11(4), 205–217. <https://doi.org/10.1076/digc.11.4.205.8874>
- Pento, G. (2020). Fondamenti e didattica delle attività motorie per l'età evolutiva. Apprendere il movimento nella scuola dell'infanzia e nel primo ciclo di istruzione [Motor activity fundamentals and education for developmental age. Learning movement in pre-school and first cycle education]. Cleup.
- Piroi, M., Manolino, C., Armano, T., Taranto, E., & Capietto, A. (2023). Teacher professional development via a MOOC on assistive technology for visually impaired students learning mathematics. *Journal of Mathematics Education*, 16(1), 59–78. <https://doi.org/10.26711/007577152790164>
- Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For The Learning of Mathematics*, 30(2), 2–7.
- Reddy, G. L., & Sujathamalani, J. (2006). *Children with disabilities: Awareness, attitude and competencies of teachers*. Discovery Publishing House.

- RoboBraille.org. (2016). *Teaching the blind: Mathematics. Teaching guides*. Erasmus+ IncluTech. <https://www.robobraille.org/resources/teaching-guides/>
- Rule, A. C., Stefanich, G. P., Boody, R. M., & Pfeiffer, B. (2011). Impact of adaptive materials on teachers and their students with visual impairments in secondary science and mathematics classes. *International Journal of Science Education*, 33, 865–887.
- Schweikhardt, W., Bernareggi, C., Jessel, N., Encelle, B., & Gut, M. (2006). LAMBDA: A European system to access mathematics with braille and audio synthesis. In K. Miesenberger, J. Klaus, W. L. Zagler, & A. I. Karshmer (Eds.), *Computers helping people with special needs. ICCHP 2006* (pp. 1223–1230). Springer. https://doi.org/10.1007/11788713_176
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.
- Simon, M. A. (2022). Contributions of the learning through activity theoretical framework to understanding and using manipulatives in the learning and teaching of mathematical concepts. *The Journal of Mathematical Behavior*, 66, 100945. <https://doi.org/10.1016/j.jmathb.2022.100945>
- Stylianidou, A., & Nardi, E. (2019). Tactile construction of mathematical meaning: Benefits for visually impaired and sighted pupils. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd conference of the International Group for the Psychology of mathematics education* (Vol. 3, pp. 343–350). PME.
- Tseng, M. H., & Cermak, S. A. (1993). The influence of ergonomic factors and Perceptual–Motor abilities on handwriting performance. *American Journal of Occupational Therapy*, 47(10), 919–926. doi:<https://doi.org/10.5014/ajot.47.10.919>.
- Turella, C. F., & Conti, K. C. (2012). Matemática e a deficiência visual: Atividades desenvolvidas com o material Dourado [Mathematics and the visually impairments: Activities developed with gold material]. *Benjamin Constant*, 52, 2.
- Vygotsky, L. S. (1993). The blind child. In R. W. Rieber, A. S. Carton, & A.S. (Eds.), *The collected works of L.S. Vygotsky* (pp. 97–109). Springer. https://doi.org/10.1007/978-1-4615-2806-7_6

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Learning Trajectory for the Measure of Length: How to Move Toward the Sense of Measure



Elisabetta Robotti and Alessandra Boscolo

Introduction

The Italian National Guidelines for Kindergarten (ages 3–6) encourage children to develop measuring skills. This process does not necessarily rely on standard measuring tools, such as rulers, meters, and balances, but rather on any tools that help children to develop the meaning of measure, as a product of the action of measuring with a measurement unit. From this perspective, our research focuses on a learning trajectory to develop the meanings of length measurement in kindergarten and early years of primary school.

Research shows that measurement can be effectively introduced in preschool education (Robotti, 2022). For example, Clements and Stephan (2004), argue that children's understanding of measurement has its roots in the preschool years. Preschoolers recognize continuous attributes like length and weight, although they do not know how to measure them accurately. Three-year-olds often rely on perceptual cues, such as visually comparing objects to determine which is longer (Van den Heuvel-Panhuizen & Buys, 2008). By ages four to five, however, most children can learn to move beyond perceptual cues and start developing more accurate measuring skills. As Boulton-Lewis et al. (1996) suggested, when children directly compare two objects and recognize equality or inequality, they are ready to learn how to measure by connecting numbers to quantities. In this context, measurement can be defined as the process of assigning a number associated with a unit of measurement, to continuous quantities. These researchers describe measurement as involving two key aspects: identifying a unit of measurement and subdividing (both mentally and physically) the object into that unit by placing it alongside (or on top of) the object to be measured. To better understand how students engage in the physical activity of measuring length, Clements and Stephan identify six foundational concepts:

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partitioning (mentally dividing an object into equal-size unit), *unit iteration* (repeatedly placing the unit of measure along the length to be measured), *transitivity* of measure, *conservation* (understanding that moving an object does not change its length), *accumulation of distance* (understanding that as you iterate a unit along the length of an object and count the iteration, the number words signify the space covered by all units counted up to that point) and *relation to number* (understanding that the measurement is expressed as a number combined with the chosen unit of measurement). Kamii and Clark (1997) emphasize that comparing lengths is central to developing the concepts of conservation, transitivity, and unit iteration. For this reason, they recommended beginning instruction with comparisons using non-standard units. As well, several studies highlight that measurement is a complex set of concepts and skills that takes time to master. Children need to confront real measurement problems to build and internalize the concept of measurement.

In alignment with the previous research, in our work, we conceptualize a learning trajectory (LT) for the measure of length. The idea of elaborating a LT for the concept of measurement is not new. Szilágyi et al. (2013) investigated the development of this concept in students from prekindergarten through second grade. Specifically, the researchers conceptualize a LT for length measurement as a description of how children think and learn in that specific mathematical domain. As they suggest, one effective approach to creating a LT begins by building a developmental progression, or sequence of thinking levels, through which most children progress in learning the mathematical topic. In their view, developmental cognitive progressions can guide teachers in making sense of students' understandings of a concept and in choosing appropriate instructional activities that help children move along these progressions (Sarama et al., 2022). From this perspective, Smith III et al. (2011) conjectured a route through a set of tasks designed to stimulate those mental processes or actions to move children through a developmental progression of levels of thinking about length measurement. The idea of combining cognitive and didactic elements to create a LT is also discussed in the works of Clements and Stephan (2004) and Smith III et al. (2011). In particular, Clements and Stephan (2004) primarily explore how children develop a conceptual understanding of measurement, starting from initial intuitions (e.g., comparing lengths) and progressing toward systematic conceptualization (e.g., identifying a unit of measurement). Thus, their LT outlines progressive cognitive phases, such as the transition from visual perception ("this is longer") to understanding the iteration of equal units and associating a numerical value to measurement. To achieve this goal, the researchers suggest planning practical activities and transitioning from non-standard units to standard units of measurement. Similarly, Smith III et al. (2011) introduced the concept of LT as a path of conceptual development based on phases that lead to systematization and generalization. While Clements and Stephan's (2004) work places greater emphasis on the cognitive development path and how to support it through instructional strategies, in this article we balance cognitive and didactic aspects, highlighting how LT can be used as a tool for instructional design.

Unlike the LT, which focuses on assessing acquired skills in terms of cognitive progression, in our work we are interested in investigating how children can

progress along a trajectory toward the meaning of measurement by tackling problem-solving situations. Therefore, the presented LT is framed upon didactical and epistemological considerations. Similarly to Clements and Stephan (2004), the function we attribute to the LT is to guide and support instructional choices. However, we do not conceive of the LT as being composed of progressively increasing cognitive levels. Indeed, the process that initiated our LT is quite different from the ones adopted by the other researchers. Our LT is the result of the work of an action-research group composed of preschool and primary school teachers working in synergy with two researchers (the authors), from an inquiry perspective (Jaworsky, 2006). Therefore, as a result of this collaborative research, our LT is framed by the epistemological and didactical considerations shared in the community and, in accordance with the aim of the group, it is designed to serve as a tool for guiding the development of learning activities in teaching practice. Additionally, since the work in the community develops vertically at both school levels, the designed LT refers to a vertical learning path. However, unlike a traditional curriculum addressed to specific school grades, it does not progress with placing steps in specific developmental years. This means that it can be undertaken by students from the early years of kindergarten to primary school and will be realized through different problem-solving situations where children are asked to engage. Moreover, even if the designed LT appears as a series of successive stages that develop from a foundation toward the learning objectives and competencies, each stage can not be considered as a step. Children can move forward and backward, as well as cycle more times around some questions, guided by their intellectual necessity (see, in this regard, the concept of “folding back” described by Pirie and Kieren (1994)).

In this chapter, alongside the LT, we will present exemplary activities, showing how children can move from one step to the next along the trajectory when placed in a well-defined problem situation, within a specific context and provided with appropriate tools. Each problem situation poses questions specifically designed to guide the children toward constructing an understanding of length measurement. The progression is generated through a principle of necessity—with reference to the Necessity Principle defined by Harel (2008)—and the evolution is reflected in the relationship between conceptual development and advancements in the unit of measurement, related to tools. According to the knowledge development premise (Harel, 2008), problem-solving should not only be a goal but also a means for learning mathematics. Problem-solving is typically defined as engaging in a task where the solution strategy is not known in advance. However, even when this definition is met, many of the situations students encounter in school often fail to qualify as authentic problem-solving because the situations are not intrinsic to students’ experiences and lack meaningful relevance to them. In conceiving experiences alongside the trajectory, a relevant application of the Necessity Principle is to ensure that students’ learning of measure grows out of problems that are meaningful to them. In other words, students must experience an intellectual need for it (Harel, 2008). This intellectual need is posited as a necessary condition for significant learning of measurement to occur and, therefore, for defining a LT.

Theoretical Perspective

For this study, we adopt the theoretical perspective of the DNR-based instruction system—Duality, Necessity, and Repeated-reasoning principles (Harel, 2007)—proposed as a way to think about the teaching of mathematics. This framework allows us to describe “conditions for achieving the critical goals of provoking students’ intellectual need to learn mathematics [in our case, the measure concept], helping them acquire mathematical ways of understanding and thinking, and ensuring that they internalize and retain the mathematics they learn” (Harel, 2008, p.263). DNR is a system with three components: (a) premises (assumptions underlying DNR claims), (b) determinants (*Mental Acts*, *Ways of Understanding*, and *Ways of Thinking*), and (c) instructional principles (*Duality*, *Necessity*, and *Repeated Reasoning*).

Among the eight premises, which are the explicit assumptions underlying the DNR system, only two are specifically needed in this study: the *Knowledge Development premise* concerning learning and the *Teaching premise* concerning teaching. The first suggests that knowing involves a continuous tension between accommodation and assimilation (we interpreted it as “adapting to the context”), while the second asserts that scientific knowledge construction is not spontaneous. This highlights the difference between learning with guidance and without, echoing Vygotsky’s Zone of Proximal Development (1978).

The key components of the DNR framework are the instructional principles: the *Duality Principle* (DP), the *Necessity Principle* (NP), and the *Repeated Reasoning Principle* (RP), which together form the basis for effective and meaningful mathematical activities. We adopt these components to guide instructional practices.

Duality Principle

The Duality Principle (DP) asserts that mathematical knowledge comprises both students’ understanding of specific mathematical content and their thinking about the process of doing mathematics. To clarify the process of learning, Harel (2008) presented the relation between these concerning the notion of a mental act, which includes activities such as interpreting, conjecturing, proving, explaining, generalizing, and problem-solving. A way of understanding refers to “a specific cognitive product of a mental act carried out by a person” (p. 490), whereas a way of thinking is a cognitive characteristic of a person’s ways of understanding associated with a particular mental act. For instance, in the LT, a key component is the mental act of problem-solving, particularly in finding the measure of a length. A solution to this problem represents a way of understanding (as it is a specific cognitive product), while a general problem-solving strategy, applicable across multiple similar problem situations, represents a way of thinking (as it is a cognitive characteristic of a

person's approach to solving a specific problem with respect to the particular mental act of problem-solving).

In the example we will provide in the following section (How much? An exemplary activity.), one of the solutions to the problem of measuring a length is to use LEGO bricks of the same length, align them to cover the length to be measured, and then count the number of bricks used to solve the task. The way of thinking arising from this mental act of problem-solving is to define a unit of measurement and the act of measuring with it: cover the length aligning units of measurement with the same defining characteristics and then count them. Another way of thinking is to consider a single LEGO brick as the unit of measurement and properly juxtapose the chosen unit of measurement by counting how many times the operation repeats in order to cover the length to be measured. In short, ways of understanding are the outcomes of the mental acts (ways to solve the problem “find the measure of a length”), while ways of thinking reflect the broader characteristics of those acts (general rules to solve the problem: measuring through the juxtaposition of a single unit or alignment of multiple units).

A key premise of Harel's model of mathematical knowledge is the reciprocal relationship between ways of thinking and understanding: students develop ways of thinking through the construction of ways of understanding, and the ways of understanding they generate are shaped by the ways of thinking they already possess. Thus, a way of thinking, when applied to a new situation, can be interpreted as a way of understanding. Similarly, repeated occurrences of ways of understanding can lead to the emergence of patterns that evolve into ways of thinking. This relationship lies at the heart of the DP and, consequently, at the core of learning.

Necessity Principle

The Necessity Principle (NP) points out that “for students to learn what we intend to teach them, they must have a need for it, where ‘need’ is meant intellectual need, not social or economic need” (Harel, 2008, p. 274). Therefore, an important application of the necessity principle is to ensure that the mathematics students' learning grows out of problems intrinsic to them. Harel states that students encounter an intellectual need when they face an “intrinsic problem” that is a situation that is incompatible with their existing knowledge.

Such an encounter is intrinsic to the learners for it stimulates a desire within them to search for a resolution or a solution, whereby they might construct new knowledge. [...] Knowledge they construct is meaningful to them since it is integrated within their existing cognitive schemes as a product of effort that stems from and is driven by their personal, intellectual need. (Harel, 2007, p.279)

In the example related to finding the measure of a length, we refer to the intellectual need to measure for solving a particular problem (see the section Why measuring? An exemplary activity.). Therefore, the NP ensures that children engage with the

problem situations because they stem from questions defining their cultural needs. This allows them to advance in the LT.

Repeated Reasoning Principle

The design of meaningful and intrinsic activities for children is based on the premise of knowledge development because it creates a continuous tension between accommodation (seeking an unknown solution) and assimilation (building stable knowledge). In other words, the design of activities requires what we define as “adaptation to the context”. The Repeated Reasoning Principle (RP) claims that “Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking” (Harel, 2008 p. 277). The RP complements the other two principles by aiming for students to internalize the ways of understanding and thinking that they acquire through the application of those principles. When students engage in repeated reasoning to solve intrinsic problems, the application of these ways of understanding and thinking becomes both autonomous and spontaneous.

Integrating DNR Principles in the Learning Trajectory

The LT consists of a chain of questions arising from the need to find effective solutions to problem situations. The progression along the trajectory occurs by moving from one question to the next (see Fig. 1). The DP activates the learning process

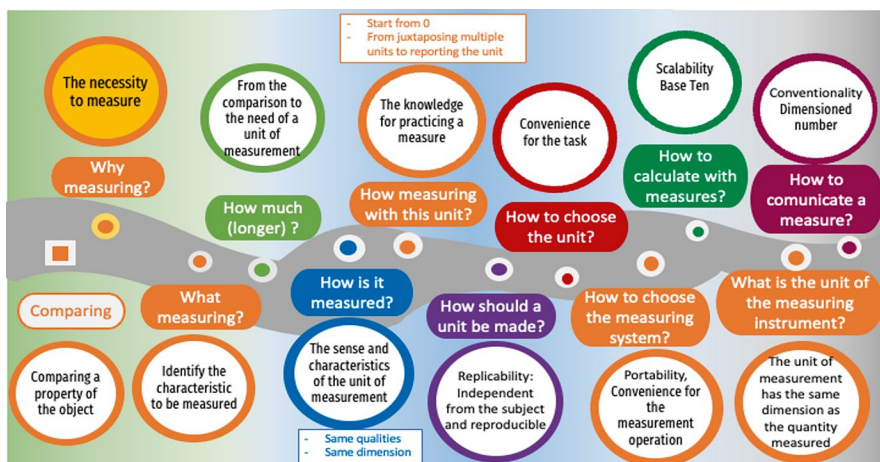


Fig. 1 The learning trajectory

through a correlation between ways of thinking and ways of understanding. Students' conceptualizations of the measurement evolve along with the construction of ways of understanding the act of measuring, and this intertwined evolution is linked with the tools involved and activated by a multimodal approach. Aligning with Nemirovsky's perspective (2003), understanding and thinking, including mathematical thinking, are perceptuo-motor activities, which become more or less active depending on the context. This means that by exploiting perceptual-motor components, the body becomes essential in the learning processes. In this perspective, the term multimodality is used to underline the importance and mutual coexistence of a variety of cognitive, material and perceptive modalities or resources in teaching-learning processes and, more generally, in the construction of mathematical meanings (Radford et al., 2009). Finally, the idea underlying the RP aligns with the kindergarten-level learning approach, where learning advancement is not made through subsequent levels but rather through stages that can, and should, be revisited multiple times for consolidation. This is an essential instructional condition for allowing students to internalize, organize, and retain desirable ways of understanding and thinking as far as research has demonstrated that repeated experience or practice is a critical factor in these cognitive processes (e.g., Cooper, 1991).

Research Context

The LT resulted from the work carried out in the Italian communities of inquiry (Jaworsky, 2006) EduMath Vallée and ZenaMath,¹ affiliated with the Mathematics Education Research Laboratory at the Department of Mathematics of the University of Genoa, involving kindergarten and primary school teachers along with researchers in mathematics education. The aim of the communities is to promote the personal and professional growth of its members, while creating tools addressing emerging needs resulting from reflective practice. Specifically, the work within the communities promotes the development of both theoretical and implementation tools, such as co-designed educational pathways within educational settings. The LT and the associated exemplary activities presented in the following sections are the result of a systematic reorganization of three years of work (2021–2024) on the topic of measurement within the two research communities. Over time, starting from the epistemological consideration and the theoretical background presented in the previous section, activities were collaboratively designed, implemented in the classroom by multiple teachers, and refined through systematic reflection on classroom practices across diverse contexts. At a later stage, the activities were systematically organized into a table, structured around their epistemological foundations and the focal points of the tasks involved in the activities. This process culminated

¹<https://sites.google.com/view/labddm/home>, visited the 10/10/2024

in the creation of the current learning trajectory, which incorporates prototypical activities that had been experimentally validated.

Learning Trajectory for the Measure of Length

The LT (Fig. 1) is the evolution toward constructing the meaning of measurement. By measurement, we mean the result of the operation that determines how many times a certain unit of measurement physically fits into a quantity to be measured. This result is a dimensioned quantity, i.e., a quantity accompanied by the chosen unit of measurement. Children progress along such a trajectory by answering questions posed as problem situations. The responses are milestones to achieve, that follows from epistemological considerations, while the order and selection of questions along the trajectory also depend on pedagogical considerations. Therefore, the LT is conceptualized as a frame within which additional directions are possible, beyond the pre-established objectives. The problem-solving process is activated through a multimodal approach exploiting the use of tools stimulating ways of thinking and understanding the task. We can particularly see its importance in dealing with the questions in the orange circle (Fig. 1). The evolution along the trajectory occurs thanks to the instructional principles defined by Harel (2008).

Here, we make this evolution explicit by showing how the sequence of problem-solving situations can be defined through the sequence of questions (Fig. 1). Those questions go around the sense of measure (understanding why we measure and what measuring means, in Fig. 1 the green part at the beginning), the sense of the unit of measurement (choosing the unit of measurement and using it, in Fig. 1 the blue part in the middle) and the sense of the measuring instruments and conventional system of measure (choosing objects and methods used for measuring, in Fig. 1 the gray one in the last part).

In creating the LT, we assume the children has already acquired the ability to compare the lengths of several objects, in accordance with the Clements and Stephan's (2004) research findings. Comparison of lengths, "It is longer than..., it is shorter than..., it is as long as...", arises from the specific needs of the children (e.g., to show who found the longest stick in the forest or a stick as long as the teacher's one). Below, we will describe the sequence of questions in the first green part, the subsequent blue part, and the final grey part.

The Sense of Measurement

The core question which starts the LT toward the sense of measurement, is "Why measuring?". It concerns, specifically, the intrinsic reason why measuring is necessary (e.g., the need to deal with a disadvantageous situation such as an unfair game, see Fig. 2), as a preparatory step toward focusing on the act of measuring itself. This



Fig. 2 Unfair game (on the left) and fair game (on the right)

question is related to situations in which is activated the NP. Constructing the sense of measurement is the evolution starting from the competence of comparing the same characteristic of certain objects, for example the length, going beyond the simple comparison. This can be represented by the question “How much (longer) is?”, referring to the selected characteristic of the object. To answer this question, children have to move from merely comparing lengths to measuring them. This also depends on the question, “What is being measured?” through which the property to be measured is identified (e.g., the height or length of a given object, even one on which it was not so easy to make comparisons by juxtaposition or alignment).

The Unit of Measurement

The question “How much longer is?” leads toward a second part (in Fig. 1, the blue one), where the sense of the unit of measurement can be established. Indeed, this question opens to reflections on what is needed in order to answer. Here, children can simply intuit the need for an instrument to measure, e.g., using some sticks. Moreover, at this stage, the sense of the unit of measurement may overlap with the sense of the measuring instrument. The next question arise from practical issues, related to the act of measuring: “How to measure with this unit?”. By focusing on an instrument and a unit of measurement to answer these two questions, children might use something relating to their bodies, for example their steps or feet, or they might realise that they need to use some external object, like sticks. They might have chosen a single stick or multiple sticks and, in this second case, these might not be “the same”. In particular, sticks may differ in irrelevant characteristics (e.g., the colour) or relevant ones, such as having different lengths. In the second case, this choice affects the measurement. Alternatively, children may select identical objects (such as LEGO bricks of the same size), or they may select only one object (e.g., a LEGO brick or a stick). In addition to the unit(s) selected, they might or might not use an accurate arrangement, that is what Szilágyi et al. (2013) referred to as measurement error due to “holding the track with the finger”.

This represents a key point in the LT, children are establishing the characteristics defining a unit of measurement. A unit of measurement has to be replicable, that is

subject-independent and reproducible. The use of the selected unit of measurement (e.g., the LEGO brick) is characterized by different procedures, which can be assimilated into ways of understanding (i.e., methods of solving the problem). Each procedure reflects different conceptualizations of the action of measure (i.e., way of thinking in relation to the measurement problem). For example, the procedure of incrementally moving the unit of measurement to cover the length to be measured, while counting the number of times the unit is repeated to cover the entire length, involves a precise method of measurement, counting how many times the chosen unit physically fits the measured length. Differently, aligning identical objects (considered as repetitions of the unit of measurement) until covering the length to be measured, and counting them only at the end of this procedure, is an operation similar to the construction of a ruler. This situation effectively illustrates the dual relationship between ways of understanding and ways of thinking in the learning process, further being an example of how the development of the instruments in use and the conceptual development (understood as the evolution along the learning path on measurement) are linked.

Moreover, this example shows the non-linearity of the learning path. The idea that the LT is not linear aligns with the concept of folding back proposed in Pirie and Kieren's (1994) model of growth in mathematical understanding, which distinguished eight levels of understanding. The concept of folding back in this model highlights the nonlinear nature of mathematical learning, when students encounter problems or obstacles, they can "come back" to a previous level of understanding to deepen and enrich their knowledge. This process goes beyond simple repetition, as suggested in Harel's (2008) RP and involves enhancing prior actions considering new insights. Although the principle of folding back may resonate with our idea of the LT non-linearity, our LT does not follow progressive cognitive levels, as outlined in Pirie and Kieren's model (1994). In the context of measurement, if a student struggles with the conceptualization of measuring, they can come back to the act of measuring—possibly by changing the unit of measurement used and how to use it—to create more effective mental images.

Measuring Instruments and Conventional System of Measure

The use of an instrument and an associated unit of measurement may raise questions regarding such choices and the relations between such choices. These considerations open to the last part of the LT (in Fig. 1, the gray one), concerning the meaning of the measuring instrument in relation to the unit of measurement. In this last phase, core questions are "How to choose the unit?", which concerns the convenience of the chosen unit for dealing with the specific problem situation, and "How to choose the measuring instrument?", which is a question referring more to practical criteria than conceptual issues (e.g., the portability of the object). Dealing with measurement activities, children can decide to change the instrument and/or the unit of measurement on the basis of the criteria mentioned above. The transition of

instruments is related to the interaction between the problem-solving situation (which is the way of thinking, comprising mental acts such as calculation, inference, structuring, conjecture, etc.) and the specific contextual understanding of the measurement scenario: measuring something in a certain way—ideally the most effective way—and using a particular instrument.

Another evolution toward the sense of the unit of measurement and the measuring instrument concerns the scalability of the instrument, based on a base 10. In particular, the scalability of a measuring system is an answer to the need to calculate with measurements. Moreover, at this stage, a specific question regards the relationship that coexists between the unit of measurement and the selected measuring instrument, concerning both the communication of the result of the measurement process and the measurement process itself. If we measure with a “LEGO-meter” (see Fig. 3b), we consider both the LEGO-meter and each LEGO block involved in this process as having a specific characteristic: the length of the same edge. If the length to be measured is “one LEGO-meter and three blocks, that is 13 blocks”, this means: the length of one LEGO-meter, which corresponds to 10 times the length of a specific edge of a block, plus three additional times the length of that same edge of a block. In other words, the total length is equal to 13 times the length of a specific edge of a block, not generally 13 blocks.

A further key consideration concerning the choice and the related usability of a unit of measurement regards the conventionality of a system of measurement, i.e., considering a unit of measurement as a conventional tool. The conventionality specifically arises from the need to communicate measures. Answering the question “How to communicate a measure?”, children can recognize the importance of choosing a unit of measurement whose associated measurements can be understood by others. In particular, we can distinguish between the idea of conventionality within the class group (thus, a narrower social group) or conventionality within a broader social group (such as the international system).

We conclude the presentation of the LT with two observations. In the learning path, a key node is represented by the distinction between the act of measuring and

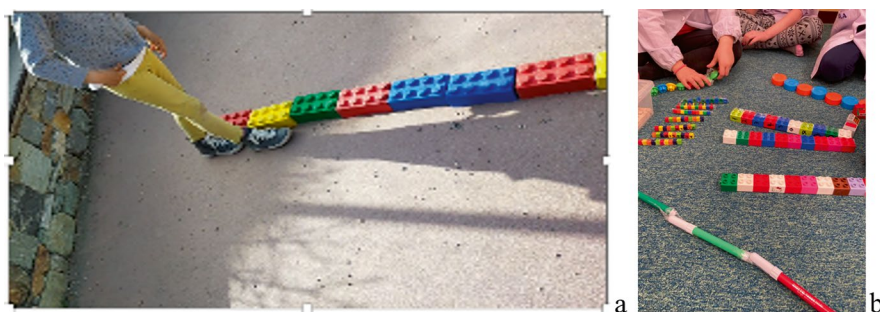


Fig. 3 (a) measuring the shadow through LEGO bricks. (b) LEGO-meter (foreground, on the right), Wooden-meter (to the left), Caps-meter (background, on the right), Felt-tip Pens-meter (foreground, on the left)

its result, that is measurement as a dimensioned quantity. What is a measure? The outcome of the act of measurement is a dimensioned quantity, a number, directly derived from the selected unit of measurement answering the question “How many units comprise the measured quantity?”. We, therefore, recognize, as the final stages in the measurement conceptualization, children’s ability to make connections between measurements acquired using different units of measurement. Along the trajectory, individuals also articulate a relationship between measurements, affirming the principles of indirect proportionality (Carpenter & Lewis, 1976) according to which the larger the unit of measurement, the smaller the value associated with the measurement.

Prototypes of Learning Activities

In this section, we will provide three activity prototypes for illustrative purposes, each associated with one question of the LT. The first example focuses on the NP in response to the question “Why measuring?”, while the second one addresses the question “How much smaller/larger/longer?”. The third example allows us to illustrate how, in response to the last question, children intuit the need to refer to a tool. Students initially refer to parts of their own body before progressing to external objects, providing evidence of the construction of the concept of a unit of measurement (Carpenter & Lewis, 1976). The question “How do I choose the unit of measurement?” responds to the necessity of a reproducible as well as a conventional, convenient and portable unit of measurement. By showing these examples, we aim to illustrate what is intended by providing a specific situation, in a specific context, and with specific artifacts to address the NP. Further, we aim to illustrate how the LT allows us to identify, further than the primary objectives of the activity itself, the possibility for further directions to develop within them.

Why Measuring? An Exemplary Activity

The measurement, as a necessary step to overcome the mere comparison of lengths, is crucial in progressing along the LT. To illustrate how this transition moves by the NP, we will present an exemplary activity that took place in a class with 4 and 5-year-old children. The problem situation, depicted in Fig. 2 (on the left), arises from the “Capture the Flag” game. The teacher suggests playing Capture the Flag in a situation where one of the two teams is disadvantaged. The children of that team point out this disadvantage and request a fair situation. The teacher asks the children what they mean by a *fair* situation. Children answered that the child with the red flag (Fig. 3, on the right) should stay “in the middle”. Therefore, the issue shifts to defining middle: “middle means half”, say the children, it also means “dividing into two parts”. The teacher (in the position of the child with the red flag) maintains an

unfair position, so the children declare that “for it to be in the middle, the two parts must be equal”. Therefore, the characterization of half adds to the two conditions “to be divided into two parts” and “the two parts must be equal”. Summarizing, in the first the children’s answers simply refer to a point in between the two teams. Only later, they specified the condition that the length between the point and each of the two teams must be equal.

The characteristic of having equal lengths addresses the question of “Why measuring?”. Once this aspect is internalized, the teacher can shift the focus toward “how” to construct two equal lengths (how to measure for verifying the equalities of the lengths). This is an example of how the activity, designed with the aim of emerging the necessity of “measuring”, provides the starting point of potential further directions generated from the problem situation. The necessity of a fair game, that is finding the “equal lengths”, moves attention towards the response to the questions: “How measuring?” (way of thinking) and “What do I need to measure?”, that open to the problems of adopting a suitable unit of measurement and how to proceed for measuring.

As a first attempt children involve their own bodies, as measuring with steps. However, children’s steps are not a replicable unit of measurement: for instance, the independence from the subject who measures is missing. Consequently, they suggest using just a child’s step, but this also lacks replicability, because the reproducibility of the same unit for the entire measuring process, or further measuring process, is missing. Then, they suggest using an external object, one child’s shoes, placing them one after the other for the entire length to be covered, counting the times they do this operation. In this activity, the necessity of employing the same unit of measurement, in this case, the child’s shoes, for the process is acknowledged, and the essential property of the unit of measurement is gradually recognized by children.

The action of measuring is carried out by counting how many times the unit of measurement physically fits within the length to be measured. The solution to the problem of finding the midpoint was developed through two strategies (ways of understanding) that operationalize the choice of the unit of measurement (e.g., the child’s shoe) in response to the question “How long is the length?”. The first consists of counting the same number of feet or tiles starting from the two opposite extremes of the carpet until the children meet. The second consists of measuring the entire length of the carpet and cutting it into half (counting the number of total feet or number of tiles, and then splitting it into two equal parts).

How Much? An Exemplary Activity

In this exemplary activity, we show how the selection of measurement tools changes the space of possible implementable strategies. In the presented problem situation, the context (e.g. the tools available) influences not only the solution to the problem (the way of understanding) but also the *way of thinking* about the solution. Answering

the question “How much?” (more precisely, How many units of measurement fit into the length?) children’s answers are strongly influenced by the chosen unit of measurement.

The activity took place with 3 and 5-year-old children. The problem arises from the children’s curiosity: determine the length of a pole’s shadow. The children chose two units of measurement: a girl’s steps (Fig. 3a) and LEGO bricks (Fig. 3b). The strategies for measuring implemented by the children varied depending on the chosen unit of measurement. The measurement process involves a form of direct or indirect counting of the repetitions of the unit of measurement within the length. Counting how many LEGO bricks cover the entire length to be measured involves direct counting, an operation similar to using a ruler. Conversely, counting how many steps cover the entire length appears as indirect counting, as it does not imply the preexistence of a cardinality, since the steps are not predefined. In both cases, the result of the measuring action is a dimensioned length: the length of the shadow of the pole in the courtyard is 13 LEGO bricks or 13 Ilaria’s feet.

Since the way of understanding (dealing with the problem of “how to measure”) is influenced by the resources at disposal, we can say that the measuring instrument used to define the unit of measurement is a thinking tool (it is functional to the construction of the problem-solving process). Simultaneously, the same tool influences the conceptualization of measurement, emphasizing the need for consistently equal measurement units throughout the measuring process. Hence, the need to modify the tool itself by considering all LEGO bricks as identical.

In this example we find the reification of Harel’s (2008) DP. The thinking process resulting from redefining the measuring tool (and thus, the chosen unit of measurement) leads to the conceptualization of measurement as a dimensioned quantity and the operation of measuring as determining how many times the unit of measurement physically fits into the quantity being measured. Furthermore, in the evolution of the activity, the insights that may arise surpass its initial objectives. From merely intending to answer the question “How much?”, concerning activating the process of measuring, children shift to questions concerning the properties needed for an object to be a unit of measurement and how to measure with the selected unit.

How Do I Choose the Unit of Measurement? An Exemplary Activity

The choice of a unit of measurement (and then a measuring tool) involves four criteria: replicability (independence from the subject and reproducibility), convenience (for the specific problem-situation), practicality (concerning the measuring operation), and conventionality (concerning social agreement). Therefore, this choice cannot be indifferent to either the problem situation or the dimension to be measured, for instance the length. Moreover, during a problem-solving process, further needs may arise that require modifying or changing the measuring tool and, thus, the chosen unit of measurement. As seen in the previous example, the instrument

can influence the problem-solving process but, at the same time, the process can affect the redefinition of the instrument to be used for measurement.

This example concerns the problem of choosing a convenient unit of measurement, more precisely, the most convenient in the given situation. In the class from the previous example, children who had been using LEGO bricks as a unit of measurement for measuring a large length (e.g., the length of the classroom wall), may have been led by the situation itself to seek a unit of measurement that reduced the number of actions on the length to be measured. Indeed, a larger unit of measurement is needed (NP in act). Addressing this need, the concept of the LEGO-meter is born: a unit of measurement that combines 10 LEGO bricks (Fig. 3b).

The LEGO-meter is a tool designed and created by the children. The reason why they chose to connect exactly this numerosity (10 LEGO bricks) with tape is not explicit. Our hypothesis is to consider the epistemological construction of numbers previously realized in that class: the children have always worked with their fingers to structure the composition and decomposition of the number 10 (Robotti, 2021).

Due to the structure of the LEGO-meter itself, it is possible to define measurements corresponding to multiples of the tool as well as to submultiples of the tool. The scalability of the system of measurement is explored. The transition to the submultiple was not initially obvious. In the first time, children described in terms of “half” LEGO-meter whatever length not covering entirely the LEGO-meter (e.g., the measure of a length is described as a certain number of LEGO-meters and “half”, where the “half” represents a number from 1 to 9 bricks). Soon after, they redefined the measurement as “2 LEGO-meters and a bit”. The focus then shifted to defining the measurement of the “little piece”, this was 3 LEGO-bricks. In the first time, some children expressed length measurement as 23 LEGO-bricks, and no one was able to express the measurement in terms of 2 LEGO-meters and 3 LEGO bricks. It implies that a child did not naturally manage multiples and submultiples but tended to convert them into the same unit of measurement. We could still observe the intertwined roles of the mental acts related to measuring (inferring, counting, interpreting...) and the ways of understanding, closely linked to the use of the tools involved. It was also evident how the process of constructing the sense of measurement is constituted by back-and-forth movements of these two aspects.

Once the appropriate unit of measurement, the most convenient for the situation, is chosen, the focus shifted to the necessity to adhere to the principle of conventionality. Why was it necessary to agree on which unit of measurement to use? In other words, why should all children (at least in the class) use the same unit of measurement? Using the same unit ensures uniformity and coherence in the results obtained. If each child in the class were to use a different unit of measurement, it would be extremely difficult to compare and understand the measurements made by others. Consequently, there is a need for ease of communication: adopting a common unit of measurement simplifies communication among individuals. To communicate, children must use the concept of a dimensioned quantity, and they agree with each other on the measurement unit used working all together on the same problem-solving situation.

Conclusions and Further Directions

This study presents a learning trajectory (LT) of length measurement that integrates epistemological and didactical considerations. It emerges from a three-year action-research process involving preschool and primary school teachers. The LT is a tool designed to support teachers in planning problem-solving situations that could guide children toward constructing the meaning of measurement, articulated as a progression of key questions. Embedding the theoretical principles of Harel's (2008) framework into a path framed on problem-solving activities, the LT aims to support a teaching-learning characterized by meaningful learning experiences. Some examples of activities associated with three key questions toward the sense of measurement are provided, which constitute prototypes for illustrating how to build learning activities about the core questions of the LT for developing the meaning of measurement. They may also represent an example of how children can progress in their learning by dealing with problem-solving situations, following the NP. Another key aspect, highlighted by the examples, is the intertwined evolution of the measuring tools and conceptual understanding (e.g., their relationship with the unit of measurement and the measurement itself) dealing with problem situations through the learning path, in accordance with the DP. Finally, the examples show that moving along LT is not necessarily linear: it is possible to revisit the conceptual nodes of the questions, even when dealing with different experiences, following the PR.

With the LT, teachers can design learning activities aimed at constructing the meaning of measurement, focusing on length. However, our future research aims to investigate whether what is learnt can be extended to other quantities, such as capacity and time. Finally, to fully leverage its potential, future research should focus on evaluating the LT's usability and effectiveness when adopted by teachers outside the communities involved in its co-construction. Additionally, professional development opportunities should be explored to ensure teachers have the knowledge and support necessary to use the LT effectively. Indeed, implementation issues cannot be underestimated (Boscolo, 2024). The LT acts as a systematic design tool, prompting critical questions about its utility: How practical do teachers find it? To what extent does it support their instructional planning? What additional resources or training might they require? Answering these questions will help refine the LT and ensure its broader applicability in diverse educational contexts.

References

- Boscolo, A. (2024). The implementation of active, bodily experience mathematics learning activities: The perspective of Italian primary and secondary school mathematics teachers. *Implementation and Replication Studies in Mathematics Education*, 4(2), 177–214. <https://doi.org/10.1163/26670127-bja10022>
- Boulton-Lewis, G. M., Wilss, L. A., & Mutch, S. L. (1996). An analysis of young children's strategies and use of devices of length measurement. *Journal of Mathematical Behavior*, 15(3), 329–347.

- Carpenter, T. P., & Lewis, R. (1976). The development of the concept of a standard unit of measure in young children. *Journal for Research in Mathematics Education*, 7(1), 53–58.
- Clements, D. H., & Stephan, M. (2004). Measurement in PreK-2 mathematics. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics* (pp. 299–320). Lawrence Erlbaum Associates.
- Cooper, R. (1991). The role of mathematical transformations and practice in mathematical development. In L. Steffe (Ed.), *Epistemological foundations of mathematical experience*. Springer.
- Harel, G. (2007). The DNR system as a conceptual framework for curriculum design and instruction. In R. Lesh, J. Kaput, & E. Hamilton (Eds.), *Foundations for the future in mathematics education* (pp. 263–280). Lawrence Erlbaum Associates.
- Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, part I: Focus on proving. *ZDM Mathematics Education*, 40, 487–500.
- Jaworsky, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9, 187–211.
- Kamii, C., & Clark, F. B. (1997). Measurement of length: The need for a better approach to teaching. *School Science and Mathematics*, 97(3), 116–121. <https://doi.org/10.1111/j.1949-8594.1997.tb17354.x>
- Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of mathematics education* (Vol. 4, pp. 113–120). PME.
- Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, 26(2), 165–190. <https://doi.org/10.1007/BF01273662>
- Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics*, 70, 91–95. <https://doi.org/10.1007/s10649-008-9172-y>
- Robotti, E. (2021). How spatial structuring abilities support the development of long term effective mental calculation strategies. In J. Novotná & H. Moraová (Eds.), *Proceedings SEMT21* (Vol. 1, pp. 357–366), Prague.
- Robotti, E. (2022). Building the meaning of measurement in kindergarten: the case of length. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of 12th congress of the European Society for research in mathematics education* (pp. 2258–2265), Bozen-Bolzano, Italy. Hal-03750250.
- Sarama, J., Clements, D. H., Barrett, J. E., Cullen, C. J., Hudyma, A., & Vanegas, Y. (2022). Length measurement in the early years: Teaching and learning with learning trajectories. *Mathematical Thinking and Learning*, 24(4), 267–290. <https://doi.org/10.1080/10986065.2020.1858245>
- Smith, J. P., III, van den Heuvel-Panhuizen, M., & Teppo, A. R. (2011). Learning, teaching, and using measurement: Introduction to the issue. *ZDM Mathematics Education*, 43(5), 617–620. <https://doi.org/10.1007/s11858-011-0369-7>
- Szilágyi, J., Clements, D. H., & Sarama, J. (2013). Young children's understanding of length measurement: Evaluating a learning trajectory. *Journal for Research in Mathematics Education*, 44(3), 581–620.
- Van den Heuvel-Panhuizen, M., & Buys, K. (Eds.). (2008). *Young children learn measurement and geometry*. Sense Publishers.
- Vygotsky, L. S. (1978). In M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds. & Trans.), *Mind in society: The development of higher psychological processes*. Harvard University Press.

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Fostering Logical Reasoning with the “Betta-the-Bee” Game



Raffaele Casi, Cristina Sabena, and Carlotta Soldano

Introduction and Theoretical Background

Playing is a fundamental activity in all cultures, as it has been discussed by various scholars, starting from Huizinga’s (1971) seminal work. According to Bishop (1988), play constitutes one of six ‘universal’ activities that can be characterized as mathematical activities. In particular, playing is popular and ubiquitous in early childhood education, including mathematics education. As Radford pointed out in his plenary lecture at the POEM4 Conference, in early mathematics education literature play is usually considered as a source of development or as a window through which adults may observe signs of children’s development (Radford, 2020). To these two main strands, he added a vision of play in the production of subjectivities in preschool. In our research, we align with the first and last conception and consider playing a rule-based game as a suitable context in which children may encounter mathematical ideas, and specifically logical reasoning, namely forms of reasoning typical of logics. Here the reference is to classical logics, or propositional logics, in which propositions are declarative sentences that can be either true or false but not both.

Research in mathematics education at the secondary level has focused on logical reasoning skills for decades, primarily linking them to problem-solving and argumentation and proving (e.g., Durand-Guerrier et al., 2011; Schoenfeld, 1985). Different forms of reasoning have been studied in these processes, such as explorative, abductive, and deductive reasoning. Arzarello and Soldano (2019) highlighted that these different forms of reasoning are not separate from each other, but they interact dynamically during problem-solving. They also showed that questioning plays a crucial role in facilitating mathematical inquiry and in fostering logically sound critical thinking.

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Asking and answering questions are crucial processes in mathematical activities. They are closely related to the delicate process of planning, decision-making, and control in problem-solving, as well as to argumentation. As introduced by Schoenfeld (1985), control involves “global decisions regarding the selection and implementation of resources and strategies” (p. 15). In other words, it seems to play a primary role, encompassing actions such as planning, monitoring, assessing, decision-making, and conscious metacognitive acts. Grounding on this general meaning of control, Arzarello and Sabena (2011) illustrated how secondary school students processes in the context of argumentation and proof activities are managed and guided through two intertwined modalities of control, namely semiotic and theoretical control. Semiotic control pertains to knowledge and decisions primarily involving the selection and implementation of semiotic resources (e.g., choosing an algebraic formula to represent a problem). It includes semantic control, which involves the ability to manage or regulate the meaning and interpretation of signs. Theoretical control necessitates explicit reference to the theoretical aspects of mathematical activity, such as when a student consciously uses a specific property or theorem to support an argument. In addition to these categories, we can refer to logical control as the ability to apply logical reasoning, rules, and principles to make decisions, solve problems, and regulate thought processes.

The development of different types of control is generally considered a learning objective for secondary school students—especially with regard to theoretical aspects and explicit knowledge of the principles that underly them. However, it is our belief that, in an appropriate educational environment, it is possible to start much earlier, as early as preschool. Featured within a Vygostkian perspective on learning, emphasizing social interaction and cultural context in learning from early childhood (Vygotsky, 1931/1978), our research is grounded in the hypothesis that structured, rule-based games that encourage them to formulate and answer questions, and analyze the consequences of their choices, can be exploited by teachers and educators in creating such environments.

To this purpose, we designed a card game in which children can engage with and apply key aspects of logical reasoning in mathematics: the “Betta-the-Bee” game. In this chapter, we present and analyse the design of the game. We also provide some insights from the first observations of children engaging in playing the game, in the presence and under the guidance of one or more adults.

The Design of the ‘Betta-the-Bee’ Game

The Betta-the-Bee game uses 24 flower cards that show a flower in a pot, surrounded by some butterflies and by a caterpillar.¹ The cards differ in the number of petals, the colour of the pot, the position and number of butterflies, the position of the

¹Cards are retrievable at the following link: <https://www.dfedidamath.unito.it/home-page/giochi-di-carte/betta-lapetta>

caterpillar and the position of the leaf in relation to the stem. All other aspects of the cards remain the same. This choice was made to promote the noticing of the variables by observing the characteristics that change and those that remain the same in the cards (Fig. 1).

The game dynamics are inspired by the commercial games “Who is it?” and “Oudordodo”. The play begins when a player (hiding player) hides Betta under a flower card without being observed by the other players. The aim of the opposing player (seeking player) is to find out where Betta is hidden by asking questions about the characteristics of the cards (e.g. “Is Betta-the-Bee under a card with a red pot?”). The cards that do not have the required characteristic are removed by the players and another question can be asked until only the card under which Betta is hidden is left on the table.

All variables are represented in two or three modalities. We use the term modality to denote the different qualitative or quantitative values that a variable can take on.

The following table details the variables and modalities used in the Betta-the-Bee game and outlines the design choices made.

As it can be seen in the table, all the variables are displayed in three modalities, with the exception of the position of the leaf and the position of the butterflies, which can only be on the right or the left.

The 24 cards are laid out face up on the table. As already mentioned, the participating players take on two roles: the hiding player and the seeking player(s). The hiding player asks the seeking players to close their eyes and hides the Betta-the-Bee card under one of the flower cards. When Betta is hidden, the seeking players can open their eyes and start asking questions that may lead to finding Betta. Only questions that relate to the characteristics of the cards are allowed. Each question may relate to only one modality and must be formulated in such a way that only “yes” or “no” is possible as an answer. After the seeking player has answered “yes” or “no”, the players can discard the cards in the following way: if the answer is “yes”, all cards without this modality are discarded (e.g. all cards without a red pot); if the answer is “no”, all cards with this modality are discarded (e.g. all cards with a red pot). After the cards have been discarded, another seeking player formulates another question, the hiding player answers and other cards are discarded. When

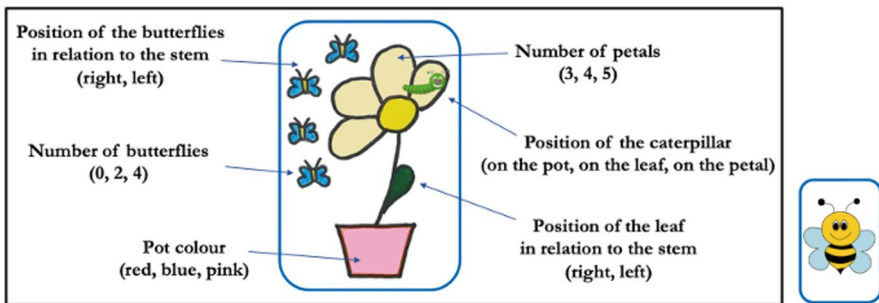


Fig. 1 One of the cards of the Betta-the-Bee game with a description of variables and modalities (left), Betta-the-Bee card (right)

there is only one card left on the table, Betta's location is discovered and the round ends.

The game dynamics described above emphasize three key subsequent phases within each game round. After hiding Betta, the first phase is the *formulating questions* phase, carried out by the seeking players. The second phase consists of *answering questions*, a role assigned exclusively to the hiding player, who performs it throughout the game. Finally, in the *discarding cards* phase the seeking players (and often the hiding player too) collaboratively remove cards according to the received answer.

Each of these three phases requires the players to carefully observe and compare the cards on the table, with particular attention to the variables and modalities present on the displayed cards. As emphasized by Mason (2002), noticing is a fundamental process in learning mathematics. In a sense, indeed mathematical understanding grows through awareness of patterns, structures, and relationships that require to be noticed. In the process of answering questions in the Betta-the-bee game, the hiding player must carefully observe the card under which Betta is hidden and establish if the asked modality is displayed on it, in order to decide whether to answer "yes" or "no". Finally, careful observation is required during the card removal phase. Each player must examine the cards to decide which to keep and which to discard, depending on whether the modality mentioned in the question is present or not. In addition, the hiding player must make sure that the card under which Betta is hidden is not accidentally removed.

The dynamics of the game stimulate different and rich mathematical aspects, that we term 'Mathematics-in-the-Game' (MiG). They are highlighted in the left column of Table 2, with reference to the three main game phases.

The first mathematical aspect of the MiG, shown in Table 2, concerns the law of the excluded middle. This aspect appears in all the three game phases. Applied to the game, the law of the excluded middle states that Betta can only be hidden under a card that either has a certain modality or does not have that modality. For example, Betta can only be hidden under a card in which the pot is blue or under a card in which the pot is not blue. As a result, questions with "yes" or "no" answers ensure that the game comes to a conclusion. During the question formulation process, players can only ask questions that can be answered with "yes" or "no", as other types of questions may prevent the game from progressing. When answering questions, the hiding player can therefore answer "yes" or "no" with absolute certainty. Finally, when removing cards, the law of the excluded middle uniquely determines which cards must be discarded and which must be kept.

The second aspect of MiG refers to the logical operation of negation. This aspect can be seen in the card removal process: if the answer to a question is "no", this means that Betta is not hidden under a card with the specified modality. Consequently, all cards with this modality must be discarded. If, on the other hand, the answer to a question is "yes", this means that Betta is hidden under a card with the specified modality, so all cards without this modality must be discarded. In other words, if the answer is "no" the cards belonging to the subset that has the modality specified in the question are discarded; if the answer is "yes" the cards belonging to the complementary subset are discarded.

In this context, the use of the logical operation of negation is combined with the application of material implication. For example, if the answer to the question “Is Betta-the-Bee hidden under a card with a red pot?” is “no”, the reasoning that leads to the removal of the cards can be made explicit as follows: IF Betta is not under a card with a red pot, THEN I can remove all cards with a red pot. Conversely, if the answer to the same question is “yes” the reasoning for removing cards would be as follows: IF Betta is under a card with a red pot, THEN it is not under a card without a red pot. THEN I can exclude all cards without a red pot.

Finally, the aspect of case-based reasoning appears both in the formulation of the question and in the answer to the question. This type of reasoning ensures that, even if players focus on a single modality at a time in questions and answers, they will be able to finish the game by formulating their questions appropriately. This is ensured by the fact that each variable on all 24 cards of the game is represented by different modalities.

Besides the Mathematics-in-the-Game, the game also provides other kind of mathematical inputs, which are strictly connected on the cards in the deck. We term ‘Mathematics-in-Deck’ (MiD) the mathematical features as presented in Table 1. In the ‘Betta-the-Bee’ design, MiD encompasses classical basic mathematical skills usually considered in early mathematics education, such as number sense, and spatial abilities. However, in our view the most interesting aspect of the deck from the

Table 1 Description of variables, modalities and design choice of the Betta-the-Bee game

Variable	Type	Modalities	Design choices
Pot	Colour	Red Pink Blue	The colour of the pot is the only variable that does not have a direct connection to mathematical aspects. This variable was chosen for two reasons. On the one hand, it allows children who might have difficulty identifying mathematically oriented variables to participate in the game by focusing on a primarily perceptual aspect. On the other hand, questions concerning this type of variable may help reduce the set of available cards right from the initial stages of the game, thereby enabling the formulation of questions about mathematical variables with a smaller number of cards remaining on the game table.
Petals	Number	Three Four Five	The decision to use small numbers and to arrange the petals side by side was made to allow children easily quantify by means of the process of subitizing, or the counting process.
Butterflies	Number	Zero Two Four	We remind that the inclusion of the number zero transforms this variable from numerical to existential.
Butterflies	Position	Left Right	The choice of these three positional variables enables a focus on spatial and topological aspects, including both above-below and right-left relationships.
Leaf	Position	Left Right	
Caterpillar	Position	On the petal On the leaf On the pot	

mathematics education perspective is not so much the specific mathematics content of the cards. It is the fact of presenting to children a context in which different variables with different modalities are present at the same time, and children may be engaged in the process and in the challenge of noticing them (in the sense of Mason, 2002) while playing.

MiD is integrated into the variables that are present in the cards and the modalities in which these variables appear. In order to keep the difficulty of the game suitable for young children, each variable is either dichotomous or trichotomous, i.e. it appears only in two or three modalities, respectively. The butterfly variable is existential, as it may or may not be present on the cards. Quantity variables include petals and butterflies. Identifying them requires referring to what Lemke (2003) calls ‘typological meaning’, namely a type of meaning that involves categorizing phenomena into distinct types or categories. Position variables are the leaf and the caterpillar. These variables, which potentially could be closer to a ‘topological meaning’ (involving gradual, continuous variation), are indeed treated in the game as typological ones, as the leaf may be left or right (two positions) and the caterpillar may be on the petals, on the leaf, or on the pot (three positions). This typological nature of the game is coherent with the typological nature of the semiotic resource through which the game develops, namely verbal language.

To take advantage of the rich stimuli associated with the MiD in the Betta-the-Bee card context, we suggest introducing a preliminary discovery phase before the main game begins. This phase should help players to effectively notice variables and modalities. During the discovery phase, the adult facilitating the game (e.g. teacher, parent or caregiver) introduces the 24 flower cards that are on the table (usually displayed in three or four rows) and guides the players to explore them by asking carefully worded questions. Examples include: “Have you looked carefully at all these cards? What do they represent?”, “Can you see any similarities between some of the cards? What about the differences?”, or similar prompts designed to encourage observation and perception, and ensure that all children know the names of the observed features of the cards (i.e. all modalities and variables). Once the players have familiarised themselves with the cards, especially the MiD, the hiding player can hide the Betta card under one of the flower cards, which begins the game phase.

Methods and Research Questions

The Betta-the-Bee game was developed within a research group comprising mathematics education researchers (the authors of this chapter), kindergarten and primary school teachers, and pre-service kindergarten teachers. Overall, the design phase for the game lasted 3 years and the current version has been published on the research group website (<https://www.dfedidamath.unito.it/>). The game is freely accessible under the Creative Commons CC-BY-NC-ND 4.0 license.

The design process was guided by principles inspired by Educational Design Research (EDR; McKenney & Reeves, 2019). This method is designed to uncover the underlined mechanism rather than to evaluate the efficacy of a proposed educational session. According to this approach, we followed and reiterated in a cyclical way three phases: A) research group meetings in which the game was designed; B) experimental educational sessions in the kindergarten classrooms or in home setting; and C) research group meetings in which the experimental educational sessions were discussed. Each experimental educational session in kindergarten was conducted in the participating teachers’ classrooms. During the sessions, a researcher was present in the classroom as an observer. The experimental educational sessions conducted in home settings involved children familiar to the researchers. Each session comprised gameplay activities with the participation of one or two children, one of the authors as a researcher, and a caregiver (parent). All sessions were video- and audio-recorded, and the audiovisual data, together with field notes from teachers and researchers, were analyzed during subsequent design meetings to refine both the game and the associated educational sessions.

In the previous section of this chapter, we presented the design of the game and the analysis of the associated MiD and MiG components. These considerations are the result of reflections that we elaborated mainly during the group meetings of phases A and C, nurtured with direct and video-observation of the experimental sessions (phase B). We proposed the game and conducted empirical observations with various groups of kindergarten children and different teachers. The second or the third author were always present as participant observers and one video-camera filmed each session, with the consensus of parents. Children engaged in the game through teacher-organized play (also known as guided play or structured play, Reikerås, 2020), where the teacher sets the rules and initiates the play. Children could also play the game during their free time if they wished, as the cards were made available to them. Additionally, the third author carried out three interviews with children aged 3, 5, and 6, in home setting.

Despite phase B was primarily meant to refine the design of the game according, it allowed us also to get first data on how children faced the MiD and MiG components when being introduced to the game in different environments when playing the game and on how the various kinds of control may intervene. As, according to a Vygotskian perspective, the adults play an important role in the children’s encounter with knowledge—which in our case is embodied in the cards (MiD) and in the game rules (MiD)—the children interplay with the teachers or more generally the adults was also of interest to us.

More precisely, the results presented in the next section of the chapter will be orientated to answering the following research questions:

RQ1a: How is the process of observing the MiD carried out by children during the discovery phase of the game?

RQ1b: How may adults (teachers, parents or caregivers) effectively support children in this process?

RQ2a: What challenges do the children face in the different phases of the game (formulating questions, answering questions and discarding cards)?

RQ2b: How may adults (teachers, parents or caregivers) effectively support children in overcoming these challenges?

Results

We will briefly report and discuss the results of this first phase of empirical observation. This phase of the research allowed us to refine the design of the game and further explore aspects of the game that were highlighted in the a priori analysis as detailed in the previous section. The presentation of findings is organized around three selected episodes.

Episode 1. They All Look the Same, But They Have So Many Differences

To help children to identify the variables of the MiD, before presenting the rules of the game, some teachers chose to initiate the exploration by asking to children to describe what they were seeing on the cards and helping them noticing the variables. The following episode serves to address RQ1a and RQ1b. It reports the discovery phase involving a group of four 4-year-old students and their teacher (all names are pseudonyms):

1. Teacher	Take a good look at the cards, as they all look the same, but they have so many differences
2. Marghe	Because this one has 1, 2, 3, (<i>counting the petals of one card</i>) 1, 2, 3 (<i>counting the petals of another card</i>) 1, 2, 3, 4, 5 (<i>counting the petals of another card</i>)
3. Teacher	What do they have? What are they?
4. Marghe	Two are less and one is more
5. Teacher	But what are these that you counted? They are...?
6. All	Petals
7. Teacher	The petals of a flower
8. Fede	A daisy
9. Teacher	What we see is a daisy and the daisy is in all the cards and it's a daisy inside a pot, the pot is in all the cards too
10. Marghe	Some don't have a caterpillar, but some do
11. Teacher	Some have a caterpillar
12. Michi	This pot is red, and this one is blue (<i>pointing first to a card with a red pot and then to a card with a blue pot</i>)
13. Teacher	The pots have different colours, good! And what do you also notice that is different?
14. Fede	That it has two butterflies (<i>looking at a card with two butterflies</i>)
15. Teacher	That there are butterflies
16. Marghe	More! 2 (<i>pointing to a card</i>) 1, 2, 3, 4 (<i>counting the butterflies on a card with 4 butterflies</i>)
17. Teacher	Some 2, some 4 and some

18. Fede	5
19. Teacher	And some? Do you see them here (<i>pointing to a card with no butterflies</i>)?
20. All	No
21. Teacher	And some don't have butterflies at all
22. Marghe	Then here no, here no, here yes ... (<i>pointing to one by one all the cards</i>)
23. Teacher	And then also try to observe something else
24. Fede	That here there are 5 and here 3 (<i>pointing first to a card with five petals, and then to a card with three petals</i>)
25. Teacher	What? What are the names of these that are 3, 4 and 5? The ...
26. All	Petals
[...]	
27. Michi	The caterpillar is not always on the leaf
28. Teacher	It's not always on the leaf! Try to observe the caterpillar
29. Marghe	One is here, one is here...
30. Zen	One is on the pot
31. Teacher	On the pot or?
32. Michi	On the petals
33. Teacher	On the petals or?
34. Marghe	On the stem
35. Fede	On the leaf.

The observation process is initiated by the teacher, who invites the children to look at the cards and to pay attention to differences among them (line 1). In a dialogue involving words and pointing gestures to the observed cards, a process of identification and naming of the main variables and modalities of the cards begins. Having shared names for the observed things, namely a semantic control over the MiD, is a necessary condition for the game, which is based on questioning and answering. The experimenting teachers appear conscious of the importance of developing such control. In the reported episode, the teacher guides children in naming all the elements of the cards (e.g. see lines 5–9, 25–26).

In the episode, children identified the cards variables and modalities in two different ways. One way is enacted by Michi, who notices both the blue and the red pot and recognises the variable by contrasting the two modalities in which it appears (line 12). Namely, she contrasts two cards and notices that they show different modalities (red and blue) of the same variable (pot). Another way involves two or more players looking at two different cards at the same time. A child notices one specific feature on one card and states it loudly. The other looks at a different card and notices that the same feature (the variable) appears differently (a different modality) and states it loudly. This is the case of the dialogue between Fede and Marghe in lines 14 and 16: Fede observes two butterflies on one card and states “it has two butterflies”, while Marghe (line 16) points out another card with a different number of butterflies and counts them. The search for differences is carried out by children in a close collaboration, which entail observing what the other child is observing and pointing to, taking it into account and selecting a card with a different modality of the same variable.

In both cases it is interesting to take a closer look on how the teacher inserts in the dialogue: after that some modalities have been identified by the children, the teacher soon mentions the relative variable. It happens in line 13, when the teacher comments “The pots have different colours, good!” right after Michi has said “This pot is red, and this one is blue”. And in line 15, when the teacher says “That there are butterflies” after that Marghe has looked at a specific card and said “That it has two butterflies”. In the game, the variables always occur in each card in one particular modality. In other words, it is not possible to observe the variables if not through the modalities. This may explain why despite the initial explicit prompt to children to look for differences (instantiated in the different modalities), the teacher’s subsequent effort is directed to making them noticing the variables connected to the mentioned modalities. Such a process, based on the comparison between modalities, is mainly carried out through verbal language. At the end of the episode, the teacher exploits the word “or” to connect all the positions in which the children have observed the caterpillar (lines 30–35).

In our data, we have found that the first approach to contrast modalities is used by children mostly for the “colour of the pot” and occasionally for the “number of petals”. The second approach occurs more frequently for other variables related to number and position (see Table 1). A possible interpretation regards the perceptual nature of the variable “colour of the pot”, which can be identified with a glance, or to the fact that the variable “number of petals” can be easily captured by subitizing, as the petals are prominently displayed on the card. On the other hand, in order to notice the different modalities in which the variable “number of butterflies” appears, it requires very often a counting process. We also found that the children generally notice the presence of the caterpillar immediately, but only after a while do they notice the different modalities in which it appears on the cards. The same applies to the left/right modalities of the leaf variable. In other words, the spatial position seems to be more difficult to notice or name than the number of petals/butterflies and the colour of the pot. According to our observation, no child noticed the difference in the position of the butterflies. In the above specific episode, the students have noticed all the variables except the position of the leaf in relation to the stem. The position of the leaf may be to the right or left of the stem, which was found to be quite a difficult distinction to manage for children. We will come back on this aspect in the next episode as well.

Episode 2. Is the Leaf on This Side Or on This Other Side?

To address questions RQ2a and RQ2b, investigating children challenges and how the teachers provided support in overcoming them, we rely on the a posteriori analysis conducted after several experimental educational sessions related to phase B of the research. These sessions were conducted both in the kindergarten and in the home environment. The playing processes investigated are those shown in Table 2, namely formulating questions, answering questions and discarding cards.

In relation to the phase of **formulating questions**, we observed children having difficulties mostly in the two situations: (i) verbalizing and articulating the question and (ii) identifying the variable or modality on which to base the question.

Table 2 Relation between MiG and game phases

		GAME PHASES		
		Formulating questions	Answering questions	Discarding cards
MiG	Law of excluded middle	✓	✓	✓
	Negation			✓
	Material implication			✓
	Case-based reasoning	✓	✓	

Some students simply pointed to a card and asked if Betta was under it. In these moments, the teacher typically suggested some variables and suggested questions in the following way: “Would you like to ask about the number of petals? Or perhaps the butterflies?”. Another challenge emerged when only few cards remained on the table during the game. In these cases, we noticed that some children asked inefficient questions instead of choosing the most suitable variable.

The following episode offers a brief excerpt from a game played by the previously mentioned group of four 4-year-old students, who were introduced to the Betta-the-Bee game on the same day. It focuses on the final question asked during the third match, when only two cards are left on the table. These cards are identical except for the position of the leaf in relation to the stem (Fig. 2). Fede is the hiding player.

In this extract, we can observe both challenges (i) and (ii) related to the phase of question formulation. Both in the first lines and in the last part of the excerpt, Marghe has difficulties in verbalizing and articulating the question (i). At the very beginning Marghe struggles with formulating her question and uses “above” instead of “below” (line 36). She receives assistance from her friend Fede to correct the mistake (line 37). In the last part, it is the teacher that steps in to support Marghe, engaging in a sort of word game where they alternate saying one word each (lines 48–52).

Moreover, we can also spot a challenge in identifying the variable on which to base an efficient question (ii). Initially, Marghe asks a question about the colour of the pot (line 38). The answer is yes, but the children struggle in the phase of turning cards, as we may infer from the seconds spent in silence. After making explicit that no card may be discarded as the pots are both red (line 42–43), the teacher prompts the students to look if they may find something different (line 44). After another long moment of silence, Michi identifies the modality for the question (line 45). Again, we can observe that he faces difficulties in wording the entire question, involving the terms “left” and “right”. In several cases we observed that children tended to avoid left-right questions unless absolutely necessary, i.e. when only two cards did remain and they differed solely in the position of the leaf.

Since we understood that the process of noticing variables underlies all the phases of the game, we decided to conduct individual interviews immediately following the play sessions in the home environment. These interviews aimed to deepen our understanding of how variables become apparent to the children. The

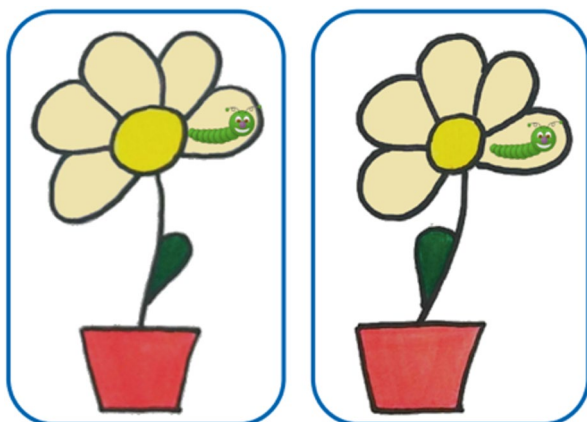


Fig. 2 Last two cards on the table

36. Marghe:	But Betta the bee is by chance above...
37. Fede:	No, under...
38. Marghe:	Well, under a red pot (<i>pointing to one of the two cards</i>)?
39. Fede:	Yes
40. Marghe:	So we turn...
<i>(4 second of silence)</i>	
41. Teacher 1:	Which one do you turn over?
42. Marghe:	None!
43. Teacher 1:	They are both red!
44. Teacher 2:	Michi, you may try to look if you find something different.
[...]	[...] <i>(4 seconds of silence)</i>
45. Michi:	This (<i>pointing to one of the two cards</i>) is a leaf on the other side and the other (<i>pointing to the other card</i>) leaf is on the other side
46. Teacher 2:	Bravo! So let's repeat, this (<i>pointing to one card</i>) is to the ...? (<i>raising her arm in the same direction as the leaf</i>)
47. Chorus:	Left
48. Teacher 2:	And that one (<i>points to the other card</i>) to the right. So, which is the difficult question to ask? Is Betta the bee...
49. Marghe:	Under...
50. Teacher 2:	The card...
51. Marghe:	That has...
52. Teacher 2:	The leaf...
53. Marghe:	Right (<i>lifting the left arm</i>)?
54. Fede:	No
55. Marghe:	Left (<i>keeping the same arm raised</i>)
56. Teacher 2:	Does it have the leaf on the right?
57. Fede:	No
58. Teacher 2:	So let's turn the [card with the] leaf to the right (<i>turning the card over</i>). Let's see... Marghe, try if it [Betta] is there... (<i>Marghe lifts the card and finds Betta</i>).

interviews were conducted with children who had not participated in the educational sessions at the kindergarten. In contrast to the kindergarten sessions, the exploration phase was omitted in order to make the noticing of variables more visible through the phase of questions formulation.

When interviewing Jack, a 5-year-old child, we observed that the first two questions he asked in his initial game were about the colour of the pot and the number of butterflies (see transcript in subsection: Episode 3. Yes is a tricky answer). As the game progressed, Jack asked the following three questions before he found Betta: Does it have two butterflies? Does it have one caterpillar? Does it have five petals? All the questions, except for the question about the colour of the pot, refer to numerical or existential variables. Even when Jack does notice the caterpillar, he does not attempt to ask a spatial question about the position of the caterpillar. We can interpret these observations through two different lenses. The first suggests that it is easier for Jack to distinguish between numerical quantities of variables in the MiD than between spatial features. The topological nature of spatial features as opposed to the typological way in which they are treated (essentially through language) could potentially help explain these difficulties (Lemke, 2003). The second perspective focuses on the specific graphical features of the cards and their perceptual apprehension. The variables that Jack refers to in his questions are those prominently displayed. For example, the flower with the different number of petals is at the top and is the largest graphic element on the card. The butterflies, which are also at the top, stand out with their contrasting colours and are well separated from the other graphic elements. The variables relating to the spatial position, on the other hand, are located in the middle of the card, are smaller and, in the case of the caterpillar, always overlap with other graphic elements.

In relation to the phase of answering questions, we observed that children faced two main challenges: (i) remembering under which flower card Betta is hidden, and (ii) connecting the verbalization of a modality (usually spatial) with its graphical representation.

Regarding the first challenge, we observed that some hiding players occasionally forgot under which card they had hidden Betta. As for the difficulties in verbalization, which occurred mainly with the variable “leaf position”, the adult took the opportunity to reinforce the terms right and left by associating the two terms with arm movements. This was done after ensuring that all players were on the same side of the table in terms of the orientation of the cards.

Episode 3. Yes Is a Tricky Answer

In relation to the discarding cards phase, we observed that children face the following two challenges: (i) discarding the cards after receiving a “yes” answer, and (ii) discarding the cards after receiving an answer to an inefficient question.

In the discarding cards phase, players have to remember the questions asked and the answers received, then to combine this information to decide which cards to eliminate. If the answer is “yes”, the cards lacking a specified characteristic—namely cards showing other specific characteristics have to be removed. We noticed early on, from the very first classroom session, that “yes” is a difficult answer to receive.

Acting correctly after receiving the “yes” answer requires the use of the logical operation of negation in conjunction with material implication, that is to say to exert logical control in combination with semantic control (see also the previous discussion in the section: The design of the “Betta the Bee” game). This challenge is reflected in the longer time necessary to children to discard the cards when the answer is “yes” compared to when the answer is “no”. Although we observed this aspect in numerous play sessions, we could not find evidence of this challenge beyond the time required or the greater involvement of the adult in supporting the process. We, therefore, decided to rely on an individual interview to explore this issue further. To this end, we report the transcript of the first Jack’s gameplay while he is playing in the role of the seeking player. Alongside Jack is a researcher explaining the game and acting as the hiding player, and Jack’s mother observing and supporting him in the game.

59. Jack:	Does it have a pink pot?
60. Researcher:	Is Betta under a card with a pink pot? Yes!
	(Silence)
61. Researcher:	So now we have to figure out what we have to turn...so Betta could be for example here, here, here... (<i>pointing to the cards with pink pot</i>) so these are all the cards that we have to keep. And so, which are the cards where Betta-the-Bee is definitely not hidden?
62. Jack:	Does it have 4 butterflies?
63. Researcher:	Uh, no, wait, wait, let’s reason about the pot
64. Mom:	You asked the question “is it under a pink pot?”, Carlotta [the researcher] said “Yes, it is under a pink pot”, then we have to discard all the pots that...
65. Jack:	that are pink
66. Mom:	That are pink? [...] But if Betta-the-bee is under a pink pot, do you have to keep the pink pots or do you have to discard them?
67. Jack:	You have to keep them.
68. Researcher:	Very good then let’s keep all the pink pots and discard the others!

In order to focus Jack’s attention on the consequences of having received a “yes answer”, Jack’s mother makes two interventions. First, she summarizes the question and the received answer and asks him to draw the conclusion on the cards to be discarded (line 64). But Jack fails (lines 65). At this point she reformulates the information received with the yes answer—namely that Betta-the-bee is indeed under a pink pot (line 66) and asks the child a direct question on whether the pink pot cards are to be kept or discarded. The child is now able to answer. The key passage here is that the focus of attention is shifted from the cards to be discarded, which Jack is not able identify, to what to do with the pink pot cards (keeping or discarding them). The fact that keeping the pink pot cards means discarding all the others is made explicit by the researcher immediately after (line 68) and the game may continue. According to the game design, this requires activating logical control on the complementary set and applying the law of the excluded middle.

Discussion and Conclusions

The careful design of the *Betta-the-Bee* game is driven by the belief that, in order to enhance learning opportunities for all children, pedagogies must intentionally focus on fundamental and specific aspects of mathematical reasoning from early childhood education. Specifically, the game is designed to create a suitable context in which children can engage with logical forms of reasoning inherent in mathematical activities. However, these forms of reasoning are not considered in isolation but rather as embedded in fundamental processes activated through the three phases of the game: examining the cards with specific goals in mind, asking questions to make decisions, and discarding cards according to the received answer. Carefully observing a diagram or figure, asking questions, evaluating answers and taking decisions are all essential processes in mathematical problem-solving and proof-related activities, as they require the integration of semiotic (including semantic) and theoretical (including logical) control (Arzarello & Sabena, 2011).

In playing the game, answering a question has generally proven to be easy for children. It requires the hiding player to exercise semantic control, linking the asked feature to their perception of the chosen card—provided, of course, that they remember which card they selected to hide Betta. As researchers, our focus has therefore been on the processes of asking questions and discarding cards.

Although the law of the excluded middle permeates all phases of the game, it is particularly evident in the constraint that only yes/no questions are allowed. Our observations indicate that the main challenges children face in this regard involve selecting the modality for the variable to consider and formulating the question appropriately. Playing the game in a non-competitive setting may enable children to support one another in these processes (as it happens in Episode 2).

In subsequent ongoing experimental cycles (not covered in this chapter), we introduced a “fortune wheel” displaying various modalities as a support tool for children struggling to choose which question to ask. This artefact contributed making reasoning by cases more apparent to children.

The teacher’s intervention has proven necessary in cases where inefficient questions were asked—namely, questions that do not allow for the elimination of any cards. With primary school children (grades 1 and 2), we are exploring also the issue of inefficient questions as an explicit topic of discussion. More generally, the challenges children face in formulating questions are leading us to place greater emphasis on the linguistic aspects involved in the questioning phase. Considering research on how the Italian language develops in young children may help refine our analysis of semantic control over the MiD and its role in game dynamics, particularly in relation to MiG.

The management of negation has proven to be a challenging issue, particularly in relation to material implication during the discarding phase after a “yes” answer. We observed that children take longer to discard cards, frequently make mistakes, and sometimes even hesitate. In fact, correctly responding to a “yes” answer requires applying the logical operation of negation alongside material implication, demanding both logical and semantic control. As noted by Horn

(1989), languages often develop lexical and syntactic strategies to facilitate the processing of negation (e.g., negative concord in some languages). In Episode 3, we saw how Jack's mother transformed the "problem of the yes answer" into a simpler question—focusing instead on whether a certain type of card should be kept or removed.

The process of observing cards permeates all phases of the game. Our data clearly show that the teacher plays a crucial role in guiding children's observation of the MiD, encouraging them to identify and name the variables and their modalities. This process unfolds through dialogue and pointing gestures, fostering a shared semantic understanding essential for the game questioning and answering dynamics. Children primarily use two strategies to identify differences: contrasting modalities within a single variable or collaboratively comparing different cards.

The teacher may strongly support this process by reinforcing variable recognition through verbal prompts, helping to strengthen children's analytical skills. There appears to be a general consensus among the authors of the POEM5 Conference that "the teacher has a key role as a mediator of mathematical objects because play environments do not provide sufficient support for children to explore new mathematical meanings on their own" (Palmér et al., 2023, p. vii). When playing alongside children, the teacher serves as a model, extending their influence beyond the mathematical content itself. As Radford (2020) notes, "in interacting with the children in classroom activity, teachers bring to the fore, and make available to children, features of knowledge and being that are relevant in teaching and learning" (p. 55). Further research is needed to explore the teacher's role in managing games (included rule-based games as Betta-the-bee) as educational activities.

References

- Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics*, 77(2), 189–206. <https://doi.org/10.1007/s10649-011-9291-9>
- Arzarello, F., & Soldano, C. (2019). Approaching proof in the classroom through the logic of inquiry. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 221–243). Springer. https://doi.org/10.1007/978-3-030-15636-7_10
- Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. D. Reidel.
- Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2011). Argumentation and proof in the mathematics classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 349–367). Springer.
- Horn, L. R. (1989). *A natural history of negation*. University of Chicago Press.
- Huizinga, J. (1971). *Homo ludens: A study of the play-element in culture*. Beacon Press (Original work published 1938).
- Lemke, J. L. (2003). Mathematics in the middle: Measure, picture, gesture, sign, and word. In M. Anderson, A. Sáenz-Ludlow, S. Zellweger, & V. V. Cifarelli (Eds.), *Educational perspectives on mathematics as semiosis: From thinking to interpreting to knowing* (pp. 215–234). Legas.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.
- McKenney, S., & Reeves, T. (2019). *Conducting educational design research*. Routledge.

- Palmér, H., Björklund, C., Reikerås, E., & Elofsson, J. (Eds.). (2023). *Teaching mathematics as to be meaningful—Foregrounding play and children’s perspectives: Results from the POEM5 conference, 2022*. Springer.
- Radford, L. (2020). Play and the production of subjectivities in preschool. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years: Results from the POEM4 conference 2018* (pp. 43–60). Springer.
- Reikerås, E. (2020). Relations between play skills and mathematical skills in toddlers. *ZDM—Mathematics Education*, 52, 703–716. <https://doi.org/10.1007/s11858-020-01152-1>
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Harvard University Press (Original work published 1931).

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Mathematical Modelling in Early Years of School



Trude Fosse, Troels Lange, and Tamsin Meaney

Introduction

Thirty years ago, Carpenter et al. (1993) wrote:

The construction of a model or representation of a problem is one of the most fundamental problem-solving processes. Many problems can be solved by representing directly the critical features of the problem situation with an equation, a computer program, or a physical or pictorial representation. Modeling also turns out to be a relatively natural problem-solving process for young children. There is an extensive body of research documenting that even before they receive formal instruction in arithmetic, young children can solve a variety of different types of addition and subtraction word problems by directly modeling with counters the action and relationships described in the problems. (s. 428)

Although Carpenter et al. (1993) highlighted the value of modelling for younger students, its depiction in academic literature remains scarce, unlike modelling for high school students (English & Watters, 2004). Since modelling and exploration constitute core elements of mathematics in the current Norwegian curriculum (Kunnskapsdepartementet, 2019; abbreviated to LK20), even the youngest students in school are expected to learn about these practices. Typically, mathematical modelling involves identifying critical features of a real-world situation and generating a model applicable to solving specific problems (Doerr & Pratt, 2008). This understanding is reflected in the statements in LK20 about what a model is and what the aims for the students are, “a model in mathematics is a description of reality using mathematical language. The pupils shall gain insight into how mathematical models are used to describe everyday life, working life and society in general” (Kunnskapsdepartementet, 2019, Core elements section).

Although modelling in the first years of school could provide opportunities for students to engage with various mathematical concepts across different contexts,

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very little research has been done on this (English & Watters, 2004; Stohlmann & Albarracín, 2016). Chan (2009) focussed on mathematical modelling as a contributor to developing students' problem solving, but did not consider problem posing. Instead, the contexts of the modelling tasks were decided by the researchers.

Although there is limited research on young children's mathematical modelling, understandings about its processes can be related to what is known about students in the first years of school solving word problems. Word problems have been investigated extensively (Verschaffel et al., 2020), particularly addition word problems. Verschaffel et al. (2020) described earlier research in which problem-solving models "assumed that the most critical steps in the solution of a word problem were the construction of a mental representation of the problem situation (i.e. situational model) and the transformation of that situational model into a mathematical model" (p. 4). These presumptions led to a focus on the reading skills of students so they could develop appropriate situational models, which could be mathematised into mathematical models that would contribute to them identifying a correct answer. Yet, subsequent criticism about the lack of consideration of the influence of content and context as well as students' informal solution strategies, such as directly modelling the problems with concrete materials or fingers, has led researchers to focus on how word problems can contribute to children moving from informal strategies for solving specific problems to general strategies that can be used for a range of problems, often involving formal arithmetic calculations. However, Verschaffel et al. (2020) noted that knowing how to facilitate students to make such a move is still open for debate. Chan's (2009) investigation suggested that mathematical modelling supported Grade 6 students to focus on the process of problem solving by reasoning their way towards a solution, which included arguing about any flawed reasoning which appeared in the discussion. This could be considered to show how children could be moved from informal strategies to general strategies.

To contribute to this debate, we discuss how children's number stories, known as *regnefortelling* (-er in plural) in Norwegian can give insights into mathematical modelling. *Regnefortellinger*, produced by young students, are multimodal, narrative texts that pose a problem involving quantities, sometimes self-chosen and sometimes supplied by the teacher, as well as a solution to the problem (Fosse & Meaney, 2020). The students choose the context for the problems and decide how to communicate them and their solutions through writing, symbols, and drawings. Unlike textbook word problems, *regnefortellinger* are based on children's own realities, including their imaginations. Hence, they connect to familiar situations and afford children the opportunity to mathematise aspects of their everyday lives (see for example Fig. 1).

As we assume that students construct mental representations of problems while they author their *regnefortellinger*, the story in a *regnefortelling* with its setting and events can be considered a situation model in alignment with Verschaffel et al. (2020). Therefore, the creation of a *regnefortelling* indicates that a mental representation, that is a situational model of the settings and events defining the posed problem, precedes the solving of the problem, rather than being produced as children interpret the problem.

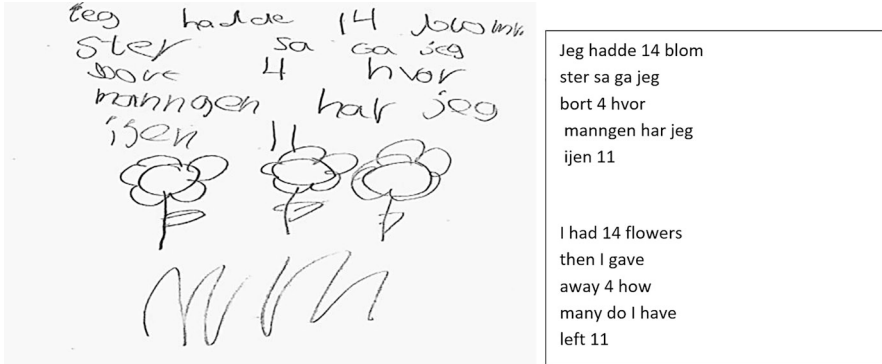


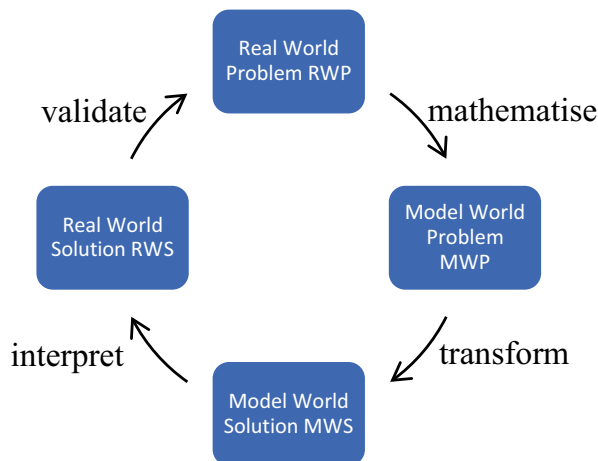
Fig. 1 Nr. 2-65—example of regnefortelling (it is a literal transcription of the Norwegian text but with the translation being into standard English)

We use mathematical modelling to investigate regnefortelling created by second-grade, seven-year-old children, that involve two or more arithmetic operations, which can be described as two-step problems. Constructing mental representations of two-step word problems is considered to demand greater mathematical cognition than one-step problems (Hickendorff, 2021). Viterbori et al. (2017) found that because adjustments to determine the appropriate arithmetic operations are needed in two-step operations, higher demands are made on students’ working memories. This is likely also to be the case with regnefortelling, as one-step problems in our data were by far the most common. As Alibali et al. (2014) found that developing two-step problems was challenging for intermediate-grade children, it was interesting to find regnefortelling in our data set which included two-step problems. Hickendorff (2021) had found that by year 6, students no longer make situation models, when solving word problems. As experienced problem solvers, they instead looked for cues that gave insights into which operation to apply to the numbers in the problem. Thus, analysing these examples of two-step problems, considered generally to be beyond the possibility for most seven-year olds, can provide insights into the relationship between situation models, mathematical models, and their connections to the contexts that the children have chosen in the regnefortelling. By understanding the mathematical modelling processes that contributed to these two-step problems, it would be possible for teachers to support other young children to develop more complex problem posing and general strategies for problem solving.

Theoretical Framework

Mathematical modelling is typically perceived to be a cycle of processes. Various versions exist, but as in Fig. 2 most modelling cycles involve identifying a “real world” problem (RWP), mathematising it into a mathematical problem or a “model world” problem (MWP), transforming the model world problem to produce a model

Fig. 2 Mathematical modelling cycle



world solution (MWS), *interpreting* the solution in relation to the original problem to produce a “real world” solution (RWS), and validating this solution against the original RWP (Doerr & Pratt, 2008). Extra-mathematical problems, i.e., problems that are not inherently mathematical, count as belonging to the “real world”. Hence, the children’s real world in regnefortelling can include a fantasy element or be entirely imaginary.

In relationship to regnefortelling, Fig. 2 indicates that the RWP comes into being during the process of conceiving the story with its setting and events. This situation model (a mental representation) is materialised in the actual artefact, the regnefortelling. Mathematising involves identifying relevant quantities and mathematical operations and combining them in an algorithm, which is the MWP. This may or may not be explicitly included in the regnefortelling. Carrying out the algorithm, such as calculating the value of an arithmetic expression, transforms the MWP into the MWS. Interpreting the MWS as a statement about the real-world results in the RWS. Validating this against the original RWP involves evaluating the reasonableness of the result, subject to real world conditions.

The modelling process is likely to be more complex when the problems, that the students are working with, require two or more steps. This is because although the context and the RWP are the same, the processes of mathematising, transforming, interpreting and validating may be interconnected in different ways. For example, it could be expected that the MWS to the first step could become information that is mathematised into the MWP of the second step. By reconstructing regnefortelling, which involve two-step problems, as mathematical modelling cycles, it is possible to gain insights into the mathematical thinking of these seven-year-old children. This can provide insights for teachers who want to develop students’ problem-posing capabilities as well as the mathematical modelling competencies. In Norway, it will also provide some insights into how the LK20 requirement to use mathematical modelling in the younger years could contribute to the common mathematical activity of constructing regnefortelling.

Method

In 2018, we collected 209 regnefortellinger from three grade 2 classes in a school in Western Norway. Producing regnefortellinger were part of the usual mathematics classroom activities and the teachers were not given any specific requirements for this research, but were asked to continue their normal practices. Nevertheless, there were differences between the classes. In one class, students drew a card with numbers less than 20, from two different decks and created a regnefortelling with these two numbers, using addition and/or subtraction. In the other classes, the students chose the numbers themselves. In both classes, the students chose the context, the settings and the events, for the regnefortellinger and how to convey the problems and their solutions in drawings, symbolic representations and the written descriptions. All the students worked individually.

The data included a variety of regnefortellinger, some of which have been published elsewhere (Fosse et al., 2023; Fosse & Meaney, 2020, 2021). In this paper, we focus on the 16 regnefortellinger that included two-step problems, in which the solution for one part of a problem was included into a subsequent part of the problem. These regnefortellinger were classified according to the relation between the problem parts using the categories in Table 1.

None of the regnefortellinger included drawings and only three included algorithms (two CP and one RD problem). Only one problem, a CP, did not include a solution, although in this case the solution was zero, suggesting the student perhaps knew the answer, but decided it was not necessary to include.

Following this classification, we then considered how each of the regnefortelling could be related to the processes in the modelling cycle (see Fig. 2). As the RWP is situated within a context, the first step of the analysis is to identify the RWP by noting context and if there was a stated problem. Often, the stated problem was a MWP as it focused a very narrow aspect of the context, such as buying a specific item. This meant there was limited information about the RWP and the mathematising process. The next step in the analysis was to identify how the MWP was transformed into

Table 1 Classification of two step regnefortellinger

Category	Number of regnefortellinger	Description	Example
Running total (RT)	11	Problems in which amounts were added and/or subtracted a step at a time from an original amount	Figs. 4–6
Continuous problem (CP)	3	A series of addition or subtraction part problems are described but without amounts being specifically calculated between each part-problem	Figs. 5 and 7
Related but different (RD)	2	One part of the problem could be about money being spent on buying objects, while the second step could be about eating one of the objects	Fig. 3

MWS. If a symbolic expression was provided it was easy to see how the students had transformed the problem into the solution. A clearly stated answer provided the MWS. If this was relayed as an amount in real world terms, for example a number of units rather than a unit-less number, then we considered that a RWS had been provided. However, the interpreting process was often not explicit. Similarly, the validating process could only be identified implicitly, because of the limited information about the initial mathematising of the RWP.

In the next section, we present four examples of regnefortellinger, three RT problems and one CP. They provide insights into different aspects of the relationship between the contexts that the students use and mathematical models.

We have not included an example of an RD problem, because, as can be seen in Fig. 3, switching the objects in focus produces two separate one-step problems rather than a two-step problem.

Results

The first example (Nr 1-7), depicted in Fig. 4, is about buying ice cream and requires the use of subtraction three times. Thus, it is an example of a RT problem. It provides insights into how the context of money, one frequently used in mathematics textbook problems although often in unrealistic situations (see for example, Queiroz et al., 2018), affects the relationship between RWP and MWP. The RWP is to determine how much money is left from an initial amount of 100 kr, after three consecutive purchases of ice cream by Pål and his friends. The remaining amount of money is calculated after each purchase, before the next purchase is made.

The figure shows handwritten student work on the left and two boxed versions of the problem on the right. The handwritten text is in Norwegian and reads: "Peter hadde 9 kr", "men han kjøpte", "3 epler å et eple", "kostet 3 kr vor mange", "penger har han igjen?", "— kr å han", "spiste et eple", "vor mange", "epler har", "han igjen? 2 epler". The first boxed version is in Norwegian: "Peter hadde 9 kr", "men han kjøpte", "3 epler å et eple", "kostet 3 kr vor mange", "penger har an igjen?", "___ kr å han", "spiste et eple", "vor mange", "epler har", "han igjen? 2 epler". The second boxed version is in English: "Peter had 9 kr", "but he bought", "3 apples and one apple", "costs 3 kr how much", "money did he have left?", "___ kr and he", "ate one apple", "how many", "apples does he have", "left? 2 apples".

Fig. 3 Nr 1-12—Peter buys apples

Pål Skall på butikken for og kjøpe is
han har 100 kr og isen koster 15 kr
hvor mange penger har han igjen
85 kr. og så kommer Gjermund han vil også
ha en is for 10 kr hvor mange kr har han igjen
75 kr og så kommer Håvard og han vil
ha en is for 20 kr hvor mange kr har han
igjen 55 kr

<p>Pål skall på butikken for og kjøpe is han har 100 kr og isen koster 15 kr hvor mange penger har han igjen 85 kr og så kommer Gjermund han vil også ha en is for 10 kr hvor mange kr har han igjen 75 kr så kommer Håvard og han vil ha en is for 20 kr hvor mange kr har han igjen 55 kr</p>	<p>Pål is going to the shop to buy ice cream he has 100 kr and the ice cream costs 15 kr how much money does he have left 85 kr and then comes Gjermund he also wants an ice cream for 10 kr how many kr does he have left 75 kr then comes Håvard and he wants an ice cream for 20 kr how many kr does he have left 55 kr</p>
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Fig. 4 Nr 1-7—Pål and his friends buy ice cream

There is no explicit mathematisation in the regnefortelling. Likewise, the MWP, the MWS, and process that transforms the first to the second, are implicit since there is no mention of mathematical operations and no symbolic algorithms provided. Nevertheless, we consider that the regnefortelling involves three “runs” through of the modelling cycle, one for each purchase. In the first run, the MWP is a subtraction $100 - 15$, which, having been transformed into the difference of 85, is the MWS. This is implicitly interpreted as the RWS, 85 kr, and implicitly validated as a legitimate answer to the original RWP. This amount is then the start amount for the next RWP, the modelling process of which can be reconstructed in the same way. A similar case can be made for the third run of the cycle. Thus, in this regnefortelling, we can reconstruct a modelling process consisting of three consecutive cycles, where the RWS from one cycle feeds into the RWP for the next. Alternatively, the reconstruction could posit one cycle with three sub-cycles between the model world problem and the model world solution, that is as a sequence of subtractions where the minuend is the amount of money available, the subtrahend is the expense, and the difference is the amount of money after the purchase. The difference in one subtraction becomes the minuend in the next. In symbolic terms: $100 - 15 = 85 \rightarrow 85 - 10 = 75 \rightarrow 75 - 20 = 55$. However, this alternative reconstruction is less reasonable given that the remaining amount of money after each purchase of ice cream is explicitly stated.

The absence of explicit mathematisation is not surprising since money acts as the universal medium for exchange of commodities by quantifying—and hence mathematising—the use value (of ice cream) to the exchange value (the money equivalent of the ice-cream) (Savard et al., 2020). It is a prime example of Skovsmose’s

(1994) point that “mathematics is formatting our society” (p. 43). Therefore, real world money problems are set in an already mathematised world with the effect that the distinction between the RWP and the MWP disappears. Thus, situational models revolving around events in which money is exchanged leads to regnefortellinger in which the RWP is already mathematised to the MWP. It may be that regnefortellinger which include money, by being derived from this particular class of situational models, can contribute to children developing complex problems which in turn may facilitate the development of general strategies. However, the loss of the distinction between RWP and MWP may restrict their understanding of the mathematising process if not discussed explicitly. Similarly, MWS and RWS are easily conflated in such settings, thus making distinguishing between MWS and RWS seem trivial, pointless and neglectable.

The regnefortelling (Nr 3-21) depicted in Fig. 5 is also a RT problem about money, although it involves more events and types of transactions than the previous one in Fig. 4. Nr 3-21 provides an example of how young students can combine different operations while staying within the same context. As such, it provides insights

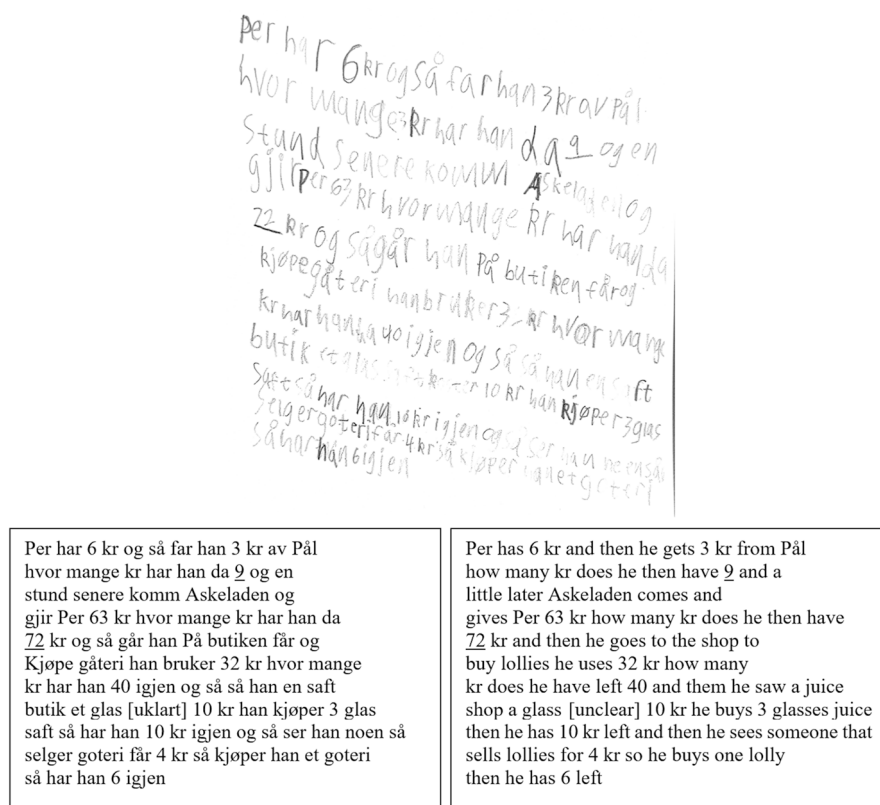


Fig. 5 Nr 3-21—per gains money and buys things

into how young children can move beyond specific informal strategies to developing general strategies for solving problems.

Per, so the story goes, has 6 kr, Pål gives him 3 kr and Askeladden gives him 63 kr. Per then uses 32 kr on lollies, buys three glasses of juice that are 10 kr each, and finally buys lollies for 4 kr. After each of the events, it is stated how much money is left, in the first three cases as the answer to the explicit question “how much has he now/left?” and in the last two as a statement “then he has [amount] left”. The five events in Nr 3-21, Fig. 5, are handled consecutively, suggesting mathematical modelling in 5 cycles, in which the RWS of one cycle feeds into the RWP of the next. As was the case in the first example, there is no explicit mathematisation. The reconstructed MWP for the first two cycles are additions $6 + 3$ and $9 + 63$, and subtraction for the third and fifth, $72 - 32$ and $10 - 4$. The unit, kr, is included in the RWS for cycle 2 (72 kr) and implicit in the others, 9 (kr), 40 (kr) and 6 (kr) respectively.

The fourth cycle is more complex, suggesting a move from a RT to a CP. The RWP is about the amount of money which remains from 40 kr after Per buys 3 glasses of juice, each costing 10 kr. Unlike the other exchanges, this is handled as one event rather than three consecutive events, one for each of the glasses of juice. MWP can be reconstructed in three different ways, either as consecutive subtraction $40 - 10 - 10 - 10$ or subtraction with a composite subtrahend consisting of a sum $40 - (10 + 10 + 10)$ or a product $40 - 3 \times 10$. It is not possible to tell from the regnefortelling which reconstruction is the most appropriate, but nonetheless it is a far more complex MWP than Nr 1-7 in Fig. 4, and regnefortelling with this complexity were rare in our data set.

Carpenter et al. (1993) asserted that children can solve various types of addition and subtraction problems when they mathematised the operations by directly modelling them. In this example, the problem is not modelled by counters, but is part of a self-authored story formed from a situational model in which the problem is articulated. The use of money as stated in regard to the first example facilitates this self-authoring. By controlling the setting of the story and the events in it, the child embedded the RWP, ensuring it made sense to them. It may be that using the names, Per, Pål and Askeladden, from familiar Norwegian folk tales supported the sense-making and the possibility to show their mathematical understandings. It seems that putting their self-authored problem into a familiar setting provided a possibility for this Grade 2 student to engage with quite complex mathematics.

Figure 6 (Nr 3-25a) depicts a regnefortelling about cards, which is somewhat different to the previous two. In it, the context allows for incorporation of anti-social behaviour, indicating that children’s self-chosen contexts provide opportunities to make sense of the real world with mathematics. There are two events in the regnefortelling. First, Peter, who has nine cards, and Mina, who has eight cards, collect them into a pile, and then Jon steals 4 cards from the pile. After each event, there is a question, asking how many cards there are, and the answer, making this a RT problem.

Compared to the two previous examples, the setting for Nr 3-25a is different, as it is about piling up and stealing cards rather than receiving and using money to purchase items. However, the reconstructed mathematical modelling processes are

Peter har 9 kort
 Mina har 8 kort
 så setter de bunkene
 sammen kor mange
 kort har de til sammen? 17
 så stjeler Jon
 4 kort kor mange
 kort igjen? 13

Peter har 9 kort mina har 8 kort så setter de bunkene sammen kor mange kort har de til sammen? <u>17</u> så stjeler Jon 4 kort kor mange kort igjen? <u>13</u>	Peter has 9 cards Mina has 8 cards then they put the piles together how many cards do they have together? <u>17</u> then Jon steals 4 cards how many cards left? <u>13</u>
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Fig. 6 Nr 3-25a—the card problem

similar. In this case, there are two cycles where the RWS of the first feeds into the RWP of the second. The mathematisation strips the amounts of cards of their unit, producing the MWP. The mathematical operations are addition and subtraction with small numbers, in the first MWP $9 + 8$ and in the second $17 - 4$. The MWS “17” in the first cycle is not explicitly interpreted to form RWS (which would be “17 cards”), but this is implicit both in the question (“how many cards do they have together”) and in the statement of the second RWP since this involves stealing “4 cards”. Similarly, the MWS (“13”) in the second cycle is not explicitly interpreted to a RWS (“13 card”) but this is implied in the question (“how many cards left”).

The antisocial behaviour of Jon, who steals cards, distinguishes this regnefortelling from the previous ones. Regnefortellinger that included anti-social behaviour did appear occasionally in the data set (Fosse & Meaney, 2020). This suggests that the situational model is affected by the educational context of producing a regnefortelling, in that the situational model is not just dealing with the mathematics but also testing the boundaries of what is possible to raise in a mathematics lesson from the wider society. Whereas most research on word problems have focused on children’s mathematical understandings (Verschaffel et al., 2020), the RWP and RWS in the modelling cycle require consideration of the context that children use in their regnefortelling. Especially, the validation of RWS provides opportunities for a discussion that goes beyond the mathematical understandings to talk about the real-world experiences that inform the children’s regnefortellinger. English and Watters (2005) highlighted that mathematical modelling in the early years of school provided opportunities for children to draw on their informal, contextual knowledge. Regnefortellinger, which allow students to use imaginary scenarios, seem to also provide them with opportunities to play with social aspects of the contexts, which is not possible when they are predetermined by the teacher.

The regnefortelling (Nr 2-29) in Fig. 7 is brief: Noah has 88 kr, buys a toy car for 80 kr, and finds 4 kr. A question is asked about how much it is, and an algorithm

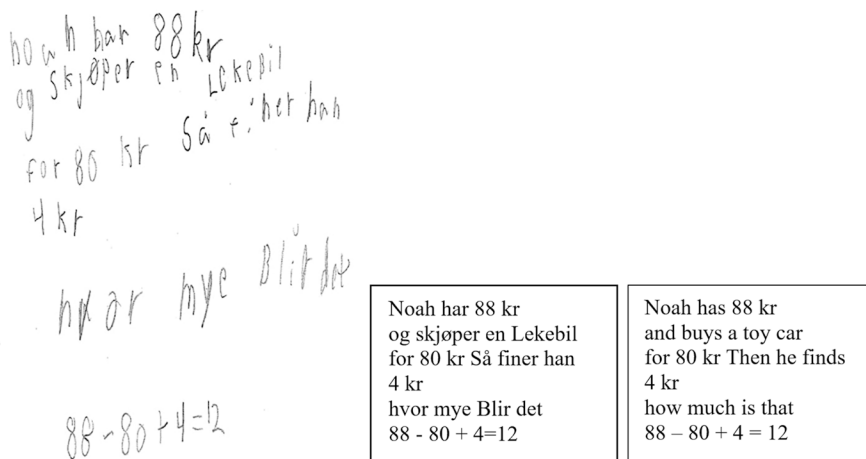


Fig. 7 Nr 2-29—Noah buys a toy and finds some money

with an answer is provided. This is an example of CP as the problem is presented as a whole without each step being calculated individually. Among two-step problems in our sample, CP were rarer than RT problems (see Table 1), suggesting that they are more complex than RT problems. However, it seems that CP problems can shift the modelling process so that interpreting the MWS to become a RWS is even less likely to be explicit. This seems to be because the algorithm takes prominence and the RWP becomes just a vehicle for producing it.

As was the case with Nr 3-25a in Fig. 6, there are two events, but unlike the previous examples, there is only one RWP. The two events are combined in one modelling cycle, with elements of explicit mathematisation. As in the previous example, a taken-for-granted mathematisation involves stripping the amounts of their unit, which never reappears, as no explicit RWS is included. MWP and its transformation into MWS is combined in one algorithm $88 - 80 + 4 = 12$. As was the case with Nr 1-7 in Fig. 4 and Nr 3-21 in Fig. 5, there is no explicit RWS, as the 12 is not provided with an indication that money (kr) is the real-world setting, and there are no signs of validation. The regnefortelling demonstrates that this student can combine two events into one modelling cycle and can use formal arithmetic symbolisation as a mode of expression. The irony, perhaps, is that the student seems to consider the RWP as merely a prompt, such as the case with word problems in textbooks, to focus on the calculation. This has similarities with Hickendorff (2021) who found that intermediate students looked for numbers and cues in word problems to determine what calculation they were expected to use, rather than relating to the context of the problem. In so doing, the situational models drawn from the word problems, similar to what author does in creating Nr 2-29 in Fig. 7, are reduced to focus only on the MWP and its transformation to MWS.

Discussion

In this chapter, we have reconstructed modelling cycles for four regnefortellinger, which were representative of the two-step problems from a large data set. Two-step problems are considered complex for children of this age but are also recognised as providing better opportunities for supporting them to develop mental models in their solution strategy (Viterbori et al., 2017). Our analysis of these regnefortellinger provides insights into different aspects of interactions and connections between the setting and events in situational models and mathematical modelling, which indicate that creating regnefortellinger can contribute to the development of mental representations of complex problem posing contexts. Although two-step problems have been used to investigate children's problem solving (Viterbori et al., 2017), reconstructing the mathematical modelling processes, when problem posing is combined with problem solving as in regnefortellinger, provides more information about the role of contexts. English and Watters (2005) found that sometimes young students' informal contextual knowledge impeded their work on mathematical modelling tasks whereas at other times it supported them. Our results show that the choices made about the contexts for their regnefortelling did have an impact on specific mathematical modelling processes but in different ways. We also found that creating CP, while seemingly mathematically more advanced, could result in the real word steps, RWP and RWS, and the mathematical modelling processes that connect the real world to the mathematical world, mathematising and validating, being ignored as the author focuses on producing an algorithmic mathematical statement.

In our analysis, we reconstructed the mathematical modelling for the arithmetic problems posed and solved in the regnefortellinger. These reconstructions indicated that the situation model was co-created with the RWP and consequently that the context was important for what problems the students could pose, make sense of and solve. Being authored by the children, regnefortellinger put them in control of the modes of expression. Hence, it can be assumed that these are in direct contact with their perception of the setting and events in their story (Johnsen-Høines, 2020). Thus, the choice of context affected what was possible within the posing of the problem and also its solution. Analysing the regnefortellinger from a mathematical modelling perspective, therefore, extended Chan's (2009) findings about problem solving to provide insights into how problem posing could contribute to the construction of situational models, when children chose their own contexts.

Whereas English and Watters (2005) found that young students' informal knowledge affected their interpretation of the modelling situations, in our analysis, it seemed that the situation model could reduce the distinctions between the different mathematical modelling processes. For example, contexts such as money seem to result in the conflation of the RWP with the MWP because money is already an abstract representation of an exchange linked to the monetary value of different items. As well, it seemed that when symbolic algorithms were provided in the regnefortelling, the children could have been channelled into ignoring the need to

validate their MWS to gain an RWS, as was the case with Nr 2-29 (Fig. 7). The algorithm may have been considered sufficient and, therefore, the validating process of the modelling cycle was no longer deemed important in the students' authoring of the regnefortellinger. This has similarities with Chan's (2009) results in her research on mathematical modelling and problem solving.

The two-step problems included in the regnefortellinger highlighted the mathematics that the children were interested in, although the mathematical choices would also be determined by what they were aware of. English and Watters (2005) found that children worked with mathematical ideas that they had not yet been formally introduced to. In the regnefortellinger, the mathematics included not only understandings about addition and subtraction but also equality, a crucial component needed for early algebra (Sumpter & Löwenhielm, 2022). When children work with a set of related problems, concerns have been raised about the use of "equality-strings" where strings of calculations are concatenated with equal signs (Sumpter & Löwenhielm, 2022). For example, Nr 1-7 (Fig. 4) could have been set out incorrectly as $100 - 15 = 85 - 10 = 75 - 20 = 55$. Sumpter and Löwenhielm (2022) warned that it is common for children to view the equal sign as an indicator to "complete the calculation" rather than equality, which is why these equality strings seem to be a common way for children to reason about how to express what they were calculating. However, in our small set of two-step regnefortellinger, there were no equality-strings. It may be that by authoring these more complex regnefortellinger using familiar modes of expression, the children are more able to understand the relationship between the different steps so that, when they are ready, they produce an algorithm, such as that in Nr 2-29 (Fig. 7), in which they use the mathematical symbols appropriately.

Conclusions and Implications

Our findings provide understandings about the affordances and challenges of having children produce complex regnefortellinger. Supporting teachers to use mathematical modelling as a framework for understanding students' regnefortellinger may contribute to their facilitation of young children's arithmetic knowledge and skills, so that they can move from specific informal strategies to more general strategies. It would also provide teachers with possibilities to use existing practices about regnefortellinger to fulfill curricular requirements about integrating mathematical modelling into their teaching (Kunnskapsdepartementet, 2019).

For a regnefortelling to be connected to all the processes of the modelling cycle, it seems that the student needs a purpose, perhaps from a more interesting context than in N2-29's in Fig. 7. A context in which children are interested may support them to make explicit the RWS, rather than just focusing on finding an MWS. Thus, there could be a need for teachers to have children discuss the inter-connections between the context and the problem in their regnefortellinger. This could relate to the mathematical ideas and to the societal issues that they raise through the choice

of context. It may be that textbooks provide only limited kinds of examples that students can draw upon for producing their own regnefortellinger, suggesting a need for children to see alternative examples of mathematical problems, which could include two-step problem-solving. This could include problems in which the mathematization is not already embedded, and hence the distinction between the real world and the mathematical world obscured, as typically is the case with money problems.

Regnefortellinger can also support young students to gain insights into modelling as a process, as required by the Norwegian curriculum (Kunnskapsdepartementet, 2019), since they are situated in children's everyday experiences, including their imaginations. However, to maximise these possibilities, teachers may need more information about the four steps of the modelling cycle, as depicted in Fig. 2. In particular, they may need support to devise prompts which could challenge the children to develop more complex regnefortellinger, involving richer settings and events, and provide opportunities to discuss the processes involved in mathematical modelling. In their research, Chan (2009) found that the intervention of the teacher could support the students to engage in further validation processes, that took them beyond a recognition of a plausible solution. It seems that some of the issues with using regnefortellinger to involve young children in mathematical modelling could be overcome if the teacher engaged the students in discussions about what they have represented and the processes that they went through.

There were few two-step problems in the collection of regnefortellinger. However, the analysis of these suggests that they are not beyond the reach of young students more generally. Making more explicit connections to the modelling cycle, could increase the students' mathematical understandings more generally, extend the range of complexity they can handle, and reduce the possibility of students simply looking to identify relevant numbers in word problems as they grow older (Hickendorff, 2021).

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References

- Alibali, M. W., Stephens, A. C., Brown, A. N., Kao, Y. S., & Nathan, M. J. (2014). Middle school students' conceptual understanding of equations: Evidence from writing story problems. *International Journal of Educational Psychology*, 3(3), 235–264. <https://doi.org/10.4471/ijep.2014.13>
- Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education*, 24(5), 428–441. <https://doi.org/10.2307/749152>
- Chan, C. M. E. (2009). Mathematical modelling as problem solving for children in the Singapore mathematics classrooms. *Journal of Science and Mathematics Education in Southeast Asia*, 32(1), 36–61. <https://hdl.handle.net/10497/15726>

- Doerr, H. M., & Pratt, D. (2008). The learning of mathematics and mathematical modeling. In M. K. Heid & G. W. Blume (Eds.), *Research on technology in the teaching and learning of mathematics: Synthesis and perspectives* (Vol. 1 Research syntheses, pp. 259–285). Information Age.
- English, L. D., & Watters, J. J. (2004). Mathematical modelling with young children. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of mathematics education* (Vol. 2, pp. 335–342). PME.
- English, L. D., & Watters, J. J. (2005). Mathematical modelling in the early school years. *Mathematics Education Research Journal*, 16(3), 58–79.
- Fosse, T., Lange, T., & Meaney, T. (2023). Lek med problemer [Playing with problems]. *Nordisk Barnehageforskning Tidsskrift*, 20(2), 89–109. <https://doi.org/10.23865/nbf.v20.280>
- Fosse, T., & Meaney, T. (2020). Using problem posing to bring real-life into the mathematics classroom: Can it be too real? *For the Learning of Mathematics*, 40(3), 40–45.
- Fosse, T., & Meaney, T. (2021). Milieus of learning in a Norwegian mathematics textbook. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Häikiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future: Proceedings of Norma 20 the ninth Nordic conference on mathematics education Oslo, 2021* (pp. 81–89). SMDF Svensk Förening för MatematikDidaktisk Forskning Swedish Society for Research in Mathematics Education.
- Hickendorff, M. (2021). The demands of simple and complex arithmetic word problems on language and cognitive resources. *Frontiers in Psychology*, 12, 1–12. <https://doi.org/10.3389/fpsyg.2021.727761>
- Johnsen-Høines, M. (2020). *Begynneropplæringen: Matematikdidaktikk—barnetrinnet. [Beginner's education: Mathematics didactics—Primary school]*. Caspar forlag AS.
- Kunnskapsdepartementet. (2019). *Læreplan i matematikk 1.–10. trinn (MAT01–05)* [Curriculum for Mathematics year 1–10 (MAT01–05)]. Fastsatt som forskrift. Læreplanverket for Kunnskapsløftet 2020. <https://www.udir.no/lk20/mat01-05?lang=eng>
- Queiroz, M. R. P. P. D., Barbosa, J. C., Noss, R., & Hoyles, C. (2018). The gap between the financial mathematics expressed in textbooks and that practiced in banks. *Acta Scientiae*, 20(2), 96–116. <https://doi.org/10.17648/acta.scientiae.v20iss2id3816>
- Savard, A., Cavalcante, A., Turineck, L. P., & Javaherpour, A. (2020). Some considerations towards the concept of money in elementary school. *For the Learning of Mathematics*, 40(3), 22–24. <https://www.jstor.org/stable/27091167>
- Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Kluwer Academic Publishers.
- Stohlmann, M. S., & Albarracín, L. (2016). What is known about elementary grades mathematical modelling. *Education Research International*, 2016(1), 5240683. <https://doi.org/10.1155/2016/5240683>
- Sumpter, L., & Löwenhielm, A. (2022). Differences in grade 7 students' understanding of the equal sign. *Mathematical Thinking and Learning*, 1–16. <https://doi.org/10.1080/10986065.2022.2058160>
- Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: A survey. *ZDM*, 52(1), 1–16. <https://doi.org/10.1007/s11858-020-01130-4>
- Viterbori, P., Traverso, L., & Usai, M. C. (2017). The role of executive function in arithmetic problem-solving processes: A study of third graders. *Journal of Cognition and Development*, 18(5), 595–616. <https://doi.org/10.1080/15248372.2017.1392307>

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Assessment of Early Arithmetical Skills in Natural Learning Situations: Two Assessment Situations and a First Validation



Hedwig Gasteiger and Christiane Benz

Early Arithmetical Learning in Natural Learning Situations

Mathematics learning in schools and beyond is based on a solid basic understanding of numbers and quantities at an early age. Conversely, if a basic understanding at an early age is lacking, there are gaps which—especially if undetected—can become larger and larger and lead to serious difficulties in learning arithmetic in the first years of school. These findings have been increasingly supported by numerous national and international studies over the last 20 years (e.g., Nguyen et al., 2016; Skopek & Passaretta, 2020). There is now a strong consensus that children can acquire this basic understanding of numbers and quantities as well as basic arithmetic skills, such as counting or dealing with quantities, in situations that encourage them in their natural learning processes (“natural learning situations”—Gasteiger, 2014, p. 276), like everyday activities or play situations (e.g., Gasteiger, 2012, 2014; Pramling et al., 2019; Seo & Ginsburg, 2004; van Oers, 2010).

However, we know that it cannot be taken for granted that children actually acquire early arithmetic skills in these play and everyday situations. In order to support children in their development, early childhood professionals have to moderate and accompany the learning process in these natural learning situations (Björklund et al., 2018; Zosh et al., 2018). Therefore, situational observation and perception is a central competence facet of the mathematical didactic competence of early childhood professionals (Gasteiger & Benz, 2018). This competence is

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necessary in order to recognize the individual abilities of children in their actions and expressions and to be able to assess their level of competence (Gasteiger & Benz, 2018). Early childhood professionals require a comprehensive understanding of a child's existing knowledge and skills. This foundation enables the professionals to plan future steps that will guide the child into the zone of proximal development (Vygotskij, 1978) or to act spontaneously in the moment with learning stimuli, based on the situation. A professional assessment is the basis for deciding which stimuli can help the child to develop further. The aim of this assessment is to guide the next steps with the child, in a goal-oriented manner. Assessment and learning support are therefore inextricably linked and should be understood as a kind of continuous cycle (Hußmann & Selter, 2013; Mulligan & Mitchelmore, 2018).

Prerequisites for an Action-Guiding Assessment of Early Mathematical Skills

For adequate pedagogical and didactic action, the early childhood professional needs to have a clear understanding of the arithmetic competences of each child. To achieve this, an assessment of early mathematical skills is essential. However, this assessment can only be successful when the early childhood professional is familiar with “the general developmental trajectory of, and individual differences in, children's understanding and learning of mathematics. This knowledge is essential to guide instruction for the group and tailor it to the individual” (Ginsburg, 2016, p. 941). Because it is knowledge that drives pedagogical action, Gasteiger and Benz (2018) refer to this as essential “explicit knowledge”.

As “knowledge of content and students” (Ball et al., 2008, p. 401) or a part of the competence of individual-related diagnosis and fostering (Gasteiger & Benz, 2018) early childhood professionals need to know the developmental processes of mathematical abilities, for example developmental steps in learning to count or to compare quantities—which as well is seen as a part of the essential “explicit knowledge” in the competence model of Gasteiger and Benz (2018).

Based on this knowledge, professionals are able to realize which individual mathematical skills their children have (a part of the competence facet situational observation and perception). In other words, early childhood professionals are only able to support children adequately in their developmental process, if they know the learning status (Where is the learner right now?), the learning goal (Where will/should the learner go in the next step?) and the steps towards the learning goal (How can the learner get there?) (Ramaprasad, 1983). Empirical studies have shown that a continuous assessment of children's performance with the aim of using diagnostic information to improve teaching and learning (formative assessment), can have a positive impact on children's performance (Hattie, 2012; William et al., 2004). However, the way in which the assessment is carried out seems to be important (Schütze et al., 2018).

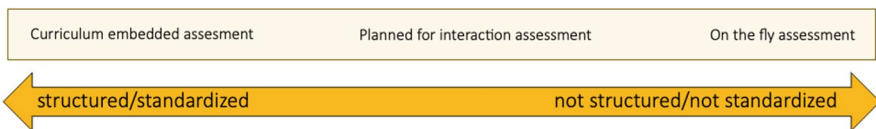
Possibilities for or Forms of Action-Guiding Assessments in Kindergarten

The aim of formative or action-guiding assessment can be described as obtaining information about the children’s competencies, as a starting point for future learning or for the adaptation of stimuli and support offered during a learning process (Philipp, 2023). In kindergarten (as well as in school), this information can be obtained in different ways, whereby the forms of assessment can be differentiated according to the degree of standardization and structuring, both in the acquisition of information and in the documentation (Fig. 1; Benz et al., 2024).

The most structured, standardized or formalized form of formative assessment is formal and embedded in curriculum (Shavelson, 2006). It can be implemented through a standardized test with a detailed manual. It has to be said that these tests are mostly used as a summative assessment or assessment *of* learning, which means to get results that can be compared to norms and not as assessment *for* learning, where the focus is on assessing the learning status with the goal to plan the next steps. Another example of a curriculum embedded assessment is a diagnostic interview conducted in a one-on-one setting in which specific skills are asked about. Less formalized is the planned-for-interaction (Shavelson, 2006) assessment which can take place in everyday life within the framework of planned action (Ertle et al., 2016) and play situations. Children’s skills and competencies are observed in this planned interaction. The third form, an on-the-fly assessment (Shavelson, 2006), can be understood as unplanned observations during everyday daycare activities. This is the least standardized form.

Tests can be experienced as an artificial situation in kindergarten. Due to test quality criteria, the way the child is addressed is usually precisely specified. Therefore, to ensure objectivity, in some test manuals, repeating tasks and answering questions are not allowed. Diagnostic interviews as another form of curriculum--embedded assessment are often not so prescriptive as tests but also take place in a structured and formal one-to-one situation. When evaluating children’s answers to the questions in the interview, not only products (as in tests) but also solution processes are taken into account. Thus, further aspects can be observed and documented which can be useful for further pedagogical and didactical decisions.

The planned-for-interaction assessment can take place in everyday situations within the framework of planned action and play situations in kindergarten, because learning situations and assessment situations do not have to differ in terms of the



Benz et al., 2024; Shavelson, 2006; Shavelson et al., 2008; Schütze et al., 2018

Fig. 1 Differences of assessment forms, due to standardization

task. Only the interaction between the early childhood professional and the child may be organized differently depending on the respective objective. In contrast to a learning situation, an assessment goal focuses less on helping the child to find a solution. If a child names an incorrect counting result, a stimulus in a learning situation could be, “Tap each marble as you count so that you don’t get mixed up”, while a stimulus in an assessment situation could be, “Can you explain or demonstrate how you got this result?” In this way, natural learning situations can also become assessment situations by gathering information about the child’s stage of development.

On-the-fly assessments in kindergarten can be understood as unplanned observations during everyday activities. This is the least standardized form of gathering information and requires a high level of expertise from early childhood professionals (Bruns et al., 2020).

In an evaluation of the practicability of different assessment tools in Germany (Benz et al., 2024), early childhood professionals rated a curriculum-embedded assessment, such as a one-to-one interview, more highly than other forms. the on-the-fly assessment was rated the lowest. Positive aspects of the curriculum-embedded assessment were mentioned, especially the accuracy, the clear mathematical focus in the situation as well as the organization and usability of the documentation. The early childhood professionals felt that they could train their assessment competence in a given situation of reduced complexity and, thus, would have the opportunity to acquire knowledge not only about relevant mathematical content, but also about which mathematical processes were important and how these could be assessed. They regarded assessment as an opportunity for professional development. This indicates that the interviewed professionals require and actively seek structure and precision in assessing children’s developmental progress. However, the early childhood professionals also highlighted negative aspects of the assessment embedded in the curriculum, viewing it as an artificial, highly controlled situation that children may perceive as a test. The on-the-fly assessment was reported to be quite demanding because of the complexity of everyday situations. In particular, the organization of the documentation was described as difficult, as the focus of the observation only emerges in the situation and is not known from the outset.

Observation and Assessment in Natural Learning Situations

As early mathematical education is not aimed for school-based contexts, but is more focused on natural learning situations like play and everyday situations, it seems sensible to assess the children’s developmental steps on-the-fly (Shavelson, 2006), through everyday observations and when a child shows mathematical skills spontaneously. This form of assessment is highly demanding (see above), as the early childhood professional must know in the situation, which developmental steps were taken and which were not (yet). This in turn requires that these developmental steps

are part of the explicit knowledge of the early childhood professionals and that this knowledge can be activated in the situation.

Therefore, it may be less demanding to pre-plan assessment situations in natural learning situations with the child (planned-for-interaction, Shavelson, 2006). This reduces the challenge for the early childhood professional to be spontaneously alert in every situation in order to not miss anything.

Especially when early childhood professionals have had little opportunity in their vocational training to acquire the explicit knowledge that is essential for an on-the-fly assessment (as it is the case in Germany—see Blömeke et al., 2017), a planned-for-interaction assessment can be helpful. This could be in the form of a game or a pedagogical situation, such as looking at a picture book, with detailed stimuli and questions and a documentation instrument tailored to this as a tool to assess children's mathematical skills and competence.

Two situations, which can be classified as natural learning situations (a game and reading a picture book), are described below that can be used as planned assessment and observation situations. The game situation has a stronger focus on strategies and processes to determine quantities, whereas the picture book reading situation has a broader focus on basic arithmetic skills. For the assessment situations in a natural learning context, early childhood professionals are provided with stimuli and questions that are assigned to different levels of arithmetic skills so that children's individual basic arithmetic skills and learning processes can be identified and observations can be documented.

Playing a Memory Game with Egg Cartons and Eggs

The classic memory game can be played with egg cartons and different quantities of eggs in the carton. The predetermined structure of the egg carton particularly stimulates the structuring perception of quantities and opens up the possibilities of different strategies to determine the quantity.

Like the classic memory game, the cartons are closed and only two cartons are allowed to be opened, with each turn. With a typical memory game, only two cartons with the same quantity and same arrangement (identical representation) would be a solution to finding a pair. In order to create more learning opportunities, the assessment-version of the memory game only require pairs with the same quantity. Two cartons are a pair if they contain the same quantity of eggs, even if the eggs are arranged differently. This rule enables many different learning opportunities (Bäckman, 2016).

The pairs of the memory game can be created together with the children. While creating the pairs different arrangements of one quantity can be discussed with the children. Here, different structures of arrangements and therefore different part-whole relations can become visible. Also, different possibilities to determine the quantity can be used, such as count all, count on and non-counting strategies (derived fact, composing, decomposing).

After the cartons are prepared, the game can start. The aim is to find pairs and explain why it is a pair. When children find pairs when playing the game and explain why the different arrangements have the same quantity, they can show different abilities concerning counting and non-counting strategies to identify the quantity.

The mathematical focus of the game lies on different processes for quantity perception and number determination based on structures in quantity representations. This emphasizes a process-oriented view of mathematics. Early childhood professionals direct their questions and actions not only toward the outcome, but, more importantly, toward understanding children's thought processes and the pathways they took to arrive at their conclusions. Rather than simply inquiring, "How many are there?", the emphasis shifts to questions like "Why are there seven?", "Why did you know that?", "How did you know there were seven?" or "How did you know that these boxes make a pair?"

Various situations arise during the game that can be used as assessment situations, but also as learning opportunities. This also applies to the situation of preparing the pairs (see above). However, the role and action of the early childhood professional is slightly different depending on whether the focus is on assessing children's competencies or on fostering competencies. Based on the observation in the assessment situation the early childhood professional can support the learning process in an adaptive way. Therefore, similar situations or even the same situations and questions can be used to help children to the next step in developing different strategies for determining and comparing quantities.

In order to enable precise observations and structured documentation in this assessment situation, an observation and documentation sheet with questions and stimuli was developed for the professionals, which provides possible observation categories and learning stimuli (see Fig. 2).

Despite the structured nature of the assessment and observation, the playful character of the game should be retained. For this reason, the questions do not have to be asked in any particular order, but can be chosen according to the play situation.

The practicability of the documentation is currently being tested by professionals in Germany. Following this initial test, the observation form has to be evaluated for validity (similar to the validation process described in the next section).

Reading a Picture Book: Pepe's Adventure¹

When reading a picture book that encourages mathematical interactions, there are many opportunities to observe and assess mathematical skills. Experts could use this pedagogical situation for assessment purposes even without additional materials. However, if early childhood professionals are to be supported in observing in a structured way and in directing their attention precisely to the most important

¹ Developed by Sonia Knochel and Hedwig Gasteiger as part of an exam thesis at the LMU Munich.



Finding pairs with egg cartons – Play Situation:
NAME _____

Arithmetical competences	Questions/stimuli focusing on the process	Quantity	Observation	Learning stimuli for different strategies to perceive, determine and compare quantities
The child ...				
Determination of quantity	(After only one box is opened) <i>How many eggs should be in the other box? Why do you know that?</i>		...says the correct number(s)	<i>Counting strategies-</i> - Put every single egg in the lid after you've counted it. - Touch one egg and use a number word. <i>Non-counting strategies</i> - Can you say how many eggs are in the box without touching them? - How did you know that? - Can you see, how many eggs are in the box? - How did you see that?
			Strategy	
			...counts all the eggs /touches them	
			...counts all the eggs without touching them	
Comparing of quantities	<i>Is the same quantity of eggs in both boxes? Why? Why not?</i>		...says the correct answer	<i>Comparing strategies</i> - Count both quantities. Which is the bigger number? - Take at the same time one egg of each carton in the lid. Stop, if it is no longer possible to take an egg of each carton. In which cartons are some eggs left? - Can you show me, how you must replace the eggs, in order to have the same arrangement?
			Explanation	
			...counts all eggs	
			...describes structures (Which?):	
Perception of structures	<i>How did you know that these two boxes are a pair? (same quantity but different arrangement)</i>		Explanation	<i>Perceiving and using structures</i> - Please show me, where did you see X eggs? - How did you see that these are X eggs? (Asking for the named parts.) -What pattern do you see?
			... says number without explanation	
			... counts	
			... uses structures (Which?)	

Fig. 2 Documentation tool for memory game

developmental steps, it is useful to provide more specific questions and prompts for these natural learning situations. In an optimum scenario, once early childhood professionals have used these structured questions and prompts several times, they can dispense with it because they are then automatically focusing on the essential knowledge and skills of children.

For this purpose, the picture book *Pepe's adventure* was developed, which, in addition to the illustrated pages (see Fig. 3), also opens up opportunities for action, for example by allowing bananas to be counted and distributed, which are attached to the picture book pages with magnets.

The early childhood professionals can note down their observations using an observation grid, which refers to the basic arithmetic skills that the book could make available to children, and which can be marked with a cross (excerpt see Fig. 4). Additionally, the observation grid gives some guidance about how children can be led to the next step of development or how the early childhood professional can help the child to react on the question or prompt (column on the right). There is also space for the teacher to make additional notes.

In the following, the story of the picture book *Pepe's Adventure* is described together with the basic arithmetic skills that can be observed by reading the picture book:

Pepe the monkey likes to jump from tree to tree (resultative counting: How many jumps can he do today?). But he actually wants to jump much further today. He wants to jump as far as the child can count (Counting skills, number word sequence: How far can you count?). Then a small accident happens. Pepe has not held on



Fig. 3 Scenes from the book *Pepe's Adventure*: Pepe's jumps & the game of hide and seek (pictures Andrea Maier)

<p><i>page 2 and 3 (jumps):</i> The child counts _____ jumps.</p> <ul style="list-style-type: none"> <input type="checkbox"/> the child points to pictures while counting <input type="checkbox"/> jumps were omitted <input type="checkbox"/> some jumps were counted several times <input type="checkbox"/> incorrect number word sequence: _____ _____ 	<p><i>stimuli to assist the child/for next steps of learning:</i></p> <ul style="list-style-type: none"> <input type="checkbox"/> pointing to jumps <input type="checkbox"/> count to 3 together with the child
<p><i>page 9 (hide and seek):</i> Child counts backwards: _____ _____</p> <p><i>Option:</i> counting backwards from 20 done <input type="checkbox"/> yes <input type="checkbox"/> no</p> <p>Child counts backwards: _____ _____ _____</p>	<p><i>stimuli to assist the child/for next steps of learning:</i></p> <ul style="list-style-type: none"> <input type="checkbox"/> counting backwards to 8 together with the child ('10, 9, 8 and now it is your turn') <input type="checkbox"/> Option: counting backwards from 20: together with the child to 18 ('20, 19, 18 and now it is your turn')

Fig. 4 Documentation tool for Pepe's adventure (excerpt)

tightly and falls to the ground. He injures himself (resultative counting: How many wounds does he have?). By chance, Pepe's friend Maika comes by and treats him. She moves on. As a thank-you, Pepe wants to collect bananas for Maika (resultative counting: How many bananas do you see?). Pepe collects eight bananas, which the child counts again and puts in a basket. Coco—Pepe's monkey friend—and her little brother Toni come along. They have also collected bananas (compare quantities: Who has more?). Pepe is happy to meet Coco and Toni and suggests playing hide and seek. Toni looks after the two baskets of bananas and Coco and Pepe play hide

and seek. Pepe is supposed to look for Coco and first has to count backwards from ten (Counting skills, number word sequence backwards: How does Pepe have to count?). In the meantime, Toni plays with the bananas and counts them (Counting skills, number word sequence & counting on: Show Toni how far you can count. Can you start counting from three?). Oh dear, Toni has got the bananas all mixed up (Comparing: Who has more bananas in the basket now?). Pepe has won the hide-and-seek game. Now he realizes that he’s really hungry. He eats four bananas (Calculating with countable elements: How many does he have left in his basket?). Pepe remembers that he wanted to give Maika 6 bananas - now he has eaten too many... (Calculating: How many is he still missing?). So, he jumps up and collects four more bananas (Calculating: How many does he have?). He looks for Maika and finds her in the forest. Finally, he can thank her properly and give her the bananas. Maika is happy and invites Pepe to sit under the tree and to eat the bananas together.

In order to check whether the picture book can be used to validly record early arithmetic skills, a sample of 26 German kindergarten children ($n = 14$ female, $n = 12$ male, between 3 years 6 months and 6 years 6 months old—average 4 years 11 months; $SD = 11$ month) was assessed with the Pepe picture book. Additionally, the OTZ (Osnabrücker Test zur Zahlbegriffsentwicklung [OTZ—a test of number concept development]; van Luit et al., 2001) was carried out with the children in individual situations to check the validity with correlations ($N = 25$, the data of one girl is missing). The OTZ contains 40 items and consists of eight subscales. Four subscales focus on counting skills and calculating, while the other four subscales address more basic mathematical skills such as serialization, classification and one-to-one mapping.

The Pepe picture book assessment contains 13 items with arithmetical skills, relevant for further learning, representing four subscales of different arithmetical content (see Table 1). First of all, a reliability test was carried out on the entire Pepe picture book assessment.

There were 13 items related to arithmetic content, including resultative counting, number word sequence, comparison, and calculation, which formed a reliable scale

Table 1 Subscales and items of the Pepe picture book assessment

Arithmetical content	Context in the story/details	Number range/items
Resultative counting	Jumps Wounds Bananas	Until 9 Until 4 Until 9
Number word sequence	Counting forward (jumps) Counting forward (hide and seek) Counting backwards (hide and seek) Counting backwards (option) Counting on (hide and seek)	As far as the child can count As far as the child can count From 10 From 20 Forward starting with 4
Comparison	Bananas in two baskets Bananas in two baskets	8 vs 5 6 vs.7
Calculating	Eating bananas Looking for bananas Collecting bananas	8–4 $4 + \underline{\quad} = 6$ $4 + 4$

Table 2 Descriptive data from the Pepe picture book and the OTZ (Solution rate: average of the points achieved divided by the maximum number of points to be achieved)

	<i>M</i>	<i>SD</i>	Min	Max
OTZ (40 items)	.40	.23	.05	.95
OTZ (20 items)	.28	.24	.00	.90
Pepe	.67	.21	.13	1.00

(Cronbach's alpha.792). The OTZ also showed good reliability values in the sample (Cronbach's alpha.924) for the whole scale with 40 items. When only the 20 OTZ-items which assess number word sequence, counting skills and calculating (comparable to the skills assessed in the Pepe picture book situation) are taken into account, the items also show good reliability (Cronbach's alpha.879).

Looking at the descriptive data (see Table 2), we can see that the OTZ was clearly more difficult for the children than answering the questions when looking at the Pepe picture book. The mean value at the OTZ was 40% for all 40 items and 28% for the 20 items assessing number word sequence, counting skills and calculating—compared to a mean value of 67% for the 13 picture book items.

To explain the results, it must be considered that the OTZ is normed for an age range of 4 to 7 years and our sample of assessed children was younger (3.6 to 6.6 years). Moreover, the OTZ includes counting and calculating in the number range up to 20. The counting tasks in the picture book situation can also be completed in a higher number range, but the calculation tasks are limited to the number range up to 10 (see Fig. 4).

To test the validity of the Pepe picture book assessment, we looked at the correlation between OTZ (in both configurations) and the Pepe picture book items. We calculated a Pearson correlation for the scale with all OTZ-items related to the Pepe picture book items. Both assessment tools correlated strongly and highly significantly with $r = .77$ ($p < .001$; Pearson). As the data collected with the 20 OTZ-items representing counting skills and calculating was not normally distributed, we analyzed the relation between the Pepe picture book assessment and OTZ with 20 items with a Spearman-Rho correlation. The correlation coefficient $r = .73$ ($p < .001$, Spearman-Rho) also shows a strong relation between the scale of the 20 OTZ-items representing counting skills and calculating and the Pepe picture book assessment. The Pepe picture book is, therefore, an assessment and observation tool that can gather information on children's basic arithmetic skills reliably in kindergarten.

Discussion and Conclusion

In this article, two natural learning situations—as situations which support children in their natural learning processes—were presented (a game and a picture book reading situation). These situations focus on the observation and detailed

assessment of arithmetic skills by early childhood professionals, through a structured approach, with deliberately-prepared stimuli. In this way, a precise, mathematically focused and systematic observation and assessment is possible. In addition to assessing a child's developmental stage, it is also possible to observe: What can a child achieve and perform with support? In what developmental process is the child currently engaged? The information provided through these situations highlight how it is possible to combine assessment with individual support for children and the stimulation of learning processes.

With the prepared stimuli, professionals do not need to possess the same level of explicit knowledge about the learning processes of children as is required in spontaneous, situational (on-the-fly) observations. Still, by linking questions and observations to arithmetic skills while engaging with the picture book or playing the memory game, key arithmetic concepts for further learning emerge, thereby enhancing the professional competence of early childhood professionals.

Furthermore, the Pepe picture book assessment tool based on a natural learning situation has been empirically proven. Basic arithmetic skills are validly recorded with this instrument. In both situations—the game and the picture book—the early childhood professionals were provided with an observation grid in order to meet their desire for simple organization and usability of the documentation (Benz et al., 2024). Based on the information gathered with the presented opportunities for assessment, the early childhood professionals can plan the next steps of learning. This could be counting children in the morning, counting the plates for lunchtime, simulating a rocket start by counting backward and many other things. The information can also be used to offer children specific board games or picture books with appropriate learning possibilities.

The observation grid and the accompanying stimuli can also serve as valuable tools for professional development, as they enhance early childhood professionals' awareness of the mathematical potential inherent in natural learning situations. Additionally, they help professionals recognize the various levels of mathematical skills that children may demonstrate. As it is mentioned above, working with these prepared and structured stimuli can help the early childhood professionals to get ideas about situations in which they could ask questions, to provide further learning opportunities and to assess children's skills. This is because learning support without prior assessment is not adaptive and can be unspecific, whereas assessment without subsequent support does not help children in their development and can lead to stigmatization. It is the strong link between assessment and learning support (Hußmann & Selzer, 2013; Mulligan & Mitchelmore, 2018) that provides a good basis for sustainable mathematical learning processes.

More extensive testing of both planned-for-interaction-situations in practice will provide further insights into their feasibility and usability. If the situations described in this chapter prove to be effective, observation grids and accompanying stimuli can also be developed for other natural learning situations, such as dice games or everyday situations like setting the table.

References

- Bäckman, K. (2016). Children's play as a starting point for teaching shapes and patterns in the preschool. In O. Helenius, T. Meaney, T. Lange, A. Wernberg, & M. Johansson (Eds.), *Mathematics education in the early years* (pp. 223–234). Springer. https://doi.org/10.1007/978-3-319-23935-4_12
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407. <https://doi.org/10.1177/0022487108324554>
- Benz, C., Reuter, F., Maier, A., & Zöllner, J. (2024). Kindergarten professionals' perspectives on observing children's mathematical competencies. In H. Palmér, C. Björklund, E. Reikerås, & J. Elofsson (Eds.), *Teaching mathematics as to be meaningful—Foregrounding play and children's perspectives* (pp. 171–183). Springer. https://doi.org/10.1007/978-3-031-37663-4_13
- Björklund, C., Magnusson, M., & Palmér, H. (2018). Teachers' involvement in children's mathematizing—Beyond dichotomization between play and teaching. *European Early Childhood Education Research Journal*, 26(4), 469–480. <https://doi.org/10.1080/1350293X.2018.1487162>
- Blömeke, S., Jenßen, L., Grassmann, M., Dunekacke, S., & Wedekind, H. (2017). Process mediates structure: The relation between preschool teacher education and preschool teachers' knowledge. *Journal of Educational Psychology*, 109(3), 338–354. <https://doi.org/10.1037/edu0000147>
- Bruns, J., Carlsen, M., Eichen, L., Erfjord, I., & Hundeland, P. S. (2020). Situational perception in mathematics (SPiM). Results of a cross-country study in Austria and Norway. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years* (pp. 317–332). Springer. https://doi.org/10.1007/978-3-030-34776-5_19
- Ertle, B., Rosenfeld, D., Presser, A. L., & Goldstein, M. (2016). Preparing preschool teachers to use and benefit from formative assessment: The birthday party assessment professional development system. *ZDM*, 48(7), 977–989. <https://doi.org/10.1007/s11858-016-0785-9>
- Gasteiger, H. (2012). Fostering early mathematical competencies in natural learning situations—Foundation and challenges of a competence-oriented concept of mathematics education in kindergarten. *Journal Für Mathematik-Didaktik*, 33(2), 181–201. <https://doi.org/10.1007/S13138-012-0042-X>
- Gasteiger, H. (2014). Professionalization of early childhood educators with a focus on natural learning situations and individual development of mathematical competencies: Results from an evaluation study. In C. Benz, B. Brandt, U. Kortenkamp, G. Krummheuer, S. Ladel, & R. Vogel (Eds.), *A mathematics education perspective on early mathematics learning between the poles of instruction and construction* (pp. 275–290). Springer. https://doi.org/10.1007/978-1-4614-4678-1_17
- Gasteiger, H., & Benz, C. (2018). Mathematics education competence of professionals in early childhood education—A theory-based competence model. In C. Benz, A. S. Steinweg, H. Gasteiger, P. Schöner, H. Vollmuth, & J. Zöllner (Eds.), *Mathematics education in the early years* (pp. 69–91). Springer. https://doi.org/10.1007/978-3-319-78220-1_4
- Ginsburg, H. P. (2016). Helping early childhood educators to understand and assess young children's mathematical minds. *ZDM*, 48(7), 941–946. <https://doi.org/10.1007/s11858-016-0807-7>
- Hattie, J. (2012). *Visible learning for teachers: Maximizing impact on learning*. Routledge.
- Hußmann, S., & Selter, C. (2013). *Diagnose und individuelle Förderung in der Lehrerbildung. Das Projekt dortMINT*. Waxmann.
- Mulligan, J. T., & Mitchelmore, M. C. (2018). Promoting early mathematical structural development through an integrated assessment and pedagogical program. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (pp. 17–33). Springer Nature. https://doi.org/10.1007/978-3-319-73432-3_2

- Nguyen, T., Watts, T. W., Duncan, G. J., Clements, D. H., Sarama, J. S., Wolfe, C., & Spitler, M. E. (2016). Which preschool mathematics competencies are most predictive of fifth grade achievement? *Early Childhood Research Quarterly*, 36, 550–560. <https://doi.org/10.1016/j.ecresq.2016.02.003>
- Philipp, K. (2023). Diagnostische Kompetenz von Mathematiklehrkräften. Einblick in verschiedene Forschungsperspektiven. In A. S. Steinweg (Ed.), *Grundlegende Kompetenzen sichern: Lernende und Lehrende im Blick* (pp. 9–24). University of Bamberg Press.
- Pramling, N., Wallerstedt, C., Lagerlöf, P., Björklund, C., Kultti, A., Palmér, H., Magnusson, M., Thulin, S., Jonsson, A., & Pramling, S. I. (2019). *Play-responsive teaching in early childhood education* (1st ed.). Springer. <https://doi.org/10.1007/978-3-030-15958-0>
- Ramaprasad, A. (1983). On the definition of feedback. *Behavioral Science*, 28(1), 4–13. <https://doi.org/10.1002/bs.3830280103>
- Schütze, B., Souvignier, E., & Hasselhorn, M. (2018). Stichwort—Formatives assessment. *Zeitschrift für Erziehungswissenschaft*, 21(4), 697–715. <https://doi.org/10.25656/01:16754>
- Seo, K.-H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements & J. Sarama (Eds.), *Engaging Young children in mathematics. Standards for early childhood mathematics education* (pp. 91–104). Lawrence Erlbaum Associates.
- Shavelson, R. J. (2006). On the integration of formative assessment in teaching and learning: Implications for new pathways in teacher education. In F. Oser, F. Achtenhagen, & U. Renold (Eds.), *Competence-oriented teacher training: Old research demands and new pathways* (pp. 63–78). Sense. https://doi.org/10.1163/9789087903374_006
- Shavelson, R. J., Young, D. B., Ayala, C. C., Brandon, P. R., Furtak, E. M., Ruiz-Primo, M. A., Tomita, M. K., & Yin, Y. (2008). On the impact of curriculum-embedded formative assessment on learning: A collaboration between curriculum and assessment developers. *Applied Measurement in Education*, 21(4), 295–314. <https://doi.org/10.1080/08957340802347647>
- Skopek, J., & Passaretta, G. (2020). Socioeconomic inequality in children's achievement from infancy to adolescence: The case of Germany. *Social Forces*, 100(1), 86–112. <https://doi.org/10.1093/sf/soaa093>
- Van Luit, J. E. H., van de Rijdt, B. A. M., & Hasemann, K. (2001). *Osnabrücker Test zur Zahlbegriffsentwicklung: OTZ*. Hogrefe.
- Van Oers, B. (2010). Emergent mathematical thinking in the context of play. *Educational Studies in Mathematics*, 74, 23–37. <https://doi.org/10.1007/s10649-009-9225-xs>
- Vygotskij, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press. <https://doi.org/10.2307/j.ctvjf9vz4>
- William, D., Lee, C., Harrison, C., & Black, P. (2004). Teachers developing assessment for learning: Impact on student achievement. *Assessment in Education: Principles, Policy & Practice*, 11(1), 49–65. <https://doi.org/10.1080/0969594042000208994>
- Zosh, J. M., Hirsh-Pasek, K., Hirsh-Pasek, K., Hopkins, E. J., Jensen, H. N., Liu, C. M., Neale, D., Solis, S. L., & Whitebread, D. (2018). Accessing the inaccessible: Redefining play as a spectrum. *Frontiers in Psychology*, 9. <https://doi.org/10.3389/fpsyg.2018.01124>

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Early Childhood Student Teachers Designing and Discussing Mathematics in Games



Annette Furnes, Sigurd A. Haaland, Magni Hope Lossius, and Elena Severina

Introduction

Bennett (2005) distinguished between two types of approaches to Early Childhood Education and Care (ECEC). In contrast to the school readiness approach, the pedagogical approach underlying ECEC in Norway, like other Nordic countries, is the social policy pedagogy approach, which focuses on the child's holistic development, being "firmly play-based, with movement, choice and child autonomy in evidence" (Bennett, 2005, p. 11). To achieve this, "[s]taff are trained to work in open framework contexts", which "allow them to experiment with different pedagogical approaches" (Bennett, 2005, p. 11). To support children's mathematical learning in play, ECEC teachers need to recognise mathematical content in the moment; meaning that kindergarten teachers' mathematical knowledge is an important prerequisite for being able to do this (Oppermann et al., 2016).

Games are one type of play, valued by mathematicians for their rule-governed behaviour, mirroring how they work with mathematics itself (Bishop, 1988a). In teacher education, there is a growing awareness of the value of game-based approaches to teaching and learning of mathematics (Kangas et al., 2016; Russo et al., 2021; Oflaz, 2023). Research has shown the potential of mathematical games to enhance mathematical thinking (Smith & Golding, 2018). There are a few studies concerning mathematics and game design, for example Pilten et al. (2017). However, in research on game-based learning of mathematics in teacher education, game design comprised only 2% of studies (Moon et al., 2024). In Pilten et al.'s (2017) study, the mathematical game design skills of primary school student teachers were investigated in relationship to design principles. As games are recognised as mediators of mathematical content and spaces for mathematical learning (Bayeck, 2020),

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being able to design games could also be useful for future ECEC teachers. However, there is a lack of research from ECEC using a social policy pedagogy approach, that have holistic and playful settings connected to mathematics and games. Our study explores the design of games by early childhood student teachers aiming to contribute to understandings about how to recognise mathematics in such settings.

The Norwegian Framework Plan (Ministry of Education, 2017) for ECEC, describes mathematics in the learning area “Quantities, spaces and shapes”, and indicates that kindergarten teachers “shall use [...] games [together with other tools and materials] to inspire the children’s mathematical thinking” (Ministry of Education, 2017, p. 54). “Quantities, spaces and shapes” builds upon Bishop’s (1988a) six fundamental mathematical activities, which he considered to be universal for all cultures. One of the fundamental mathematical activities is Playing, defined as “[d]evising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by” (Bishop, 1988b, p. 183). Studying problem-solving and problem-posing in the Norwegian kindergarten context, Fosse et al. (2020) built on Bishop (1988a, 1988b) and expanded the fundamental activity of Playing to include imagination and hypothetical thinking. By modelling reality, all players practice making assumptions, predictions, guesses and estimations. Therefore, problem-solving can be related to both Playing, by the question *how to*, and another of the six fundamental activities, Explaining, related to *why*.

There is limited research available on how teacher education supports student teachers to include mathematical content in holistic, play-based learning situations. Therefore, our research question is: *What mathematics can be identified in the games student teachers design?*

As the timeframe of the teaching session was limited and the student teachers were new to game design, some of their valuable ideas may not have been implemented as intended. Therefore, we analysed the student teachers’ process of designing, playing and discussing the games. We first describe some connections between game design features (Schell, 2019) and mathematics, and then present Artefact-Centric Activity Theory (ACAT), the theoretical framework of our study. We identify mathematical content in the games, based on Bishop’s fundamental activities, while ACAT is used to reveal the mathematical potential for young children of the game and the student teachers’ understandings about mathematics for young children.

Game Design and Mathematics

As noted previously, mathematics in the Norwegian Framework Plan (Ministry of Education, 2017) is understood in terms of Bishop’s (1988a, 1988b) fundamental mathematical activities. These activities are: Counting (systematically identify, organise, compare and explore quantities of objects), Designing (using mental images to design a shape or an object in the environment), Measuring (quantifying

qualities for purposes of comparison and ordering), Locating (exploring one's spatial environment and conceptualising and symbolising it), Explaining (finding ways to account for the existence of phenomena) and, previously described, Playing. Bishop (1988a, p. 48) associates the fundamental mathematical activities with questions: Counting would correspond to human activities focusing on *how many*; Measuring—*how much*; Locating—*where*; Designing—*what*, Playing—*how*, and Explaining—*why*.

To understand the game design process, which is non-linear, creative and complex, Schell (2019) suggested using different “lenses”. The lens of elemental tetrad describes game design as consisting of four interrelated elements: Mechanics (game procedures and rules), Story (sequence of events), Aesthetics (sensory appeal) and Technology (materials and interactions). The lens of player indicates the importance of always having in mind who is the player and what is a suitable gameplay type (individual, competitive, collaborative, etc.).

Games, often considered to be tangible real-life objects, have possibilities for conveying mathematics, that are closely related to how game design elements are interconnected. For example, the fact that the Technology of the game has a die is not enough to conclude that the game is about Bishop's Counting, because the Aesthetics component could define that the die has colours or geometrical shapes instead of dots or number symbols. The rules decide how this die can be used. For example, a die with shapes combined with the rule “find a similar shape” challenges a child to recognize the shape, therefore engaging in Designing, and not Counting. Likewise, unclear rules will require players to negotiate, creating opportunities for problem-solving and, therefore, Playing. Aesthetics affects visualisations of the mathematical content, while Story influences the possibility to choose different actions, instead of blindly relying on chance. Therefore, we carefully describe the game design elements before identifying its mathematical content.

Game Design and Artefact-Centric Activity Theory

To discover how student teachers incorporated mathematics into designing games for young children, we choose to use ACAT. Originally ACAT was developed to support the development of digital games for children to be used on multi-touch devices (Ladel & Kortenkamp, 2011) and further developed to assess digital apps (Larkin et al., 2019). ACAT is a theoretical model based on Cultural-Historical Activity theories originated in the 1930s as part of Soviet psychological activity theory and further developed by Leontiev (1972/81) and Engeström et al. (1999). According to Larkin et al. (2019), ACAT can be a “methodological tool to understand and structure the complex, causal network of relationship that comprises the interacting activity system of teaching and learning” (p. 60). Both digital games (apps) and analogue games (e.g. board games) are cultural artefacts which mediate children's learning of the mathematics which is in focus (see Fig. 1).

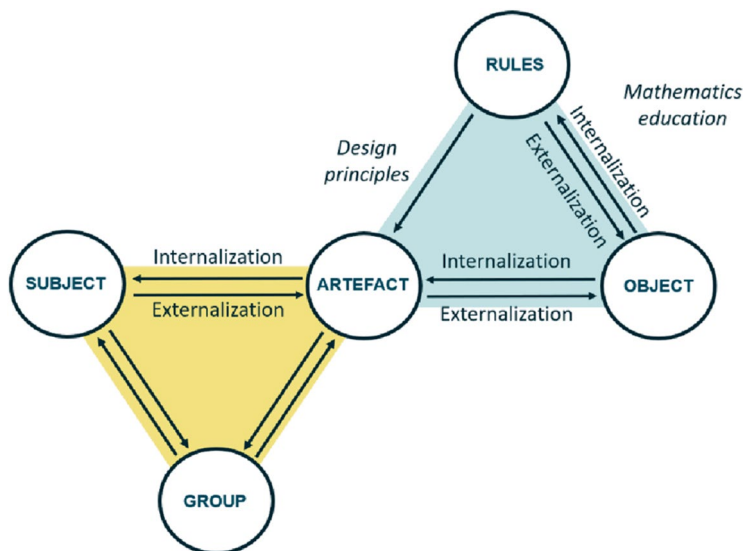


Fig. 1 Adopted ACAT model (Ladel & Kortenkamp, 2011, p. 66): blue—the right triangle nodes, yellow—the left triangle nodes

The ACAT model (Fig. 1) consists of five nodes: subject, artefact, object, rules, and group (Ladel & Kortenkamp, 2011; Larkin et al., 2019). The relationship between these is described by the intertwined processes of internalization and externalization, operating continuously (Engeström et al., 1999). In ECTE context, Vee and Meaney (2022) used ACAT to identify the potential relationship between the child (subject) and the mathematical content (object) mediated by apps (artefact) based on the student teachers' app evaluations. Although children were not participating in the study, the student teachers focused on them as the subjects when making their evaluations, because they would be expected to be learning mathematics (object) through interacting with apps (artefact).

In alignment with Schell's (2019) elemental tetrad, we consider that game procedures and rules (element of Mechanics) are related to the rules node as they dictate how the game is played. The game's elements of Aesthetics and Technology belong to the artefact node because they are tools to visualise the Mechanics and make it tangible for players. The gameplay type and element of Story frame allowed interactions among players, making it relevant for the group node. The group node represents players' interaction with each other while playing, including the required scaffolding of children's learning, with and without the support of the artefact, which is linked to the rules node.

The ACAT model can be split into three substructures: subject–artefact–object (horizontal axis), artefact–object–rules (right triangle), and artefact–subject–group (left triangle) (Ladel & Kortenkamp, 2011; Larkin et al., 2019). The horizontal axis gives information about the mathematical concepts the games make available for players. The right triangle is linked to the design process and may provide insight

into how the student teachers viewed mathematics and expected it to be learnt. The left triangle is linked to the gameplay and may indicate how the student teachers thought children would engage with mathematics through the opportunities made available by the game, such as by discussing it with others or by silently taking each turn to move around the board. Analysing each substructure helps to understand the potential relationships that the student teachers considered that there were between the subject (child) and object (mathematical content), mediated by the artefact (games designed by student teachers).

According to Ladel and Kortenkamp (2011), the subject–artefact–object-line (Fig. 1) is the primary substructure for studying interaction. Here, learning and understanding of the object by the subject through the artefact are viewed as ongoing, intertwined processes of internalization and externalization. When studying game design processes in an educational context, the artefact–object–rules-triangle (Fig. 1, right triangle) provides insights into the interaction between the game (artefact) and the mathematical content (object) through the game’s rules and procedures (rules). “Through the rules we define the object formally (externalizing it) and the rules are made to capture the nature of the object in the best way possible (internalization)” (Ladel & Kortenkamp, 2011, p. 4). The rules externalize abstract mathematical concepts (e.g. one-to-one correspondence), making them concrete (e.g., through the provision of a dotted die, rather than one with the digits representing the amounts) while supporting the subject’s internalizing of these expectations (e.g. about using the provided die) within the physical components of the game. The artefact-subject-group-triangle (Fig. 1, left triangle) emphasises how the subject is influenced by the potential group who would play the game. The interaction with and around the game depends on who is playing, which could influence which mathematical objects were engaged with, which may differ from those which were the focus of the game.

Methods

In Norway, early childhood teachers complete a three-year bachelor program, in which “Language, Text and Mathematics” is a mandatory course in the first year of study, specifically focusing on mathematics and Norwegian language. Playful learning is one of the course’s central topics, as well as Bishop’s (1988b) six fundamental mathematical activities. In this chapter, we use data collected in a small-scale design research project on playful learning from this course. As part of the project, teacher educators, including the authors, organised a 6 hour workshop in which student teachers designed games suitable for children aged one to 6 years. The guidelines indicated that the game should facilitate experiences of at least one of Bishop’s (1988b) activities, which had previously been introduced in the course. The first-year student teachers designed the games during an interdisciplinary play workshop in five classes (two in autumn 2023 and three in spring 2024). As well as designing the games, the student teachers played the games and either made a sales pitch to

present their game to fictitious buyers or made a game page in Book Creator (a digital app). Each group of student teachers was taught by two teacher educators and was observed by at least two other teacher educators. The authors attended workshops as either teachers or observers.

Data from the workshops includes unstructured, individual field notes on student teachers' game design processes, photos, as well as 21 games. For this paper, we analyse the games and the field notes on student teachers' actions and dialogues from designing, presenting, playing and discussing games in the five classes. Due to the unstructured nature of the field notes, the amount of data about each game varied. The project followed ethical requirements for research in Norway.

We conducted the data analyses in three steps. In the first step, we categorized the games according to the elemental tetrad and gameplay type (Schell, 2019) to reveal the diversity of game designs. In the second step, we identified the mathematical content, according to Bishop's (1988a) six universal activities. The third step involved identifying how the student teachers considered that young children would engage with the mathematics, using ACAT.

Initially, three of the authors worked together to become familiar with the games and field notes. We then extended the field notes for each game with a description of the game's design, based on the elemental tetrad and types of gameplay (Schell, 2019). Most of the games had clear rules (20 of 21), a linear story (19 of 21) and involved competitive gameplay (16 of 21). For further analysis, we chose one game with open rules, one with a branching story and one with collaborative gameplay to ensure that we gained insights from the range of game design types.

In the second step of the analysis, we identified the mathematical content of the game. In operationalising Bishop's mathematical activities in relationship to each game, we agreed that *Counting* could be identified in rules that encouraged players to use elements of aesthetics (tally objects, number symbols, number line or relevant drawings on the cards/game board) and technology (e.g., task cards, dice) to find out *how many*. *Measuring* could be identified in rules that encouraged players to use elements of aesthetics (physical objects or functional visualizations on the gameboard and/or task cards), technology (e.g., task cards, rope, natural material) and story (unfolding of the route) to use comparison words and find out *how much*. *Locating* could be identified in rules that encouraged players to use elements of aesthetics (functional visualizations), technology (e.g., task cards), story (unfolding of the route) or other ways to identify the route to find out *where*. *Designing* could be identified in rules that encouraged players to use elements of aesthetics (functional visualizations like shapes) and technology (e.g., task cards, puzzle pieces) to find out *what*. *Playing* could be identified in rules that encouraged players to use elements of aesthetics (functional visualisations), technology (e.g. task cards, puzzle pieces) and story (unfolding of the route and related dependencies) to engage in problem-solving to find out *how*. *Explaining* could be identified through rules that encouraged players to use elements of (functional visualisations), technology (e.g. task cards) or story (unfolding of the route and related dependencies) to classify or define something and find out *why*. Collaborative games might increase the number of situations where *why*-questions are posed and where players are engaged in

Table 1 Classification of our data into different ACAT nodes

ACAT node	Description of the node and type of data that belongs to it	Example from our data
Object	Bishop's activities. Data included explicit naming of mathematics in the game by the student teachers or the mathematics made available by features of game design	Counting was evident in field notes which captured student teachers during the game design discussing counting in the game or the use of number symbols on the board
Rules	Rules of the game. Data orally or written describing rules and/or gameplay of the game	The student teachers describe the rules: " <i>Roll the dice, count the number of dots on the dice and move the corresponding pieces forward</i> "
Artefact	Explicit aspects of the game design made available through the game itself or student teachers' remarks, e.g. aesthetics	The design of the game (Fig. 2) might incorporate counting through aesthetics like number symbols on the gameboard and technology through a die with dots etc.
Subject	Children. Data documenting student teachers' discussions of the learning potential for a child from playing the game	Student teachers said during the sales pitch, " <i>in this task, they [children] will explain to each other how to cross the water, for example by using "left" and "right"</i> ". Here student teachers explain the game's potential for children to learn about locating
Group	Group of players. Field notes documenting student teachers discussing the potential for interaction between children and/or ECEC teachers using the game or other comments about groups as players	Student teachers said during the sales pitch: " <i>Now we have one hunter. It might be that the children want two hunters. That is open for discussion. ... that's where we can play with the children's imagination. They can participate, we can have multiple wolves, hunters</i> " this fieldnote describes how the student teachers identified possible interactions in a group related to the object playing

problem-solving together, hence gameplay type may influence engagement in *Explaining* and *Playing*.

In the final step, we used the ACAT model (Ladel & Kortenkamp, 2011) to analyse the games and corresponding field notes. The mathematical content identified in step two was considered to be the object. Then the data was further analysed to identify their relevance to other nodes (see Table 1). If several of Bishop's mathematical activities were evident in the data, each was considered a separate object and analysis was repeated for each object.

Results

To answer the research question, "*What mathematics can be identified in the games student teachers design?*", we present the results from three games to illustrate how the students teachers identified how the potential for mathematical learning opportunities was integrated into the variety of game designs.

Fig. 2 Game “One, Two, Three—Eat Little Red Riding Hood”



Game “One, Two, Three—Eat Little Red Riding Hood”

The game “One, two, three—eat Little Red Riding Hood” (Fig. 2) was selected for analysis as it was the only game where the design included rules which were open for negotiation. It was based on the fairy story “Little Red Riding Hood”, which is well known in Norwegian ECEC institutions. This game was designed, played, and presented by the student teachers to a potential buyer by holding a sales pitch.

The gameboard had a predefined route, with progress determined by rolling a die. According to student teachers in the sales pitch presentation, this was a competitive game where children would be encouraged to play with an adult. Like 10 of the 21 games designed by the student teachers, this game was inspired by “Snakes and Ladders” with the game space being restricted to in and around the board. Players started at the gate (bottom right corner of Fig. 2) and move forward along spaces 1–19 (coloured circles on the board in Fig. 2). There were two places where the player should go backwards (arrow) or down (ladder). To complete the route, the players had to cross a pond in front of Little Red Riding Hood’s house by throwing a ball into a free-standing egg-box. The game elements and player pieces are thematically adapted to the fairy story.

Analysis

We identified Locating, Measuring, Counting, Playing in the game and the field notes about the student teachers’ design. Locating and Measuring were not as evident in the game design and related processes. Locating, Measuring and Playing were recognised in the rule “throw the ball into the egg-box”, indicating that the player must throw a ball into a free-standing egg-box to cross a pond. Estimating the distance to the egg-box involves judgments related to *where* and *how long*. However,

freedom to position the egg-box, invited players to consider *how* to achieve the desired outcome, indicating that Playing could be considered part of the object of the game. Counting could be recognised through the rule to use a dotted die to define *how many* steps to move a player piece forward on the board.

Studying the processes of internalization and externalization between the game (artefact) and the child (subject) offers information about the opportunities that the student teachers made available for *Counting* activities in the game. One-to-one correspondence could be stimulated by interpreting the die and moving the game piece around the board accordingly. The numbered circles on the game board are an externalization of a number sequence that could invite the players to internalize the symbols and/or numerical sequence 1–19. However, the student teachers' explanation of the rules during the sales pitch "you start at the gate, roll the dice, and move as many steps as the dice shows in one direction (except in positions 5 and 12)" indicates that they may not have identified how children could internalize other Counting components other than one-to-one correspondence.

Playing (object) was initially identified in relation to the rule about the pond crossing. The student teachers commented during the sales pitch, "They (the children) have to agree where to place it (the egg-box). They can adjust the difficulty by placing it closer." We interpreted this as the student teachers were highlighting the children's possibilities for problem-solving, through the provision of open rules. During the sales pitch, student teachers defined their game as "multifunctional", meaning it was also considered to act as a set of toys, revealing a slightly different understanding related to Playing as an object:

This game allows for negotiations because now we have one hunter. It might be that the children want two hunters. That is open for discussion. Perhaps they want to include grandma? We don't have grandma here, but it is possible. We have certain rules that are not completely determined, we can change the game as we go along. The hunter, who will be him? Who will get eaten? Little Red Riding Hood or the hunter? That is where we can play with the children's imagination. They can participate, we can have multiple wolves, hunters.

In this utterance, the student teachers indicated that they valued the rules of the game being open, as they provided opportunities for the children to negotiate and therefore to engage in problem-solving. The focus on negotiation also highlighted possible interactions between the *group* (children and kindergarten teachers), the *subject* (the player) and the *artefact* (the game), which are part of ACAT's left triangle. Several hypothetical scenarios are presented by the student teachers, highlighting that they focused on the mathematical object Playing, by considering problem-solving and rule negotiation. The student teachers raised different ideas about what the children might wonder about, indicating that the student teachers were open to listening to the children, making them aware of children's possible internalization processes. They suggested that children might include new, previously unidentified elements in the game (grandma, an extra hunter) and encourage rule negotiations among children. If this happens the realisation of such possibilities could be considered examples of the externalization of children's internal understandings, as Engeström et al. (1999) described that an internal representation can become externalized through speech. All these ideas were open to different degrees

of complexity in the gameplay and thus possible developments in the mathematical objects and children's learning.

Measuring and *Locating* were externalized as objects through the rule "throw the ball to the egg-box"-part of gameplay and using game board design with a free-standing egg-box. The student teachers suggested that children might be interested in "measuring how long it is from start to end [on the game board]". This indicates that the presence of the objects *Measuring* and *Locating* in game design could appear, but that student teachers might need further assistance to identify possibilities in playing games with children that are more realistic.

According to the student teachers' sales pitch, the presence of the geometrical shapes (square, rectangle, circle, or triangle) on the back of wolf-figures could expose children to *Designing*. However, Bishop's definition of designing included "creating a shape or design for an object" (1988b, p. 183), which the game design did not seem to prompt the children to do when playing the game. Rather, the use of shapes to differentiate the wolves seemed to only act as labels. Therefore, the student teachers may need further insights into what is involved in the mathematical activity of *Designing*.

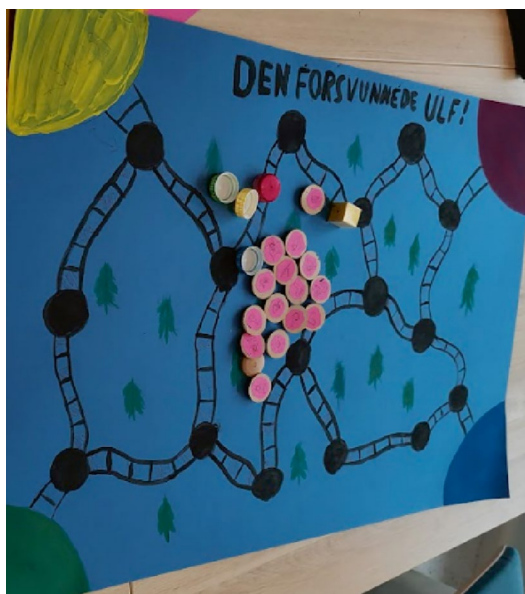
To summarise, four of Bishop's (1988a) mathematical activities were identified as objects of the game based on the collected data about the student teachers' design process: *Measuring*, *Counting*, *Locating* and *Playing*. Although student teachers' comments suggested that they considered that young children could learn about *Designing* from playing the game, we found their explanation not to be in alignment with Bishop's definition. The student teachers paid attention to these mathematical foci (objects) while designing the game. When student teachers described the possible learning opportunities for children provided by the game, they did not focus on *Locating* or *Measuring*, but elaborated opportunities explicitly connected to *Counting*, implicitly (through rules negotiation and problem-solving) related to *Playing*, and *Designing*, although this may need further development. Our analyses of the left and right triangles of ACAT helped us to identify nuances in what mathematics the student teachers noted and what they still needed support to recognise.

The last peculiarity of the game is related to its collaborative gameplay. Studying the rules of the game in the right triangle of the ACAT model, it seems to be a competitive game for repetitive activities about counting. However, analyses of the data related to the group in the left triangle of ACAT revealed that the game could encourage children to problem solve by negotiating rules, add characters and construct new ideas.

Game "Ulf That Disappeared"

The game "Ulf That Disappeared" (Fig. 3) was selected for analysis of a particular feature of the route choice that seemed to support the emergent development of the story. The game supports two to four individually competing players and has a clearly defined set of rules. According to the student teachers' explanation, the game was inspired by the game "The Lost Diamond". Each player starts in the corner

Fig. 3 Game “Ulf That Disappeared”



diagonally opposite to the colour matching their piece. The objective of the game is to find Ulf, who is lost in the forest (represented by the game board), and return from the forest to any starting point. The move allowance in each round is determined by rolling a dotted die, but otherwise, players are free to choose their route and direction. Wooden forest markers are randomly placed on the circular spaces at the start of the game. When landing on a forest marker space, the player turns the marker, revealing either a magnifying glass (check any marker on the board without sharing the information), boots (roll the die to move one more time), a hot air balloon (move to any space on the board), a treasure (worth two points), a wolf (worth one point, forcing the player back to the nearest starting point), or Ulf (unique marker, worth three points). When the Ulf-marker is found, all players must exit the forest through a starting point (worth two points if the starting point matches your piece colour). The player with the most points after everyone exits the forest wins. This game was designed, played and presented in Book Creator.

Analysis

We identified Counting, Playing and Locating as the mathematical objects of the game. Counting was identified in the inclusion of the rule to use a dotted die to define *how many* steps to move a player piece forward on the board and by comparing *how many* points. Locating and Playing were prompted through the inclusion of rules requiring players to navigate in the forest while also maximising the collection of points. Locating was identified from the implicit requirement that children should read and use the forest map, presented through the aesthetics of gameboard, in order

to decide *where* to move next. Playing was identified through the children being required to engage in problem-solving, related to *how* to choose the next destination to win the game.

To see how *Counting (object)* was included in the design, we studied the horizontal axis of ACAT. The processes of internalization and externalization between Counting and the player through the game could be stimulated by interpreting the dots on the die and moving the game piece the same quantity of spaces which indicated a need for one-to-one correspondence. These requirements suggested that the children had possibilities to internalize counting concepts and processes such as the cardinal principle and the number sequence, by keeping track of their points. Utilisation of the hot-air balloon had the possibility to facilitate the children's internalization of the number sequence and the cardinal principle, in order to answer the question of "*how many forest markers may I get*".

When *Playing* was the object, we identified the important role of the free route choice as an externalisation of the relationship in the right triangle of ACAT. The student teachers mentioned that when collecting forest markers, some routes may provide more markers than others, and that after Ulf was found, each player had the opportunity to decide to either collect more markers or exit as quickly as possible. To solve the problem of *how* to win the game, these decisions were relevant. Therefore, we interpret these as examples of the potential problem-solving opportunities that children might engage in. At the same time, focus on gaining the most markers required utilisation of Counting, while orientation on the gameboard required Locating.

In the rules, it seemed that the children could engage with *Locating* by identifying the routes made available from utilising the magnifying glass and the hot-air balloon markers. Additionally, if they chose to act strategically they could gain an extra two points to match the exit and the player piece's colour. From the student teachers' perspective, Locating was a central mathematical object of the game because the child needed to orient themselves in the forest. Superficially, this makes sense, as the game offered children possibilities for their route choice on the map, with the goal of the game being to locate Ulf. However, the analysis of the game using ACAT indicated that the route choice was more likely to promote children to engage in Playing (strategies and hypothetical reasoning) and Counting to a significantly higher degree than Locating.

After the student teachers had designed their game, they presented it in Book Creator. The aim of this part of the task was to make the student teachers aware of the potential relationship between young children, the mathematical content in the game, and how their game might support learning of the mathematical content. However, this group of student teachers used Book Creator to show the design process and pictures of the games, rather than writing reflections on what they should be aware of when the game will be played with children. Therefore, there was insufficient data to complete an analysis of the left triangle of ACAT. This was typical for those groups of student teachers who were assigned to use Book Creator rather than present a sales pitch about their games.

Game “Ulf Goes Out”

In the game “Ulf Goes Out” (Fig. 4), all the players had to collaborate to get Ulf home and the student teachers positioned it from the start as a “problem-solving” game. This made it unusual both because of the expectations that the children were to work together and in regard to the mathematical content. The rules were clearly defined, even though one of the student teachers stated that all games played in ECEC should be open to rule change to adapt to age groups. The player piece is moved along a specific path with a stop at five places to solve a task. Answers or heuristics were not provided. If Task 1 is solved, the players are allowed to move on. Solving Task 2 allows the group to take a shortcut and skip Task 3. This provides an alternative scenario in the game, but not the route choice. The players meet several characters from the picture book “Ulf is Unwell” (Wexelsen & Kuhn, 2020) while playing, and the game unfolds almost exclusively on the gameboard. This game was designed, played and then presented in Book Creator.

Analysis

We identified Counting, Locating, Playing and Explaining as the potential mathematical objects that children would engage with if they played the game. Locating was identified through the rule requiring players to navigate the player piece along a road containing aesthetics components like tasks, a bridge and a ladder. Additionally, task 5 challenges players to cross a pond using direction markers on the “hills”. This requires orientation in the game board landscape and use of the direction terms. Counting is a potential object because of the set order of the tasks (task 3 cannot be done before task 1), also Task 4 requires the use of a tally on the game board to answer “*How many flowers are there in the garden?*”. Playing is

Fig. 4 Game “Ulf Goes Out”



identified through Task 2, answering *how* a snake might climb a ladder, while Explaining is also connected to Task 2 as the collaborative gameplay type facilitates discussions concerning *why*.

To consider how *Counting* was included in the design, we focused on the right triangle of the ACAT model. The student teachers painted six bushes with various amounts of roses on the game board, and Task 4 required the players to find how many flowers there were in the garden. Hence the game could be considered to externalize Counting concepts such as one-to-one correspondence, the number sequence, and the cardinal principle through the presentation of the rose bushes on the game board. The horizontal axis of ACAT suggests how children could internalize and externalize these Counting concepts by pointing their fingers at each rose while counting. The student teachers explained that the artefact did not include answers because “this is a prototype” and someone, presumably an adult, should read and provide feedback to the children. Studying the left triangle of ACAT, the collaborative format of “Ulf goes out” could provide more situations where externalization of Counting is possible. As the children must work together, it is most likely that they would share their heuristics and answers providing their understandings to others about the mathematics they were engaging with.

The game potentially encourages the internalization and externalization of *Locating* concepts and processes by promoting positioning and orientation in the route design, starting at the “Start” space and ending at “Home”, following arrows on the road and stones on the way. The student teachers explained, “players have to move their player pieces along the route”. In Task 5, the players would need to use direction terms such as left, right, forward to cross a pond scattered with stones. When explaining the game design, a student teacher stated “in this task, they will explain to each other how to cross the water, for example by using ‘left’ and ‘right’.” This suggests that the student teachers were aware of how children might internalise and externalise the use of these terms, through hearing others use them and using them themselves. However, the student teachers’ reflections did not discuss if and how Locating could be internalized by players. There is also some uncertainty whether direction terms would actually be used to solve the problem in Task 5.

The most prominent mathematical objects of the game, are *Playing* and *Explaining*. In Task 2, players are required to figure out how a snake would climb a ladder. The location of Task 2 is accompanied by a model of a snake and the task stated, “I have no legs nor arms; will I still be able to climb the ladder?”. According to the student teachers, the players are expected to discuss the difference in how a snake moves in contrast to humans, restrictions in the way a snake moves, linking to for example an earthworm climbing a stick, enquire about snakes’ general ability to climb, explore the model of the snake, come to an agreement and conclude if and how a snake would climb a ladder. According to Bishop (1988a, p. 48), Explaining answers the question *why* while Playing answers *how to*. In this task, these two objects are intertwined involving problem-solving through *how to* while exploring the model snake and answering the *why* question through discussions. The rules do not dictate how to solve the tasks nor provide specific answers. The rules, however, make the game collaborative and thus facilitate Explaining in the group node. Hence, the group node in the left triangle of ACAT indicates where there may be

opportunities for interaction, which could contribute to engaging with Explaining, building on rules of the game which require the children to cooperate which is discussed in the right triangle of ACAT.

The game design had some limitations, as young children might struggle with the lack of clarity concerning how the tasks were to be solved which could lead to them not engaging with the mathematical objects that the student teachers had identified. Also, the game had only five tasks, so children might find it uninteresting to play more than once. However, the student teachers seemed to recognise some of the limitations when they stated, “This [the game] is a prototype”, indicating that several things could be improved.

Discussion

This study aimed to get insights into the mathematics identified in games designed by student teachers. Mathematics was expressed in terms of Bishop’s (1988a, 1988b) mathematical activities. Most of the student teachers’ games were competitive board games, with unambiguous rules and a linear storyline, along a predefined route. The results of the study provide some insights into the mathematics the student teachers integrated into their games so it became available for the players. From the findings, it was clear that some student teachers still needed teacher educators’ support to expand their understanding of how games they design could offer children mathematical experiences.

Across the different games, the student teachers identified Bishop’s six mathematical activities (1988a, 1988b), although to different degrees. In earlier research, Vee (2023) found that student teachers main focus was on Counting when evaluating apps (digital games), with some apps also considered as providing opportunities to support Locating and problem-solving, as part of Playing. Similarly, the student teachers in our study also seemed to focus on Counting and Playing by mostly including opportunities to engage with these in their games. Counting (Bishop, 1988a) was the most frequently identified mathematical activity because of the children were expected to use a dotted die to determine the number of steps along the routes set out in each of the games. The student teachers also identified Counting in the task cards, which they included in their games, which were also considered as providing problem-solving activities. The integration of Counting in the games by the student teachers was in alignment with Bayeck’s (2020) literature review which indicated that games can be effective in fostering number sense among young children. Overall, it seems that designing games can support student teachers to identify different aspects of mathematics in play situations. However, given that not all of the six activities were in focus, there is a role for teacher educators to develop the student teachers’ understandings about other mathematical activities, such as Designing and Measuring.

The mathematical activity of Playing (Bishop, 1988a) was identified by the inclusion of different kinds of problem-solving. In “Ulf That Disappeared” and “Ulf Is Going Out”, problem-solving was prominently expressed through the game design elements such as Mechanics and Aesthetics. However, in “One, Two,

Tree—Eat Little Riding Hood”, the student teachers discussed aspects of Playing in their sales pitch, revealing a different understanding of gameplay, where children are invited to modify Rules and Aesthetics, like adding characters, of the game.

The use of ACAT highlighted how the fundamental mathematical activities (1988a, 1988b) identified in the designing of the games was not always evident in the student teachers’ reflections during testing, playing, and presenting their games. For example, in the game “One, Two, Three—Eat Little Red Riding Hood,” it was possible to identify Locating and Measuring in the Aesthetics element of the game (design of the pond), but the student teachers did not discuss these two activities as related to game design during their reflections. The analysis using ACAT provided a nuanced picture of the possible interactions between the subject and the object. The right triangle offered insights into what student teachers paid attention to when designing their games, while the left triangle showed how the student teachers considered that the group might affect the mathematics promoted by the game. As was the case in other research with student teachers which analysed how they recognised mathematics (e.g., Vee & Meaney, 2022; Vee, 2023), ACAT facilitated the identification of the mathematical content. It also expanded our understanding of how the student teachers’ game design process, was manifested in the game features.

As the student teachers were in their first year, it was to be expected that mathematical awareness during the design process would still require some development, as it is dependent on their mathematical understanding and their experiences of playing games with children. According to Oppermann et al. (2016), mathematical knowledge is essential for ECEC teachers to use their pedagogical repertoire in play settings, of which playing games is one example. Ofiaz (2023) highlighted the role of game design and reflection in the development of student teachers’ skills to adopt games as a pedagogical tool in their practice. In this study, we could see that the game design required mathematical knowledge so that they could recognise how to include mathematics into the game’s design (Schell, 2019) and fulfil their aim for the games. The student teachers’ reflections highlighted how they considered how the mathematical content might appear when the game was played with children. The didactical considerations during the game design concerned the predicted development of the player. Therefore, the ECEC teacher educators may need to provide ongoing support to assist student teachers to make didactical decisions throughout the game design process.

Conclusions

This study investigated the mathematics that could be identified in the games that student teachers designed during their compulsory course that included mathematics education in their ECEC teacher education. We identified differences between what the student teachers identified as mathematics in the game design and the potential mathematics which could be available to children from playing the games. Our findings provide feedback on the use of game design as a teaching and learning

activity in ECTE. Hence, this study contributes to a knowledge gap about how ECTE can support student teachers to recognise the mathematical potential of games to be designed or played. Potential practical strategies to enhance student teachers' awareness could be reflection exercises highlighting the differences between mathematical knowledge and didactical application of mathematical knowledge.

Several limitations are identified. The current study is limited by the sample size, restricted to only one institution, one point in time and without follow-up later in their education. Due to these limitations, we suggest future research in other institutions, with follow-up studies in the third year of study. We see a need for further research on how to support the student teachers' awareness of the potential of games for introducing young children to mathematics. Mathematical game design could be explored further to expand student teachers' experiences and further development. It would also be interesting to study experiences with actual playing of games together with children.

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References

- Bayeck, R. Y. (2020). Examining board gameplay and learning: A multidisciplinary review of recent research. *Simulation & Gaming, 51*(4), 411–431. <https://doi.org/10.1177/1046878119901286>
- Bennett, J. (2005). Curriculum issues in national policy-making. *European Early Childhood Education Research Journal, 13*(2), 5–23. <https://doi.org/10.1080/13502930585209641>
- Bishop, A. J. (1988a). *Mathematical enculturation: A cultural perspective on mathematics education*. Kluwer.
- Bishop, A. J. (1988b). Mathematics education in its cultural context. *Educational Studies in Mathematics, 19*(2), 179–191. <http://www.jstor.org/stable/3482573>
- Engeström, Y., Miettinen, R., & Punamaki, R.-L. (1999). *Perspectives on activity theory*. Cambridge University Press. <https://doi.org/10.1017/CBO9780511812774>
- Fosse, T., Lange, T., & Meaney, T. (2020). Kindergarten teachers' stories about young children's problem posing and problem solving. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years: Results from the POEM4 conference, 2018* (pp. 351–368). Springer. https://doi.org/10.1007/978-3-030-34776-5_21
- Kangas, M., Koskinen, A., & Krokfors, L. (2016). A qualitative literature review of educational games in the classroom: The teacher's pedagogical activities. *Teachers and Teaching, 23*(4), 451–470. <https://doi.org/10.1080/13540602.2016.1206523>
- Ladel, S. & Kortenkamp, U. (2011). An activity-theoretic approach to multi-touch tools in early Maths learning. *Paper presented at the activity-theoretic approaches to technology-enhanced mathematics learning orchestration symposium (ATATEMLO)*, Paris, France
- Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT framework to evaluate the design of two geometry apps: An exploratory study. *Digital Experiences in Mathematics Education, 5*(1), 59–92. <https://doi.org/10.1007/s40751-018-0045-4>

- Leontiev, A. (1972/1981). The problem of activity in psychology. In J. Wertsch (Ed.), *The concept of activity in soviet psychology* (pp. 37–71). M.E. Sharpe. <https://doi.org/10.4324/9781003575429-2>.
- Ministry of Education. (2017). *Framework plan for kindergartens: Regulation on the content and tasks of the kindergarten*. Norwegian directorate for Education and Training. <https://www.udir.no/contentassets/7c4387bb50314f33b828789ed767329e/framework-plan-forkindergartens%2D%2Drammeplan-engelsk-pdf>
- Moon, J., Yeo, S., Si, Q., & Ijeluola, A. S. (2024). A scoping review of game-based learning for mathematics teacher education. *International Journal of Mathematical Education in Science and Technology*, 1–29. <https://doi.org/10.1080/0020739X.2024.2337934>
- Oflaz, G. (2023). Evaluation of educational games prepared by mathematics teacher candidates according to game design key model. *Educational Policy Analysis and Strategic Research*, 18(1), 145–174. <https://doi.org/10.29329/epasr.2023.525.7>
- Oppermann, E., Anders, Y., & Hachfeld, A. (2016). The influence of preschool teachers' content knowledge and mathematical ability beliefs on their sensitivity to mathematics in children's play. *Teaching and Teacher Education*, 58, 174–184. <https://doi.org/10.1016/j.tate.2016.05.004>
- Pilten, P., Pilten, G., Divrik, R., & Divrik, F. (2017). Evaluation of mathematical game design skills of pre-service classroom teachers. *International Electronic Journal of Elementary Education*, 10(2), 255–264. <https://doi.org/10.26822/iejee.2017236120>
- Russo, J., Bragg, L., & Russo, T. (2021). How primary teachers use games to support their teaching of mathematics. *International Electronic Journal of Elementary Education*, 13(4), 407–419. <https://doi.org/10.26822/iejee.2021.200>
- Schell, J. (2019). *The art of game design—A book of lenses* (3rd ed.). CRC.
- Smith, E., & Golding, L. (2018). Use of board games in higher education literature review. *MSOR Connections*, 16(2), 24–29. <https://doi.org/10.21100/msor.v16i2.624>
- Vee, M. (2023). How do beginning preservice kindergarten teachers' evaluations of mathematical apps provide insights into their understandings about bildung (danning) for young children? *Nordisk barnehageforskning*, 20(2), 51–69. <https://doi.org/10.23865/nbf.v20.268>
- Vee, M. K., & Meaney, T. (2022). Kindergarten preservice teachers evaluating mathematical apps. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva Kivisto, J. Radišić, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future: Proceedings of Norma 20, The ninth nordic conference on mathematics education, Oslo 2021* (pp. 249–256). Swedish Society for Research in Mathematics Education.
- Wexelsen, R., & Kuhn, C. (2020). *Ulf er uvel [Ulf is unwell]*. Ena Vigmostad & Bjørke.

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On the Way to Pattern-Making: Young Children's Self-Initiated Non-Pattern Creations



Miriam M. Lüken

Introduction

Research on young children's patterning competency has shown its importance for their overall mathematical development (Rittle-Johnson et al., 2019; Wijns et al., 2019). Beyond that, creating patterns seems to be an activity young children pursue out of their own interest. Reports on pattern-making in free-play environments and on children-made patterns surface repeatedly in early childhood research literature. For example, Waters (2004) observed a young girl who created an AB pattern with shapes from a hammer and nails construction kit. Seo and Ginsburg (2004) reported a young girl creating a necklace with an AB pattern from plastic beads, and a young boy building a pattern with three-dimensional reflection symmetry from rectangular and triangular prisms. In a nursery classroom, Threlfall (1999) observed a string threaded with alternate discs and tubes, and an egg box containing small pieces of material—strips and rounds—placed alternately in the depressions. From these unsystematic reports, we can gather that young children use everyday objects (e.g., beads) as well as geometrical play materials (e.g., building blocks) to create patterns, and that they create patterns with different structures (e.g., AB patterns with a translational symmetry, three-dimensional patterns with a reflectional symmetry).

Ginsburg et al. (1999, 2003) found that children spend 21–36% of their free-play time engaging in pattern and shape activities. In their studies, pattern and shape activities refer to detecting, predicting, or creating patterns or shapes, whereby patterns include simple translation patterns as well as line symmetry and elaborate reflection symmetries in three dimensions. From the arrangements made while creating with geometric construction material, Schuler (2001) found 66% to be patterns (i.e., symmetrical designs). Lüken (2023), with a larger and younger sample, found only 38% of all creations made in a free-play environment to be patterns; ten

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percent of the creations resulted from sorting behavior (i.e., children sorting objects along a self-imposed rule). My interest is in the other 52% of creations. I assume them not to be mere “faulty” patterns but creations with their own value for children’s patterning development. The aim of this chapter, therefore, is to take a closer look at these non-pattern creations, which result from self-initiated play with geometrical play materials (like shapes, cubes or pattern blocks) and everyday objects (like buttons or spoons), and to link experiences with arranging material in a geometric way to pattern-making.

Theoretical Background

To focus on non-pattern creations, first, an understanding of a mathematical pattern has to be established. With the properties of a pattern in mind, non-pattern creations are defined in a second step and clearly distinguished from patterns. For analyzing the non-pattern creations, we need to look into research that has studied children playing and creating with objects. The extensive work on construction play offers valuable insights into how children arrange objects, and it describes different kinds of children-made constructions which can inform the later analysis of the data.

Mathematical Patterns

Children create patterns – but what is a “pattern”? In this study, a geometrical arrangement is understood as a pattern if it can be reduced to a smallest subset (the unit of repeat) that repeats or transforms in such a regular way, that it consequently makes the pattern predictable (Sawyer, 1964; Threlfall, 1999). A regular, geometric repetition of the unit of repeat can be carried out by any congruence mapping. Shifting the unit along a line results in a translation pattern (e.g., ABAB) which seems to be the predominant pattern type in early childhood education. However, the “concept of ‘pattern’ goes far beyond sequential repeated patterns” (Clements & Sarama, 2009, p. 190). A regular, geometric repetition can also be carried out by mirroring (reflection pattern) or rotating (rotation pattern) the unit of repeat. Consequently, and in line with Clements and Sarama (2009), I consider all symmetrical arrangements as *repeating patterns*. Furthermore, similarity transformations (uniform scaling), which enlarge or shrink an object (respectively the repeating figure) during its repetition in a regular way, result in *growing* (or *shrinking*) *patterns* and can be considered as yet another type of pattern. Often, growing patterns are numerical patterns in a geometric representation, with a specific number of objects added from stage to stage, while the overall shape stays the same. To summarize: all kinds of children-made symmetric figures and objects as well as growing patterns are considered as patterns in this study. Consequently, *non-pattern* creations might be as aesthetic, complex, or in some way regular, for example elements

ordered in a line. In contrast to repeating or growing patterns, however, non-pattern creations lack the unit of repeat (respectively the repeating figure) and its geometrically or numerically regular repetition or transformation.

Research on Construction Play

Creating arrangements with material can be considered as construction play. Research on construction play, in particular research on blockplay, goes back a long way (Guanella, 1934; Hanfmann, 1930; Hetzer, 1931; Johnson, 1933/2015) and touches on a variety of related topics (see for an overview Gura & Bruce, 1992; Henschen, 2020). For the theoretical background of this paper, I focus on open exploratory blockplay, different kinds of children-made constructions, and the development of construction play.

“Constructive play occurs when children use play materials to build something” (Christie & Johnson, 1987, p. 439). Play materials for constructing usually involve materials like building blocks, Lego, Lokon, and boxes, but also clay, sand, paper, and puzzles (Christie & Johnson, 1987; de Moor, 2008). Blockplay as a subcategory of constructive play has been particularly well researched and most findings, which are presented in the following, stem from open exploratory block building activities. Building blocks are wooden solids, mostly differently sized cuboids and triangular prisms.

Looking at different kinds of children-made constructions, researchers from different eras and countries consistently describe similar constructions that can be roughly grouped into abstract block-structures, patterns (i.e., symmetrical structures), and representations of the real world (Gura & Bruce, 1992; Hanfmann, 1930; Henschen, 2020; Hetzer, 1931; Johnson, 1933/2015; Sarama & Clements, 2009). The origins of block-constructing can be seen in holding blocks next to each other, creating heaps of material, and simple stacking and lining activities (Gura, 1992; Hanfmann, 1930; Hetzer, 1931; Schuler, 2001; Shotwell, 1979). For more elaborate constructions, Gura (1992, p. 54) suggests “basic block forms”, which she groups in a first step with regard to spatial dimensions. For one-dimensional (linear) and two-dimensional structures, she further differentiates between vertical and horizontal constructions, and for two- and three-dimensional structures, she makes a distinction between enclosures and areal/solid structures (completely covering a surface or volume).

Regarding the developmental aspect of blockplay (Guanella, 1934; Gura, 1992; Hanfmann, 1930; Johnson, 1933/2015; Sarama & Clements, 2009), children between the ages of 1 and 7 years seem to progress from the placement of unconnected blocks and the connection of two blocks at a time to linear arrangements of numerous blocks vertically and horizontally. They proceed to simple combinations of vertical and horizontal lines into first crosses and enclosures, and then into covering a plane (also horizontally and vertically). Early spatial configurations are later combined as more complex block constructions, in particular three-dimensional

arrangements. Furthermore, developmental trends can be seen in an increase in the constructions' complexity and in a decrease in the number of creations with growing age (Hanfmann, 1930; Gura, 1992).

From the review on blockplay research, two relevant issues seem to be missing. First, children creating *patterns* are mentioned in the literature cited above but are not thoroughly studied. Hanfmann's (1930, p. 259) categories of children's constructions for example, comprise a *symmetrical structure* category, which seems to consist only of reflection patterns. However, examples of other categories (e.g., *single buildings*) also show creations with a reflectional symmetry, whereas creations with a translational symmetry are integrated in several categories. Gura's (1992) very thorough system of basic block forms does not touch upon the aspect of symmetry at all. Bruce et al. (1992), when looking at patterns, focus on patterns with translational symmetry only. Second, children create and construct not only with building blocks but also with other didactical materials like shapes or even with everyday objects like stones, sticks, beads, or paper strips (Hetzer, 1931; see spontaneous patterning examples from the introduction). Schuler (2001) in her try-out of building materials for her own Montessori classroom, and Lüken (2023) in her recent study on self-initiated patterning, specifically looked into children creating with diverse materials and compared pattern creations (i.e., symmetrical constructions in Schuler's study) to non-pattern creations. Schuler (2001) found that the designs of older children (5- and 6-year-olds) were more complex and, in addition, that the percentage of symmetrical work (i.e., patterns) increased with age. In the context of self-initiated creating in a free play environment, Lüken (2023) found that the total number of patterns varied largely between individual children and that not all children created patterns on their own accord. Similar to Schuler, the extent of pattern-making significantly increased with age. Looking specifically at the materials that children chose for creating, the materials' external properties and number of different objects that make up a set of materials influenced pattern creation (Lüken & Jaeger, 2024). Both studies, however, did not analyze children's *non-pattern creations*, which make up between 34% and 52% of all creations. It is unclear to date, how far the basic forms of blockplay and the development in building-structures are applicable to creations with materials other than building blocks, and how the non-pattern creations relate to or possibly even precede pattern-making.

The Current Study

The current study investigates creations young children made from a range of geometrical play materials and everyday objects in a free-play environment. Photos of the creations were taken and used for categorization. In a first approach, the types of children-made creations were categorized, in particular differentiating between pattern and non-pattern arrangements (Lüken, 2023). For this paper, the aim was to

further focus on the *non-pattern creations*, to describe the different types of non-pattern creations, and to link self-initiated creating experiences to pattern-making.

The specific research question is:

- What types of non-pattern arrangements do young children create of their own accord?

With the results for this question in mind, I then offer some considerations in the discussion section regarding the overarching question, in how far non-pattern creations reflect meaningful experiences on the way to pattern-making.

Method

Participants

The sample consisted of 84 children from 1 year and 5 months to 6 years and 9 months of age (37 girls) from five early childhood centers in Germany. Clustered per year, the distribution of age in the sample was as follows: 1-year-olds (3), 2-year-olds (17), 3-year-olds (19), 4-year-olds (26), 5-year-olds (11), and 6-year-olds (8).

Design, Materials, and Procedure

The data collection took place during 40 hours of unstructured, child-directed play-time per classroom (referred to as free play). Every morning of data collection between breakfast and lunch, the children were free to choose their activity and their play partners for roughly two hours. All children were available for observation during this time-slot as they were supposed to stay in the area of their own classroom and they had no other assignments. A team consisting of the author and one of four bachelor students collected the data in each classroom. The researcher communicated with the children, but neither did she initiate any activity nor did she attempt to influence a child's choice of material or creation.

In order to make the five free-play environments more comparable and inviting for constructing activities, we offered a wide range of different materials and added geometrical play materials, and everyday objects to the early childhood environment (pattern blocks, paper clips, Cuisenaire rods, cubes, shapes, Hama beads, Lego, pegboards, building blocks, plastic spoons, beads and strings, and buttons). All of the materials came in multiple colors. Naturally, children also made creations from the classrooms' other materials, such as plates, chairs, craft materials, chestnuts, etc. All materials were readily accessible to all children. An early childhood educator supervised very young children handling small objects.

Children's creations were documented by digital photography. For this, the researcher moved through the classroom during free play, pausing to observe groups of playing children. When a child finished creating, the researcher took a photo of their creation.

Data Analysis

For data analysis, all photographed creations were interpreted by means of content analysis (Mayring, 2008). In this study's case, it meant assigning pictures to categories. In an iterative procedure, categories were developed deductively and inductively, and the creations coded accordingly. Consistent with my definition of the term pattern, I started with the distinction between non-pattern and pattern creations, inductively adding classification (i.e., creations resulting from sorting behavior) as the third main category during the first round of coding. After several rounds of coding the entire data set for main categories, the non-pattern creations were divided into subcategories. I started this process inductively, developing the subcategories distributing, penciling, and aligning, again with several subcategories each. With reference to the theories of Gura (1992) and Hanfmann (1930), I deductively regrouped the subcategories of aligning several times (see Table 1 for final non-pattern subcategories with prototypical creation pictures). After all categories had been developed and creations coded accordingly, a second coder double coded 25% of the creations. The inter-rater reliability calculated as the percentage of answers which received the same category from both coders was high (96.3% agreement). Linear regression analysis was used to model the relationship of the main categories with age. For the description of the data in the graph (Fig. 1), age was categorized in six groups, whereas the age was kept continuous in the regression analysis to allow for a better modeling of the relationship between the outcomes and age. Finally, the number of creations in each category was quantified and compared.

Results

A total of $n = 972$ creations were made by the 84 children during data collection. The relative frequencies of the main categories were non-pattern 52%, pattern 38%, and classification 10%. Working with individual proportions of these categories per child and resulting from the linear regression analysis, age is seen as a major determinant for non-pattern creations. The proportion of non-pattern creations decreases every 10 months by 6% (95% confidence interval $[-1.0, -0.19]$; $p = 0.005$), see Fig. 1 for mean individual proportions per main category and age groups).

The frequency distributions of the non-pattern subcategories are distributing 17%, penciling 8% and aligning 75% (see Table 1 for further details). The category distributing contains arrangements with no regular relations between single objects

Table 1 Non-pattern subcategories (relative frequencies based on n = 506 non-pattern creations)


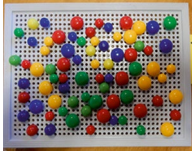
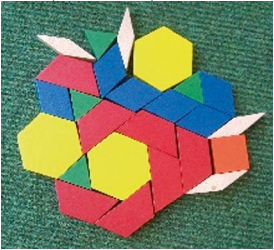

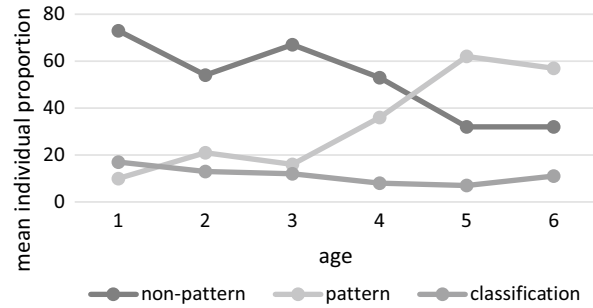
<p>Distributing 17%</p>	<p>Heaping 1.19%</p> 	<p>Spreading 8.1%</p> 	<p>Selecting 7.31%</p> 
<p>Penciling 8%</p>	<p>Drawing 4.94%</p> 	<p>Writing 2.96%</p> 	
<p>Aligning 75%</p>	<p>Laying 51.19%</p> <p>Row & chain 29.64%</p>  <p>Enclosure (horizontal) 6.13%</p>  <p>Area (horizontal) 15.42%</p> 	<p>Stacking 24.31%</p> <p>Tower 12.06%</p>  <p>Enclosure (vertical & 3-dim.) 1.19%</p>  <p>Area (vertical) 3.16%</p>  <p>Building (3-dim.) 7.9%</p> 	

Fig. 1 Mean individual proportions per main category and age groups (Lüken, 2023)



and it was further subdivided into the categories heaping (material was piled; 1.19%), spreading (material was distributed flat on the surface; 8.1%) and selecting (a small selection of material (number of objects <10) was picked out; 7.31%). In the category penciling, the material was used by the children to create the pen-lines in drawing (e.g., animals, vehicles, or flowers on grass with sky, sun, and clouds; 4.94%) and writing (letters and words; 2.96%). The creations from the category aligning were regularly ordered arrangements (without a repeating unit or regularly transforming figure). This largest category was further subdivided into creations that resulted from laying activities (50.19%), working exclusively in the horizontal plane, and on the other hand resulted from stacking activities (24.31%), working (additionally) along the vertical. The creations from both subcategories were further categorized according to arrangements in a line (row & chain, tower), covering an area (horizontal, vertical), enclosing an area (horizontal, vertical & three-dimensional), and three-dimensional buildings (only for stacking).

Discussion

The current research analyzed the self-initiated non-pattern creations of young children. In the following, I first answer the research question and then link the creating experiences that resulted in non-pattern creations to pattern-making.

What Types of Non-Pattern Arrangements Do Young Children Create of Their Own Accord?

Besides patterns and classifications, more than half of all creations were arrangements with no discernable unit of repeat or regularly transforming figure. These non-pattern creations were categorized into the subcategories distributing, penciling, and aligning, which reflect three fundamentally different ways of interacting with the material. Each of the three categories is discussed in turn.

Distributing The arrangements of the category distributing have no regular relations between their objects. The children seemed to simply distribute material in an arbitrary way on their working space. They either took handfuls of material and piled it on one or more heaps (heaping), spread the material out flat (spreading), or picked out a small selection of material, often just one or two objects, and laid them down as their creation (selecting). Hanfmann (1930) and Gura (1992) also observed young children creating heaps of materials in their studies on blockplay. Gura, in contrast to Hanfmann, points out that the pile may be indeterminate in form, but “the placement is not random” (Gura, 1992, p. 53). I agree with this statement for the spreading and selecting categories, as the children purposefully placed the materials apart from each other or decisively chose particular objects for their creation. Still, because of their spatially unstructured arrangement, the creations have a random vibe.

Penciling In the category penciling, the creations look as if they were drawn or written with a pencil or crayon. This category is different from the overall distinction between representational creations of the real world and abstract or patterned creations in blockplay research (Henschen, 2020). As I did not ask the children in my study if their creations were meant to represent something, it is not possible to make the distinction between representational and abstract. I, therefore, categorized representational creations (e.g., creations that looked like houses, flowers, garages, stars, or other real-world objects) according to their structure and not according to their external appearance. However, for nearly 8% of all non-pattern creations, the children's interaction with the material as a writing and drawing device was so pointedly different from the other creations that the category penciling was created.

Aligning In contrast to the random piling or spreading of material, three quarters of all non-pattern creations were somehow ordered arrangements (category aligning). Children made the objects touch on one or more sides, aligning them more or less neatly. This created a sense of order with (parallel) lines and (regular) angles. The category includes both creations from simple lining and stacking activities, and more elaborate constructions, which can also be found in the category systems of Gura (1992), Hanfmann (1930), and Schuler (2001). In contrast to Gura and Hanfmann, and in line with Schuler, symmetrical creations or creations with a regularly transforming unit of repeat were coded for the separate category pattern. I subdivided the category aligning into creations that on the one hand resulted from laying activities, working exclusively in the horizontal plane, and on the other hand resulted from stacking activities, working (additionally) along the vertical. As most patterning activities are carried out in the plane, this subdivision makes sense from a pattern point of view. With half of the creations in the aligning-laying and a near quarter in the aligning-stacking categories and with regard to Gura (1992), the creations of both subcategories were further categorized according to dimensionality. This differentiation helped in connecting experiences from aligning objects to the making of different pattern types (see below).

In How Far Do Non-Pattern Creations Reflect Meaningful Experiences on the Way to Pattern-Making?

The answer to this overarching question is not inferred from the empirical data. Instead, I want to offer some theoretical considerations on pattern-making and link them to the empirical data on non-pattern creations. First, however, I would like to point out that construction play provides a range of meaningful experiences for children outside of experiencing patterns. In creating with material, children can, for example, experience the concept of balance, characteristics of geometric solids (Henschen, 2020), or simply the sheer joy of creating. Studies even show that early construction competencies have an influence on later mathematical learning (Verdine et al., 2014; Wolfgang et al., 2001). Coming back to my focus of creating patterns and understanding the structure of patterns, non-pattern creations might additionally involve meaningful experiences for or even be precursors to pattern-making. It would go beyond the scope of this chapter to link all non-pattern subcategories to pattern-making. Thus, I limit the discussion to creations from the broadest subcategory aligning-laying.

Over half of all non-pattern creations resulted from laying activities. Children put the material's objects on the plane of their working space and aligned them, i.e., created regular spatial relationships between the objects. As diverse objects instead of repeating units or regularly transforming figures were aligned, the creations are not patterns. Still, they show strong references to operations from the symmetry of the plane, such as creating a row or chain from material, like buttons in a line next to each other, which have some semblance of simple translation patterns, just without the unit of repeat. It is the operation of shifting along a line, which is repeated over and over again. The data set includes multiple examples of rows of material spanning the length of the early childhood classroom, somehow touching on the idea of infinity of translational repeating patterns. For a mathematical translation, however, the very same geometrical figure is shifted. The aligned elements in the non-pattern creations are different objects, however, most often from the same category of material (e.g., buttons, albeit in varying orders of color or size). The same holds true for shifting along two dimensions. Instead of creating regular tessellations, children covered a surface with a mix of geometric shapes or with the same geometric shape in a random order of color. It seemed as if the children needed to experience the covering of a surface without overlaps or gaps before they focused additionally on regularity regarding the objects (creating a unit of repeat or a regularly structured figure). Analogous to shifting actions, a number of creations from the aligning-laying category indicate flipping actions, e.g., turning over an object before laying it down or reversing a sequence. Other creations suggest circular rotation movements around a point of rotation. Again, the focus seemed to be on the movement (flip/turn, rotate) and not on the regularity of the objects' characteristics. The latter is focused in classifying activities in which objects are purposefully selected according to specific characteristics. The ability to classify, in particular discerning the features of objects (Björklund, 2016, p. 277), might be seen as a

necessary competency to pattern-making for all pattern types because the same set of objects has to be selected multiple times in order to create the repeating units or the transforming figure. Then again, the aligning-laying activities might provide meaningful experiences for later positioning the unit of repeat/figure in a spatially regular way, e.g., aligned in a row, flipped over next to each other, or moved around in a circle.

Conclusion

Self-initiated creating with material provides young children with experiences in a variety of mathematical topics, including both patterns and basic distributing and aligning competencies on the way to pattern-making.

References

- Björklund, C. (2016). Playing with patterns: Conclusions from a learning study with toddlers. In T. Meaney, O. Helenius, M. Johansson, T. Lange, & A. Wernberg (Eds.), *Mathematics education in the early years: Results from the POEM2 Conference* (pp. 269–287). Springer. https://doi.org/10.1007/978-3-319-23935-4_15
- Bruce, T., Gura, P., & Froebel Blockplay Research Group. (1992). Being mathematical. In P. Gura & T. Bruce (Eds.), *Exploring learning: Young children and blockplay* (pp. 75–91). P. Chapman.
- Christie, J. F., & Johnson, E. P. (1987). Reconceptualizing constructive play: a review of the empirical literature. *Merrill-Palmer Quarterly*, 33(4), 439–452.
- Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Routledge. <https://doi.org/10.4324/9781003083528>
- De Moor, E. (2008). Domain description geometry. In M. van den Heuvel-Panhuizen & K. Buys (Eds.), *Young children learn measurement and geometry: A learning-teaching trajectory with intermediate attainment targets for the lower grades in primary school* (pp. 115–144). Freudenthal Institute, Utrecht University. <https://doi.org/10.1163/9789087903985>
- Ginsburg, H., Inoue, N., & Seo, K.-H. (1999). Young children doing mathematics: Observations of everyday activities. In J. Copley (Ed.), *Mathematics in the early years* (pp. 88–99). National Council of Teachers of Mathematics.
- Ginsburg, H. P., Lin, C., Ness, D., & Seo, K.-H. (2003). Young American and Chinese children's everyday mathematical activity. *Mathematical Thinking and Learning*, 5(4), 235–258. https://doi.org/10.1207/S15327833MTL0504_01
- Guanella, F. M. (1934). Block building activities of young children. *Archives of Psychology*, 174, 1–92.
- Gura, P. (1992). Developmental aspects of blockplay. In P. Gura & T. Bruce (Eds.), *Exploring learning: Young children and blockplay* (pp. 48–74). P. Chapman.
- Gura, P., & Bruce, T. (Eds.). (1992). *Exploring learning: Young children and blockplay*. P. Chapman.
- Hanfmann, E. (1930). Über das Bauen der Kinder. *Zeitschrift für Kinderforschung*, 36(3), 255–334.
- Henschen, E. (2020). *In Bauspielen Mathematik entdecken: Aktivitäten von Kindern mathematikdidaktisch analysieren und verstehen*. Springer.
- Hetzer, H. (1931). *Kind und Schaffen: Experimente über konstruktive Betätigungen im Kleinkindalter*. Fischer.
- Johnson, H. M. (2015). *The art of blockbuilding*. Facsimile. (Original work published 1933).

- Lüken, M. M. (2023). Young children's self-initiated pattern-making during free play. *Mathematical Thinking and Learning*, 1–21. <https://doi.org/10.1080/10986065.2023.2276798>
- Lüken, M. M., & Jaeger, L. S. (2024). Unprompted pattern-making with everyday objects and didactical materials in the early childhood classroom. *Conference paper presented at the 15th international congress on mathematical education (ICME)*, Sydney, Australia.
- Mayring, P. (2008). *Qualitative Inhaltsanalyse. Grundlagen und Techniken*. Beltz.
- Rittle-Johnson, B., Zippert, E. L., & Boice, K. L. (2019). The roles of patterning and spatial skills in early mathematics development. *Early Childhood Research Quarterly*, 46, 166–178. <https://doi.org/10.1016/j.ecresq.2018.03.006>
- Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge. <https://doi.org/10.4324/9780203883785>
- Sawyer, W. W. (1964). *Vision in Elementary Mathematics*. Penguin Books.
- Schuler, J. (2001). Symmetry and young children: Observation and analysis of construction in a Montessori early childhood classroom. *Montessori LIFE*, 13(2), 42–48.
- Seo, K., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education: Lessons from new research. In D. H. Clements & J. Sarama (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 91–104). Lawrence Erlbaum.
- Shotwell, J. (1979). Counting steps. *New Directions for Child Development*, 1979(3), 85–96.
- Threlfall, J. (1999). Repeating patterns in the early primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–30). Cassell.
- Verdine, B. N., Golinkow, R. R., Hirsh-Pasek, K., & Newcombe, N. S. (2014). Finding the missing piece: Blocks, puzzles, and shapes fuel school readiness. *Trends in Neuroscience and Education*, 7(1), 7–13. <https://doi.org/10.1016/j.tine.2014.02.005>
- Waters, J. (2004). Mathematical patterning in early childhood settings. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the 3rd millennium: Towards 2010—Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia* (pp. 565–572). MERGA.
- Wijns, N., Torbeyns, J., Bakker, M., De Smedt, B., & Verschaffel, L. (2019). Four-year olds' understanding of repeating and growing patterns and its association with early numerical ability. *Early Childhood Research Quarterly*, 49, 152–163. <https://doi.org/10.1016/j.ecresq.2019.06.004>
- Wolfgang, C. H., Stannard, L. L., & Jones, I. (2001). Block play performance among preschoolers as a predictor of later school achievement in mathematics. *Journal of Research in Childhood Education*, 15(2), 173–180. <https://doi.org/10.1080/02568540109594958>

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Multimodal Mappings of a Child-Created Model Pattern Within an Embodied Activity



Morten Bjørnebye

Introduction

Educational research highlights early pattern knowledge as a foundation for both current mathematics proficiency and future mathematical achievement (Fyfe et al., 2019; Rittle-Johnson et al., 2019; Zippert et al., 2019). The term pattern is defined as any replicable regularity (Papic et al., 2011). Repeating patterns, growing patterns and spatial-structural patterns are common types of patterns encountered by young children (Papic et al., 2011; Rittle-Johnson et al., 2013). Repeating patterns consist of smaller repeating units and have a cyclical structure (e.g., AABAABAAB...), where the repeating part (e.g., AAB), known as the unit of repeat, must occur at least twice. Growing patterns involve a systematic increase or decrease of elements in a sequence (e.g., AB, ABB, AB BB, AB BB, ...). Spatial-structural patterns refer to the invariant relationships between different properties (e.g., collinearity, number, size, configuration/spacing) of geometric shapes (Mulligan et al., 2020; Papic et al., 2011), and can take the form of repeating patterns (e.g., ○○□○○□...), or growing patterns (e.g., ○□○□□○□□□...).

Children's early patterning experiences involve a range of activities, such as creating, copying, translating and extending regularly arranged objects in their environment. These experiences arise in free play (Lüken, 2023), construction activities (Wijns et al., 2020) and guided instruction (Zippert et al., 2021). Copying or extending patterns using the same materials is considered easier than tasks that involve recognising the cyclical feature of repeating patterns, particularly the unit of repeat (Rittle-Johnson et al., 2013). To assess this skill, children are often asked to replicate a pattern's structure with different materials, a process referred to as

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generalising (Wijns et al., 2019), abstracting (Rittle-Johnson et al., 2013), translating (Lüken & Sauzet, 2021) or transferring (Junker et al., 2024).

From a multimodal perspective, O'Halloran (2015) describes mathematics as the mapping of relations and patterns, suggesting that patterns are experienced through language, gestures and movement, and concrete, visual and auditory modalities. Building on this, the process of replicating a pattern's structure involves the concept of multimodal mapping, understood as the mapping of the repeating unit across and within various modalities, with consistency achieved when all modalities align with the unit of repeat. This aligns with the embodied perspective on task design, which emphasises creating learning environments that allow learners to use their hands or body to map spatial entities and relationships in a way that maintains consistency between the movement modality and the mathematical learning objective (DeSutter & Stieff, 2017; Skulmowski & Rey, 2018). Examples of consistent multimodal mappings of an AAB pattern, where the repeating unit is represented by a model tower of cubes in "blue, blue, red," include sequences of upper-body movements (e.g., "hard clap, hard clap, soft clap") and full-body movements (e.g., "parallel jump, parallel jump, left leg step"). These mappings may involve interaction with spatial entities represented by visual elements, configured either linearly (e.g., jumping on "square, square, triangle" arranged in a row), or non-linearly as spatially dispersed elements on the ground. This dynamic, multimodal embodied approach to repeating pattern experiences contrasts with commonly used assessment tools and learning activities that emphasise mapping of linear configurations of elements within similar modalities. Additionally, with some exceptions (e.g., Junker et al., 2024; Lüken & Sauzet, 2021), educational research on patterning tends to focus on the end product, while the process of children's exploration of patterns, particularly within a multimodal context involving non-linear configurations of the unit of repeat, remains less understood.

Therefore, this paper poses the research question: "What dimensions of variation can a child discern in an activity designed to support multimodal mappings of a self-made model pattern?" This question guides the analysis of a case involving a five-year-old participant in a task-based interview conducted after a design study of pattern learning. It explores which aspects emerge as dimensions of variation during the child's creation of an AABBC model pattern, followed by the multimodal mapping of its repeating unit across various modalities and spatial configurations of visual elements that incorporate upper- and full-body movements.

Review of the Literature

I begin by reviewing studies of early pattern learning, focusing on the spatial configurations of visual elements in tasks designed to assess and foster repeating pattern knowledge. Then I explore intervention research on multimodal approaches to pattern learning.

The taxonomy of Rittle-Johnson et al. (2013) for the acquisition of repeating pattern knowledge among kindergarteners consists of four progressively complex cognitive levels: exact copying, continuation, abstraction with varied materials, and unit recognition. Using linearly configured coloured geometric shapes and Unifix cubes, they assessed four-year-olds' abilities. Most children could duplicate and extend patterns, and some recreated patterns with new materials, but few identified the unit of repeat. Rittle-Johnson et al. (2013) suggest that while children may copy and extend patterns through visual matching, abstraction requires understanding the pattern's structure rather than its surface features.

Zippert et al. (2018) developed the Teacher-Based Patterning Assessment (TBP) to assess children's proficiency with repeating patterns, using linear configurations of pictures (e.g., apple, sun, umbrella) for tasks involving pattern extension, abstraction, and identification of missing elements. Subsequent research by Zippert et al. (2020) applied the TBP to evaluate preschoolers' understanding of patterns, finding that these skills predicted future mathematics achievement. Further, Zippert et al. (2021) used the TBP, among other measures, to investigate the impact of focusing on patterns in objects and numbers on preschoolers' mathematical skills. This study, comparing five conditions, showed that tutoring in repeating patterns and numeracy most effectively improved children's understanding of these patterns. Both studies suggest that the TBP is a useful tool for evaluating the impact of early math interventions on patterning skills.

Lüken and Sauzet (2021) explored the patterning strategies of children aged three to five. Like the TBP, they used linear configurations to assess pattern translation, but instead of the TBP's use of visuals, they utilised coloured cubes, with which children replicated a given pattern using different colours. Their findings suggest that although patterning strategies and skills naturally develop in three to five-year-olds, five-year-olds often display a limited focus on the pattern's structure. As a result, Lüken and Sauzet (2021) recommend that early mathematics interventions should place greater emphasis on children's understanding of the structure of patterns. In a subsequent study, Lüken (2023) explored the structure of patterns created by one-to-six-year-olds in their self-initiated play. Focusing on geometric patterns with concrete materials, the study revealed that children performed various congruence mappings, such as translation, reflection and rotation. Out of the 972 creations examined, 38% were patterns: 12% involved translation (linear structure), 19% reflection, 5% rotation, and 2% were growing patterns. Thus, over two-thirds of the patterns were non-linear (26% of the total creations), with reflection symmetry occurring most frequently.

Junker et al. (2024) conducted a multimodal analysis to explore similarities and differences in the success rates and incorrect strategy usage among six-year-old students with low and high number sense when solving tasks involving pattern duplication, extension, transfer and unit identification. The analysis focused on verbal, spatial and gestural expressions in tasks involving linear configurations of coloured geometric tiles or horizontal stacks of coloured cubes. In the ABB transfer activity, the A and B components of the pattern differed in both colour and shape, and students were challenged to replicate the same type of linear configured pattern

using tiles of different colours and shapes. The findings demonstrated that students with a high number sense consistently outperformed their peers with a low number sense, achieving success rates of 99% vs. 97% in duplication, 36% vs. 9% in extension, 78% vs. 73% in transfer, and 27% vs. 12% in unit identification. The multi-modal analysis revealed that success in duplicating and transferring patterns was typically achieved by matching one tile at a time from left to right. This systematic, linear approach could be observed alongside verbal expressions such as “This one is that one” or phrases like “big, small, big, small,” where the words “big” and “small” distinguished the A and B elements of the pattern. These verbalisations were sometimes synchronised with gestures, such as finger-pointing at the respective elements, and rhythmic variations in vocal pitch to emphasise the pattern’s structure. Junker et al. (2024) found that in unsuccessful trials, students often employed recursive strategies, focusing on individual elements rather than the repeating unit (e.g., producing ABAABB instead of ABBABB).

Flynn et al. (2020) evaluated the patterning abilities of children aged four to six by comparing the use of abstract language (such as the letters AAB or numbers 112) with non-abstract instructions (such as “This part, this part”). The findings showed that children exposed to the abstract language of patterns were more able to recognise and create patterns involving objects and numbers. Wijns et al. (2021) examined the impact of a 20-week intervention on five-year-olds’ patterning and numerical abilities. This intervention included sessions focused on repeating and growing patterns and their underlying structure. While most pattern activities emphasised linear configurations using concrete materials, symbols, and pictures, two activities adopted alternative approaches: one involved copying and creating movement patterns (e.g., “high, high, low, high, high, low”; AAB pattern), and the other used music in singing a pattern song (e.g., “up, down, up, down”; AB pattern; Wijns et al., 2021, p. 5).

Warren and Miller’s (2010) intervention study, involving patterning activities for four-year-olds, demonstrated improvements in the children’s ability to copy, extend, and create repeating patterns, but there was no advancement in their skills to fill in missing elements. The intervention included activities featuring both linear and non-linear pattern configurations, together with mapping the repeating unit across modalities. For example, students engaged in movement activities, such as a sequence of jumping and clapping, which was then translated into tangible materials. However, the assessment tools employed used linear configurations of cards depicting the patterns, with students being provided with concrete materials (e.g., lighthouses and ladybugs) to complete the tasks.

In summary, the review highlights the importance for preschoolers of mastering repeating patterns and suggests that interventions could facilitate this development. With some exceptions, however, most studies focus on the outcomes of children’s engagement with repeating patterns through linear configurations of visual or concrete materials (e.g., Rittle-Johnson et al., 2013; Zippert et al., 2018). Considering the growing evidence of a connection between pattern recognition, structural understanding and abstract mathematical thinking (e.g., Lüken & Sauzet, 2021; McCluskey et al., 2018; Mulligan et al., 2020), adopting a process perspective can

provide insights into understanding how multimodal mapping across various spatial configurations of repeating units supports early pattern learning.

Variation Theory of Learning

The variation theory of learning characterises learning as the recognition and understanding of previously unnoticed aspects of a specific phenomenon (skill, knowledge, learning objective; Marton, 2014). Within this framework, the elements that learners must identify to acquire the intended skill or knowledge, the object of learning, are identified as critical aspects. Learning is, thus, seen as a qualitative change in how a learner perceives these critical aspects, which entails discerning variation against a background of invariance (Marton, 2014). In a teaching context, variation theory emphasises the importance of making the critical aspects of a learning objective visible and discernible (Kullberg et al., 2024). To achieve this, a dimension of variation is defined as an aspect, whether critical or not, that is varied within a context centred on a specific object of learning (Kullberg et al., 2024). When repeating patterns are the object of learning, students are encouraged to explore different ways of copying, extending, repairing, and mapping repeating units of varying length, using a variety of materials, modalities, and structures. It is through exploring the dimensions of variation within a learning objective that students develop powerful ways of seeing (Marton, 2014), leading to a deeper, more flexible and sustainable approach to understanding repeating patterns.

As highlighted in the reviewed literature, a common approach to conceptualising the generalisation and abstraction of pattern knowledge involves assessing children's ability to map repeating units using different materials, often framed within single or dual modalities across linear configurations of elements (e.g., Rittle-Johnson et al., 2013; Wijns et al., 2019). From the perspective of variation theory, however, generalisation is understood as a pattern of variation. For generalisation to occur, the critical aspects must remain invariant while other aspects are varied (Kullberg et al., 2024). Consequently, if students have already experienced linear pattern mapping within concrete modalities, this aspect of multimodal mapping may no longer be essential for their learning. Therefore, it is argued that, in the learning of repeating patterns, children should experience novel variations in the mapping of a pattern (the invariant unit of repeat) across multimodal linear and non-linear configurations of elements (the variations). For example, when children use a linear AABC colour pattern with cubes to model non-linear sequences of steps on geometric shapes of different colours on the ground (e.g., circles, squares, triangles), they engage with visual configuration, full-body movement, and auditory modalities as dimensions of variation, which enhances their recognition of the invariant unit of repeat. This context highlights the importance of providing diverse, multimodal experiences that allow children to explore the underlying structures of patterns in novel forms, thereby deepening their pattern understanding.

Method

Task Design of Early Learning of Patterns

The case examined in this paper is part of a Norwegian design study of pattern learning involving ten children aged four to 5 years, conducted with informed consent from their legal guardians. The study took place in an embedded design near their kindergarten, and included five 1 hour intervention sessions, followed by individual and group-based post-testing through task-based interviews, all of which were videotaped. The interventions were collaboratively conducted by kindergarten teachers and two researchers, with one of the researchers (the author of this paper) conducting the task-based interviews. Informed by the multimodal embodied perspective on task design (DeSutter & Stieff, 2017; O'Halloran, 2015; Skulmowski & Rey, 2018), the principles (DPs) that guided the cyclical process of task development, evaluation, and refinement (Gravemeijer & Cobb, 2006) were:

- DP1: Consistent multimodal mappings of repeating units.
- DP2: Aesthetic and expressive aspects of full-body performance.
- DP3: Collaboration and multimodal dialogue.
- DP4: Guided play.

The interventions enabled the children to create, explore and map repeating patterns in both linear and non-linear configurations of visual elements, using diverse modalities and materials. These activities included dancing and stepping on spatially dispersed coloured dots sprayed on the ground, building with blocks, and participating in playful and cooperative settings.

Reflecting the nature of case studies as in-depth examinations of phenomena within defined temporal and spatial boundaries (Yin, 2018), this paper presents insights from a video excerpt featuring five-year-old Wanda (pseudonym) during individual testing post-intervention. The goal of case studies is to achieve theoretical (analytical) generalisation by confirming, expanding and refining existing theories, which requires selection of cases with the potential to reveal something new and different (Eisenhart, 2009). This paper aims to contribute to the discourse on a process-oriented approach to children's pattern learning and assessment (e.g., Junker et al., 2024; Lünen & Sauzet, 2021), with a focus on the notion of multimodal mapping of the repeating unit. The case of Wanda is used to illustrate dimensions of variation in her multimodal pattern mapping through her engagement in the activity described below.

The Embodied Pattern Activity

The activity, aimed at enhancing the creation of a model pattern and the mapping of its repeating unit across different modalities (using the same colours), unfolds over five stages.

Stage 1: Freely choose red, white, and blue cones to fill a linear array of five empty circles, creating a referential model pattern (Fig. 1).

Stage 2: Multimodal mapping of the model pattern using coloured blocks (10 cm × 10 cm × 5 cm) to build a pattern tower (Fig. 2).

Stage 3: Multimodal mapping of the model pattern using Unifix cubes (Fig. 3).

Stage 4: Step alongside the cone pattern, accompanying the movement with verbalisation of colours, then enter a large circle (d = 5 m) with 300 randomly arranged dots in red, white, and blue. While navigating across the circle, map the model pattern repeatedly through verbalised physical interactions by walking on dots (Fig. 4).

Stage 5: In a six-dotted matrix featuring pairs in red, blue and white, map the model pattern repeatedly into a verbalised dance (Fig. 5).

Fig. 1 Cone model
(Stage 1)



Fig. 2 Tower model
(Stage 2)



Fig. 3 Unifix model
(Stage 3)



Fig. 4 Pattern walk
(Stage 4)



Process of Analysis

Drawing on variation theory, it is assumed that examining variations in how Wanda discerns the repeating unit throughout the pattern activity will facilitate an analysis of her understanding of creating a model pattern and mapping it multimodally. Table 1 details the modalities analysed and describes the primary characteristics of each.

In coding, the concept of a multimodal bundle is used to represent the integration of various modes to convey a specific idea (Abdu et al., 2021). These multimodal bundles have both static and relational characteristics, allowing for synchronic and diachronic analyses. Synchronic analysis examines the simultaneous use of multiple modal resources and can identify whether a multimodal bundle maintains inter-modal consistency or not (Abdu et al., 2021). For example, consistency is observed when the spoken word “blue” aligns with an audible stepping on a blue dot (coded

Fig. 5 Pattern dance (Stage 5)



Table 1 Modalities with descriptions of characteristics

Modalities	Descriptions of characteristics
Verbal	Verbal elements (e.g., spoken colours)
Upper-body movement	Use of hands to manipulate concrete elements (e.g., cones, blocks, cubes)
Full-body movement	Physical elements produced by stepping beside cones or on dots on the ground
Auditory	Aural elements produced by gait or stepping
Visual	Visual elements, including coloured concrete objects (e.g., cones, blocks, cubes) and coloured dots or cones tagged through physical interactions (e.g., stepping)

as a full-body movement-verbal-auditory-visual bundle, whereas inconsistency occurs if the verbal cue does not correspond to the colour of the dot.

As a complementary approach, the diachronic analysis examines the meaning created by connected multimodal bundles over time, with each modality contributing distinct information to the pattern mapping process. For instance, the auditory modality may convey the structure of the unit of repeat (e.g., “hard step, hard step, soft step”; AAB) or, as demonstrated in this research, represent its length and beat through the number of aural elements produced in conjunction with verbal expressions of colours during a stepping sequence (Table 1). The diachronic analysis of the visual modality examines the spatial configuration of connected visual elements, whether static (e.g., a linearly configured AAB-coloured pattern tower) or dynamic (e.g., a non-linear stepping sequence of dots in the order “blue, blue, red”), and their role in maintaining consistency within the mapping process (Table 1). For instance, in a sequence of verbal-full-body movement-auditory-visual bundles, the visual modality captures the spatial configuration of the produced elements and assesses alignment with the targeted unit of repeat through its colour cues, while the verbal modality reinforces this alignment.

The reliability of identifying and distinguishing temporal modalities (e.g., verbal, auditory and full-body movement) and static modalities (e.g., visual elements created using cones, blocks or cubes) was ensured through multiple reviews of video recordings to validate consistency in observations. The inclusion of images from the video aimed to enhance the trustworthiness and depth of the analysis. Analysing connected multimodal bundles across the activity's five stages enables the identification of dimensions of variation in pattern mappings related to the invariant properties of the model pattern. The video excerpt, lasting 4:09 minutes, was first transcribed in Norwegian and then translated into English. The transcript is divided into numbered sections, each describing an individual's multimodal actions, with non-verbal actions detailed in brackets and verbal expressions italicised.

Results and Analysis

The excerpt begins as the researcher facilitator finishes explaining the first stage of the activity.

(1) Wanda: [Uses hands to arrange cones, creating an AABBC pattern; Fig. 6]

(2) Facilitator: *What a nice pattern, Wanda! How long is your pattern?*

(3) Wanda: [Finger pointing at the cones] *One, two, three, four, five.*

(4) Facilitator: *Yes. Can you make the same pattern using these blocks* [Pointing at a box with $10 \times 10 \times 5$ cm blocks in white, blue, and red]? *You should make a tower.*

(5) Wanda: [Uses hands to create an AABBC-block tower, looking at the cone-pattern after placing the two blue blocks]

Fig. 6 Design of the AABBC model pattern



Fig. 7 Counting the length of the tower pattern

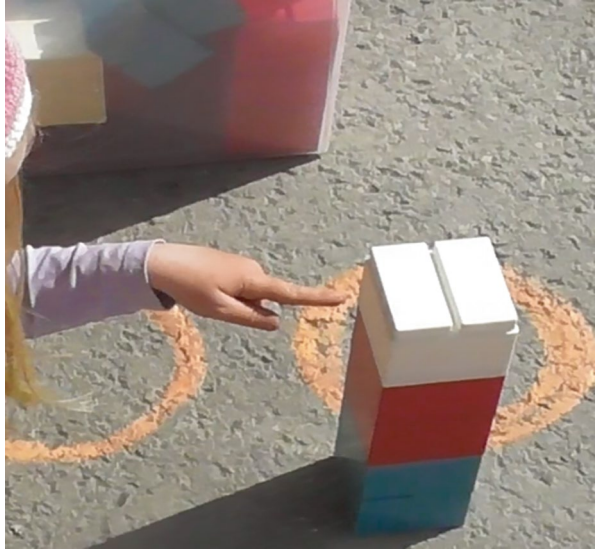
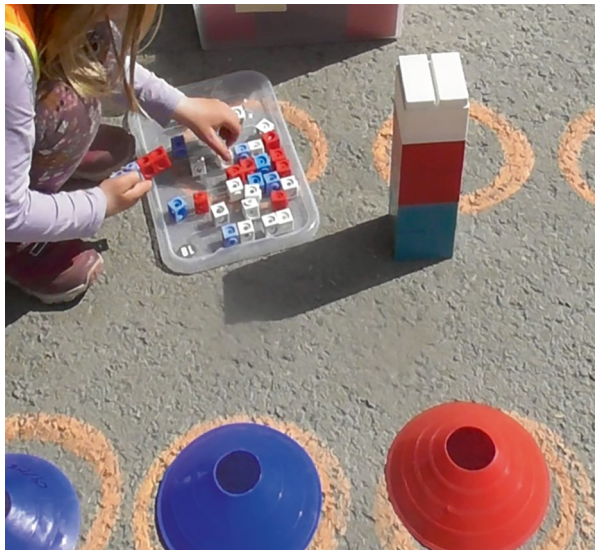


Fig. 8 Constructing the Unifix pattern



(6) Facilitator: *How long is your pattern?*

(7) Wanda: [Upwards finger pointing at the blocks; Fig. 7] *One, two, three, four, five.*

(8) Facilitator: *Yes. Can you use these smaller pieces to make the same pattern?*

(9) Wanda: [Uses hands to construct the AABBC pattern with Unifix cubes, looking at the cone pattern twice during the process; Fig. 8. Shows the finished Unifix-pattern to the facilitator]

Fig. 9 Walking alongside the cone-pattern



Fig. 10 “Blue, blue” (turn for visual support in the cone-pattern)



(10) Facilitator: *Great. Can you walk the pattern here* [Pointing at the cones], *and continue* [Pointing inside the 300 dotted circle], *until you get to the other side* [Pointing]?

(11) Wanda: *Should I bring this* [Shows the Unifix pattern]?

(12) Facilitator: *Yes.*

(13) Wanda: *Blue, blue, red, red, white* [Stepping alongside corresponding cones; Fig. 9. Stops]. *And then into the circle?*

(14) Facilitator: *Yes.*

(15) Wanda: *Blue, blue* [Stepping on two blue dots, halts, turns around and looks at the cone-pattern; Fig. 10]. *Red, red, white. Blue, blue, red, red, white. Blue, blue,*

Fig. 11 “Blue, blue (pause), red, red (pause), white (pause)”



Fig. 12 Dancing “Blue, blue, red, red, white”



red, red, white. Blue, blue, red, red, white. Blue, blue, red, red, white [Stepping on corresponding dots, rhythmically dividing the pattern into three parts, AA (short pause) BB (short pause) and C (short pause); Fig. 11]

(16) Facilitator: *Then you are going to dance the same pattern here* [Pointing at the six-dotted matrix]

(17) Wanda: *Blue, blue, red, red, white. Blue, blue, red, red, white. Blue, blue, red, red, white.* [Rhythmic verbalised stepping on dots; Fig. 12. Briefly pausing while balancing on one leg in between each cycle; Fig. 13. Not looking at the Unifix pattern during the process]

Table 2 presents a summary of the analysis, outlining the modalities involved in connected bundles, the spatial configuration of visual elements, the frequency and

Fig. 13 Balancing on one leg in between AABBC-cycles



Table 2 Modalities in bundles, frequency, configuration, mapping (in)consistency and observations in excerpt

Stage	Modalities involved in connected multimodal bundles	Frequency/configuration of visual elements/ mapping (in)consistency	Line
1	Visual (linear configuration of coloured cones), upper-body movement (hands to arrange cones in an array)	1/linear/not applicable	1
2	Visual (linear configuration of coloured blocks), upper-body movement (hands to stack blocks into a tower)	1/linear/consistent	5
3	Visual (linear configuration of coloured Unifix cubes), upper-body movement (hands to stack Unifix cubes into a tower)	1/linear/consistent	9
4	Visual (gait alongside linearly arranged coloured cones and non-linear gait onto coloured dots in the 300-dotted circle), full-body movement, auditory, verbal (speech of colours)	1/linear/consistent5/ non-linear/consistent	1315
5	Visual (non-linear configuration of stepping onto coloured dots in the six-dotted matrix), full-body movement, auditory, verbal (speech of colours)	3/non-linear/consistent	17

the (in)consistency in Wanda’s mapping of the AABBC model pattern across the different stages of the activity.

Table 2 shows that Wanda performed 11 consistent multimodal mappings of the AABBC model pattern she created. In the first three stages, Wanda interacted with concrete materials, manipulating cones, blocks, and Unifix cubes. In contrast, in Stages 4 and 5, she combined distinct auditory temporal movements within visual configurations of elements (the 300-dotted circle and the six-dotted matrix; Table 2).

In these two stages, she demonstrated rhythmic mappings of the AABBC pattern, where the physical tagged elements were arranged in a non-linear spatial configuration. Initially in Stage 4, Wanda relied on the reference cone pattern in her mapping process, but after the first repetition in the 300-dotted circle, she internalised the repeating unit. While navigating across the circle (15; Fig. 11), her movement-based mapping was divided into three synchronised parts, following a 2–2–1 beat structure and articulated as “Blue, blue”, “red, red” and “white”. Similarly, within the six-dotted matrix, Wanda created a fluent, non-linear sequence of verbalised and audible steps, where the integration of these temporal modalities reinforced the five-beat rhythm of the AABBC pattern. After completing each AABBC sequence, she paused, balancing on one leg before continuing her dance, which reinforced the perception of the pattern’s boundaries (17; Fig. 13). Table 2 shows that the first three stages involved tactile manipulation of objects in the mapping of linear configurations of the repeating unit (i.e., bundles of two modalities), whereas the latter two stages involved full-body movement, incorporating bundles of four modalities. In these final stages, Wanda synchronised her articulation of colours with auditory movements, highlighting rhythm, beat, and the non-linear configuration of physically tagged visual elements as dimensions of variation in her pattern mapping.

Discussion and Final Remarks

The review of studies of repeating patterning in early years reveals three main insights: the dominance of linear configurations in patterning tasks, a certain rigidity in the use of modalities, and a need for a deeper understanding of how children create and map patterns. To address these insights, I explored the case of five-year-old Wanda’s engagement in a five-stage activity focused on the multimodal mappings of the repeating unit in her self-created model pattern (AABBC). The diachronic analysis highlights key differences in how Wanda mapped the model pattern at each stage, emphasising her recognition and application of the repeating unit across various modalities and visual configurations of elements. The initial stages (1 to 3) primarily involved unarticulated material manipulation, while the latter stages (4 and 5) were characterised by rhythmic full-body movements combined with the articulation of colours. This shift highlights the integral role of verbalisation in mathematical learning (e.g., Peng et al., 2020) and the intrinsic connection between rhythm, mathematical thinking (Radford, 2015), and pattern learning (Junker et al., 2024). In Junker et al.’s (2024) study of six-year-olds, a rhythmic, linear approach to pattern mapping was observed, with verbal expressions such as “big, small, big, small” synchronised with gestures like finger-pointing and vocal pitch variations. Building on these findings, the present study extends the understanding of rhythm in pattern mapping by highlighting how full-body movement contributes to the synchronisation of multimodal elements. For example, as Wanda danced within the six-dotted matrix, her pattern mapping combined rhythmic, non-linear visual configuration of audible steps with synchronised verbalisations of the colours, making the five-beat rhythm

evident through her multimodal interaction. This illustrates how the verbal and auditory modalities, with their sequential characteristics, support the perception and production of rhythm, while the visual modality, shaped by movement trajectories, can adapt to non-linear configurations according to the spatial arrangements of the visual elements. Despite the increased demands on working memory posed by the temporal nature of movement-based pattern mappings compared to static visual modalities, the cyclical rhythm of such mappings may shift focus to the repeating unit and its boundaries, rather than individual elements, thereby reducing reliance on error-prone recursive strategies, a common approach observed in unsuccessful trials (Junker et al., 2024).

Unlike other assessment tools that focus on linear mapping of elements within a single modality (e.g., Fyfe et al., 2015; Rittle-Johnson et al., 2013), the initial stages of the activity offered a wider variety of concrete objects in terms of size and shape (cones, blocks and Unifix cubes), although limited to the same three colours. Stages 4 and 5 encouraged Wanda to navigate between non-linear and linear visual configurations of patterns, engaging in mappings of the unit of repeat that integrated full-body movement and auditory modalities. The non-linear pattern configuration aligns with the study by Lüken (2023), which found that one-to-six-year olds frequently produced non-linear patterns during their self-initiated play with materials. Furthermore, the multimodally rich experiences with patterns emerging from Wanda's initial construction of the AABBC pattern resonate with the activities proposed by Wijns et al. (2021) and Warren and Miller (2010), which combined pattern creation and translation with movement and auditory elements.

As noted earlier, educational design and assessments of pattern knowledge often exhibit a bias towards mapping linear configurations of visual or concrete elements, aiming to streamline the evaluation process for educators (e.g., Zippert et al., 2018). However, from the perspective of variation theory, the validity of the experiences provided by such assessment tasks and pattern activities must be weighed against novel ways of mapping patterns as a learning objective. Marton (2014) emphasises that discerning dimensions of variation within an object of learning can shape a learner's recognition of critical features in new contexts. In response, this results from my study demonstrate how a child's creation of a model pattern can support multimodal mappings of its repeating unit across different visual configurations of elements, incorporating full-body movements. Thus, rather than focusing on the end-product and the notion of "using different materials" to conceptualise a child's ability to copy, extend, repair and abstract a given linear pattern (e.g., Rittle-Johnson et al., 2013; Wijns et al., 2019), this research shifts attention to examining dimensions of variation in how multimodal mappings of a self-created model pattern occur. In this study, Wanda's multimodal mappings of the model pattern do not meet the criteria for repeating pattern generalisation (Wijns et al., 2019), or abstraction (Rittle-Johnson et al., 2013), as the mapping process relied on the same colours. However, moving beyond these cognitive perspectives, variation theory offers a different approach by conceptualising generalisation as a pattern of variation perceived through experience (Kullberg et al., 2024), where critical aspects remain invariant, while others change. This approach provides insights into learners' discernment and

multimodal understanding of pattern knowledge. These findings, therefore, reinforce the importance of targeted interventions to support children's pattern creation and their comprehension of pattern structures (e.g., Lüken & Sauzet, 2021). By acknowledging multimodality and embodiment as powerful ways of experiencing mathematics (O'Halloran, 2015; Skulmowski & Rey, 2018), educators can design learning activities that not only foster understanding and facilitate the generalisation of pattern knowledge, understood as a pattern of variation, but also address both the learning process and the assessment of repeating pattern knowledge in early childhood education.

This research extends previous studies on multimodal experiences with pattern structures (e.g., Junker et al., 2024; Lüken & Sauzet, 2021; Wijns et al., 2021) by providing insights into variations in a child's multimodal mappings of the repeating unit in a self-created model pattern, emphasising the role of physical interaction with different spatial configurations of visual elements. Future studies should test the refined concept of multimodal mapping of patterns (O'Halloran, 2015), exploring differences and variations to assess its generalisability (Eisenhart, 2009) with a focus on how movement shapes children's ability to discern the repeating unit across properties such as beat, shape, size, collinearity, and configuration of elements. In conclusion, while acknowledging the limitation of focusing on one single case, this paper advocates for a broader understanding of how a multimodal approach supports pattern learning in early childhood education.

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References

- Abdu, R., van Helden, G., Alberto, R., & Bakker, A. (2021). Multimodal dialogue in small-group mathematics learning. *Learning, Culture and Social Interaction*, 29, 100491. <https://doi.org/10.1016/j.lcsi.2021.100491>
- DeSutter, D., & Stieff, M. (2017). Teaching students to think spatially through embodied actions: Design principles for learning environments in science, technology, engineering, and mathematics. *Cognitive Research: Principles and Implications*, 2(1), 22. <https://doi.org/10.1186/s41235-016-0039-y>
- Eisenhart, M. (2009). Generalization from qualitative inquiry. In K. Ercikan & W.-M. Roth (Eds.), *Generalizing from educational research: Beyond qualitative and quantitative polarization* (pp. 51–66). Routledge.
- Flynn, M. E., Guba, T. P., & Fyfe, E. R. (2020). ABBABB or 1212: Abstract language facilitates children's early patterning skills. *Journal of Experimental Child Psychology*, 193, 104791. <https://doi.org/10.1016/j.jecp.2019.104791>
- Fyfe, E. R., McNeil, N. M., & Rittle-Johnson, B. (2015). Easy as ABCABC: Abstract language facilitates performance on a concrete patterning task. *Child Development*, 86(3), 927–935. <https://doi.org/10.1111/cdev.12331>
- Fyfe, E. R., Rittle-Johnson, B., & Farran, D. C. (2019). Predicting success on high-stakes math tests from preschool math measures among children from low-income homes. *Journal of Educational Psychology*, 111, 402–413. <https://doi.org/10.1037/edu0000298>

- Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In T. Plomp & N. Nieveen (Eds.), *Educational design research* (pp. 72–113). SLO.
- Junker, A., Nortvedt, G. A., & Farsani, D. (2024). Patterning strategies in grade 1 students with low and high number sense proficiency. *Educational Studies in Mathematics*. <https://doi.org/10.1007/s10649-024-10341-5>
- Kullberg, A., Ingerman, Å., & Marton, F. (2024). Planning and analyzing teaching: Using the variation theory of learning 1st ed. Routledge. <https://doi.org/10.4324/9781003194903>.
- Lüken, M. M. (2023). Young children's self-initiated pattern-making during free play. *Mathematical Thinking and Learning*, 1–s. <https://doi.org/10.1080/10986065.2023.2276798>
- Lüken, M. M., & Sauzet, O. (2021). Patterning strategies in early childhood: A mixed methods study examining 3- to 5-year-old children's patterning competencies. *Mathematical Thinking and Learning*, 23(1), 28–48. <https://doi.org/10.1080/10986065.2020.1719452>
- Marton, F. (2014). *Necessary conditions of learning*. Routledge. <https://doi.org/10.4324/9781315816876>
- McCluskey, C., Mulligan, J., & Van Bergen, P. (2018). Noticing mathematical pattern and structure embodied in young children's play. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Making waves, opening spaces: Proceedings of the 41st annual conference of the mathematics education research group Australasia* (pp. 535–542). MERGA.
- Mulligan, J., Oslington, G., & English, L. (2020). Supporting early mathematical development through a 'pattern and structure' intervention program. *ZDM Mathematics Education*, 52(4), 663–676. <https://doi.org/10.1007/s11858-020-01147-9>
- O'Halloran, K. L. (2015). The language of learning mathematics: A multimodal perspective. *The Journal of Mathematical Behavior*, 40, 63–74. <https://doi.org/10.1016/j.jmathb.2014.09.002>
- Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education*, 42(3), 237–268. <https://doi.org/10.5951/jresmetheduc.42.3.0237>
- Peng, P., Lin, X., Ünal, Z. E., Lee, K., Namkung, J., Chow, J., & Sales, A. (2020). Examining the mutual relations between language and mathematics: A meta-analysis. *Psychological Bulletin*, 146(7), 595–634. <https://doi.org/10.1037/bul0000231>
- Radford, L. (2015). Rhythm as an integral part of mathematical thinking. In M. Bockarova, M. Danesi, D. Martinovic, & R. Núñez (Eds.), *Mind in mathematics: Essays on mathematical cognition and mathematical method* (pp. 68–85). LINCOM GmbH.
- Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development*, 14(3), 376–396. <https://doi.org/10.1080/15248372.2012.689897>
- Rittle-Johnson, B., Zippert, E. L., & Boice, K. L. (2019). The roles of patterning and spatial skills in early mathematics development. *Early Childhood Research Quarterly*, 46, 166–178. <https://doi.org/10.1016/j.ecresq.2018.03.006>
- Skulmowski, A., & Rey, G. D. (2018). Embodied learning: Introducing a taxonomy based on bodily engagement and task integration. *Cognitive Research: Principles and Implications*, 3. <https://doi.org/10.1186/s41235-018-0092-9>
- Warren, E., & Miller, J. (2010). Exploring four year old indigenous students' ability to pattern. *International Research in Early Childhood Education*, 1(2), 42–56.
- Wijns, N., De Smedt, B., Verschaffel, L., & Torbeyns, J. (2020). Are preschoolers who spontaneously create patterns better in mathematics? *British Journal of Educational Psychology*, 90(3), 753–769. <https://doi.org/10.1111/bjep.12329>
- Wijns, N., Torbeyns, J., De Smedt, B., & Verschaffel, L. (2019). Young children's patterning competencies and mathematical development: A review. In K. M. Robinson, H. P. Osana, & D. Kotsopoulos (Eds.), *Mathematical learning and cognition in early childhood: Integrating interdisciplinary research into practice* (pp. 139–161). Springer. https://doi.org/10.1007/978-3-030-12895-1_9
- Wijns, N., Verschaffel, L., De Smedt, B., De Keyser, L., & Torbeyns, J. (2021). Stimulating preschoolers' focus on structure in repeating and growing patterns. *Learning and Instruction*, 74, 101444. <https://doi.org/10.1016/j.learninstruc.2021.101444>

- Yin, R. K. (2018). *Case study research and applications: Design and methods* (6th ed.). Sage.
- Zippert, E. L., Clayback, K., & Rittle-Johnson, B. (2019). Not just IQ: Patterning predicts preschoolers' math knowledge beyond fluid reasoning. *Journal of Cognition and Development, 20*(5), 752–771. <https://doi.org/10.1080/15248372.2019.1658587>
- Zippert, E. L., Douglas, A.-A., & Rittle-Johnson, B. (2020). Finding patterns in objects and numbers: Repeating patterning in pre-K predicts kindergarten mathematics knowledge. *Journal of Experimental Child Psychology, 200*, 104965. <https://doi.org/10.1016/j.jecp.2020.104965>
- Zippert, E. L., Douglas, A.-A., Tian, F., & Rittle-Johnson, B. (2021). Helping preschoolers learn math: The impact of emphasizing the patterns in objects and numbers. *Journal of Educational Psychology, 113*, 1370–1386. <https://doi.org/10.1037/edu0000656>
- Zippert, E. L., Loehr, A. M., & Rittle-Johnson, B. (2018). *A new teacher-based assessment of preschoolers' patterning skills [Poster]*. Society for Research on Educational Effectiveness.

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Collaboration in Primary Students' Collective Mathematical Reasoning



Peter Markkanen, Helena Eriksson, and Lovisa Sumpter

Introduction

Mathematical problem solving is one of the key aspects of mathematical understanding, serving not only as a central component but also as a crucial element in preparing students to function in a modern society (Gravemeijer et al., 2017). Research has shown that students' ability to solve problems can be enhanced through collaborative problem-solving processes (e.g., Harding et al., 2017) and that students construct collective meaning-making in collaboration with each other (Roschelle & Teasley, 1995). One of the difficulties when working together is creating a collaborative environment, where what is considered collaborative can vary from superficial interactions to demanding interactions where one can find content-oriented learning (Krummheuer, 2007). The ability to work collaboratively and solve problems in groups are also recognised as essential twenty-first century skills (Felmer, 2023).

However, working with problem solving is not always straightforward, and students may experience difficulties in identifying relevant elements for solutions or visualising appropriate solution paths in the situation (Klang et al., 2021). Students can also have unstructured habits in problem solving, which significantly impact their ability to solve mathematical problems and often prevent them from fully understanding the problem at hand, which, in turn, hinders a group from developing and implementing effective plans to find solutions (Widodo et al., 2019). In a successful collaboration, group members pay attention to and respond to what others do and say (Sjöblom & Meaney, 2021). They provide space for each other's contributions and monitor how they relate to the problem-solving goal (Webb, 2009). This means that the collaborative problem-solving space depends on both the individuals

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and the social context where the problem solving takes place (Seidouvy & Schindler, 2019). One key aspect of productive collaborations is students' ability to engage in mathematical argumentation (Stein et al., 2008). Collective reasoning stems from social interaction and depends on the justifications given for different decisions (Sumpter, 2016). Thus, students must develop the capacity to articulate mathematical reasoning for themselves and their peers (Hansen, 2021). Reasoning ability is a crucial skill that is closely linked to problem solving. Promoting students' experiences with mathematical reasoning and problem solving has been identified as a key objective in mathematics education (Lithner, 2017).

In addition, the nature of the task will affect the problem solving and the mathematical reasoning (Granberg & Olsson, 2015; Hestner & Sumpter, 2018). It means that certain tasks that might be more open and allow different types of solutions can generate a variety of mathematical arguments (Eriksson et al., 2023). Therefore, teachers must not only be able to create or choose appropriate tasks and activities but also pose questions that allow students to reflect and develop their reasoning (Gravemeijer et al., 2017; Webb, 2009). There is limited research on how students' collaboration and reasoning are interconnected (Sidenvall, 2019). The studies that have been conducted predominantly focus on older students (e.g., Granberg & Olsson, 2015) or collaboration in relation to other specific abilities, such as spatial reasoning (Duff et al., 2021). Therefore, this study seeks to address this gap by investigating how primary school students' problem-solving space can be created in collective reasoning.

Background

The chosen theoretical frameworks employed in the study are a framework for collaboration (Roschelle & Teasley, 1995; Szabo et al., 2024), and one about collective mathematical reasoning (Sumpter, 2016). Starting with collaboration, one can divide research, according to Seidouvy and Schindler (2019), into two overarching perspectives: (a) research focusing on the outcomes of student collaborations, a product-focused approach and (b) research directing attention to the process in student collaborations. There are varying perspectives on what collaborative problem-solving entails. In PISA 2015, it was defined as the ability of an individual to effectively participate in a team of two or more members by sharing understanding, achieving consensus, determining solutions, and uniting collaborative knowledge, skills, and actions to solve problems (OECD, 2017). Similar definition is provided by Lahann and Lambdin (2020) who stated that collaboration involves a group of students working together, sharing ideas, solving problems, and achieving a common goal. What both these definitions have in common is the joint process, a key feature identified by Cao (2024); that collaborative problem solving should be viewed as a social process in which individuals reach a high level of cognition through interaction with their social environment. Gestures, such as pointing, moving arms and hands, play an important role in young students' reasoning, that will

help them to understand better each other's mathematical arguments (Wathne & Carlsen, 2022). The understanding of each argument is crucial (Sjöblom & Meaney, 2021), especially since collaboration entails a shared responsibility for the group's actions (Klang et al., 2021). In the present study, we align with Roschelle and Teasley (1995) when they define collaboration as "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (p. 70). A core feature of collaboration, therefore, is the notion of a shared conception of the problem and social interactions in the context of a problem-solving activity, which are described in the theory of Joint Problem Space (JPS).

A JPS is considered a socially negotiated, shared knowledge structure that emerges among students through the external mediational framework of shared language, situation, and activity (Roschelle & Teasley, 1995). As such, it covers several aspects of problem solving, social interactions, and mathematical reasoning. In this study, we have chosen a reduced version of JPS that has been utilised in previous studies (e.g., Szabo et al., 2024). Then, the focus is on the following situations: (1) Understanding the problem and creating a goal, that is, initiating (I) the reasoning within the group and thereby a JPS; (2) Creating, implementing and evaluating problem-solving strategies (S); (3) Observing and negotiating (N) conceptions as well as deviations from the goal; and (4) Solving the problem and thereby reaching the goal (G). Conception is defined as a mental structure within the individual, serving as an umbrella concept comprising beliefs, meanings, concepts, propositions, rules, mental images, and preferences (Philipp, 2007). To build a JPS, participating students must in a collaborative process introduce and accept knowledge in the JPS, monitor ongoing activity for evidence of disagreements, and repair deviations that may hinder progress. It encompasses turn-taking, giving and receiving suggestions, asking questions, and providing explanations, all central aspects of a collaborative problem-solving process, when reasoning with each other (Clark & Schaefer, 1989; Granberg & Olsson, 2015; Sjöblom & Meaney, 2021).

Continuing with collective mathematical reasoning, it can be viewed in different ways (Sumpter, 2016). In this study, mathematical reasoning is defined as a meaning-making process that emerges from collective joint work, which can be structured into chains of arguments (Sumpter, 2016). This is inspired by Lithner's (2008) description of reasoning as "the line of thought adopted to produce assertions and reach conclusions in task solving" (p. 257). Although reasoning in mathematics education is often characterized as high-quality thinking, there is limited clarity about what this entails or how it can be effectively demonstrated and described in young children's reasoning (Sumpter, 2016). To provide structure to the reasoning process, the following steps are utilised: (1) a (sub)task is explored (TS); (2) a strategy choice is made (SC); (3) the strategy is implemented (SI); and (4) a conclusion is suggested (C). For each step, different arguments can be expressed (Eriksson & Sumpter, 2021; Lithner, 2008). Here, the focus is on the steps and not the arguments or the content of the arguments. In contrast to studies on collaborative reasoning (e.g., Granberg & Olsson, 2015), which often focus on specific individuals or particular characteristics of reasoning, such as types of imitative reasoning, we explore

how students engage in collaborative problem solving through their collective mathematical reasoning. While decisions and arguments in this context are shaped by a group, the emphasis remains on the collaborative processes involved (Sumpter, 2016), and learning and teaching are seen as ongoing changes in human behaviour (Sumpter & Hedefalk, 2018). In regard to reasoning in mathematics, students often require support to explore and develop their reasoning skills, both individually and collectively (Hansen, 2021). Thus, the teacher can be viewed as an agent who can provide targeted support to the students as needed.

In this study, we are interested in how a problem-solving space can be created through collective reasoning. The aim is to explore how students create collaborative problem-solving in their collective mathematical reasoning. The research questions posed are: (1) How is joint problem-solving a part of collective mathematical reasoning? (2) What contextual factors can be identified as contributing to the process of creating a joint problem space?

Method

The data were video recordings of students' work with three cases from a sequence of six (for information about all six cases, see Sumpter and Hedefalk (2023)). The work with the cases was part of a larger project exploring the progression of students' reasoning using a cross-curricular approach between mathematics education and sustainability education. In Case 2, the students were asked to share four paper biscuits among three animals (soft-toys). In Case 5, they were instructed to distribute 12 biscuits with the additional condition that one of the three animals was really hungry. Finally, in Case 6, the students had to share eight biscuits among three animals, one of whom was portrayed as both hungry and sad. Sharing, is an appropriate mathematical topic since it is a fundamental aspect of division and, thereby, fractions (Lamon, 2020; Vonk et al., 2024). However, compared to division, where the fundamental principle is that all recipients receive equal amounts, in reasoning about sharing, arguments can be based on understandings about fairness and unequal parts (Watson, 1997). The selected cases presented the students with ethical dilemmas (Sumpter & Hedefalk, 2023) and were intentionally designed to facilitate diverse solutions while actively engaging the students (e.g., Hestner & Sumpter, 2018). As a result, the concept of fair sharing remains open to interpretation.

The teacher, Kim (pseudonym), was instructed to encourage students to express arguments by asking them to explain their proposed ideas and compare their respective ideas and reasoning processes. Hence, the encouragement of expressing arguments and listening to each other arguments were central (e.g., Eriksson & Sumpter, 2021; Sjöblom & Meaney, 2021). Three students worked together as a single group—Ida (6y 1 m), Nova (6y 2m) and Adam (6y 6m). All students were born in Sweden but had another first language than Swedish. The students' parents have signed a letter of consent, thereby complying with Swedish ethical guidelines. In addition to the teacher, the second author was present, responsible for operating the camera.

The data analysis followed a five-step process. In the first step, the video recordings were transcribed verbatim, supplemented with descriptions of both students' and the teacher's actions—capturing not only verbal exchanges but also relevant nonverbal interactions related to their engagement with the task (e.g., Mergenthaler & Stinson, 1992). Such transcriptions give rich descriptions of the data, permitting a micro-analysis to be conducted. The second and the third step of the analysis adopted a theory-driven approach. Step two involved organising the data in accordance with the structure for collective reasoning (Lithner, 2008; Sumpter, 2016), comprising: Task Situation (TS), Strategy Choice (SC), Strategy Implementation (SI), and Conclusion (C). This step entailed an analysis of how reasoning was structured, beginning with the identification of the task situation (TS), wherein students or the teacher defined the problem and discussed which problem needed to be solved. It is noteworthy that, at times, students revisited the task situation (TS) during the reasoning process, although they had already agreed on what was the task and problem to solve. The analysis further examined the students' strategy choices (SC), that is, the specific approaches proposed for solving the task. Subsequently, the analysis focused on strategy implementation (SI), identifying instances where students applied either their own or their peers' strategies. Finally, conclusions (C) were identified when the students argued why their delivery of the biscuits to the recipients was fair, and all the animals were content. The organisation of the data in this structure allows us to identify where in the reasoning certain processes take place.

In the third step of the analysis, the data were examined through the lens of the Joint Problem Space (JPS) framework (Roschelle & Teasley, 1995), which focuses on: Identification (I), Strategy (S), Negotiation (N), and Goal (G), identified as situations for collaborative problem solving (Szabo et al., 2024). The third step began by identifying instances where students recognised and articulated the problem to be solved (I). These moments typically occurred at the beginning of each task, although students occasionally revisited this phase during the problem-solving process to re-evaluate or reaffirm the nature of the problem. The analysis then focused on students' strategy use (S), including the creation, implementation, and evaluation of approaches to solve the task. Next, we examined negotiation (N), which captured moments of discussion around conceptual challenges, divergent understandings, and differing perspectives, particularly regarding what constituted fairness in the context of the task. Finally, the analysis addressed how students articulated the resolution of the problem and reached a shared understanding of the goal (G). Step two and three were initially carried out independently by two researchers. They then convened to systematically compare and evaluate their respective interpretations. The individual analyses showed a high level of consistency, and in cases of discrepancies, the relevant data was jointly reviewed and categorised. The structure of the analysis to this point is presented in Table 1.


To exemplify steps two and three of the analysis, empirical data are utilised, as shown in Table 2.

As illustrated in Table 2, after some time trying to understand the task where the students collaboratively moved the biscuits around, one child, Adam, decides to half

Table 1 The structure of the analysis

Time	Person	Data	Reasoning	JPS
Time codes from the video film	Participants	Verbal and nonverbal conversations	Organisation of the reasoning, including content (TS), (SC), (SI), (C)	Situations of collaboration (I), (S), (N), (G)

Table 2 Illustration of the analysis through students work with Case 6

Time	Person	Data	Reasoning	JPS	
03:00	All students	[The students have been moving the biscuits around for a while as they try to understand the task]		TS	I
03:08	Adam	[Cuts one biscuit in half, then distributes seven biscuits (with one left over), so that the hungry and sad animal gets 3.5 biscuits, while the other animals receive 2 and 1.5 biscuits respectively]		SC → SI	S

a biscuit, and experiments with distributing them. The exploration of the task, although a silent activity, is categorised as TS. Subsequently, Adam takes action, which we interpreted, as a strategic decision, reflecting reasoning in SC mode. We also assess that Adam’s action may be seen as a step towards reasoning in SI mode, since he has started to implement a strategy. In the third step of analysis, we categorised the collaboration using the JPS framework. As seen in Table 2, the students, although silent, are engaged in an activity and thereby they initiated their collaborative work. We categorised that their actions initiated the establishment of a JPS (I). When Adam takes action, we interpreted this as a situation in which he creates and test strategies, hence a situation when students implementing a strategy (S).

In the fourth step of the analysis, the coding outcomes at the different steps of the reasoning structure (Sumpter, 2016) were compared with the various situations described in the JPS framework (Szabo et al., 2024). By studying the columns horizontally, it became possible to identify how situations within joint problem-solving are part of collective mathematical reasoning.

Table 3 Two examples of data interpretation, codes, and resulting themes

Data		Interpretation	Code	Theme
Ida	One, two [Give half of a biscuit to the hungry animal, and the other half to the other two animals]	Fairness can mean that everyone gets the same amount, or that a hungry animal receives more	Perspectives on fairness	Challenges related to the task
Nova	What were you thinking here? [Points at the two animals that together received half a biscuit]			
Nova	[Places the eighth biscuit next to the hungry animal]	Non-verbal support of reasoning	Actions with tools	Artefacts
Adam	Oh, no! [Takes the eighth biscuit and cuts it in half]			

Step five was thematic, where data was analysed inductively. First, the entire transcribed material was carefully reviewed, and situations in which the context appeared to influence students' collaboration during the problem-solving process were marked. Since students themselves—their thoughts and perspectives—are a central part of the context, attention was also given to how they asked questions, expressed agreement or disagreement (either verbally or through body language), or contributed in other collaborative ways (e.g., Clark & Schaefer, 1989). During this coding process, we identified significant situations within the data and assigned codes to these instances based on their characteristics. These codes were then systematically organized into overarching themes (e.g., Nowell et al., 2017). Table 3 presents examples of data interpretation, codes, and the resulting themes.

The thematic analysis led to the identification and construction of four themes, which form the basis for presenting the results.

The method was first tested with only one Case (Case 6) in a pilot study (Markkanen et al., 2024). Compared to the results from the pilot, besides validating previous themes, we found an additional theme. To ensure the trustworthiness (Nowell et al., 2017) of the themes, we also analysed additional data from the same students' work on two other cases, following the five steps described above. This approach aimed to confirm the credibility and dependability of the findings through cross verification.

Results

In our analysis, we identified four themes concerning contextual factors that influence the process of creating a joint problem space. The themes are: (1) artefacts; (2) challenges related to the nature of the task; (3) students' conceptions; and (4) teacher's contributions.

Artefacts

The first theme is artefacts, which appeared to play a role in students' reasoning in establishing a JPS. The excerpt, on which Table 2 is based, occurred when the three students are gathered around a table, with three animals and eight biscuits (Case 6) and illustrates the importance of artefacts in the initial phase of setting up a JPS. The students quickly engaged with the task, by moving around the eight biscuits, negotiating. In so doing, they could be considered to be showing their understanding of the problem (Table 4).

In Table 4, the biscuits played a central role for the students as they tried different solution paths (I) during their group exploration (TS), thereby creating a JPS (00:38). Despite the students' silence, it becomes evident how they are initiating (I) the reasoning (TS), Ida uses the biscuits, as an external mediational artefact in the JPS (00:40). Regardless their efforts in shuffling the biscuits back and forth, they do not manage to move further in the reasoning process.

However, within the strategy choice (SC), the artefacts (the paper biscuits) become part to the problem-solving process (Table 5).

In the strategy choice (SC) outlined in Table 5 (02:09), the students initiated (I) and created a JPS by focusing on how to distribute resources among the three animals. They identified the problem and made a strategy choice: to allocate resources to the recipients. However, this strategy choice came with the realisation that equal distribution of whole biscuits is impossible. All three students utilised and emphasised the physical materials as tools to support thinking and reasoning, something that facilitated an JPS and encouraged collaborative efforts in tackling the challenge of distributing biscuits fairly. As illustrated in Table 5, the students' implementation of a strategy (SI) was a potential approach to understanding the problem (I): the strategy was created, and evaluated as a problem-solving strategy (S) in the JPS. The artefacts were also important to conclude (C) that the number of biscuits should be equal, in this situation, independent of size (02:24). In the JPS, the students evaluated their conclusion and implemented their problem-solving strategies (S).

During the students' work on Case 2, the biscuits also were a central artifact in developing a JPS as the students use scissors to implement the strategy of dividing a biscuit. One of the students, Ida, suggests that the fourth biscuit should be cut into smaller pieces and shared among the three animals. While Ida is responsible for cutting, both Adam and Nova are actively involved in the process, as summarised in Table 6.

The artifact is used by the students as they implement (SI), the strategy choice (SC), by dividing the biscuit into three parts. In the JPS, Ida measures with the scissors over the biscuit (N), and Adam affirms this (N). The solution for distribution is carried out with an additional cut, and the conclusion (C) is confirmed by the other students (N) after 3 min and 8 s.


Table 4 Students work with the physical artefacts in Case 6

Time	Person	Data	Reasoning	JPS
00:10–00:40	All students	[All three students engage in rearranging the biscuits, attempting to allocate them to various animals. Despite their efforts in shuffling the biscuits back and forth, they do not achieve an even distribution]	TS	I
00:40	Ida	[Letting the hungry and sad animal pretend to eat one biscuit. Identifies that the biscuits are to be distributed]	TS Care about the hungry one {0, 0, 1} remained 7	I → S

Table 5 Students work with the physical artefacts in Case 6

Time	Person	Data	Reasoning	JPS
02:09	Adam	[Cuts one of the biscuits from the hungry and sad animal in two. Then puts the two biscuit parts back among the others]	SC Cannot be divided equally by whole numbers	I
02:11	Kim	How many biscuits are there now? [Arranges the biscuits so every biscuit is to be seen]	Probing for SI	N
02:12	Adam	Nine [Points and counts the biscuit pieces regardless of size]	C Sequential counting	S
02:14	Ida and Nova	[Points and counting sequentially]	C	S
02:16	Adam	Nine	C	S
02:24	Ida and Adam	1, 2, 3, 1, 2, 3, 1, 2, 3 It is nine. [Points and counts the number of biscuit pieces]	C Equality (equal number but not equal size)	S

Table 6 Students work with physical artefacts in Case 2

Time	Person	Data	Reasoning	JPS	
02:53	Ida	[Ida positions the scissors, first aiming for the middle of the biscuit and then slightly off to the side]	SC → SI	N	
03:03	Adam	[Adam nods toward the biscuit that Ida is holding in her hand]	SI	N	
03:03	Ida	[Ida cuts, resulting in one piece slightly larger than half and another slightly smaller than half. Ida then cuts the piece that is slightly larger than half once more, roughly in the middle. She places the three biscuits pieces on the table]		C	G
03:08	Nova	Stop, no more cutting	C	N	
03:08	Adam	[Nods again]	C	N	

Challenges Related to the Nature of the Task

As described earlier, the tasks were designed to allow different solutions and provide stimuli to open up discussion about what fair share could be, by including extra information, such as one animal was very hungry. The different perspectives about what is “fair” were an important aspect when students started to elaborate on the problem, thereby initiating (I) the reasoning within the group. This led to, a JPS being created. This can be seen in the students’ work with Case 6, in which all three students were engaged in a collaborative problem-solving session that lasted for 2 min. During this time, they deliberated on different approaches to fairly

Table 7 Students' challenges related to the nature of the task (Case 6)

Time	Person	Data	Reasoning	JPS
02:24	Adam	A) Two here and two here. [Pointing in turn one, two of the animals (not the, hungry and sad one). However, his actions do not match his words, as he gives 1 and 2 biscuits respectively to these animals, for a total of 3 biscuits]	SC	S
		B) And this should have three pieces. [Points on the, hungry and sad animal] And then there should be a piece here and a piece here. [Points first to the, hungry and sad animal and then to the two others. After that he points to the two biscuits in front of him]	SI	S
02:45	Nova	No, I have another idea	TS → SC New TS, when fair is equal distribution	N
02:51	Kim	Do you think it should get more biscuits?	Probing for SI	N
02:54	Nova	As many	SC Equality	S → N

distributing the eight biscuits among the three animals. Despite their efforts, a unanimous resolution was not agreed upon, as two interpretations of fairness emerged. Adam contended that the hungry animal deserved a larger share, while Nova advocated for an equal distribution among the three animals. Table 7 started with a strategy choice suggested by Adam.

In Table 7, Adam both predicted (SC) and implemented (SI) the allocation of resources to the recipients, indicating that fairness involves providing more resources to those in need. Using the terminology from JPS, the situation is maintained as a JPS by Adam through strategic implementation and evaluation (S). However, at the end of the summary of the extract in Table 7, Nova indicated that she disagreed with Adam's suggestion of fairness in this situation. She suggested "as many" (02:54), thereby creating an alternative strategy-choice (SC). Nova's argument contributes further to the JPS, indicating that the ongoing process involves a negotiation (N) of different viewpoints regarding fairness.

In the students' work on Case 2, there were also indications that the challenges within the task created conditions for the students to create a JPS in their reasoning. At 00:27, the students suggested discarding the fourth and final biscuit. They argued differently that it would be fair if each of the three animals receives one biscuit. The fourth biscuit complicated the situation. For example, see (01:25) in Table 8.

The challenge of being unable to distribute an equal amount of biscuits to the animals, without splitting one, required the students to use fractions in the solution. This enabled the students to create a JPS in their reasoning. The students implemented (SI) the strategy choice (SC) of cutting, and in the JPS, they demonstrated that the fourth biscuit is the issue that needs to be solved (S).

Table 8 Students' challenges related to the nature of the task (Case 2)

Time	Person	Data	Reasoning	JPS
00:27–01:25	All students	[For almost a minute, the students propose various solutions to get rid of the fourth biscuit. Kim, the teacher, questions whether it is a good idea to dispose of the biscuit by throwing it away]	SC → SI	N
01:25	Nova	[Nova holds the fourth biscuit in her hand and looks mischievously at Kim]	SC → SI	Implicit asking for S
01:31	Ida	Let's cut it then. [Sighs]	SI	S

Table 9 Conceptions related to problem solving in Case 6

Time	Person	Data	Reasoning	JPS
00:10–00:47	All students	[The students initially moved the biscuits around for a while]	TS	N
00:47	Nova	Wait, wait we all have to...think [All three students move all eight biscuits around in a mode of action, as they think]	TS → SC	N

Table 10 Conceptions related to mathematics in Case 2

Time	Person	Data	Reasoning	JPS
03:55	Kim	How did you know that?	Questioning C	N
03:59	Adam	No, I’m just thinking it through. We’re not just counting together	C	N

Table 11 Conceptions related to care (students’ preferences) in Case 5

Time	Person	Data	Reasoning	JPS
01:25	Ida and Nova	[They mimic chewing sounds and pretend the animal is eating]	SI → C	S
01:25	Ida	The really hungry one must eat everything	C	G
01:30	Kim	Can it eat all the biscuits? Do the others think that’s okay?	Questioning SI	N
01:36	Adam	I’ve never seen people do that, making [chewing sounds]	Questioning SI	N
01:42	All students	[Handles the biscuits but ultimately decides the really hungry one should have all of them]	SI → C	G
02:06	Ida	When you give them, you must make sure they actually eat. [Place the biscuit under the animal as if the animal has eaten the biscuit]	C	N → G

Students’ Concepts

The third theme was that the JPS was built on students’ concepts of what mathematics entails, which was that mathematics involves thinking (Table 9).

In Table 9, we see that Nova raises that “thinking” is central to the collective reasoning, and that all of the participants need to think. It is in the transition from task situation (TS) to choosing an appropriate strategy where she puts forward the need to think, and that they all have to do it as a group. When applying JPS, the situation falls under the category negotiation (N). The students repeatedly signalled that it was important to think when working with mathematics, especially when working together, see Table 10.

In the short interaction illustrated in Table 10, the students are questioned about their conclusion (C) by the teacher, Kim. One of the students, Adam, answers that one needs to think it through. It is not just him who has to do it since “We’re not just counting together”, hence not simply considering arithmetic procedures. This situation is categorised as negotiation (N).

When the students are asked to distribute 12 biscuits among three animals, with the additional condition that one of the three animals was really hungry (Case 5), their understanding of mathematics becomes evident. The activity revealed how conceptions of vulnerability and the need for extra care was incorporated into the mathematical reasoning, becoming an integral part of JPS. For example, students Ida and Nova implicitly argued that the distribution process must also account for the recipient's acceptance of their allocated resources, see Table 11.

As illustrated in Table 11, Ida implicitly demonstrated her understanding of the importance of ensuring that the distribution aligned with the agreed decision, thereby displaying conceptions of preferences. The students develop the JPS by implementing a strategy (S) at time (01:25), which is integrated (SI) into a negotiation (N). This implementation is questioned, but ultimately confirmed as an implicit argument, forming a conclusion (C) within the solution (G).

The Teacher's Contributions

The fourth theme differs from the first three in that it focuses on how Kim, the teacher, contributed to the students' problem-solving process. In the earlier summaries of excerpts, there were several situations where the teacher contributed to or facilitated the reasoning process. For instance, the teacher's contributions included arguments that maintained the JPS while observing the children's collaborative problem-solving. As an example, in Table 5, at 02:51, the teacher facilitated the students' reasoning, by asking questions like "Do you think the same?", "What about this one [pointing to some of the animals]?" or "What do you think?". In Table 12, another example of the teacher's contribution is summarised, involving negotiations with the students about how the animals who receive a smaller share of biscuits might feel. This kind of negotiation was something that occurred frequently in the video-recordings:

Table 12 The teacher's contributions in Case 6

Time	Person	Data	Reasoning	JPS
04:15	Kim	Do these animals think it's okay that they didn't get as many pieces? [Points to the two other animals (not the hungry and sad one)]	Questioning C	G → N
04:20	Adam	That Hungry. [Emphasises that the sad and hungry animal should be given more]	C	N
04:38	Kim	Let's go back to these animals. Do they think it's fair that they have to share five pieces of biscuits while this one gets five pieces all to itself?	Maintain the TS in the conversation	N
04:43	Nova	They equal. [Makes the same gesture again, pointing to the two other animals]	C	N → G

In Table 12, the teacher contributed twice. The first example is at 4 min and 15 s, when Kim queried the students' suggested conclusion (C), which brought the solution (G) back to observing and negotiating (N). As such, Kim challenged the students' reasoning to stimulate arguments that could help the collaborative process. Twenty-three seconds later, Kim challenged the students to reflect on their suggested conclusion. However, the students decided to stick with their common solution, based on a shared sense of fairness. However, our data also included examples where students changed their solution after the teacher made a contribution. Looking at all instances, we see that Kim's contributions almost always involve providing input in the form of negotiation conceptions (N) or redirecting proposed solutions back for further negotiations.

Discussion

In this study, we have explored how students created collaborative problem-solving through their collective mathematical reasoning by combining the framework for collective mathematical reasoning (Sumpter, 2016) with the JPS framework for collaboration (e.g., Roschelle & Teasley, 1995; Szabo et al., 2024). By doing so, the results show collective mathematical reasoning in greater detail, using key concepts from the JPS framework, such as initiating, implementing, and negotiating conceptions. What we can add is *when* in the reasoning collaboration takes place. The results signal that the different situations of JPS are often related to strategy choice (SC) and the implementation of strategy (SI), or the conclusion (C). Only a few aspects of JPS were noticed concerning the first step of reasoning, the task situation (TS). This differs from previous research on collective mathematical reasoning, where the first step of reasoning and the associated arguments have been vital to understanding the task (e.g., Eriksson & Sumpter, 2021). Although previous studies signal that collaboration can vary, from superficial interactions to demanding interactions (Krummheuer, 2007), we did not expect the difference in the importance of various steps in reasoning when applying the JPS theory. The focus on strategy choice and implementation in the present study might be connected to students not having so much experience in problem solving (e.g., Widodo et al., 2019) compared to our earlier work (e.g., Eriksson & Sumpter, 2021). At the same time, there are instances when the students paid attention to the others (e.g., Sjöblom & Meaney, 2021), and providing space for a collective solution goal (e.g., Webb, 2009), meaning that there are several indications of productive group work. Given that productive collaborations are connected to students' ability to engage in mathematical argumentation (Stein et al., 2008), there are implications for teaching: what can teachers do to help students in their collective reasoning and in collaborative work?

The results from thematic analysis can provide some insight. One of the themes was about the teacher's contributions to collaborative problem-solving. Previous studies have shown how quickly teachers can take over the reasoning and limit

students' space to express arguments (e.g., Sumpter & Hedefalk, 2018). Here, the results signal that the teacher contributed to the process and the JPS framework helped to illuminate the teacher's contributions and facilitations in the reasoning process. As such, the teacher provided space for both collective reasoning and collaboration (e.g., Webb, 2009). Another result that could function as a part-explanation was about the cases. Previous studies have shown the nature of the task will affect the problem solving and the mathematical reasoning (e.g., Granberg & Olsson, 2015; Hestner & Sumpter, 2018). Our results support such a conclusion, and through the micro-analyses, we can also illustrate how the nature of the task contributes to a JPS. Here, given that what is considered 'fair' is not univocal, negotiation about fairness became important to move forward, and since the students had to negotiate on how to create fairness, it meant that both the choice of strategy and implementation of the different strategies depended on the different views of fairness. This is different compared to the mathematical progression of fairness that Watson (1997) describes where the goal is division. As such, the tasks required a successful collaboration, which has been identified as connected to students' ability to engage in mathematical argumentation (e.g., Sjöblom & Meaney, 2021). In addition, previous studies have noted that grappling with challenges inherent in mathematical tasks fosters "doing mathematics" (Stein et al., 2008). One possible implication is that tasks of this kind might help students develop and access pertinent knowledge and experiences and adeptly employ them in sophisticated thinking and reasoning processes.

The results also illustrated how students' understanding of mathematics shaped the construction of a JPS by the student group. This was sometimes due to the nature of the task, which required to some degree a shared understanding of 'fair', but also other conceptions. There is plenty of research from the past four decades on how different beliefs can affect reasoning in mathematical problem solving (e.g., Hestner & Sumpter, 2018), but not so much attention has been on the mathematical collaborative process. The results show that the students negotiated (N) how to interpret the situation, including what could be considered 'fair', which included which strategy choice (SC) to opt for. As stated earlier, there are indications that JPS and collective mathematical reasoning are closely connected. When adding conceptions (e.g., Philipp, 2007), it appears that how students interpret what mathematics is and how it should be done, forms their reasoning as well as their JPS. In particular, their conceptions of what 'group work' is stands out. Group work in mathematics encompass how to work together, how to share ideas, how to solve problems, and how to achieve a common goal (e.g., Lahann & Lambdin, 2020). Just as Seidouvy and Schindler (2019) concluded, a collaborative problem-solving space depends both on the context and individuals. What we can add is that students' conceptions of such collaborative work appear to be vital for the success, especially with respect to negotiation. The implication is that teachers need to be aware of this affective component when using collaborative work as part of mathematics teaching.

The last theme generated from the thematic analysis is about the role of the artefacts. Here, the artefacts—in the shape of paper biscuits—had a pivotal role

throughout the problem-solving process: in each step of the reasoning, these artefacts appeared to be integrated in the decisions that were made. The artefacts acted as a support and tools for the children. They also created conditions for various forms of reasoning, both verbal and non-verbal. Their non-verbal reasoning indicates that students are not entirely dependent on language to maintain JPS: the artefacts contributed to the context by facilitating students to produce actions and gestures, which supported students to present and negotiate new ideas within the JPS (e.g., Roschelle & Teasley, 1995). Together with gestures (e.g., Wathne & Carlsen, 2022), the artefacts helped the students to communicate their reasoning. Given that this study was not designed to study the role of artefacts in collective mathematical reasoning or JPS, we therefore suggest this as a topic for further studies.

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References

- Cao, Y. (2024). Research on collaborative problem solving teaching in a secondary school mathematics classroom. In Y. Cao (Ed.), *Students' collaborative problem solving in mathematics classrooms. Perspectives on rethinking and reforming education* (pp. 1–18). Springer. https://doi.org/10.1007/978-981-99-7386-6_1
- Clark, H. H., & Schaefer, E. F. (1989). Contributing to discourse. *Cognitive Science*, 13(2), 259–294. https://doi.org/10.1207/s15516709cog1302_7
- Duff, S. M., van Bergen, P., & Mulligan, J. (2021). Collaboration in spatial reasoning tasks with grade 1 and 2 students. *Australian Primary Mathematics Classroom*, 26(3), 4–8. <https://search.informit.org/doi/10.3316/informit.448527216563708>
- Eriksson, H., & Sumpter, L. (2021). Algebraic and fractional thinking in collective mathematical reasoning. *Educational Studies in Mathematics*, 108(3), 473–491. <https://doi.org/10.1007/s10649-021-10044-1>
- Eriksson, H., Hedefalk, M., Markkanen, P., & Sumpter, L. (2023). Fair sharing and division – Mathematical reasoning regarding integers and fractions in preschool and preschool class. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Herendiné-Kónya (Eds.), *Proceedings of the thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 2096–2103). Alfréd Rényi Institute of Mathematics and ERME. <https://du.diva-portal.org/smash/get/diva2:1845155/FULLTEXT01.pdf>
- Felmer, P. (2023). Collaborative problem-solving in mathematics. *Current Opinion in Behavioral Sciences*, 52, 101296. <https://doi.org/10.1016/j.cobeha.2023.101296>
- Granberg, C., & Olsson, J. (2015). ICT-supported problem solving and collaborative creative reasoning: Exploring linear functions using dynamic mathematics software. *The Journal of Mathematical Behavior*, 37, 48–62. <https://doi.org/10.1016/j.jmathb.2014.11.001>
- Gravemeijer, K., Stephan, M., Julie, C., Lin, F. L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education*, 15(S1), 105–123. <https://doi.org/10.1007/s10763-017-9814-6>
- Hansen, E. K. S. (2021). Students' agency, creative reasoning, and collaboration in mathematical problem solving. *Mathematics Education Research Journal*, 34(4), 813–834. <https://doi.org/10.1007/s13394-021-00365-y>

- Harding, S., Griffin, P., Awwal, N., Alom, B. M. M., & Scoular, C. (2017). Measuring collaborative problem solving using mathematics-based tasks. *AERA Open*, 3(3). <https://doi.org/10.1177/2332858417728046>
- Hestner, Å., & Sumpter, L. (2018). Beliefs and values in upper secondary school students' mathematical reasoning. In B. Rott, G. Törner, J. Peters-Dasdemir, & A. Möller Safrudiannur (Eds.), *Views and beliefs in mathematics education: The role of beliefs in the classroom* (pp. 79–87). Springer. https://doi.org/10.1007/978-3-030-01273-1_8
- Klang, N., Karlsson, N., Kilborn, W., Eriksson, P., & Karlberg, M. (2021). Mathematical problem-solving through cooperative learning—The importance of peer acceptance and friendships. *Frontiers in Education*, 6. <https://doi.org/10.3389/educ.2021.710296>
- Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom. *The Journal of Mathematical Behavior*, 26(1), 60–82. <https://doi.org/10.1016/j.jmathb.2007.02.001>
- Lahann, P., & Lambdin, D. V. (2020). Collaborative learning in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 94–95). https://doi.org/10.1007/978-3-030-15789-0_23
- Lamon, S. J. (2020). *Teaching fractions and ratios for understanding*. Routledge eBooks. <https://doi.org/10.4324/9781003008057>
- Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67(3), 255–276. <https://doi.org/10.1007/s10649-007-9104-2>
- Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM*, 49(6), 937–949. <https://doi.org/10.1007/s11858-017-0867-3>
- Markkanen, P., Eriksson, H., & Sumpter, L. (2024). Building a joint problem-solving space: How collaboration in collective mathematical reasoning can develop [Paper presentation]. In *The 6th POEM conference: A mathematics education perspective on early mathematics learning between the poles of instruction and construction, Bari, Italy* (Vol. 05, pp. 20–21) <http://www.diva-portal.org/smash/record.jsf?pid=diva2%3A1865268&dsid=-3369>
- Mergenthaler, E., & Stinson, C. (1992). Psychotherapy transcription standards. *Psychotherapy Research*, 2(2), 125–142. <https://doi.org/10.1080/10503309212331332904>
- Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic analysis. *International Journal of Qualitative Methods*, 16(1). <https://doi.org/10.1177/1609406917733847>
- OECD. (2017). *PISA 2015 Assessment and analytical framework: Science, reading, mathematics, financial literacy and collaborative problem solving*. PISA, OECD Publishing. <https://doi.org/10.1787/9789264281820-en>
- Philipp, R. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Information Age Publishing.
- Roschelle, J., & Teasley, S. D. (1995). The construction of shared knowledge in collaborative problem solving. In C. O'Malley (Ed.), *Computer supported collaborative learning* (pp. 69–97). https://doi.org/10.1007/978-3-642-85098-1_5
- Seidouy, A., & Schindler, M. (2019). An inferentialist account of students' collaboration in mathematics education. *Mathematics Education Research Journal*, 32(3), 411–431. <https://doi.org/10.1007/s13394-019-00267-0>
- Sidenvall, J. (2019). Literature review of mathematics teaching design for problem solving and reasoning. *Nordic Studies in Mathematics Education*, 24(1), 51–74. <https://doi.org/10.7146/nomad.v24i1.148997>
- Sjöblom, M., & Meaney, T. (2021). “I am part of the group, the others listen to me”: Theorising productive listening in mathematical group work. *Educational Studies in Mathematics*, 107(3), 565–581. <https://doi.org/10.1007/s10649-021-10051-2>
- Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10(4), 313–340. <https://doi.org/10.1080/10986060802229675>
- Sumpter, L. (2016). Two frameworks for mathematical reasoning at preschool level. In T. Meaney, O. Helenius, M. L. Johansson, T. Lange, & A. Wernberg (Eds.), *Mathematics education in the*

- early years: Results from the POEM2 conference, 2014 (pp. 157–169). Springer. https://doi.org/10.1007/978-3-319-23935-4_9
- Sumpter, L., & Hedefalk, M. (2018). Teachers' roles in preschool children's collective mathematical reasoning. *European Journal of STEM Education*, 3(3). <https://doi.org/10.20897/ejsteme/3876>
- Sumpter, L., & Hedefalk, M. (2023). När dela lika är olika [When fair share is unequal]. *Nordisk Barnehageforskning*, 20(2), 110–129. <https://doi.org/10.23865/nbf.v20.256>
- Szabo, A., Tillnert, A., & Mattsson, J. (2024). Displaying gifted students' mathematical reasoning during problem solving: Challenges and possibilities. *The Mathematics Enthusiast*, 21(1–2), 179–202. <https://doi.org/10.54870/1551-3440.1623>
- Vonk, J., Tomeny, T. S., Jett, S. E., Mercer, S. H., & Cwikla, J. (2024). A cross-sequential study of theory of mind, IQ, and fair sharing framed socially and non-socially in young children. *Cognitive Development*, 71, 101460. <https://doi.org/10.1016/j.cogdev.2024.101460>
- Wathne, U., & Carlsen, M. (2022). Third grade students' multimodal mathematical reasoning when collaboratively solving combinatorial problems in small groups. *Mathematical Thinking and Learning*, 1–20. <https://doi.org/10.1080/10986065.2022.2099611>
- Watson, J. (1997). Children's construction of 'fair' representations of one-third. *Australian Journal of Early Childhood*, 22(2), 34–38. <https://doi.org/10.1177/183693919702200208>
- Webb, N. M. (2009). The teacher's role in promoting collaborative dialogue in the classroom. *British Journal of Educational Psychology*, 79(1), 1–28. <https://doi.org/10.1348/000709908x380772>
- Widodo, S. A., Istiqomah, N., Leonard, N., Nayazik, A., & Prahmana, R. C. I. (2019). Formal student thinking in mathematical problem-solving. *Journal of Physics Conference Series*, 1188, 012087. <https://doi.org/10.1088/1742-6596/1188/1/012087>

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Diagrammatic Activity and Communication About It in a Multilingual-Responsive Mathematical Learning Environment in Kindergarten



Barbara Ott, Priska Sprenger, and Ángela Uribe

Introduction

Mathematical learning in school is always embedded in school socialization. This can be seen as a constitutive element for learning (Krummheuer & Brandt, 2001). Dörfler (2006) specifies this by explaining that mathematics learning can be understood as progressive participation in the social practice of diagrammatic activities. Early diagrammatic activities are, for example, exploring and manipulating an arrangement of a specific quantity of eggs in an egg carton, such as dividing the set of eggs into subsets (Benz et al., 2024). In addition to these diagrammatic activities, communication about them is also essential for mathematical learning (Ott & Wille, 2022; Wille, 2020). In alignment with these understandings, the curriculum for kindergarten¹ of the canton of St.Gallen, Switzerland, where this study took place, states that in mathematics lessons, “the focus is on playful exploration...discovering and representing mathematical relationships...and communicating about them” (Bildungsdepartement Kanton St.Gallen, 2020). In Switzerland, such communication is characterized by great language diversity. On the one hand, the

¹In Switzerland, children start learning mathematics at school in kindergarten. Kindergarten is part of compulsory education and is generally attended by children from the age of four for 2 years (Konferenz der kantonalen Erziehungsdirektorinnen und -direktoren, 2020).

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children learn and use Swiss High German as the language of instruction, while they speak Swiss German in their other surroundings.² On the other hand, immigration in recent decades has contributed to Switzerland increasingly becoming a multilingual country over and above the existing multilingualism. Thirty-three percent of children under 15 regularly speak two languages at home, and around 10% are even in contact with three or more different languages (Bundesamt für Statistik, 2021). This means that many children in Switzerland experience “lifeworld multilingualism” (Gogolin, 2010, p. 531) and there is a linguistic “super-diversity” (Vertovec, 2007, p. 1024) in kindergarten and school. Several languages are represented, which are only partially or not shared at all by the learners, between themselves or with the teachers (Uribe, 2024). Current research on mathematics teaching in school classes provides evidence that the different languages of learners represent a learning potential (Barwell, 2018; Redder et al., 2022; Uribe, 2024). But this diversity remains rarely used productively in mathematics learning in kindergarten. More research is needed about how learning environments can be designed for kindergarten that enable children to activate their multilingualism in a social diagrammatic practice and how the children become diagrammatically active in different language settings. This article presents such a learning environment. Two scenes from different language settings are used to reconstruct how a learner becomes diagrammatically active and communicates about it.

Diagrammatic Activity and Communication About It

In this article, diagrams are understood in the sense of Peirce, following Dörfler (2006). In Peirce’s theory, diagrams are signs with a relational character and encompass any form of inscription that is integrated into “a (conventional) system of rules of production, use and transformation” (Dörfler, 2006, p. 202; translated by the authors). In this sense, diagrams are, for example, geometric figures as well as terms like $5 + 3$. However, no diagram is a diagram per se but must first be interpreted as such. To do this, the corresponding rules of the sign system must be known (Hoffmann, 2005). The sign system also determines how the diagrams can be operated and experimented with. Such activities can be called diagrammatic activities (Dörfler, 2006; Ott & Wille, 2022). As such, Dörfler (2006) mentions the manipulation of diagrams, experimenting with diagrams and exploring their properties, investigation of relationships between different diagrams as well as the invention and design of diagrams. According to Huth (2023), not only actions but also gestures can have a diagrammatic character.

In the learning environment presented in this article, plastic eggs and egg cartons that can fit up to ten eggs are used. Even if these were not specifically designed for

²Swiss High German is very similar to High German in Germany. There are some differences regarding vocabulary and grammar. Swiss German is a collective term for the dialects spoken by people from all social classes in German-speaking Switzerland.

mathematics lessons, they can be used in such a way that they are subject to a “pre-conceived system of rules” (Vogel & Huth, 2020, p. 221; translated by the authors) and thus function as a diagram in a sign system within diagrammatic activities. The arrangement of the egg carton, in two rows of five places, gives it a structure that can be used to determine the cardinality with the help of a structural use (Sprenger & Benz, 2020). An example of a structural use would be when a child says “there are five at the top and two at the bottom, that’s seven” or “there are three at the top and two at the bottom and I know that three and two are five”. In each case, two subsets are perceived and used to determine the cardinality.

Another essential part of the “social practice” of diagrammatic activities is the communication “about and with diagrams” (Dörfler, 2006, p. 215, translated by the authors) as well as the communication about diagrammatic activities. This communication includes both verbal and gestural utterances, too (Huth, 2023). According to Wille (2020), communication about diagrams and diagrammatic activities enables the use of designations for diagrams that belong to different sign systems but can be related to each other. In this way, links can be established between diagrams of different sign systems and a deeper mathematical understanding becomes possible. Communication about diagrams and diagrammatic activities also makes it possible to interpret the observations made in the diagrammatic activity. According to Tiedemann (2021), such communication can conventionalize the actions, focus on characteristic points for the developed procedure and condense the description itself. It is not yet clear how diagrammatic activities and communication about them interact in a mathematical learning environment that is multilingual-responsive.

Multilingual-Responsive Mathematical Learning

Language is a key to building conceptual understanding. Beyond its communicative function, it plays an important epistemic role in the process of constructing meaning (Barwell, 2018; Planas, 2018). However, when multilingual learners are assumed to be monolingual, opportunities to draw on their rich linguistic repertoires for mathematical learning are lost. Research highlights that multilingual learners use a blend of linguistic and non-linguistic resources, including gestures, to construct and convey mathematical ideas, emphasizing the significance of multimodal communication in their mathematical discourse (Krause & Farsani, 2022; Ng, 2016).

A multilingual-responsive approach to teaching also offers further chances for learning. For instance, Moschkovich (2015) found that multilingual learners engage more deeply in classroom discussions, while Norén and Andersson (2016) emphasize that such an approach acknowledges and affirms learners’ agency and identity. Furthermore, it allows multilingual learners to connect classroom instruction to their everyday experiences (Domínguez, 2011), enriching their understanding of mathematical concepts.

Classroom settings involving multilingualism can be broadly categorized into two types (Uribe, 2024): settings with shared multilingualism, where learners and

teachers speak or understand the languages represented, and settings with non-shared multilingualism, where some or all learners and teachers do not speak or understand the same languages. Depending on the setting, different learning activities can be designed to support multilingual-responsive mathematics learning. These activities may include encouraging the use of multiple languages, offering multiple languages or fostering reflection on language (Uribe, 2024). While such activities have already been developed for mathematics learning at school, it is still unclear for kindergartens how appropriate multilingual-responsive learning environments can be designed and how learners in kindergarten use their multilingualism in such a learning environment.

Research Interest

The data presented here are part of the project “MATHEsprechen in kindergarten: Using language repertoires in mathematics productively right from the start”, which investigates the overarching question of the extent to which the use of multilingual resources is stimulated in kindergarten children through diagrammatic activities with materials in a multilingual-responsive learning environment. Accordingly, in the project, a learning environment was developed, that allows for and encourages the use of multiple languages by combining a setting in the language of instruction (kindergarten) and a setting in the home language (family) (Ott et al., 2024). This article investigates the following sub-question: How does a multilingual child become diagrammatically active and communicate about it in a setting that takes place in the language of instruction and in a setting that takes place in the home language?

Method

Using Design Research methodology (Gravemeijer & Cobb, 2006), we developed the initial design and investigated the research question, focusing on iterative refinement through two cycles of design, implementation, analysis, and redesign, aiming to both understand and improve the learning environment and theories. Individual selected scenes were analyzed according to Krummheuer’s interaction analysis (Krummheuer & Naujok, 1999). For this purpose, the non-German-language passages were translated. For the Spanish sections presented in this article, one of the researchers, who is multilingual with regard to these languages, took over the translation. In the interaction analysis, she was able to include linguistic differences and particularities. The following steps were carried out in the interaction analysis (Krummheuer & Naujok, 1999): Structure of the interaction unit, general description, detailed analysis of the individual

utterances—(re)construction of alternative interpretations, turn-by-turn analysis, summary analysis.

Design of the Learning Environment

In this article, we provide an insight into the second cycle of the design experiments, in which the learning arrangement was further developed based on the experiences from the first cycle. In the first cycle, the children's parents were involved in order to activate the home language. For example, it became apparent that too much openness in the activities led to parents strongly encouraging children to count and feeling pressure to think of a game situation spontaneously, which was sometimes experienced as overwhelming for them (Ott et al., 2024). These aspects have been taken into account and improved in the second cycle.

The learning environment was developed to create opportunities for multilingual children to activate their multilingual repertoires by learning the topic of determining cardinality with the help of a structural use. It is based on the following design principles:

- (1) Activating the home language by involving parents;
- (2) Encouraging mathematical activity and communication about it (Ott & Wille, 2022) within a social diagrammatic practice (Dörfler, 2006);
- (3) Connection of language of instruction and the home language through rich learning tasks with materials and natural differentiation (Krauthausen & Scherer, 2014).

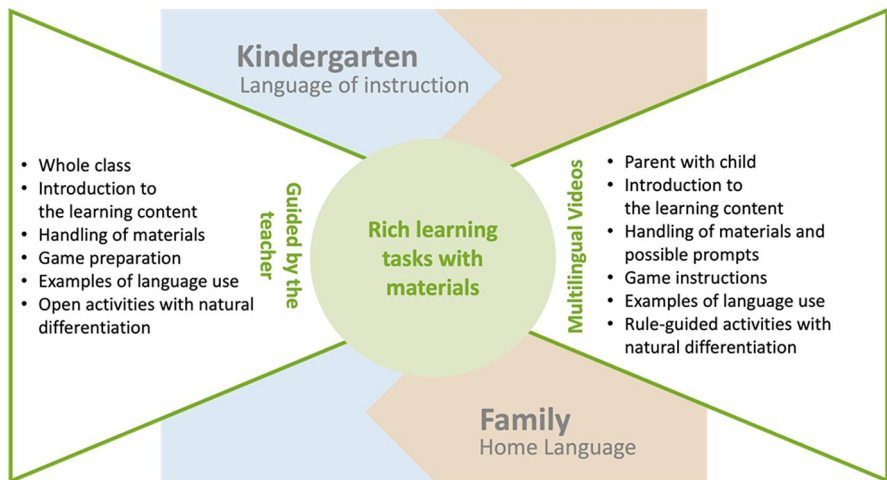


Fig. 1 Design of the learning environment

The learning environment consists of elements in kindergarten and in the family (see Fig. 1). Egg cartons and plastic eggs are the materials used and the objects of the mathematical activity. They are an essential element of the learning tasks at school as well as in the family.

The parts in kindergarten take place in the language of instruction, the use of other languages is possible, but not encouraged. The whole class participates. The children are encouraged to engage in diagrammatic activities and experience how to use the language for communication about them. An example of a learning task is: Each child has a ten-egg carton and six eggs. One child rolls a die. Each child then arranges the corresponding quantity of eggs in their carton so that one can immediately see how many eggs there are. The children compare whether they have laid the eggs similarly or differently and describe how they can immediately see the quantity of eggs. Thus, the learning tasks for the family part are prepared, too.

The activities in family are carried out by a child and one parent. Here, the learning tasks are designed as games. Parents and children are guided by multilingual videos to encourage the use of several languages. The families should use their multilingualism as they normally do. In these parent-child videos, two hand puppets, Jenny and Chrischi, introduce the learning tasks by playing the games themselves and acting as role models (see Fig. 2). A narrator draws attention to special aspects.

The parent-child videos are available in three versions: entirely in High German, the language of instruction, entirely in the respective family language, or mixed, with Jenny speaking High German and Chrischi speaking the family language. Parents and children could choose which video they wanted to watch. Additionally, in a parent video, the parents are introduced to the learning content of the learning tasks and

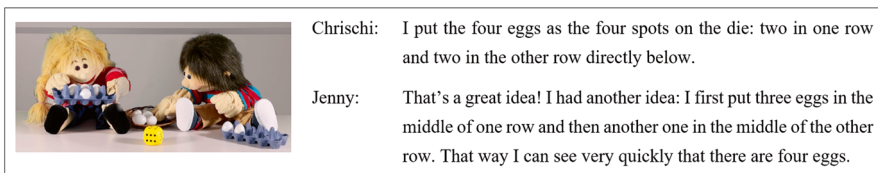


Fig. 2 Translated excerpt from the parent-child video for the learning task “Arranging eggs”

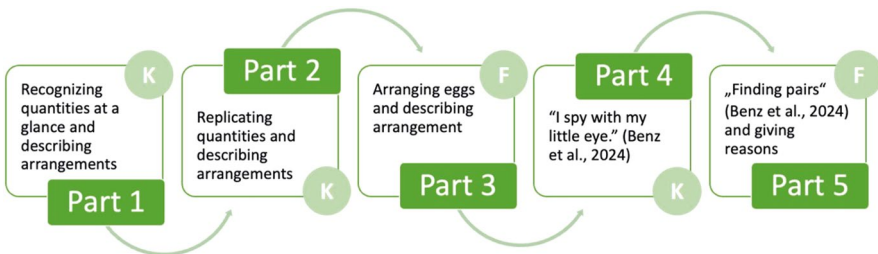


Fig. 3 The five parts of the learning environment (K: Kindergarten, F: Family)

they are given ideas on how to support their children's mathematics learning through these games. This video is available in High German or the family language.

The learning environment consists of five parts in total (see Fig. 3). Three parts were carried out with the kindergarten group, two parts in the context of the family. All parts took place on the premises of the kindergarten.

Conducting the Design Experiment

The learning environment was tested in a design experiment in September 2023 in two kindergarten groups in Switzerland with children in the last year of kindergarten. The first group consisted of ten children (average age: 5 years, 7 months), the second of six children (average age: 5 years, 3 months). Two Albanian-speaking families, one Spanish-speaking family and one Serbian-speaking family took part. The kindergarten part was carried out by a researcher. All parts were carried out in the same week: The kindergarten parts took place on Monday (part 1), Wednesday (part 2) and Thursday mornings (part 4), the family parts on Wednesday (part 3) and Thursday afternoons (part 5). For the family parts, the parents and children came to the kindergarten in the afternoon. All activities in the kindergarten and in the families were videotaped.

Results and Interpretations











In this article, we explore the research questions by analyzing Lami's case. He is 5 years and 3 months old. His home language is Spanish, and this is the only language spoken at home. Lami speaks and understands Swiss German and High Swiss German to some degree but is not fluent in them. He mixes them while speaking. His father, who conducts the family part with him, speaks or understands very little German. In the following, the summary analysis of the interaction analysis of two selected scenes is presented. To describe the position of the eggs in the carton in the transcript, the places in the carton were numbered as shown in Fig. 4.

In order to make it possible to follow the use of the different languages, the translation in the transcript excerpts (see Tables 1 and 2) underlines the passages that were originally spoken in (Swiss) High German, but not in Spanish. Passages spoken in Swiss German are also italicized. Actions are indicated in brackets.

Fig. 4 Numbering of the places in the egg carton in the transcript

E1	E2	E3	E4	E5
E6	E7	E8	E9	E10

Table 1 Transcript of the scene from part 2 of the learning environment with explaining screenshots (Swiss High German: underlined; Swiss German: underlined and in italics; Spanish: without highlighting)

Teacher:	Let's find another description. Look at it again. Lami?	
Lami:	<u>Yes</u> (moves to the egg carton) <u>this this</u> (takes the egg from E7 out of the carton and holds it in the left hand)	
	<u>give out</u> (presses with the fingers of the right hand onto E7 and then moves the hand over the egg onto E6) <u>done like this</u>	
	And <u>the other</u> (puts the egg back and presses with the fingers of the right hand on the eggs on E1 and E6)	
	<u>done like this</u> (points to Annika and then puts his right hand over the eggs on E6 and E7, index and middle finger of his left hand on the egg on E7) <u>this so many children made like this</u> (takes away the egg on E7)	
	<u>like he not</u> (points with his left hand to the empty place E7)	
	<u>so is not the same</u> (takes an egg from E3 and places it on E7)	
	<u>or so</u> or (presses the egg from his hand briefly into E3 and then places it in E8) <u>so is not also the same</u>	
	but from (turns the whole carton by 180° and presses briefly on the egg on E2 with the right hand) <u>other see it the same way</u>	
	(takes the egg from E8 in the previous view with the right hand and places it in E3 in the previous view)	
	or (turns the box 120°) <u>like this</u>	

Kindergarten Part of the Learning Environment

In the example, the scene (see Table 1) is from part 2 of the learning environment (see Fig. 3), at the end of the lesson. In the egg carton, five eggs are arranged as shown in Fig. 5. The children were asked by the teacher (researcher) to find as many different descriptions as possible for how one can immediately see that

Table 2 Transcript of the scene in the middle of the game with explaining screenshots



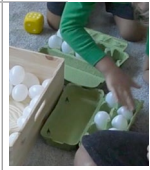
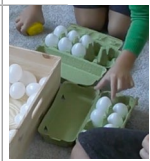
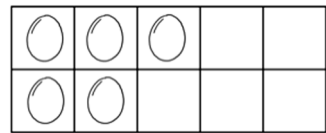
Lami:	And look here, look (puts the die on the ground and turns it so that the die face 6 is oriented as it might lie in the father’s carton) It’s like this (points to the spots on the die)	
	You placed them in (turns the die so that the die face 6 is oriented as it lies in his carton)	
	Touches the eggs in the father’s carton in the order: E5, E3, E1) one, two, three and three (touches E7–E9) Here are three (points simultaneously to the fields E5, E3, E1) and here (points simultaneously to the fields E7–E9) there are three and it’s six	
	But (takes the egg from E5 and places it in place E4) if you place them (takes the egg from E1 and places it in E2) like this, it’s the same (places the eggs back as the father had placed them)	

Fig. 5 Position of five eggs in the kindergarten scene



there are five eggs in the carton. They sit in a circle with the carton in the middle. Three children have already found three descriptions: “(Points to the top row.) There are three. (Points to the bottom row.) And there are two.”, “(Points to the bottom row.) There are two. (Points to the top row.) And there are three.” and “(Points to E1 and E6.) Here two. (Points to E2 and E7.) Here two. (Points to E3.) And here one.”

Summary Analysis

Unlike the other children, it seemed to be important for Lami to sit close to the carton, which indicates that he has planned to work with the eggs in the carton and will need more time than the other children to explain his ideas. Lami seems to have a clear idea in his explanation. It seems to be important to him to refer to what the other children have said. For example, he points to Annika or places his hand over two eggs (E1 and E6) in exactly the same way as another girl did previously. In

order to communicate his thoughts to the others and especially to the teacher, Lami combines verbal communication about the arrangement of the eggs with appropriate pointing gestures and diagrammatic activity, which is shown by taking out or placing the eggs. Lami's verbal utterances have Swiss High German parts, parts in Swiss German, and Spanish parts. The egg in position E7 plays a crucial role in his explanation. He seems to be concerned with why there are different descriptions for the arrangement of these five eggs in the egg carton. His explanation has two sections: In the first section, he explains what is "not the same". For example, the arrangement of the eggs is no longer the same when the egg on E7 is removed from this place. Likewise, it is no longer the same when the egg on E3 is moved from this place. It is possible that he has previously interpreted the other children's pointing gestures in such a way that they no longer refer to the entire quantity of the five eggs, but only to partial quantities or individual eggs, resulting in different descriptions. It is also possible that he is wondering why different descriptions are being sought for the same arrangement. In the next part of the utterance, he now explains that the arrangement of the eggs remains the same, even when viewed from different perspectives. He is presumably reacting to the previous statements, which are made in the circle opposite each other. To make his argument clear, he turns the carton and puts the eggs back in the way they were at the beginning.

Focus on Diagrammatic Activity and Communication About It

In this section of the scene, Lami is diagrammatically active in a variety of ways (Dörfler, 2006). He changes the arrangement of the eggs, adds an egg, takes it away or changes its position, and divides the set of eggs into subsets. He is not doing it here to explore the diagram, but to show and prove his argument. Presumably he wants to show that different arrangements of the eggs lead to different descriptions, but different perspectives on the same arrangement lead to the same description. He also uses gestures to manipulate the diagram (Huth, 2023), for example by highlighting individual subsets as well as to refer to statements made by other children. As an additional mathematical element, he introduces the spatial position of the carton and the perspective into the discussion. To do this, he again becomes diagrammatically active with an action, the turning of the carton, and uses this for his argumentation. Lami also accompanies his argumentation verbally and thus communicates about the diagrammatic activity ("done like this") and the resulting diagrams. He uses all the languages available to him. In Swiss German and Swiss High German, he explains the essential elements of his argument in simple words, which he repeats: What was done, who did something and whether it is the same or not. He makes the reference to the other children gesturally and verbally. It is possible that these references are particularly important to him for his argumentation. The diagrammatic activities are an essential element of his argumentation, without which it would not be comprehensible. He inserts conjunctions that are important for the logical structure of an argument in Spanish, "and" and "but from". Presumably he has not yet learned

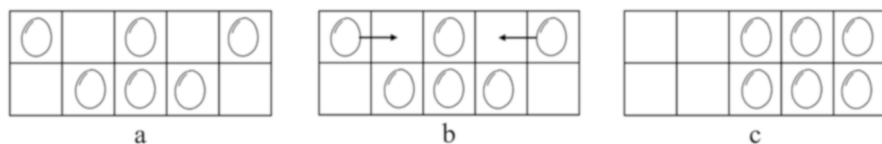


Fig. 6 Position of six eggs in the family scene: (a) carton of Lami's father, (b) Lami's reposition in his father's carton, and (c) Lami's carton

these conjunctions in the language of instruction or is so preoccupied with the diagrammatic argumentation that he uses the terms in his more familiar language.

Family Part of the Learning Environment

The scene in the example is from part 3 of the learning environment, where Lami and his father play the game "Arranging eggs" (see Fig. 3). In their egg cartons, they have arranged six eggs as shown in Fig. 6.

The scene is in the middle of the game (see Table 2) and Lami as well as his father speak Spanish throughout.

Summary Analysis

It can be stated, that for Lami, unlike his father, the previous negotiation about the arrangement of six eggs in his father's carton (see Fig. 6a) has not yet been satisfactorily concluded. It seems to be important for him to convince his father of his reasoning. The die still seem to play a crucial role for him and he seemed to focuses on this, so that his father could understand Lami's reasoning by looking at the die and the eggs in the carton. More emphatically than in a previous turn, he again asks his father to follow his reasoning by first asking him to look, "and look here, look". He then uses the die and the egg carton with the six eggs in such a way that the equality or inequality of the arrangement becomes clear by looking. His argumentation moves from the die to the egg carton, which the father should now link together. Lami first shows the father how the six dots are arranged on the die, which he has initially placed in such a way that the arrangement of the six dots on the die matches the orientation of the eggs placed in the father's egg carton, "it's like this". The statement shows that the arrangement on the die seems to be an incontrovertible fact for Lami. Then the die seem to have served its purpose with the pointing, so Lami can put it away. It is possible that he deliberately places the die next to his own egg carton so that the six on the die face is oriented in the same way as the six eggs in his egg carton. He now turns to his father's carton and explains how the eggs in this carton are arranged in contrast to what is shown on the die (see Fig. 6a). In doing so, he combines diagrammatic activity with verbal and gestural utterances. After initially tapping on the three eggs individually, the grouping of the eggs into three and

three then seems to be essential for him “here are three (points simultaneously to the fields E5, E3, E1 and here (points simultaneously to the fields E7–E9) there are three and it’s six”, as well as the total number of eggs in the carton. With the following “but”, however, he now separates the fact that there are six eggs from his argument that it does not match the arrangement of the six on the die. This time, however, he varies his argumentation as Lami becomes diagrammatically active by repositioning two eggs from his father’s carton so that the six eggs are arranged in the carton similarly to the arrangement of the six dots on the die (see Fig. 6b). He accompanies this action with an appropriate utterance, “if you place them like this, it’s the same”, picking up on his formulation from his previous arguments. By rearranging them, it can now be seen—and seen by looking—that the arrangement now matches the arrangement of the dots on the die. It seems to be irrelevant for Lami where the six eggs are placed in the carton, as it is now centered in his father’s box and right aligned in his own (see Fig. 6c). Overall, Lami brings together his argumentation that the arrangement of six eggs in the carton must match the arrangement of six dots on the die and his father’s argumentation that it is the quantity that matters. He makes clear that the quantity of eggs in his father’s carton is correct, but not the arrangement. However, this seems to be decisive for Lami in this situation.

Focus on Diagrammatic Activity and Communication About It

Lami is diagrammatically active in a variety of ways (Dörfler, 2006) in this section of the scene. He relates two sign systems to each other (Ott & Wille, 2022): the arrangement of dots on the die and the arrangement of eggs in the carton. He establishes the connection between the two sign systems by paying attention to the spatial relationships between the individual elements of the respective set in the arrangement. In addition, he changes the diagram in the egg carton according to the rules so that the new diagram has the same spatial relationships as the diagram of the six dots on the die. He also uses diagrammatically gestures (Huth, 2023) to divide the set of six eggs into subsets. He performs this decomposition in the natural numbers, too, by naming them verbally. In addition to his gestures on the egg carton, he verbally names the total number of eggs. He uses the egg carton with the eggs as the mathematical object on which he works and with which he argues and communicates about the diagrams and his diagrammatic activities. Therefore, he uses his home language, which is also his father’s language, in an elaborated way and presents a logical argument with steps that build on each other in a meaningful way, in which he adopts the father’s perspective on the eggs.

Discussion and Outlook

In this article, we use the example of Lami to address the question of how a multilingual child becomes diagrammatically active and communicates about it in a setting that takes place in the language of instruction and in a setting that takes place in the home language. The two scenes analyzed took place on the same day. They show how Lami, a multilingual child in the second year of kindergarten, becomes diagrammatically active and communicates about it in both settings. In both cases, Lami becomes diagrammatically active by acting and gesticulating (Vogel & Huth, 2020) and communicates about it and about the diagrams in the egg carton (Ott & Wille, 2022). The actions he performs are also similar in both scenes (rearranging the eggs and dividing the set into subsets). Reviewing the other scenes in the videos shows that this seems to be typical of Lami's interaction in mathematical settings. He also makes references in both scenes: in the kindergarten scene between the diagrams previously shown gesturally and the current one, and in the family scene between the six dots on the die and the diagram of six eggs in his father's carton. This aligns with the research findings on multilingual learners, which say, that multilingual learners use a blend of linguistic and non-linguistic resources to construct and convey mathematical ideas (Krause & Farsani, 2022; Ng, 2016). At the same time, this is also typical of social diagrammatic practice in mathematics lessons (Dörfler, 2006). Further research would be needed to see whether differences between monolingual and multilingual learners could be identified here and whether learners approach other learning tasks similarly or differently.

Nevertheless, differences in Lami's interaction can be observed in both settings. In the family scene, he intertwines the sign system established with the egg carton with that of natural numbers (Ott & Wille, 2022). He does not do this in the kindergarten scene, although this type of description had previously been established in the lesson and he did it there in other scenes, too. It seems not to have been important to him for his argumentation in the kindergarten or it could have overtaxed him linguistically. Overall, he is more diagrammatically active in the kindergarten scene through actions and gestures and less through verbal communication about them. In the family scene, he uses very specific actions and gestures and always combines these with verbal communication about them. It is striking how well the two go together. In summary, it can be said that acting diagrammatically with material is crucial for Lami's argumentation. In the kindergarten setting, that took place in the language of instruction (High German), his focus is on gestures and actions, partly accompanied in different languages; in the family setting, that took place in his family language (Spanish), he combines his actional and gestural reasoning with his home language. Here, the diagrammatic activity and the communication about it are of equal value. Thus, the connection between diagrammatic activity and communication about it and a possible focus on one of the two seems to depend on the setting, for example whether the child feels linguistically competent or not.

Overall, in both scenes, Lami combines diagrammatic activities and communication about them to explain mathematical relationships and to reason mathematically (Ott &

Wille, 2022). Even though both settings take place under conditions of non-shared multilingualism, Lami was able to activate and draw upon his multilingual resources and participates very actively in the interaction (Moschkovich, 2015). In the kindergarten setting (language of instruction), he explicitly activates all his linguistic resources. In the family setting (home language) with his father, he only explicitly activates his Spanish resources in this scene. This shows that the multilingual-responsive learning environment with the kindergarten and family parts was suitable for addressing Lami with all his multilingual resources. Further research should explore other design possibilities for multilingual-responsive learning environments for kindergarten.

Moving forward, further analysis of the project data should examine how the interplay between diagrammatic activity and the communication about it might vary in other multilingual settings, for example a setting of a shared multilingualism. In addition, the repertoires-in-use (Uribe et al., 2024) are examined, as will interplay between these and the diagrammatic activity and communication about it.

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References

- Barwell, R. (2018). From language as a resource to sources of meaning in multilingual mathematics classrooms. *The Journal of Mathematical Behavior*, 50, 155–168. <https://doi.org/10.1016/j.jmathb.2018.02.007>
- Benz, C., Sprenger, P., Reuter, F., Maier, A., & Zöllner, J. (2024). *Mathematik entdecken in KiTa und Grundschule. Spiel- und Lernsituationen aus der MachmitWerkstatt MiniMa [Discovering mathematics in kindergarten and elementary school. Play and learning situations from the MachmitWerkstatt MiniMa]*. Klett Kallmeyer.
- Bildungsdepartement Kanton St.Gallen. (2020, November). *Kanton St.Gallen Lehrplan Volksschule [Primary school curriculum of canton St.Gallen]*. Bildungsdepartement Kanton St.Gallen. <https://sg.lehrplan.ch>
- Bundesamt für Statistik. (2021, January 25). *Zunahme der Mehrsprachigkeit in der Schweiz: 68% verwenden regelmässig mehr als eine Sprache [Increase in multilingualism in Switzerland: 68% regularly use more than one language]*. <https://dam-api.bfs.admin.ch/hub/api/dam/assets/15384140/master>
- Domínguez, H. (2011). Using what matters to students in bilingual mathematics problems. *Educational Studies in Mathematics*, 76(3), 305–328. <https://doi.org/10.1007/s10649-010-9284-z>
- Dörfler, W. (2006). Diagramme und Mathematikunterricht [Diagrams and teaching mathematics]. *Journal für Mathematik-Didaktik*, 27(3–4), 200–219. <https://doi.org/10.1007/BF03339039>
- Gogolin, I. (2010). Stichwort: Mehrsprachigkeit [Keyword: multilingualism]. *Zeitschrift für Erziehungswissenschaft*, 13, 529–547. <https://doi.org/10.1007/s11618-010-0162-3>
- Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational design research* (pp. 17–51). Taylor and Francis Ltd.

- Hoffmann, M. H. G. (2005). *Erkenntnisentwicklung: Ein semiotisch-pragmatischer Ansatz. [Cognitive development: A semiotic-pragmatic approach.]* Klostermann.
- Huth, M. (2023). *Diagramme im Handumdrehen — der Gebrauch von Gesten beim Mathematiklernen [Diagrams in the blink of an eye - the use of gestures in learning math]*. Dissertation. Goethe-Universität Frankfurt am Main, Universitätsbibliothek Johann Christian Senckenberg. <https://doi.org/10.21248/gups.72106>
- Konferenz der kantonalen Erziehungsdirektorinnen und -direktoren. (2020, December). *Schule und Bildung in der Schweiz. [School and Education in Switzerland]*. Konferenz der kantonalen Erziehungsdirektorinnen und -direktoren. <https://www.edk.ch/de/bildungssystem/beschreibung>
- Krause, C. M., & Farsani, D. (2022). More than words: An integrated framework for exploring gestures' role in bilinguals' use of two languages for making mathematical meaning. *Canadian Journal of Science, Mathematics and Technology Education*, 22(4), 773–795. <https://doi.org/10.1007/s42330-022-00253-y>
- Krauthausen, G., & Scherer, P. (2014). *Natürliche Differenzierung im Mathematikunterricht [Natural differentiation in maths teaching]*. Klett Kallmeyer.
- Krummheuer, G., & Brandt, B. (2001). *Paraphrase und Traduktion. Partizipationstheoretische Elemente einer Interaktionstheorie des Mathematiklernens in der Grundschule. [Paraphrase and traduction. Participation-theoretical elements of an interaction theory of maths learning in primary school]*. Beltz Verlag.
- Krummheuer, G., & Naujok, N. (1999). *Grundlagen und Beispiele Interpretativer Unterrichtsforschung [Basics and examples of interpretative classroom research]*. Leske + Budrich. <https://doi.org/10.1007/978-3-322-95191-5>
- Moschkovich, J. (2015). Academic literacy in mathematics for English learners. *The Journal of Mathematical Behavior*, 40, 43–62. <https://doi.org/10.1016/j.jmathb.2015.01.005>
- Ng, O.-L. (2016). The interplay between language, gestures, dragging and diagrams in bilingual learners' mathematical communications. *Educational Studies in Mathematics*, 91(3), 307–326. <https://doi.org/10.1007/s10649-015-9652-9>
- Norén, E., & Andersson, A. (2016). Multilingual students' agency in mathematics classrooms. In A. Halai & P. Clarkson (Eds.), *Teaching and learning mathematics in multilingual classrooms: Issues for policy, practice and teacher education* (pp. 109–124). Sense. https://doi.org/10.1007/978-94-6300-229-5_8
- Ott, B., & Wille, A. M. (2022). Diagrammatic activity and communicating about it in individual learning support: Patterns and dealing with errors. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the twelfth congress of the European Society for Research in mathematics education* (pp. 4312–4319). Free University of Bozen-Bolzano and ERME.
- Ott, B., Sprenger, P., & Uribe, Á. (2024). Mehrsprachigkeitsaktivierung im Kindergarten – Design einer Lernumgebung [Multilingualism activation in kindergarten - design of a learning environment]. In P. Ebers et al. (Eds.), *Beiträge zum Mathematikunterricht 2024*. Mathematikdidaktik – Gestern. Heute. <https://doi.org/10.17877/DE290R-25091>
- Planas, N. (2018). Language as resource: A key notion for understanding the complexity of mathematics learning. *Educational Studies in Mathematics*, 98(3), 215–229. <https://doi.org/10.1007/s10649-018-9810-y>
- Redder, A., Krause, A., Prediger, S., Uribe, Á., & Wagner, J. (2022). Mehrsprachige Ressourcen im Unterricht nutzen – worin bestehen die “Ressource”? [Using multilingual resources in the classroom - what are the “resources”?]. *Die Deutsche Schule*, 114(3), 312–326. <https://doi.org/10.31244/dds.2022.03.09>
- Sprenger, P., & Benz, C. (2020). Children's perception of structures when determining cardinality of sets – results of an eye-tracking study with 5-year-old children. *ZDM Mathematics Education*, 52, 753–765. <https://doi.org/10.1007/s11858-020-01137-x>
- Tiedemann, K. (2021). Wer spricht, sieht mehr [Those who speak see more]. In K. Hein, C. Heil, S. Ruwisch, & S. Prediger (Eds.), *Beiträge zum Mathematikunterricht* (pp. S29–S36). WTM-Verlag.
- Uribe, Á. (2024). *Mehrsprachigkeit im sprachbildenden Mathematikunterricht: Gelingensbedingungen und didaktische Potenziale in sprachlich heterogenen Klassen*

[*Multilingualism in language-based Maths lessons: conditions for success and didactic potential in linguistically heterogeneous classes*]. Springer.

- Uribe, Á., Sprenger, P., & Ott, B. (2024). Exploring parent-child interactions by learning mathematics: Repertoires-in-Use within a kindergarten-family learning environment. In M. Ní Ríordáin & K. Erath (Eds.), *Proceedings of the sixteenth ERME topic conference on language and social interaction in heterogeneous mathematics classrooms* (pp. 100–107) <https://hal.science/hal-04833321>
- Vertovec, S. (2007). Super-diversity and its implications. *Ethnic and Racial Studies*, 30(6), 1024–1054. <https://doi.org/10.1080/01419870701599465>
- Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen Lernprozessen [Mode interfaces in mathematical learning processes]. In G. Kadunz (Ed.), *Zeichen und Sprache im Mathematikunterricht* (pp. 215–253). Springer Spektrum. <https://doi.org/10.1007/978-3-662-61194-4>
- Wille, A. M. (2020). Activity with signs and speaking about it: Exploring students' mathematical lines of thought regarding the derivative. *International Journal of Science and Mathematics Education*, 18, 1587–1611. <https://doi-org.ezproxy.phsg.ch/10.1007/s10763-019-10024-1>

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“I Could Cut Them in Half”: An Interview Study on First Graders Solving Different Types of Division with Remainder Problems



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Introduction

Situations that require “fair sharing” arise quite naturally in children’s lives from an early age. However, as only certain sets of numbers can be divided without a remainder, children are frequently challenged when they have to deal with a remainder in everyday life. Zorrilla et al. (2021) have identified three distinct categories of remainders, each exhibiting a unique impact on the outcome of the division task. This is due to the fact that decisions regarding the management of remainders must always be made in consideration of the particular circumstances of the situation. For example, if three donuts are shared fairly between two children, the remaining donut can be cut in half, so that each child gets one and a half donuts. However, if little toy cars are shared instead of donuts, cutting them up is not a feasible option, so a different interpretation is required. Despite their real-life relevance, such problems are rarely considered at school, especially not before division is taught. With respect to secondary students of various grades it has already been shown that students often have difficulties solving division with remainder (DWR) problems and that certain types of remainders present considerable challenges for students (Rodríguez et al., 2009; Silver et al., 1993). To date little is known about how successfully children solve real-world related DWR problems when they start school and how they justify their handling of remainders with respect to the specific contexts. The extent to which certain types of remainders might present specific challenges for students at primary level is, at present, largely unknown. In this interview study, we seek to contribute to this field of research. Sixteen German first graders were invited to work on four DWR problems and to share their thinking on their solution process. The DWR problems used in the study differ in terms of the interpretation of division (quotitive or partitive) and the type of remainder.

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Theoretical Background

For this study a classification of real-world related DWR tasks is necessary, regarding two different aspects: On one hand, two interpretations of division can be distinguished, and on the other hand, three different types of remainders can be differentiated. When looking for real-world related representations of a division problem, one numerical task (e.g. $12 \div 3$) can present two completely differently structured situations depending on the nature of the problem context. It can therefore be illustrated in two interpretations of division: partitive and quotitive division. These are regarded as prototypical mental models of the mathematical division concept, which are indispensable for a comprehensive understanding of division. Table 1 illustrates the partitive and the quotitive interpretation of the task $12 \div 3$.

Both interpretations of division have in common that the total number of items that need dividing is the same (12). However, they vary in terms of what is wanted. For partitive division, the number of groups (three equal groups) is known and the number of items per group is wanted (How many children in each group?). For quotitive division in contrast, the number of items per group is known (groups of three) while the number of groups is wanted (How many groups of three?). Depending on the real-world problem context, the two interpretations vary according to the mathematical structure as well as to the concrete actions.

Division with remainder is a fundamental theorem of number theory. Unless very specific sets of numbers are selected (e.g. by a teacher), division problems frequently leave a remainder. In an entire mathematical (symbolic) context, the solution to these types of problems is a decimal number. But when a DWR problem is set in a real-world context, “the result must be interpreted by the real-world constraints that give meaning to the problem” (Pacheco-Muñoz et al., 2022, p. 160). Consequently, the solution of such a problem does not automatically correspond to the arithmetic solution. Instead, it is necessary to interpret the remainder with respect to the specific context. Zorrilla et al. (2021) distinguish three types of situations that require a different interpretation of the remainder: In the first type of situation, the remainder is not given any consideration, the solution corresponds to the quotient without the decimal. In the second type of situations the presence of the remainder requires a recognition of the value of the quotient plus one more unit. The third type comprises situations where the result is the quotient, plus the fractional

Table 1 Partitive and quotitive division tasks for $12 \div 3$

Interpretation of division	Total number of items	Number of groups	Number of items per group
Partitive division Twelve children want to split into three equal groups. How many children are in each group?	12 children	3 equal groups	<i>Wanted:</i> How many children in each group?
Quotitive division Twelve children want to form groups of three. How many groups of three can they form?	12 children	<i>Wanted:</i> How many groups of three?	Groups of 3

Table 2 Matrix for the two interpretations of division and the three types of remainders

Types of remainders	Quotitive division	Partitive division
Type 1 The remainder is not given explicit consideration, the solution corresponds to the quotient without the decimal.	There are 7 socks in the drawer. How many pairs can you make? (QD1)	22 playing cards are on the table. Divide them equally between 4 children. How many cards does each child get? (PD2)
Type 2 The remainder requires recognizing the value of the quotient plus one more unit.	Here you see 22 children. They are to be seated at tables of four. All children want to sit down. How many tables do you need? (QD2)	Here are 13 candies. Divide all of them equally between 3 jars. How many candies are in the jar that contains the most at the end?
Type 3 The result is the quotient plus the fractional (decimal) part of the remainder.	Five liters of water are filled into 2-liter bottles. How many bottles will be full? How many liters of water are in the last unfinished bottle?	Here are 7 pretzel sticks. Divide them all equally between 2 children. How many does each child get? (PD1)

part of the remainder. Table 2 combines both classifications. It provides examples of real-world DWR problems relating to partitive and quotitive division in combination with the different types of remainder-situations described above. The examples highlighted in gray (QD1, QD2, PD1, and PD2) were used in the subsequent interview study.

The fact that not all combinations of interpretation of division and types of remainders were included in our interview study is illustrated by the examples of type 3 quotitive DWR problems and type 2 partitive DWR problems shown in the matrix. In both cases, only rather unrealistic contexts can be identified that contain extremely complex questions or no longer correspond to the characteristics of the respective interpretation of division. Therefore, in our study, we have focused on the four situations that portray realistic division contexts that relate to young children's real-life experiences.

Research Background

Previous research indicates that young students can successfully solve real-world related division problems at the beginning of school (Brown, 1992; Cheeseman et al., 2022). However, we are not aware of any studies on how successfully and in what way children at this age consider and interpret remainders in real-world division problems. Silver et al. (1993) conducted a study with 195 students in grades six to eight and found that only 45% could solve a quotitive type 2 DWR problem appropriately in terms of the respective context. Over 70% of the students in that study solved the problems using the division operation or a division algorithm.

Silver et al. (1993) therefore assume, that children who rely on a division algorithm when solving DWR problems have greater difficulties in correctly interpreting the remainder. Rodríguez et al. (2009) investigated which of the types of remainder-situations (see Table 2) are the most difficult for 45 12- and 13-year-olds to interpret. They found no difference between the between the solution rates of types 1 and 3 DWR problems. Type 2 DWR problems, on average, were solved correctly the least frequently and showed a significantly increased difficulty compared to the other types of remainder-problems. Li and Silver (2000) examined how 14 third graders solved a type 2 DWR problem, which is considered the most difficult type. They found that 11 out of the 14 participants solved the task correctly and almost exclusively used additive strategies. Li and Silver (2000) therefore argued that the use of these non-division strategies enables primary students to interpret the factual situation significantly more often than middle school students who use division strategies. In addition, Pacheco-Muñoz et al. (2022) found in their study that 50 fourth grade students in Mexico often succeeded in correctly interpreting the remainder in type 1 and type 2 DWR problems with both partitive and quotitive division. In contrast to other studies, this does not depend on whether the children used division or non-division strategies.

Method

Situations that require fair sharing occur early in children's lives. Quite often there is a remainder that needs to be dealt with to allow fair sharing. Most of the research on DWR problems focuses on secondary students, for whom type 2 DWR problems pose the greatest challenges (Rodríguez et al., 2009). Only a few studies (e.g. Li & Silver, 2000; Pacheco-Muñoz et al., 2022) focus on primary school students, i.e. third and fourth graders. The participants in our study are children at the end of their first year of primary school before any formal introduction of division. Our study seeks answers to the following two research questions:

RQ1a: How successfully do the children in the samples (n_q , n_p) solve the different tasks?

RQ1b: Does the solution rate differ between quotitive (n_q) and partitive division (n_p)?

RQ2: How do children justify their handling of the remainder in different DWR problems depending on the respective context?

A clinical interview was conducted with each of the children during and after they were solving the DWR problems. The chosen semi-standardized approach aims to understand the children's rich thinking (Kubli, 1985). The four problems chosen for the study are highlighted in gray in Table 2.

The children were part of a quantitative study ($N = 537$) on early understanding of division (Wiggelinghoff et al., [in press](#)). They were selected for the qualitative sub-study because of their drawings that required explanation ($N = 16$). Seven of these children ($n_p = 7$) worked on the two partitive division problems (PD1, PD2)



Fig. 1 Concrete material regarding PD1, PD2, QD1, and QD2 (from left to right)

portraying types 1 and 3, while nine children ($n_q = 9$) worked on the two quotitive division problems (QD1, QD2) that relate to types 1 and 2. The children were divided in two groups (partitioning and quotitioning), because the children were only given tasks for one of the types in the main study. This procedure was chosen so that children who had worked on the partitive division problems first, would not be influenced in their choice of strategy for the quotitive division problems and vice versa. The children in each group were provided with manipulatives that correspond to the respective contexts of the problems (see Fig. 1). They were encouraged to explain their thoughts and procedures when finding the solution and particularly their interpretation of the remainder. All interviews were video-recorded and transcribed including gestures and actions. The children were assigned codes from P1 to P16 in the transcripts.

The transcripts were then analyzed using qualitative content analysis (Mayring, 2015). In a first step three main categories were defined deductively based on the existing literature and the interview protocols, i.e. (1) solution strategy, (2) interpretation of the remainder, and (3) justification for the interpretation of the remainder. All transcript extracts that clearly show a child's solution strategy were assigned to the first category. Extracts showing how the children interpreted the remainder (as expected or unconventional) were assigned to the second category, while explanations and reasons for the interpretation given were assigned to the third category. In a second step, the main categories were then differentiated into subcategories based on the interview data (Table 3).

The following extract from the transcript illustrates the coding.

P5: But then there are still two [cards] left.

I: What do you want to do with the cards that are left?

P5: I put them away [...]. (*The child puts all the cards on the edge of the table.*)

The above transcript extract indicates how P5 interprets the remaining two cards and is therefore allocated to category (2). From a research point of view, it is also important to analyze how exactly the student interprets the remainder. In this example, P5 does not consider the remainder and is therefore classified as (sub-)category (2a).

Table 3 Deductively defined main categories and inductively differentiated subcategories

Main categories	Subcategories
(1) Solution strategy	(a) Grouping strategies (b) Estimation strategies (c) Sharing/dealing strategies (d) Mention without action or task knowledge
(2) Interpretation of the remainder	(a) Remainder is not considered (b) Manipulation of the remainder (c) Remainder is rounded up to the next natural number (d) Uneven distribution of the remainder
(3) Justification of the remainder	(a) Criterion of fairness (b) Number of items per group (c) Estimation, rearranging or sharing described as safe strategies (d) Everyday references and physical properties

Results

The presentation of the results is organized according to the four different tasks (PD1, PD2, QD1, and QD2). In terms of content, the presentation of results is based on the three main categories, i.e. (1) solution strategy, (2) interpretation of the remainder, and (3) justification for the interpretation of the remainder.

Sharing Pretzel Sticks (PD1): A Type 3 Partitive DWR Problem

Solution Strategy

Six of the seven children (all except P1) in sample n_p solved the task of dividing seven pretzel sticks equally between two children without any leftovers by using a grouping strategy (see Kouba, 1989), that is they place three pretzel sticks directly on each plate. P1 solved the task by dealing out pretzel sticks one by one (see Kouba, 1989) until six pretzel sticks have been used up. The fact that almost all children used a grouping strategy is initially surprising, as the number of items per group is not included in the task (see Wiggelinghoff & Peter-Koop, 2024). One explanation for this might be, that they (i.e. P2, P3, P5, P6, and P7) use suitable addition strategies to obtain the required information. P3's explanation is typical, "Because three plus three is 6 and then one is left over. [...]. I saw three plus three."

Interpretation of the Remainder

For this problem we expected the most likely appropriate interpretation would be that the remaining pretzel stick is broken in half, so that each child receives three and a half sticks (type 3, see Fig. 2). In the interviews, six out of seven children gave an explanation that corresponds to this interpretation. Only one child (P6) initially

Fig. 2 P4 breaks the pretzel stick



explained that you “can’t share seven pretzel sticks fairly.” When P6 was told that all sorts of things could be done with the stick, he also broke the remaining stick in two.

Justification for the Interpretation of the Remainder

Six children (all except P4) justified their predominantly appropriate manipulation of the remaining pretzel stick by saying that this fulfills the criterion of fairness. P2, for example, expressed, “Then every child has the same amount.” P6 explained his thought process leading to this realization

I: What made you think that you could break it up?

P6: If you put three on each plate, then it’s fair, but if you add one more, then it’s not fair. And then I came up with the idea of breaking it up.

Some children (P2, P4, P5, and P6) also explicitly described the consideration of the second criterion, that is there should be no pretzel stick left over. P4 broke the remaining pretzel stick, “so that everything is empty” and P2 stated, “not one little bit is left over!” An adequate interpretation of the remainder always depends on the specific context, because breaking the pretzel stick into two only makes sense due to its physical properties. All children with an appropriate solution have apparently thought about this possibility successfully, with two children (P4, P5) describing this explicitly. P5 drew on her everyday experience and answered the question as to why she broke the remaining pretzel stick, “Because I’ve done that before. And

they're not particularly hard, like a rock." P4, on the other hand, referred to his experience by making a comparison to sharing pencils and explained that, unlike a pretzel stick, a pencil cannot easily be broken in two like a pretzel stick.

This type 3 DWR problem, therefore, presents the children with the challenge of considering two requirements (fairness, no remainder). Six of the seven children fulfill both requirements. In their justifications, the children most often refer to the criterion of fairness and reveal that they think and judge precisely about the physical properties of the objects involved in terms of the factual context.

Sharing Playing Cards (PD2): A Type 1 Partitive DWR Problem

Solution Strategy

The same seven children (n_p) were asked to share 22 cards fairly between four children. Four students (P1, P2, P6, and P7) use the strategy of dealing them out one by one (see Kouba, 1989) until 20 cards are distributed and two are left over. Two children (P4, P5) used estimation strategies (see Bönig, 1995), in which the number of items per group is first estimated (e.g. four cards for each child) and the remaining items are then distributed or rearranged. Only one child (P3) used a grouping strategy (see Kouba, 1989) and immediately assigns five cards to each child, naming the matching addition task "five plus five plus five plus five plus five." Given the comparatively large number range for first grade students, this is quite remarkable.

Interpretation of the Remainder

For this problem it is not required to distribute all the cards and cutting the cards does not make sense in this context. Therefore, the two remaining cards are not to be considered when solving the problem (type 1). Five out of seven children (except P4 and P6) managed to interpret the remainder accordingly. P4 and P6, on the other hand, distributed all 22 cards, so that two children receive five and two children receive six cards. P6 recognized that this is not fair but wanted to distribute all the cards and does not think that there should be any cards left over. P4, on the other hand, obviously interpreted the criterion of fairness differently and stated that it is "also fair if two [children] have six [cards] and two have five."

Justification for the Interpretation of the Remainder

The five children who interpreted the remainder as expected justified this with the criterion of fairness. P7 stated that if the remaining cards were also distributed, "that would be unfair for the other two children." In addition, four children (P1, P2, P3,

and P5) also referred to physical properties of the object as well as everyday experiences in their justification. P2, for example, recognized that although it is possible to cut up cards, it would not make any sense in this context:

P2: I have an idea, but I shouldn't do that. I could cut them in half. [...].

I: Do you think it's a good idea to cut them in half?

P2: No. [...]. Because then you can't play properly, because then the card is cut in half.

P5 argued similarly, “So I shouldn't break the cards in half [...]. Because that's what you play with. [...]. Who wants to play with broken cards?” P1 and P3, on the other hand, referred to their experience with playing card games (see Fig. 3). P3 placed the remaining cards stacked in the middle and commented, “Because these belong to the deck.” P1 also considered the remaining two cards as going into the deck based on her experience playing UNO. Nevertheless, most of the first graders in our study immediately solved the type 1 partitive DWR problem as expected.

Once again, the children predominantly utilized the requirement of fairness contained in the task in their justifications. However, the children also reflected very carefully about the properties of the objects involved and the meaningfulness of breaking/cutting them in half.

Fig. 3 P3 forms a stack of cards



Forming Pairs of Socks (QD1): A Type 1 Quotitive Division with Remainder Problem

Solution Strategy

The two quotitive division problems were completed by a total of nine children (n_q). The first task was to form as many pairs as possible out of seven socks. Nine children used a grouping strategy (see Kouba, 1989) and put two socks together until only one sock remained. Since it requires two socks to make a pair, the remainder is not considered in the solution of the problem (type 1).

Interpretation of the Remainder

Eight out of nine children interpret the remainder as expected. Only P16 instead made two pairs and a group of three socks (see Fig. 4) and justified this as follows:

I: How many pairs could you form now?

P16: Two plus two plus three. [...]. Three times two is six. Then you must add one more to make seven.

Justification for the Interpretation of the Remainder

In his justification, P16 only referred to arithmetical aspects and did not consider the context of the DWR problem in any way.

The reasons given by the children who interpreted the remainder as expected, on the other hand, contain various elements. Five children (P9, P10, P11, P12, and P15) focused on the information contained in the task about the number of elements per subset (always two socks). P12 stated, “there must always be two socks in a pair.” P9 referred even more specifically to the task, “You said [...] that you can only make two pairs and not three [socks per pair].” Three other children (P8, P10, and

Fig. 4 P16 forms two pairs of socks and a group of three socks



P12) also referred to the properties of the objects and everyday experiences. P10 explained why pairs of socks are required:

I: How do you know that the answer is three pairs, and a single sock left?

P10: Well, I know that we have two feet. And then the socks are put on. And if we have two feet, then we need two socks.

P8 argued similarly and explained that you “can’t do anything with three socks per pair because you don’t have three feet.” Another child (P14), who also came up with the expected solution, argued arithmetically, because he considers that only pairs are allowed, “Because four plus three equals seven and you can’t split four and three. [...]. If you have seven, then whenever you divide [into pairs of two], one remains.”

Almost all children solve this type 1 quotitive DWR problem as expected. Some of the children based their reasoning on the number of elements per subset. Others rather referred to the properties of the object (socks) and their everyday experiences. The only child who did not interpret the remainder as expected, based his solution on purely arithmetical rather than real-life experiences.

Forming Tables of Four (QD2): A Type 2 Quotitive Division with Remainder Problem

Solution Strategy

Following the sock problem, the children (n_q) were asked to solve problem QD2, in which 22 children are to be seated at tables of four in a way that all children are seated. Again, all nine children used the grouping strategy typical for quotitive division problems (see Kouba, 1989) to find a solution.

Interpretation of the Remainder

For all children to have a seat, a sixth table is required for the remaining two children. Consequently, the solution is the value of the quotient plus one more unit (type 2). Only one of the nine children interprets the remainder as expected. P14 forms tables of four and thus adheres to the number of items per group, but also considers that all children need a seat, “Only two [children] sit at this last table then [...]. Because everyone should be seated. Otherwise, they would just stand there.” The other eight children were unable to interpret the remainder correctly. They stated that only five tables were required.

Justification for the Interpretation of the Remainder

In their explanations, the participants focused exclusively on the number of items per group and explained that two children were left over because “four should be at a table” (P10) or “because it wasn’t a four-pack” (P8). P12 explained:

I: Why can’t these two children sit at a table?

P12: They are only two. And two are not allowed at one table [...]. Because four always must sit at a table. Two would be too few.

Most students were so focused on this argument that they did not consider that there was also a requirement that all children should be seated. They failed to combine both requirements. Instead, they looked for creative solutions, to find a subjectively meaningful answer. For P8, the remaining children “could be the parents”, while P12 suggested, “They can’t go to school anymore. [...]. Because there are no more seats” (see Fig. 5).

For P13, the two remaining children could “play on their own” instead. Often these ideas no longer matched the context presented in the task but illustrated the children’s creativity in dealing with the remainder problem. When the interviewer reminded the children that each child is supposed to have a seat, four out of eight children (P8, P9, P11, and P12) subsequently changed their interpretation of the remainder by including a sixth table.

Only one of the nine children immediately solved the type 2 quotitive DWR problem as expected, while four other children required support during in the interview. Their argumentations showed that most children focused on the required number of items per group (tables of four). The second requirement, that all children should be seated, was no longer taken into consideration, and could not be reconciled with the specified number of items per group.

Discussion and Implications

The findings are pertinent to the research field. With respect to research question 1a), the qualitative data analysis revealed that most of the participating first graders could solve three of the four problems (PD1: 6 out of 7; PD2: 5 out of 7; QD1: 8 out

Fig. 5 P12 removed the “children who can’t go to school anymore”



of 9). This indicates that at least some first graders already have a high level of competence in solving DWR problems. One of the DWR problems (QD2; type 2 quotitive division), was more challenging, as only one of nine children was able to interpret the remainder adequately. A dependency of the interpretation of division could not be identified, because overall both partitive (PD1, PD2) and quotitive DWR problems (QD1) were solved as expected by most students. However, regarding research question (1b), we observed a dependency on the situation-type of the remainder with respect to the solution rates. While type 1 and type 3 DWR problems (PD1, PD2, QD1) were predominantly solved adequately, the type 2 DWR problem (QD2) was the most challenging for the first graders who were given this problem. These findings are in accordance with research results for secondary school students, who were also least likely to solve type 2 DWR problems correctly (Rodríguez et al., 2009).

For research question 2, which focuses on the students' reasoning when dealing with the remainder, we gained new insights into children's thinking. In contrast to the solution frequency, there were differences in the students' reasonings depending on their interpretation of division, which resulted from the underlying structure of partitive and quotitive division. In partitive division problems (PD1, PD2), children often explained their interpretation of the remainder with the criterion of fairness, which is typical for this interpretation. In quotitive division problems (QD1, QD2) the children often referred to the number of items per group given in the task. A second, interesting finding, however, is independent of the interpretation of division and demonstrates why the children are already solving DWR problems successfully. Their reasoning showed that the children reflected very carefully about the context and the physical properties of the objects involved in the problem. They made sensible decisions on how to deal with remainders and built on their everyday experiences. For example, the children described in a very distinguished way that pretzel sticks are easy to break, but playing cards should never be cut in half because "then you can't play properly anymore" (P2). The children also understood that pairs of socks were much more useful than "bundles of three, because you don't have three feet" (P10). The children's reasonings, therefore, demonstrate that the interpretation of DWR problems resulted from the combination of implementing the requirements of the problem (e.g. fairness or number of items per group) and their ability to make decisions about objects and their properties based on their everyday experiences. The children's difficulties with problem QD2 (type 2, forming tables of four) can also be understood based on their explanations. The problem had two requirements that the students had to interpret and act upon. On the one hand, they were asked to set tables of four, on the other hand, each child must have a seat. All first graders who did not interpret the remainder adequately, referred to the number of items per group, which for them was the important criterion. The second requirement, to provide a seat for all 22 children, receded in importance and was not compatible with the given number of items per group from the students' perspective. The structure of a quotitive division situation, in which the number of items per group is given, is apparently difficult to reconcile with a type 2 remainder, which requires the addition of a group that no longer corresponds to the given number of items per group, and therefore presented the children with the greatest challenges.

Implications for classroom teaching are twofold. To begin with, primary mathematics teachers should know that a substantial number of children can solve DWR problems successfully before the formal introduction of division, because they focus very precisely on the requirements contained in the task and think carefully about the context. Some children already have prior everyday life knowledge and might have faced remainder-situations that require strategies for fair sharing. Because of their close connection to children's everyday life, DWR problems should be considered as important out of school knowledge, when division is formally introduced. In particular, the children's rationales for dealing with remainders are frequently creative, reflecting their endeavors to identify a viable solution. Their ideas and solutions might lead to rich mathematical discussions. Secondly, teachers should be aware that certain types of remainder-situations are difficult to comprehend for many children. Therefore, type 2 DWR problems should be addressed and discussed intensively in (primary) mathematics classrooms.

A clear limitation of our study is the rather small sample size. A further study following a quantitative design would allow more substantiated statements about solution rates and difficulties regarding different types of remainder-situations. It would also be helpful to conduct further qualitative studies with larger sample sizes to better understand children's reasoning in remainder-situations to inform a subsequent larger quantitative study. In addition, it cannot be entirely ruled out that the formulation of the items and the processing of two items in immediate succession also influenced how remainders were interpreted.

References

- Bönig, D. (1995). *Multiplikation und Division. Empirische Untersuchungen zum Operationsverständnis bei Grundschulern [Multiplication and division. Empirical studies on the understanding of operations in primary school students]*. Waxmann.
- Brown, S. (1992). Second-grade children's understand of the division process. *School Science and Mathematics*, 92(2), 92–95. <https://doi.org/10.1111/j.1949-8594.1992.tb12148.x>
- Cheeseman, J., Downton, A., & Roche, A. (2022). Ideas of early division prior to formal instruction. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 131–138). PME.
- Kouba, V. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education*, 20(2), 147–158. <https://doi.org/10.2307/749279>
- Kubli, F. (1985). Jean Piaget und die Schule [Jean Piaget and school]. *Betrifft: Erziehung*, 18(1), 45–49.
- Li, Y., & Silver, E. A. (2000). Can younger students succeed where older students fail? An examination of third graders' solutions of a division-with-remainder (DWR) problem. *Journal of Mathematical Behavior*, 19(2), 233–246. [https://doi.org/10.1016/S0732-3123\(00\)00046-8](https://doi.org/10.1016/S0732-3123(00)00046-8)
- Mayring, P. (2015). Qualitative content analysis: Theoretical background and procedures. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 365–380). Springer. https://doi.org/10.1007/978-94-017-9181-6_13

- Pacheco-Muñoz, E., Nava-Lobato, S., Juárez-López, J. A., & Ponce de León-Palacios, M. (2022). Division problems with remainder: A study on strategies and interpretations with fourth grade Mexican students. *Mathematics Teaching Research Journal*, 14(5), 159–180.
- Rodríguez, P., Oliva Lago, M., Lourdes Hernández, M., Jiménez, L., Guerrero, S., & Caballero, S. (2009). How do secondary students approach different types of division with remainder situations? Some evidence from Spain. *European Journal of Psychology of Education*, 24(4), 529–543. <https://doi.org/10.1007/BF03178766>
- Silver, E. A., Shapiro, L. J., & Deutsch, A. (1993). Sense making and the solution of division problems involving remainders: an examination of middle school students’ solution processes and their interpretations of solutions. *Journal for Research in Mathematics Education*, 24(2), 117–135. <https://doi.org/10.2307/749216>
- Wiggelinghoff, L., & Peter-Koop, A. (2024). Early division prior to formal instruction: Young children explain their solution strategies. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.), *Proceedings of the 47th conference of the international group for the psychology of mathematics education* (Vol. Vol. 4, pp. 201–209). PME.
- Wiggelinghoff, L., Streit-Lehmann, J., & Peter-Koop, A. (in press). Frühes Divisionsverständnis von Erst- und Zweitklässlerinnen vor der unterrichtlichen Einführung der Division [Early understanding of division by first and second graders before division is introduced in the classroom]. *Journal für Mathematik-Didaktik*.
- Zorrilla, C., Ivars, P., & Fernández, C. (2021). Problemas realistas de división con resto. Un estudio sobre las estrategias en educación primaria [Realistic division problems with remainder. A study on strategies in primary school]. *Revista Mexicana de Investigación Educativa*, 26(91), 1313–1339.

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Preschool Children’s Collaborative Explorative Mathematical Argumentation



Friederike Reuter and Christiane Benz

Introduction

Argumentation holds great potential for knowledge construction (“arguing to learn”, e. g. Asterhan & Schwarz, 2016; Andriessen, 2006). However, the ability to successfully engage in argumentation processes has to be developed in the course of children’s education (“learning to argue”, e. g. Kuhn & Crowell, 2011; Mercer, 2009). Despite argumentation’s undisputed role in science and not least in mathematical learning, it is often not easy for students to “engage in productive scientific argumentation to propose and justify an explanation through argument” (Sampson & Clark, 2008, p. 449).

The concept of explorative mathematical argumentation (EMA) offers an approach to analyse mathematical argumentation at a young age, as a basis for the promotion of argumentative competencies in preschool children. The concept draws on a distinction between persuasive and explorative argumentation proposed by Ehlich (2014), and understands argumentation as a social, multi-modal activity of reasoning about mathematical content that aims at collaborative knowledge construction (Reuter, 2025).

Krummheuer (2007) emphasizes the crucial role of argumentation in mathematics learning, which he considers to be “argumentative learning” (2007, p. 62), even claiming that participating in argumentation marks the “constitutive social condition of the possibility of learning of a mathematical content, concept, or procedure” (p. 113). A growing body of empirical studies shows that even preschool children engage in mathematical argumentation (e.g. Salgado et al., 2024; Brunner, 2019, and with a focus on play e.g. Nergård, 2023; Sumpter & Hedefalk, 2015). Domberg et al. (2018) could show that 5- and 7-year-old children were more motivated to produce arguments in a cooperative context than in a competitive one (Domberg et al., 2018). The term explorative argumentation refers to an argumentation process

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conducted for the sake of gaining collaborative knowledge, as opposed to a competitive starting point with the aim of persuading the interlocutors of a certain claim. The concept of EMA can be used for analysing processes of young children's collaborative argumentation in mathematical learning situations (Reuter, 2023). It builds on the definition of argumentation as "a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a 'rational judge'" (van Eemeren & Grootendorst, 1996, p. 5). However, in addition to the social embedment, didactical nature and the important role of reason within the process, EMA also takes into account the multimodality of argumentation processes by explicitly including gesture and material-based action as well as verbal utterances. EMA can be defined as:

A process of collaborative knowledge construction that involves inferring conclusions from premises and supporting statements by finding reasonable justifications within a mathematical context, which means that data, warrants and conclusions are mathematically anchored. Reasoning refers to the tools applied in the process, like forming and testing hypotheses and drawing on analogies. Modalities refer to what communicative measures are taken to carry out the argumentation process. Gestures and material-based actions have to be considered as well as verbal utterances. (Reuter, 2023, p. 431)

Analyses of video material conducted in a mathematical join-in studio at Karlsruhe University of Education, Germany, have shown EMA processes in preschool children across different content areas. For an example in arithmetic, see Reuter (2023), for an example of geometric conceptualization, Reuter (2025).

The case study presented below depicts an argumentation process in which the quantities of two sets of interlocking plastic bricks are compared. The research aims at answering the question in what way collaboration takes place among preschool children and between children and learning guides in a situation of early explorative argumentation. In order to answer this question, an analysing scheme drawing on the concept of EMA was applied to several argumentative moves during the analysed argumentation process, assessing which aspects of argumentation were implemented using which communicative modality. According to van Eemeren et al. (2008), argumentative moves are "verbal moves made in argumentative discourse" (p. 16). However, in line with the growing awareness for the important role of gesture and material-based action in learning processes, moves within these modalities are also considered.

Theoretical Background

Components, Modalities and Tools of Explorative Mathematical Argumentation

Lithner (2008) claims that outside mathematics, arguments can be convincing even if they are not valid. In fact, different domains can have different requirements for argumentation. For example, argumentation in politics and jurisprudence draws to a great extent on rhetorical means (Ehlich, 2014), while mathematical argumentation,

though not necessarily merely deductive, usually deploys tools like making and communicating discoveries, drawing on analogies, developing and testing hypotheses and drawing logical conclusions (Reuter, 2023). It should be noted that assertions do not have to be mathematically correct in order to be categorized as mathematically anchored, and that individual components of the argument can remain implicit. Especially for the analysis of young children’s argumentation, who are not always able to verbalise their ideas, it is crucial to recover any implicit content (Rocci et al., 2020). An example would be the argument below:

Claim: I have seven toy cars.
 Datum: Because I have two toy cars in my backpack and five at home.
 Implicit warrant: 2 plus 5 equals 7.

Here, all three components are mathematically anchored, as they deal with mathematical content, in this case quantities. The concept of anchoring is based on Lithner (2008); see also Nordin & Boistrup, 2018). If all three constituting components are mathematically anchored, the argument is considered mathematical, even if it is not mathematically correct (e. g. “I have 55 toy cars, because there are 5 in my backpack and 5 at home, and 5 plus 5 equals 55.”).

Based on Toulmin’s (2003) model of argumentation, the components can be graphically visualized as in Fig. 1.

A datum supports a claim—chronologically, the claim can be made initially and be consequently supported by a datum or a datum can lead to a certain claim being made. The warrant explains why the datum leads to the respective claim. While the Toulmin model offers several more, optional components, these three components are constitutional for an argument. However, the warrant often remains implicit (Toulmin, 2003) and has to be inferred during the analysis (Rocci et al., 2020). Additionally, an *attack* (Homer-Dixon & Karapınar, 1989) could be performed, e.g. on the warrant, by another child claiming that “2 + 5 equals 8”. A counting strategy could then be used as a *backing* of the original claim, “No, look: 6, 7!”. Attacks can be interpersonal like in this example, or intrapersonal, when a person performing an argumentation encounters a new or unexpected realization that leads to a cognitive conflict.

Explorative mathematical argumentation (EMA) signifies a concept of argumentation adapted to early mathematics education by including different communicative

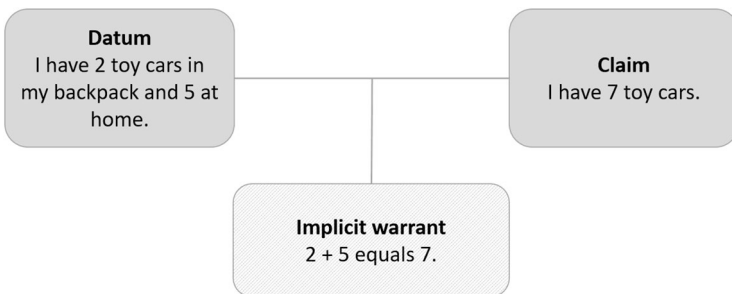


Fig. 1 Example of a mathematically anchored argument

Table 1 EMA analysing scheme (Reuter, 2023)

Tools	Modalities		
	Verbal statements	Gestures	Material-based actions
Making discoveries			
Communicating discoveries			
Challenging existing knowledge			
Finding (relational) patterns			
Using (relational) patterns			
Developing hypotheses			
Testing hypotheses			
Drawing on analogies			
Drawing conclusions			

modalities and tools for mathematical argumentation like finding and using (relational) patterns, developing and testing hypotheses, and drawing on analogies (Reuter, 2023). These tools and the three communicative modalities form the analysing scheme used for EMA processes are shown in Table 1.

EMA processes are based on the goal of collaborative mathematical knowledge construction by reasoning about mathematical content (Reuter, 2023).

By conducting explorative mathematical argumentation, children make use of certain tools of argumentation which are conducted not only on a verbal level, but also in other communicative modalities like gesture and material-based action. This places special demands on the analysis of EMA as a multi-modal processes. A filled analysing scheme will be depicted with the results. Firstly, the mathematical content of the analysed situation, comparing quantities by comparing lengths, will be presented.

Comparing Quantities of Sets by Lengths

Length as a visualization of arithmetic content is quite omnipresent. When dealing with data in the mathematics classroom, but also in everyday life, the lengths of bars or columns in a diagram serve the purpose of facilitating the comparison of numbers, visualizing relations via length comparison. Also, the number line is a typical example for the representation of numbers with the help of lengths and units.

Stacking rods with interlocking blocks can be a suitable representation of the relationship between length and cardinality (quantity of units), with the height of the single blocks representing a measurement unit (Zöllner & Reuter, 2019). With interlocking blocks of the same size, the material automatically leads to a unit iteration because no gaps and overlaps are possible. Thus, the challenge of understanding the invariance of quantity (e.g. there is still the same amount of coins lying on the table in a row, even though the length of the row was extended by leaving greater

gaps between the coins) does not arise. While the same quantity of blocks leads to rods of the same length, larger amounts of blocks lead to longer rods.

In order to make mathematically correct claims, children do not need an understanding of units and unit iteration. Therefore, such an understanding cannot be inferred from the successful completion of the task presented below. Children can argue mathematically correctly and conclude from the comparison of length to the comparison of quantity without the understanding that all blocks have to be of equal length and must stick without gaps and overlaps. However, they do need the understanding that a conclusion can be drawn from the direct comparison of length to the quantitative comparison of length units. Thus, using the one-to-one correspondence that is already inherent in the offered material, a comparison of the quantity of the two sets can be made without an exact determination of the number of bricks.

Methods

The following situation was videotaped in the mathematical join-in studio, MiniMa, at Karlsruhe University of Education in Germany. In the join-in studio, preservice teachers of childhood education and primary school education provide play-based mathematical learning environments for children from 4 to 8 years. The join-in studio is part of the long-term project “Children and Adults Explore Mathematics Together”, which combines in-service training for kindergarten and primary school professionals, pre-service university courses, development and evaluation of early mathematics learning opportunities, and research on children's mathematical development as well as professionals' competencies (Gasteiger & Benz, 2018).

Verbal and material-based actions as well as gestures were assigned to components of the Toulmin model. The model is well suited to identify and visualize an argument, while it is too rigid to depict longer argumentation processes, in which assertions often change their function and a datum may become a claim or vice versa (Gronostay, 2017; Nielsen, 2013). Therefore, individual argumentative moves were analyzed with the help of the proposed analyzing scheme to clarify which argumentative tools took place in which communicative modality.

For the following transcripts it should be noted that, unlike English, the German language has a Subject-Object-Verb structure in subordinate clauses, such as if-clauses or rationales starting with ‘because’. Several times in this sequence, a sentence is not finished verbally, but gestures and material-based actions replace the missing part of a sentence. If this leads to the omission of a verb, the transcript includes the phrase [verb omitted]. An example is found below in the student's incomplete statement “Und wenn wir die zwei...” (S-O-[V]) which translates to “And if we [verb omitted] the two” (S-[V]-O).

Case Study of Explorative Mathematical Argumentation

The first sequence occurs after a preservice teacher had read the book “Die kleine Raupe Nimmersatt” (Carle, 2018, orig. The very hungry caterpillar) to a group of nine preschool children aged 5–6 years. The task for the children was to find out whether the caterpillar had eaten more pieces of food from Monday to Friday or on the weekend. As with younger children, physical objects are used to create graphs (Clements & Conference Working Group, 2004), green and yellow plastic bricks, provided by the preservice teacher, were gathered. A green one for each piece of fruit the caterpillar ate from Monday to Friday, a yellow one for everything it ate on Saturday and Sunday. Some of the interlocking bricks had already been stacked at the start of the sequence. The 15 green bricks were stacked in towers of 1, 2, 3, 4 and 5 bricks, the 11 yellow ones in towers of 1, 1, 2, 3 and 4 (Fig. 2).

In the first excerpt, we find an argumentative move with the following components (Fig. 3):

Preservice teacher (PT): And now? How can we see better now, where did she eat more? (points successively to the green and yellow bricks with open hands)
Und jetzt? Wie können wir jetzt besser erkennen, wo sie jetzt mehr gegessen hat?

Leah: That (comes closer and points to the green bricks with the index finger).
 That one there.
... Das. Das da.

PT: And how do you see that?
Und woran siehst du das?

Fig. 2 Green and yellow brick towers

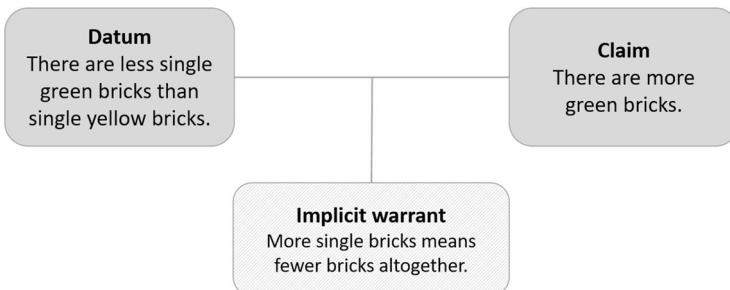
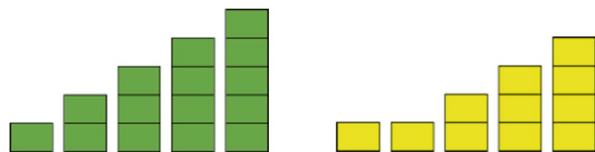


Fig. 3 Argument 1

Leah: Because here (touches upon the single green brick with middle finger) [verb omitted] only one... Look, here is... (touches upon the taller green towers with middle finger)

Weil hier nur eins... Guck, hier ist...

Ben: (Points to the green bricks with index finger) Yes, there is tall... two... taller tower. There are two one (points to the yellow towers with index and middle finger) and there is only one one (points roughly towards the green towers with index and middle finger).

Ja, da ist hö(.) zwei(.) höherer Turm. Da sind zwei eins und da ist nur eins eins.

PT: Okay.

Ben: (Points to the green bricks) Only one, but... (points shortly to the yellow bricks). That is only (points with one hand to the green bricks) one of the singles, and those are (points with two hands to the yellow bricks) two of the singles.

Nur eins, aber... Das ist nur eins von den Einser, und das sind zwei von den Einser.

The children discussed that there were more green bricks, as there are less green single bricks. The preservice teacher guiding the situation now challenges this argument by stacking the single bricks onto existing towers and asking: How does it look now? (Fig. 4).

PT: Okay. (..) And if we [verb omitted] the two... (stacks the two single yellow bricks together and onto one of the yellow towers. Then stacks the green single brick onto one of the green towers.) How does it look now?

Okay. (..) Und wenn wir die zwei... Wie sieht es, wie schaut es jetzt aus?

The children integrate this challenge (the removal of single brick towers) into their argumentation and put forward a new argumentative move. Ben and Leah both claim that there are still more green bricks and give the following reason:

Ben: Still (points to the green blocks with both feet).

Immer noch.

Leah: Still (..) that one (points shortly to the green bricks with the index finger).

Immer noch (..) das.

PT: Still that one?

Immer noch das?

Ben: Yes.

Ja.

Fig. 4 More green than yellow towers



- PT: And how did you realize that now?
Und woran habt ihr das jetzt erkannt?
- Ben: Because, because there... (points to the green bricks with the index finger)
Weil, weil den da...
- Leah: Because, look. (Points successively to all green towers with the index finger, like counting) (..) Because those over there (points to the yellow bricks)...
Weil, guck. (..) Weil die da halt eben...
- Ben: ... are three and these are four (points with his hand to the yellow, then the green bricks).
... drei sind und das sind vier.
- PT: Ah, okay. So these are three towers (points shortly to the yellow towers), at the weekend, and there (points shortly to the green towers) are four towers?
Ah, okay. Also das sind drei Türme, beim Wochenende, und da sind vier Türme?
- What happens next is quite remarkable, as the new datum is now challenged by one of the children, Denis, who confirms the data, but then changes the number of towers (Figs. 5 and 6).
- Denis: Yes. There three, there four (points with the index finger first to the yellow, then the green towers). And one can put this here on this (points successively to the two smaller green towers), then [unintelligible] it is fair.
Ja. Da drei, da vier. Und man kann das hier auf das hier stellen, dann (unverständlich) ist es gerecht.
- PT: Okay, do you want to do that?
Okay, willst du das mal machen?

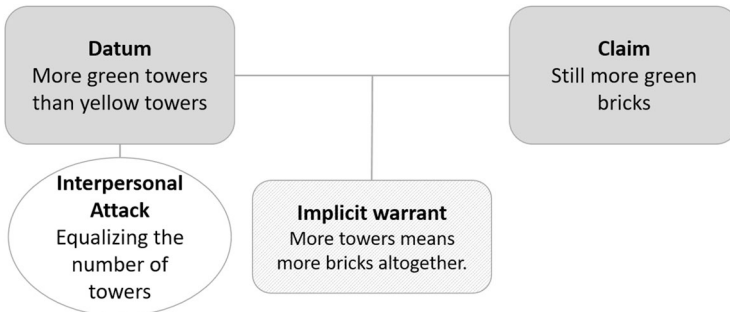
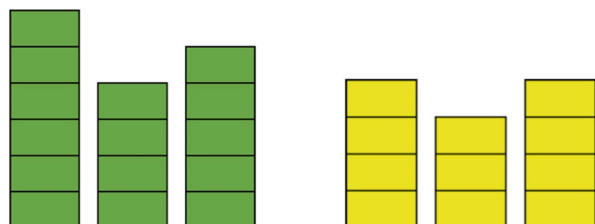


Fig. 5 Challenging the data

Fig. 6 Equal number of towers



Denis: Then that is three (points to the yellow towers), umm, that’s three and that’s three (points first to the yellow towers, then the green ones). (Stacks the two smaller green towers together)

Dann ist das drei, ähm, das drei und das drei.

PT: Okay.

It is not clear whether Denis actually has the intention to refute the other children’s reasoning, but Leah seems to interpret his intervention as a challenge (Fig. 4) and adds another datum by introducing the idea of length comparison to support the claim. Her argument is not completely valid, as she only compares one of the green towers with the yellow towers, and it is immediately challenged by Denis again. The preservice teacher contributed less, as the children took on the major part of the argumentation process (Fig. 7).

Leah: But anyway, then that tower (points to the newly built, tall green tower) is higher than those (points to the yellow towers).

Aber trotzdem, dann ist der Turm höher als die da.

PT: Ah, is that still higher then?

Aha, ist das dann immer noch höher?

Denis: Then I take one down now (stacks the top brick of the tallest green tower onto the smallest green tower, so the three green towers all have the same height now).

Dann mache ich jetzt eins runter.

Now, Leah performs a direct comparison of one of the green towers (five bricks) with one of the two taller yellow towers (four bricks). The argumentation move is executed to a great extent via material-based action while several verbal utterances remain incomplete (Fig. 8).

Leah: But now these are (repeatedly points from the yellow towers to the green towers and back)... Now they are, look (takes a green tower and holds it right next to one of the higher yellow towers).

Aber jetzt sind die... Jetzt sind die, guck.

PT: Yes.

Ja.

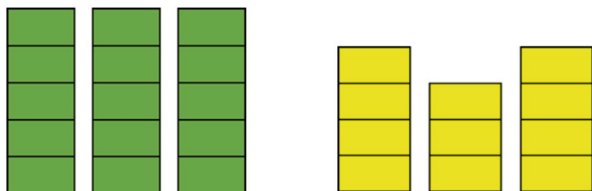
Ben: Now they are STILL... (takes the smaller yellow tower, holds it towards Denis)

Jetzt sind die IMMER noch.

Peter: Yes, still [unintelligible].

Ja, immer noch [unverständlich].

Fig. 7 Equal height of all green towers



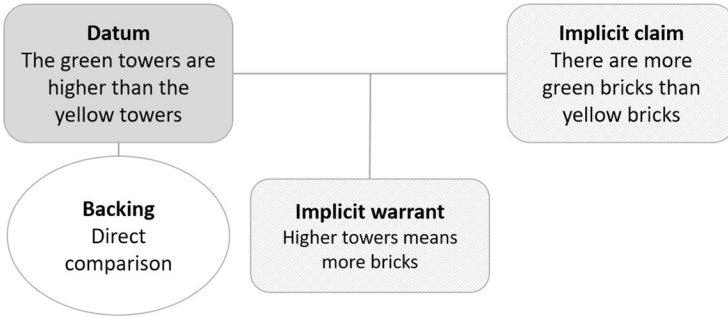
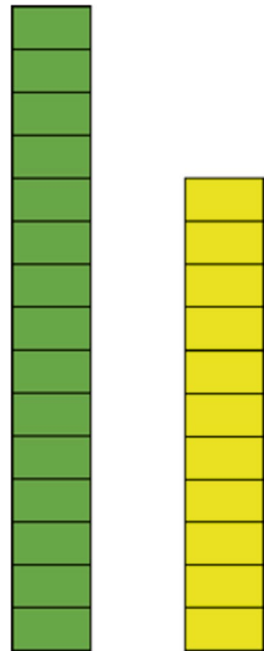


Fig. 8 Argument 3

Fig. 9 Direct comparison of two towers



Amir: The green ones are much more bricks anyway.
Die grünen sind eh viel mehr Steine.

PT: The gree... okay, how did you...
Die grü... okay, und woran hast...

The argumentation can now be considered mathematically correct. However, the children perform one last argumentation move that cannot be refuted anymore, with more children from the group chiming in (Fig. 9).

PETER: But what... but what happens when I put this one onto this here? (Stacks two yellow towers together)
Aber was... aber was passiert, wenn ich den hier drauf stelle?

- PT: Go... Stop, stop. Go back to your seats all of you... (defensive movements with both hands)
Geht mal... Stopp, halt. Geht mal alle wieder auf eure Plätze...
- Amir: And if... Now we make a... We can [verb omitted] now...
Und wenn... Jetzt machen wir ein... Wir können jetzt...
- Ben: And this... (stacks two of the green towers together)
Und das...
- Amir: (grins and stacks the two remaining green towers together) And now this one onto that.
Und jetzt der drauf.
- PT: Go back to your seats all of you.
Geht mal alle wieder zurück auf eure Plätze.
- Ben: (quickly stacks the two remaining yellow towers together). Like that! (clicks his tongue twice, points to the green tower, takes the yellow tower and holds it shortly next to the green tower, then puts it back)
So!

When asked to explain their approach, the children only verbalized this last move of the argumentation process (Fig. 10):

- Ben: I did measured which, which tower is bigger, and I saw it, that clearly that tower there is bigger (points to the green tower).
Ich hab gemesst, welche, welcher Turm größer ist, und ich hab's gesehen, dass eindeutig da der Turm größer ist.
- Denis: The green one is really very, very bigger (takes the yellow tower and holds it next to the green one).
Der grüne ist wirklich sehr, sehr größer.
- Mario: The green one of course.
Der grüne natürlich.
- Denis: (Takes both towers up from the floor, puts them on his right hand holding them with the left hand)
- Ben: See, that’s measuring.
Guck, das ist Messen.

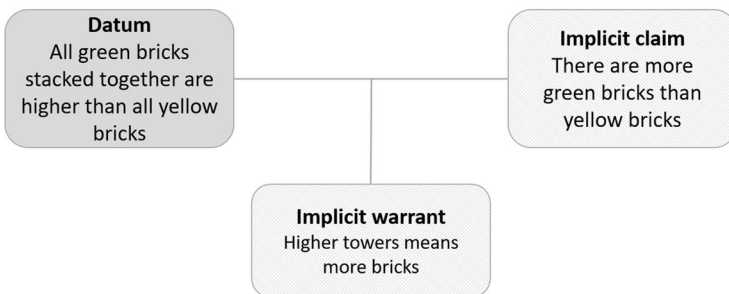


Fig. 10 Argument 4

Results

The concepts of direct comparison of length and unit (Zöllner & Benz, 2015) implicitly and explicitly play important roles in the children's approach. After initial impulses given by the preservice teacher, the argumentation chain is collaboratively developed by the children themselves, based on repeated rebutting of the data provided to support the claim. Although the preservice teacher's initial actions and verbal impulses as well as the material offered to visualize the mathematical content undoubtedly played important parts in the course of events, the presented video sequence shows that young children are able to successfully engage in play-based, collaborative argumentation processes and self-effectively develop a mathematical argumentation chain.

Although mathematically anchored, the children's argumentation is not mathematically incontestable in the beginning. Initially, the preservice teacher presented a challenge to the datum that was put forward to support the claim. After that, one of the children took over this role. The children's mathematical argumentation developed and improved with every rebuttal which was put forward. Neither the smaller amount of single bricks nor the higher number of towers are reliable indicators for the larger quantity of blocks, while the different heights of the two towers in the end visualizes the actual comparison of the amount of bricks.

Also, even with the same number of towers, the comparison of one green tower with one or more yellow towers is not sufficient. Only when all three green towers have the same length and one of them is shown to be taller than one of the two taller yellow towers through a direct comparison can the claim that there are more green blocks than yellow ones be viewed as a suitable conclusion. In the end, the children produced an argument that was not challenged anymore, by stacking two towers, one consisting of all the yellow blocks and one consisting of all the green ones, and directly comparing them to each other. They seemed to realise that this produced the most effective or convincing argument. When asked about their approach retrospectively, the children only explained this last move of the argumentation process.

The analysing scheme shows how this last argumentative move is brought forward by the children applying tools of argumentation in different modalities (Table 1).

Verbal utterances play a minor role in this process, while gestures and material-based action carried most aspects of the argumentative approach (Table 2). The hypothesis that something interesting might happen when more towers are stacked together was tested via material-based action (stacking towers together), the discovery that the green bricks make a higher tower than the yellow bricks is made and communicated by the direct comparison of the two towers, and a pointing gesture answered the question about where there were more bricks, and with that the initial question when the caterpillar ate more pieces of food. In order to keep track of the development of the argumentation process and provide suitable data and backings for their claim, the children constantly inferred the implicit information from gestures and material-based actions.

Table 2 EMA Analysing scheme

	Verbal utterance	Gesture	Material based action	Implicite content
Developing hypotheses	“But what happens when I put this one onto this here?”			Stacking towers together might be helpful
Testing hypotheses			Stacking towers together	
Making discoveries			Holding one tower close to the other for a direct comparison	All green bricks stacked together make a higher tower than the yellow bricks
Communicating discoveries		Pointing to the green tower		
Drawing on analogies				Greater length means more units, i.e. more bricks, i.e. more objects eaten

While the scheme in Table 1 focusses only on the last argumentative move, the children adopt actions from one another and adapt them to the given challenge throughout the whole process, like when the direct comparison performed by Leah is replicated by Ben on the two towers that consist of all green, respectively yellow bricks. All in all, six children contributed to the argumentation process. While Leah and Ben assisted each other in putting forward several different data to support the claim that there were more green bricks than yellow bricks, Denis provided them with challenges to the data that helped them reformulate the argumentation. Peter and Amir contributed to the final argument by stacking towers together, and Mario verbally confirmed the insight that the green tower was higher than the yellow tower, while Denis communicated this discovery on a verbal level as well as through material-based action.

The preservice teacher’s role was mainly limited to the initiation of the argumentation process, that is through reading the story book, providing the material and asking the initial question, as well as providing the first challenge by challenging the datum given by the children. After that, attacks were performed from within the group and the quality of the argumentation was collaboratively enhanced. Not only did the role of the preservice teacher become increasingly passive, but the children become so engaged in the process that they missed or ignored the preservice teacher’s request to stop and sit back. In the end, they even display the confidence to explain the mathematical concept they conducted to the preservice teacher, “See, that’s measuring”, when directly comparing the two towers.

The case study shows how preschool children engage in explorative mathematical argumentation with little teacher guidance, by putting forward a claim, several data with implicit warrants to the claim and challenges as well as backing to the data. Gesture and material-based action play an important role in the process.

Discussion

The study confirms former findings on young children involving in mathematical argumentation (Brunner, 2019; Salgado et al., 2024). The role of gesture in an EMA process has been previously shown in the context of preschoolers' geometrical conceptualization (Reuter, 2025). However, this study also reveals the role of material-based actions and sheds light on the use of argumentation in the content area of measurement and comparison. The choice of the material offered to the children to visualize the food the caterpillar ate, played an important role in the process. The same-sized bricks and the inherent unit iteration of stacking towers enable children to use the material autonomously for their argumentation, even without a fully developed concept of unit and unit iteration. Domberg et al.'s (2018) study that showed a high motivation to produce cooperative argumentation in 5- and 7-year-old children can thus be confirmed in an explicitly mathematical context. The children progressively collaborated with each other, with little guidance from the preservice teacher. With all warrants remaining implicit and data being largely provided through material-based actions, the content has to be inferred by the children in the process of collaborative engagement in the EMA process. Several different data were applied to support the claim, which itself is not changed throughout the argumentation process, until the conclusion is mathematically correct and not further challenged. Only then did the collaborative argumentation process end. This shows the children's intrinsic motivation to exhaustively go through the argumentations process, but also their ability to evaluate and adapt their own argumentative approach.

As consequences for early mathematical education practice, the results indicated that materials should be carefully chosen when planning on fostering collaborative mathematical argumentation, and gesture and material-based action should be explicitly considered in the process. Also, material that facilitates certain procedural aspects may contribute to children's ability to participate in collaborative argumentation. Depending on the intended focus, manipulatives that facilitate aspects like the understanding of units, unit iteration, and one-to-one correspondence may be less suitable for challenging and promoting measurement concepts, but convenient for the purpose of fostering argumentation processes.

Research on children's mathematical argumentation also has to take into account the modalities of gesture and material-based action and the role of content that remains implicit. The presented analysing scheme can contribute to this undertaking (Reuter, 2023).

Author contribution Friederike Reuter: Conceptualization; methodology; investigation; writing—original draft, review and editing; visualization.

Christiane Benz: Conceptualization; review and editing; supervision.

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References

- Andriessen, J. E. B. (2006). Arguing to learn. In K. Sawyer (Ed.), *Handbook of the learning sciences* (pp. 443–459). Cambridge University Press.
- Asterhan, C. S., & Schwarz, B. B. (2016). Argumentation for learning: Well-trodden paths and unexplored territories. *Educational Psychologist, 51*(2), 164–187. <https://doi.org/10.1080/000461520.2016.1155458>
- Brunner, E. (2019). Förderung mathematischen Argumentierens im Kindergarten: Erste Erkenntnisse aus einer Pilotstudie [Promoting mathematical argumentation in kindergarten: First insights from a pilot study]. *Journal für Mathematik-Didaktik, 40*(2), 323–356. <https://doi.org/10.1007/s13138-019-00146-y>
- Carle, E. (2018). *Die kleine Raupe Nimmersatt [The very hungry caterpillar]*. Gerstenberger.
- Clements, D. H., & Conference Working Group. (2004). Major themes and recommendations. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 1–72). Lawrence Erlbaum Associates.
- Domberg, A., Köymen, B., & Tomasello, M. (2018). Children's reasoning with peers in cooperative and competitive contexts. *British Journal of Developmental Psychology, 36*(1), 64–77. <https://doi.org/10.1111/bjdp.12213>
- Ehlich, K. (2014). Argumentieren als sprachliche Ressource des diskursiven Lernens [Argumentation as a linguistic resource of discursive learning]. In A. Hornung, G. Carobbio, & D. Sorrentino (Eds.), *Diskursive und textuelle Strukturen in der Hochschuldidaktik: Deutsch und Italienisch im Vergleich* (Vol. Vol. 12, pp. 41–54). Waxmann.
- Gasteiger, H., & Benz, C. (2018). Enhancing and analyzing kindergarten teachers' professional knowledge for early mathematics education. *The Journal of Mathematical Behavior, 51*(9), 109–117. https://doi.org/10.1007/978-3-319-78220-1_4
- Gronostay, D. (2017). Argumentationsanalyse à la Toulmin – Zu methodischen Problemen bei der Analyse diskursiver Argumentation [Argumentation analysis à la Toulmin – About methodological problems with the analysis of discursive argumentation]. In S. Manzel & C. Schelle (Eds.), *Empirische Forschung zur schulischen politischen Bildung* (pp. 149–159). Springer.
- Homer-Dixon, T. F., & Karapın, R. S. (1989). Graphical argument analysis: A new approach to understanding arguments, applied to a debate about the window of vulnerability. *International Studies Quarterly, 33*(4), 389–410. <https://doi.org/10.2307/2600519>
- Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *The Journal of Mathematical Behavior, 26*(1), 60–82. <https://doi.org/10.1016/j.jmathb.2007.02.001>
- Kuhn, D., & Crowell, A. (2011). Dialogic argumentation as a vehicle for developing young adolescents' thinking. *Psychological Science, 22*(4), 545–552. <https://doi.org/10.1177/0956797611402512>
- Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*, 255–276. <https://doi.org/10.1007/s10649-007-9104-2>

- Mercer, N. (2009). Developing argumentation: Lessons learned in the primary school. In N. Muller Mirza & A. N. Perret-Clermont (Eds.), *Argumentation and education: Theoretical foundations and practices* (pp. 177–194). Springer Science & Business Media.
- Nergård, B. (2023). Preschool children's mathematical arguments in play-based activities. *Mathematics Education Research Journal*, 35, 193–216. <https://doi.org/10.1007/s13394-021-00395-6>
- Nielsen, J. A. (2013). Dialectical features of students' argumentation: A critical review of argumentation studies in science education. *Research in Science Education*, 43(1), 371–393. <https://doi.org/10.1007/s11165-011-9266-x>
- Nordin, A. K., & Boistrup, L. B. (2018). A framework for identifying mathematical arguments as supported claims created in day-to-day classroom interactions. *The Journal of Mathematical Behavior*, 51, 15–27. <https://doi.org/10.1016/j.jmathb.2018.06.005>
- Reuter, F. (2023). Explorative mathematical argumentation: A theoretical framework for identifying and analysing argumentation processes in early mathematics learning. *Educational Studies in Mathematics*, 112, 415–435. <https://doi.org/10.1007/s10649-022-10199-5>
- Reuter, F. (2025). Children's Explorative Mathematical Argumentation: the Role of Gesture. *Journal für Mathematik-Didaktik*, 46(1), 1–21.
- Rocci, A., Greco, S., Schär, R., Convertini, J., Perret-Clermont, A. N., & Iannaccone, A. (2020). The significance of the adversative connectives aber, mais, ma ('but') as indicators in young children's argumentation. *Journal of Argumentation in Context*, 9(1), 69–94.
- Salgado, M., Berciano, A., & Abad-Villaverde, B. (2024). Mathematical argumentation of three-year-old children on a classroom experience based on two STEAM domains. *EURASIA Journal of Mathematics, Science and Technology Education*, 20(2). <https://doi.org/10.29333/ejmste/14187>
- Sampson, V., & Clark, D. B. (2008). Assessment of the ways students generate arguments in science education: Current perspectives and recommendations for future directions. *Science education*, 92(3), 447–472. <https://psycnet.apa.org/doi/10.1002/sce.20276>
- Sumpter, L., & Hedefalk, M. (2015). Preschool children's collective mathematical reasoning during free outdoor play. *The Journal of Mathematical Behavior*, 39, 1–10. <https://doi.org/10.1016/j.jmathb.2015.03.006>
- Toulmin, S. E. (2003). *The uses of argument. (Updated edition)*. Cambridge University Press.
- van Eemeren, F. H., & Grootendorst, V. (1996). *Fundamentals of argumentation theory. A handbook of historical backgrounds and contemporary developments*. Lawrence Erlbaum Associates Publishers.
- van Eemeren, F. H., Houtlosser, P., & Francisca, F. (2008). Identifying indicators of argumentative moves. In G. Gobber, S. Cantarini, S. Cigada, M. C. Gatti, & S. Gilardoni (Eds.), *L'analisi linguistica e letteraria* (Vol. Vol. 16(01), pp. 15–36).
- Zöllner, J., & Benz, C. (2015). The role of units in the concept of length for four- to six-year-old children. *Asia Pacific Journal of Research in Early Childhood Education*, 9(3), 67–83. <https://doi.org/10.17206/apjrece.2015.9.3.67>
- Zöllner, J., & Reuter, F. (2019). Längen als Veranschaulichung arithmetischer Zusammenhänge. Vom dreidimensionalen Material über die zweidimensionale Darstellung zur abstrakten Zahlvorstellung mit Unterstützung von Längen [Lengths as a visualization of arithmetic relations. From three-dimensional material to two-dimensional images to abstract number concepts with the help of length]. *Fördermagazin Grundschule*, 2019(4), 28–31.

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Assessing Children's Cardinality Competence Using Authentic Observation in Kindergarten



Reinert André Rinvold and Leif Bjørn Skorpen

Introduction

Authentic assessment is an approach in which information is captured through observation and other ways from children's natural or facilitated play and daily skills (Neisworth & Bagnato, 2004). This approach aligns well with learning in play and everyday situations, as advocated by, for instance, Gasteiger (2012), Gasteiger and Benz (2018), Reikerås et al. (2012) and the Norwegian Framework Plan for Kindergartens (Ministry of Education and Research, 2017). According to the Framework Plan, formative assessment continuously plays a role in both individual children's well-being and all-round development, in providing them with necessary additional support and in uncovering special educational needs (Ministry of Education and Research, 2017). In Norway, kindergartens are not organised in classrooms, and children' age range from 0–5 years old. In this study, we restrict our attention to the assessment of children from 3–5 years. The purpose of the assessment is to monitor progress and to identify children needing additional support, rather than eligibility or programme planning. According to Reikerås et al. (2012); Reikerås (2016), authentic observation of cardinality competence has been used in the Stavanger Project, a Norwegian study in which kindergarten teachers and other staff observed young children (2–5 years of age) in their usual surroundings in play and everyday situations. Mathematical skills in counting and enumeration, as well as skills from other mathematical areas, were observed. Rinvold and Skorpen (2024) presented results from another Norwegian study that built upon the Stavanger

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Project in its intent to observe everyday and play situations but which primarily focused on counting and enumeration. In the work of Rinvold and Skorpen (2024), kindergarten staff were interviewed about their observations when using this kind of observation material. Most of the observed children were around 3 years old.

In the field of developmental psychology, children's mathematical development is often studied through experiments in controlled laboratory settings. Clinical laboratory testing in developmental psychology and authentic assessment is seen as two extremes in the assessment of young children's competencies. Even standardised tests based on psychometrics have been criticised for mismeasuring children's competence, labelling children and not being useful for supporting children's learning (Bagnato, 2005; Neisworth & Bagnato, 2004). Despite these shortcomings, research in developmental psychology has achieved interesting results relevant to early mathematics education. Geary et al. (2018) importantly compared the mathematical skills or knowledge of 3–4-year-olds with their subsequent mathematical school achievement, concluding that conceptual understanding of cardinality before the age of five predicts superior school-entry number system knowledge and that delay of such understanding may indicate risk for later failure in mathematics.

In this study, we analyse the interview data of Rinvold and Skorpen (2024) using a framework for *observation contexts* based on (Neisworth & Bagnato, 2004) to measure qualities of authentic observation of cardinality-related tasks. This study contributes to a theoretically and empirically based discussion about designing tasks for authentic observation of young children's cardinality competence, how authentic observations of such competence can be conducted or whether it is possible to do at all. To strengthen the theoretical discussion and compare cardinality-related tasks intended for authentic observation with corresponding tasks from developmental psychology, we conducted a document analysis of cardinality-related tasks and observation materials. While observation contexts include who observes, how observation is conducted, where it is done and what kind of influence adults have on children's behaviour during observation (Neisworth & Bagnato, 2004), document analysis is concerned with the *observation task* itself and how it is assessed and socially valued. Neisworth and Bagnato (2004) described two *authenticity properties* that are suitable for analysing observation tasks. The first authenticity property, *functionality*, distinguishes between whether a task is assessed against the stated function it was supposed to achieve or against rules describing the form of the child's supposed standardised behaviour when performing the task (Neisworth & Bagnato, 2004). The second authenticity property, *valuableness*, relates to whether the aim of the task is worthwhile and socially valued (Neisworth & Bagnato, 2004). Comparisons with corresponding tasks from developmental psychology is covered by another kind of authenticity property, *comparability*, which is the probability that mastery of a cardinality task in the observation materials implies mastery of the corresponding task from developmental psychology. To broaden the analysis and include international data, we also analysed tasks from a German observation material called "Lerndokumentation" (Steinweg, 2006). According to Gasteiger (2012), Lerndokumentation (Steinweg, 2006) is used in everyday and play situations. While the term "authentic observation" is not explicitly used in the

material, it is based on a similar view of assessment. Lerndokumentation includes observation tasks in the mathematical areas “number and counting” and “number and structure,” but as in the Stavanger project, other mathematical areas are also covered. Neither of the authenticity properties are explicitly described in the observation materials, so they are inferred from *task descriptions* which systematizes relevant information about the observation tasks and their assessment.

The overarching research question of this study is whether and how authentic observation of cardinality competence of children aged 3–5 years can be done. This question is investigated and discussed through three research questions, the first of which is based on interview data and investigates how kindergarten staff used the observation material of Rinvold and Skorpen (2024). The second question asks for a systematic overview of relevant information of the observation materials, while the third question asks for the authenticity properties. The observation materials analysed in the second and third research questions are the same as in Rinvold and Skorpen (2024), the MIO material used in the Stavanger Project (Davidsen et al., 2008) and Lerndokumentation (Steinweg, 2006, 2015), as well as two selected tasks and how they are described in developmental psychology (Geary et al., 2018; Le Corre et al., 2006; Wynn, 1990).

Three research questions are posed:

1. In which observation contexts did kindergarten staff observe cardinality-related tasks, and how did the observation context vary between tasks?
2. How are cardinality tasks described in the observation materials and tests from developmental psychology?
3. How do the authenticity properties of functionality, valuableness and comparability of cardinality tasks vary across tasks and between the three observation materials and corresponding tasks from developmental psychology?

Theoretical Background and Framework

Psychometric test design is based on how items discriminate between children (Kline, 2015), and the items of conventional testing often only represent competences to be tested. We do not necessarily want to teach the skill asked for by the item, for example standing on one foot (Neisworth & Bagnato, 2004). A mathematical example could be counting backwards for which the aim may not be highly socially valued, but from a certain age failure may indicate later difficulties with mathematics. The competencies assessed in authentic assessment, however, are worthwhile, teachable and socially valued (Neisworth & Bagnato, 2004). We do not discuss whether tasks are teachable but focus our attention on whether the aims of tasks are worthwhile for the child and valued by adults, and these two latter authenticity properties together is denoted *valuableness*. Authentic assessment emphasises the function of behaviour rather than its form (Neisworth & Bagnato, 2004), allowing observers to recognise a child's competency in various situations rather than

looking for standardised, accepted behaviours. We interpret this as the task having an objective or aim. Assessment based on function means that accomplishing this aim is what decides mastery, not the strategy or procedure applied to achieve the objective. Assessment based on form suggests that only one strategy or procedure is accepted as mastery of the task.

Neisworth and Bagnato (2004) offered a theoretical framework for observation contexts or measurement contexts, which is a continuum from decontextualised to contextualised. “Decontextualised” and “contextualised” are also referred to as “clinical” and “natural”, respectively, with “simulated” and “analogue” being contexts between the two extremes. Each measurement context is characterised by four characteristics abbreviated by the interrogative words “where”, “what”, “how” and “who”. A simulated measurement context occurs in a place unfamiliar to the child but is purposefully made to look similar to the child’s usual environment (where); on-demand behaviour, which is measured in a way that is less free than prompted behaviour (what); using structured tests (how); and is conducted by psychologists or other certified professionals (who). An analogue measurement context takes place in the child’s usual surroundings under normal conditions (where), but some natural or typical behaviour is prompted, or some arrangement is made to increase the likelihood of observing certain behaviour (what). Analogue measurement is conducted through direct observation (how) by kindergarten staff and caregivers (who). A natural measurement context is similar to an analogue measurement context, but it measures spontaneous behaviour (what).

In the developmental psychology field, *the cardinal word principle* denotes that the last number word in a correct count of a set measures the size of the set, its numerosity or cardinality (Le Corre et al., 2006; Wynn, 1992). “Last-word-responding” refers to a situation in which a correct count of a set of items is followed; for instance, saying “there are four dogs” after counting four dogs (Fuson et al., 1985). This is often preceded by an emphasis on the last word in the count (Mix et al., 2012). The “How-many” task refers to when children are asked, for instance, “how many dogs are there” after counting the dogs (Wynn, 1990). This procedure has been criticised as possibly confusing children to think that adults ask because the children made mistakes in their counting (Le Corre et al., 2006). According to Wynn (1992), children may understand that the last number word indicates the numerosity of the set but instead prefer to answer by recounting the set. Thus, last-word-responding is not a valid measure of cardinality understanding. Le Corre et al. (2006) improved the procedure by placing the question into context and asking, “Can you tell Big Bird how many toys he has here?” Mastery of last-word-responding is insufficient for complete cardinality competence, as the child might not understand that the last counting word refers to the entire set and to the numerosity of the set (Fuson et al., 1985). Gradually a test called “Give-N” has become the most used measure, partly because of validity problems with the How-many task in developmental psychology in terms of measuring the conceptual understanding of how counting is related to cardinality (Geary et al., 2018; Wynn, 1990, 1992). While there exist some variations in how the Give-N task is described in developmental psychology, Wynn’s (1990) method is the most followed: A pile

of objects is placed in front of the child, and the child is to give a number of such objects to a puppet, a toy animal or the experimenter. Wynn's (1990) work provides an example of how the child is asked: "Could you give Big Bird two dinosaurs to play with?" One testing procedure that originated in the works of Wynn (1990, 1992) starts with Give-1 and continues to Give-6 in case of success but does not ask for a higher numerosity when a child fails on a try. Then, the child is asked once more to give the preceding numerosity. The child is asked, for instance, "Is that three?" when asked to give three, and then "Can you count to make sure?" (Wynn, 1992). The knower level is the highest numerosity in which a child succeeds at least two of three times.

While the Give-N task has been used extensively in research, it has scarcely been investigated in terms of how the task can be applied in kindergarten practice. One exception to this is the work of McDonald et al. (2021), in which educators' perceptions of implementing tasks and a test called "the Numeracy Screener" were studied. The authors determined whether the task was effective in identifying children at risk of falling behind, and how the educators used the assessment results to inform and target their instruction, finding that all respondents felt that children were engaged with the Give-N task.

Methodology

The interview data in this study, which primarily concerns the first research question, comprise 12 interviews with kindergarten staff, each lasting approximately 45 min. The central theme was how observations of children's mathematical competencies were conducted. In total, 61 children aged 3–5 years from 12 different kindergartens (K1–K12) were observed using the observation material developed by us and our colleagues (Rinvold & Skorpen, 2024). All participants gave their active informed consent. The material, inspired by the MIO material (Davidsen et al., 2008), was developed in 2019–2020 and covered counting and enumeration. Children's observed competence is not investigated in this chapter, which instead focuses on how kindergarten staff observed children and the material they used for observation. The second and third research questions are investigated by document analysis of how cardinality-related tasks and tests are described in the observation materials from our project, MIO (Davidsen et al., 2008) and *Lerndokumentation* (Steinweg, 2006, 2015), as well as the developmental psychology papers produced by Geary et al. (2018), Wynn (1990) and Le Corre et al. (2006). All three observation materials consist of an observation scheme and a handbook or manual. However, *Lerndokumentation* has a shorter manual with less detailed information about each item, instead containing general comments referring to supposed teacher knowledge. Our choice of Geary et al. (2018) is linked to their claims about the importance of early cardinality understanding, and their Give-N task is based on Wynn (1990). Geary et al. (2018) did not have a How-many task, and Le Corre et al.

(2006) was chosen because they dealt with validity problems of the How-many task and criticised earlier versions of this test.

To distinguish psychometric tests from authentic assessment, “Give-N” denotes the laboratory test, and “Fetch-N” denotes the similar everyday task of fetching N objects of a given type when prompted by the number word for N. How-many denotes tasks in which a child correctly counts a set of objects and then tells how many there are in some way, but not necessarily with last-word-responding. An adult can ask how many it is either before or after the child’s counting, but a child may also perform the task without any external prompt.

A framework based on Neisworth and Bagnato (2004) but modified to fit observations in kindergartens similar to those in Norway (Table 1), is used to analyse the interview data, and also to some extent the observation tasks. Horizontally, the framework outlines different observation contexts, and vertically, it details the *where*, *what*, and *how* of the observations. A new observation context, termed “involved”, accounts for situations in which the staff are more engaged than in merely prompting behaviour but still do not conduct structured testing with on-demand behaviour.

Additionally, we use the term “structured” instead of “simulated” when the observations are carried out by kindergarten staff rather than psychologists. For the same reason, we omit the column for clinical measurement contexts and the row for “who” from Neisworth and Bagnato (2004). In the case of tasks from developmental psychology, the terms “clinical” and “simulated” are used. Behaviours are arranged on a continuum from on-demand to spontaneous. The natural observation context refers to a play or everyday situation that is not modified in any way for observation. This kind of behaviour is spontaneous, but the entire situation is not necessarily initiated by the child. The distinction between “involved” and “analogue” lies in the degree of adult involvement. In the analogue observation context, adult interference is minimal and unobtrusive. Structured and involved observation contexts are organised by the kindergarten staff, but they differ in that the former is planned, and the latter is a spontaneous response to ongoing play or everyday situations.

Themes for descriptions of observation tasks to answer the second research question were found inductively by thematic analysis of the three observation materials: task formulation, information about assessment, example, and information about situation. The structured observation materials are more suitable for thematic

Table 1 Framework for observation contexts

	Structured	Involved	Analogue	Natural
Where?	Separate room or special conditions	Arranged circumstances	Slightly modified circumstances	Usual surroundings
What kind of behaviour?	On-demand behaviours	Facilitated behaviours	Prompted behaviours	Spontaneous behaviours
How?	Structured observation	Participant observation	Direct observation	Direct observation

Table 2 Themes for description of observation tasks

Theme	Definition
Task formulation	How the task is formulated in the observation scheme or manual
Information about assessment	Assessment criteria, including strategies that excludes mastery and strategies that must be included
Example	Examples of the task as it will be formulated to a child, or examples of a child’s mastery of the task
Information about situation	Information about how an observation situation should be arranged

analysis than the research papers from which the psychological tests are taken. However, the themes are also applied to the latter (Table 2).

The third research question concerns three authenticity properties—functionality, valuableness and comparability—as defined in the introduction and inferred from the task descriptions answering the second research question. According to (Geary et al., 2018), a child is considered to know the cardinal word principle when mastering Give-N for at least $N = 5$. The reason for this is that children may decide cardinality by subitizing when $N < 5$ and thus do not depend on counting. Similarly, also for How-Many it is not necessary to consider $N < 5$ in view of the subitizing range. Thus, to decide comparability for Fetch-N, comparison of Fetch-5 with Give-N is enough.

Results

The presentation of results is organised by research question. We first present results from the analysis of interview data for different cardinality-related tasks.

Observation Contexts from Interview Data

The findings are exemplified by excerpts from the interviews, and in the excerpts, “T” stands for a kindergarten teacher or staff member. The findings for Research Question 1 are summarised in Table 3.

Some observers found the Fetch-N task difficult to observe in play and everyday situations, leading to observation being conducted in organised rather than natural situations.

T: I found it challenging to fetch objects. Then, we brought some items into a separate room. Both of us were there so that we could observe and instruct, applying those terms. (K1)

The observation context mentioned by the teacher above was structured and can involve on-demand behaviours. The data included examples of observations in a

Table 3 Observation contexts for different observation tasks

Observation task	Observation context
Fetch-2	Structured, involved, analogue
Fetch-N, $N \geq 3$	Structured, involved
How-many	Analogue, natural
One-to-one counting	Analogue, natural
Set-construction	Natural

separate room where facilitated or prompted behaviours occurred. Additionally, the involved observation contexts were employed for Fetch-N, as illustrated below.

T: One example is that we took out the car box. Can you give me three cars? Can you give me five cars? Otherwise, we used small teddy bears of different colours. (K1)

Both toy cars and teddy bears were common in children's play, but the kindergarten staff organised an activity that resembled a laboratory Give-N task more than a slight modification of a potential play or everyday situation. Fetch-2 differs from Fetch-N for higher N values in that analogue behaviour was observed, as indicated below, meaning that the observation context depended on N.

T: Can you—there was a shortage of two—can you find two more to add on top? (K10)

Direct observation of spontaneous child-initiated behaviour was infrequent, according to interviewed kindergarten staff, although it did occur (Rinvold & Skorpen, 2024).

T: The boy who succeeded each time counted extensively. He did this independently as well. Counting became part of his activities all day long in the kindergarten, with trains and cars. He would line them up and count [on his own initiative]. (K1)

T: He wanted the adults to join him to watch or participate in parking cars in the car park, to count all the cars and how many he had room for. (K1)

The boy who, according to the kindergarten teacher, succeeded each time, created his own sets by arranging toy cars, which he then counted. The arranging task we termed "Set-construction", and the task was followed by the How-many task. Sometimes the boy initiated counting independently, but at other times he sought the kindergarten staff to prompt him to count, indicating both spontaneous and prompted behaviour, even if the prompting was wished for by the child. One-to-one counting and the How-many task are typically conducted in analogue or natural observation contexts during everyday or play situations, as suggested by the following excerpt.

T: For example, counting or one-to-one correspondence counting, or How-many, we simply did in everyday situations—the play we were engaged in, trains, cars or whatever we were doing. (K1)

As seen in Table 3, there is a clear difference between the observation context for Fetch-N and the How-many and One-to-one counting tasks. The former task was observed in structured or involved observation contexts, while the latter tasks were

observed in analogue or even natural observation contexts. An exception is Fetch-2, which also were observed in the analogue observation context.

Task Descriptions of Give-N and Fetch-N Tasks

The second research question yielded task descriptions that are used to infer the results of the third research question. This information was easily extracted from the documents. Table 4 provides the task descriptions of the Give-N task from Geary et al. (2018) and Wynn (1990).

The information about the Give-N situation including organization and that toy dinosaurs are placed in a pile, points to a clinical observation context, but Wynn (1990) indicated that the child was allowed to play with the dinosaurs before the test, and this may suggest a simulated observation context, but limited information is given in Geary et al. (2018). The testing is thought to happen in minimalistic situations in which nothing else is available for the child aside from a pile of objects of a given type from which the child is to fetch as many objects as the experimenter asks for.

Table 5 shows the task descriptions of the Fetch-N tasks from the observation materials.

Authenticity Properties of Give-N and Fetch-N Tasks

The Give-N task is functional since there are no assessment restrictions on the use of strategies (Table 5), but practically possible strategies are limited by strict organisation. To fetch a number of objects of a type, given a number word is valuable,

Table 4 Descriptions of the Give-N task from developmental psychology

	Give-N task from Geary et al. (2018) and Wynn (1990)
Give-N	N = 1, 2, 3, 4, 5, 6
Task formulation	Children are asked to give the experimenter exactly 1, 2, 3, 4, 5 or 6 objects from a pile.
Information about assessment	The score is the largest set size for which the response was correct at least two of three times. The child is considered to know the cardinal word principle with a score of five or six.
Example	Give me six pigs.
Information about situation	Fifteen toy dinosaurs are placed in front of the child in a pile. The child is first allowed to play with the toys. The child is then asked to give the experimenter 1, 2, 3, 4, 5 or 6 objects beginning with set size 1. It advances to the next size after a correct response but goes down one size after an incorrect response (Wynn, 1990).

Table 5 Descriptions of Fetch-N tasks in the observation materials

	Rinvold and Skorpen (2024)	MIO	Lerndokumentation
Fetch-N	N = 2, 3, 5, 6	N = 2, 3, possibly 5 (Set-table-5)	N = 5
Task formulation	To fetch N objects, given a number word	Fetches two (three) objects on request and sets the table for five persons	Can give you five objects simultaneously
Information about assessment	Cannot use one-to-one correspondence or visual cues	It is important to discern “will not” from “cannot”.	Simultaneous fetching is needed for mastery.
Example	Can you fetch five plates?	Five children are to be seated around this table. Please set the table.	Take three gummy bears.
Information about situation	Plenty of objects to be fetched available in a pile	Situation given for N = 5, else no	The child must take objects from a greater set.

since valuableness regards the aim of the task. This is independent of the usefulness of the context, which is less in laboratory cases.

In the Rinvold and Skorpen (2024) observation material, the Fetch-N task is formulated as “To fetch N objects”. The objects must be fetched based on information provided by a number word without relying on visual cues combined with one-to-one correspondence. These assessment restrictions in strategy use mean that the Fetch-N task is not functional, mimicking the minimalistic condition for Give-N. Since the requirement that objects of the actual kind are available in a pile is satisfied, mastery of the task implies mastery of Give-N.

The MIO material includes the Fetch-2 and Fetch-3 tasks but the task “set the table for five persons” (Set-table-5) replaces Fetch-5. All these tasks are obviously functional and valuable, but some reflection is needed to decide if mastery of Set-table-5 implies mastery of Give-N. During Set-table-5, it is possible that a child is requested to fetch five plates upon hearing the number word “five”. However, if five individuals are already seated around the table, the task might be completed through one-to-one correspondence without relying on the number word “five” as information, as the following example from our interview data demonstrates.

T: Setting the table was fun to demonstrate. It was a success. Initially, the exact number of plates and the number of people seated were placed on the table. When we sat down, we asked the child to place one plate for each person. She then set the table using pairwise matching for each seated individual. (K6)

The child demonstrated mastery of the task “Can hand out one item to each person”, which is included as an observational task in both MIO and our observation material. Mastery of this task does not imply mastery of the Give-N task, since five persons or other visual cues cannot be present according to the minimalistic condition of Give-N. The provided example in the MIO handbook, “Around this table,

Table 6 Authenticity properties of Fetch-N tasks in the observation materials

	Rinvold and Skorpen (2024)	MIO	Lerndokumentation
Functionality	No	Yes	No
Valuableness	Yes	Yes	Somewhat reduced
Comparability	N = 5, 6: highly probable N = 2, 3: subitizing possible	Not from description N = 5, probable from example	Probable

five children shall be seated. Can you set the table?” (Davidsen et al., 2008, p. 47), implies that the five individuals are not yet seated when the task begins. If this is the case and no other visual cues for five persons are available, then being able to set five plates possibly implies mastery of Give-5. When continuing with placing out knives and forks, the plates are a visual cue for five, which according to the assessment information in Table 5 must not be used in Give-5. However, the existence of one probable Give-5 situation during performance of the task is enough to show probable mastery of Give-N.

Lerndokumentation contains only Fetch-N for $N = 5$, which is above the subitizing range, so mastery of this task is a candidate for mastery of Give-N, even if the lack of $N = 6$ means that mastery of the task implies general Give-N mastery with a slightly lower probability than in our observation material. The requirement of simultaneousness should rule out that kindergarten staff assess children to master the task of Give-5 based on the use of one-to-one correspondence with visual cues. That the child must take objects from a greater set is imprecise and open to interpretation, but can mean almost the same as if the objects must be taken from a pile of objects. The restriction to simultaneous fetching implies that the task is not functional, but also a possible drawback since some children may not be able to carry with them five objects like plates. Simultaneousness may not be a natural restriction and make the task less valuable to learn.

Table 6 summarises functionality, valuableness and comparability for the Fetch-N tasks in the observation materials.

Task Descriptions of How-many Tasks

We begin with the How-many task from developmental psychology. Table 7 shows the task descriptions of the How-many task from Le Corre et al. (2006).

The How-many task from Le Corre et al. (2006) may describe a simulated observation context as the task and follow up question is formulated in a play-like situation with the intention of helping Big Bird to count. However, the use of a plastic tray which provides the toys to be counted has similarities with a clinical observation context.

Table 8 shows the task descriptions of the How-many tasks from the observation materials.

Table 7 Descriptions of How-many tasks from developmental psychology

	How-many task from Le Corre et al. (2006)
How-many	N = 2, 3, 5, 6
Task formulation	The experimenter (E) first introduced each child to a Big Bird puppet, saying, “Big Bird has a problem. He forgot how to count! Could you help Big Bird count his toys?” Le Corre et al. (2006, p. 138)
Information about assessment	The child must know the count list and answer with the correct number word when asked to do so.
Example	Not given
Information about situation	E then showed the child a tray of two, three, five, or six identical toys on a plastic tray, and the child was asked to count the toys for Big Bird. Then, E presented each set size twice in a pseudorandom order, and on one of the trials for each set size, the child was probed after the count for a cardinal response with the phrase, “OK, so can you tell Big Bird how many toys he has here?”.

Table 8 Descriptions of How-many tasks in the observation materials

	Rinvold and Skorpen (2024)	MIO	Lerndokumentation
How-many N	N not specified	5	N not specified
Task formulation	Can answer how many objects it is in a set after counting the objects	Can answer how many objects it is in a set of five after counting the objects	Counts objects correctly to...
Information about assessment	After correct counting, the child, on its own initiative says, for example, “it is four apples.”	Can answer the question “How many is it?” after a correct count to five	Unclear, but the material mentions the cardinal word principle
Example	Answers “it is four apples.”	After Gustav has counted the five cars, he says: “I have five cars!”	Not given
Information about situation	No information	No information	No information

Authenticity Properties of the How-many Tasks

The How-many task from Le Corre et al. (2006) is not functional, since the requirement of answering How-many by counting is considered a restriction in possible strategies.

In both our observation material and the MIO material, the only accepted way of mastery of the How-many task is last-word-responding. For example: “It is five cars” in response to “How many cars is it?” is interpreted as mastery based on form rather than function, as a child is not given the opportunity to demonstrate the same competence in other ways. An alternative way of answering is to count to five and

Table 9 Authenticity properties of How-many tasks in the observation materials

	Rinvold and Skorpen (2024)	MIO	Lerndokumentation
Functionality	No	No	No (strictly interpreted)
Valuableness	Yes	Yes	Yes
Comparability	Probable	Probable	Unclear

emphasise the last counting word (Mix et al., 2012). In both MIO and our observation material, one-to-one counting of sets and last-word-responding are separate tasks. In MIO, the formulation of the former task is “counting to five while correctly pointing at objects” (Reikerås, 2016, p. 66). This is also seen as an assessment based on form rather than function, since counting while using gaze is an alternative to physical pointing. However, the How-many task which includes last-word-responding can be considered a task with a socially valued aim, even if there may be only one strategy for achieving the aim. One difference between our observation material and MIO is that only the former require last-word-responding on the child's initiative. MIO does not mention such initiative explicitly, and in the given example, the child is not said to answer a question from an adult. For both our observation material and MIO, it is probable that mastery of the How-many task also implies mastery of the How-many task from Le Corre et al. (2006), as the information about assessment is similar, even if different observation contexts may have an influence.

Lerndokumentation contains the task “can count objects correctly to ...”. This requires counting, which, when strictly interpreted, means that the task is not functional. Despite this, different ways of supporting counting are allowed, not only pointing. Last-word-responding is not mentioned in the task formulation but in a supporting text for the theme “Experience of number and counting”, the cardinal word principle is an important competence. This could mean that competence is assessed based on several observation tasks, rather than just one.

Table 9 summarises functionality, valuableness and comparability for the How-many tasks in the observation materials.

Discussion

This study discusses if and how children's cardinality competence can be observed by kindergarten staff in a way that is faithful to the requirements of authentic assessment and such that sufficiently similar cardinality competence is assessed as in corresponding tasks used in developmental psychology. The first research question concerns the observation contexts in which kindergarten staff observed cardinality-related tasks and how observation context varied between tasks. The results show that overall, the How-many task was observed in more authentic observation contexts than the Fetch-N task, except for Fetch-2, which is in the subitizing range, and therefore, not relevant to understanding how counting is related to cardinality. In some participating kindergartens, they thought that the Fetch-N task was more

difficult to observe than the How-many task. This led them to observe Fetch-N mostly in more structured observation contexts (Rinvold & Skorpen, 2024), but in most cases the Fetch-N task seems to have been observed in sufficiently authentic observation contexts. Typically, this is the involved observation context in which aspects of children's play or everyday situations are starting points for facilitation. The involved observation context is a new contribution, although according to Neisworth and Bagnato (2004) observation of facilitated play can be part of authentic assessment. The application of Give-N in a kindergarten classroom in McDonald et al. (2021) occurred in a structured observation context, but even then, children were engaged according to the kindergarten teachers, in partial alignment with the work of Rinvold and Skorpen (2024). This challenges some of the advantages of authentic observation, but engagement is not necessarily enough to give the same advantages as observation in play and everyday situations. Even the natural observation context occurred, an observation context without adult organization, but observation without such organisation was rare (Rinvold & Skorpen, 2024). The natural observation context was seen for the How-many task, one-to-one counting and the Set-construction task, which involved a child who made sets of toys and either counted on his own initiative or signalled the staff to prompt him to count or to say how many toys were in the set. Our findings about observation contexts show that the framework is suitable, that observation of cardinality-related tasks can be done in a range of observation contexts, and that observation of such tasks seems to be possible in sufficiently authentic observation contexts. Our finer-grained extension of the observation context framework of Neisworth and Bagnato (2004) and the fact that the full spectrum of observation contexts appeared reinforces what is also implicit in their framework, that there are degrees of authenticity rather than a clear border between authentic and non-authentic observation contexts.

The second research question asks how cardinality tasks are described in the observation materials and tests from developmental psychology and is answered with tables in which task descriptions are divided into themes. This makes it possible to compare similar tasks in different observation materials and tests and particularly infer authenticity properties asked for in the third research question: functionality, valuableness and comparability. Both the Give-N task and the Fetch-N tasks in our observation material and Lerndokumentation are valuable, albeit less so the clinical test. The Give-N task is functional, but the Fetch-N tasks are not, since they have restrictions in allowed strategies. Set-table-5 is a valuable task that is used in place of Fetch-5 in the MIO material. This alternative composite and complex task can have Fetch-5 as a subtask, so Set-table-5 may be a valid and authentic way of assessing Give-N competence. The example in the MIO-handbook that "five children are to be seated around this table, please set the table" makes it probable that Fetch-5 appears as a subtask of Set-table-5. The comparability of the Fetch-N tasks from our observation material and Lerndokumentation is probable, as with Set-table-5 if properly organised. The How-many tasks of our observation material, MIO and developmental psychology all in some way claim that the child explicitly repeats the last counting word, and all the How-many tasks require that counting is used. These requirements mean that none of the How-many tasks are functional, but all of them are valuable. Comparability of the How-many tasks from our

observation material and MIO is probable, since the only difference from the clinical task is the observation context.

Neisworth and Bagnato (2004) decomposed the concept of authenticity into concepts as the functionality of tasks, if a task is worthwhile, teachable and socially valued, and measurement contexts or observation contexts. This chapter demonstrates how these concepts can be applied in the analysis of interview data concerning observation and the document analysis of observation materials. This study contributes to increased knowledge about the possibilities of assessing young children's understanding of cardinality through authentic observation. The analysis shows that valuableness and comparability are possible to achieve; however, combining authentic observation contexts and assessment based on function rather than form turned out to be a challenge in the investigated observation materials. For the Give-N and Fetch-N tasks, the more structured or clinical observation contexts make it possible to reduce possible strategies to choose for the child, but for the How-many task not even the clinical observation context avoids that counting and last-word-responding are required. Two possible solutions to this problem are suggested, and both should be investigated through further research. The first solution is found by determining whether it is possible to reduce the degree of authenticity to what we call the involved observation context to achieve enough functionality and validity. The kindergarten staff we studied occasionally organised the Fetch-N task in a way that made unwanted strategies less likely, but this caused the observation context to be less authentic. A pragmatic solution is to balance organisation and authenticity by using the involved observation context. Both Rinvold and Skorpen (2024) and McDonald et al. (2021) argued that children can be engaged even if the task is somewhat organised. A second suggestion is to observe composite and complex everyday tasks that are organised so that Fetch-N or How-many tasks probably appear as subtasks. Embedding cardinality tasks to be assessed into authentic composite tasks, as when Fetch-5 is embedded in Set-table-5, may solve problems with functionality. According to Neisworth and Bagnato (2004), authentic assessment requires collaboration between several everyday settings and with assessment partners. Everyday tasks such as Set-table-5 are advantageous in that parents and other non-expert adults can contribute to observation, but as we have seen, Set-table-5 does not necessarily measure Give-N competence without organisation of the tasks that make Fetch-5 probable to occur. Both solutions seem to depend on the competence of the kindergarten staff, but this needs further research. A task that was not in our observation material, the Set-construction task, appeared as a child-initiated task combined with one-to-one counting and the How-many task. The validity problems of the How-many task pointed out by Wynn (1992) do not apply to this combination of tasks. Set-construction undertaken on the initiative of a child may only occur with children who are not at risk of not understanding the cardinality-word principle. It is a question for further research whether the Set-construction task followed by the counting of constructed sets can be used to assess children's cardinality competence. As far as we know, this task, when not initiated by a Fetch-N or Give-N task in which it is common to say that a set is produced, created or constructed (Baroody & Lai, 2022), has not been previously described.

This study is limited in that the practical application of only one observation material is investigated and that we use only interview data, rather than observations of children and kindergarten staff. This limitation is particularly relevant to the German Lerndokumentation, as its cultural context differs from the Norwegian one, with which we are more familiar. Another limitation is that many variables, such as the age and competence of the children and the competence of the kindergarten staff, may influence the distribution of observation contexts and the functionality of tasks. Kindergarten practice and research are recommended to use our results and analysis methods to design or improve observation materials for authentic assessment of young children's mathematical competence and to investigate the competence required for kindergarten staff using these materials.

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References

- Bagnato, S. J. (2005). The authentic alternative for assessment in early intervention: An emerging evidence-based practice. *Journal of Early Intervention*, 28(1), 17–22. <https://doi.org/10.1177/105381510502800102>
- Baroody, A. J., & Lai, M. (2022). The development and assessment of counting-based cardinal number concepts. *Educational Studies in Mathematics*, 111(2), 185–205. <https://doi.org/10.1007/s10649-022-10153-5>
- Davidson, H. S., Løge, I. K., Lunde, O., Reikerås, E., & Dalvang, T. (2008). *MIO-Matematikken—Individet-Omgivelsene. Håndbok*. [MIO—The mathematics, the individual and the environment. Handbook] Aschehoug.
- Fuson, K. C., Pergament, G. G., Lyons, B. G., & Hall, J. W. (1985). Children's conformity to the cardinality rule as a function of set size and counting accuracy. *Child Development*, 56, 1429–1436. <https://doi.org/10.2307/1130462>
- Gasteiger, H. (2012). Fostering early mathematical competencies in natural learning situations—Foundation and challenges of a competence-oriented concept of mathematics education in kindergarten. *Journal für Mathematik-Didaktik*, 33(2), 181–201.
- Gasteiger, H., & Benz, C. (2018). Enhancing and analyzing kindergarten teachers' professional knowledge for early mathematics education. *The Journal of Mathematical Behavior*, 51, 109–117. <https://doi.org/10.1016/j.jmathb.2018.01.002>
- Geary, D. C., vanMarle, K., Chu, F. W., Rouders, J., Hoard, M. K., & Nugent, L. (2018). Early conceptual understanding of cardinality predicts superior school-entry number-system knowledge. *Psychological Science*, 29(2), 191–205. <https://doi.org/10.1177/0956797617729817>
- Kline, P. (2015). *A handbook of test construction (psychology revivals): Introduction to psychometric design*. Routledge.
- Le Corre, M., Van de Walle, G., Brannon, E. M., & Carey, S. (2006). Re-visiting the competence/performance debate in the acquisition of the counting principles. *Cognitive Psychology*, 52(2), 130–169. <https://doi.org/10.1016/j.cogpsych.2005.07.002>
- McDonald, J. A., Merkley, R., Mickle, J., Collimore, L. M., Hawes, Z., & Ansari, D. (2021). Exploring the implementation of early math assessments in Kindergarten classrooms: A research-practice collaboration. *Mind, Brain, and Education*, 15(4), 311–321. <https://doi.org/10.1111/mbe.12293>

- Ministry of Education and Research. (2017). *Framework plan for kindergartens: Content and tasks*. Norwegian Directorate for Education and Training.
- Mix, K. S., Sandhofer, C. M., Moore, J. A., & Russell, C. (2012). Acquisition of the cardinal word principle: The role of input. *Early Childhood Research Quarterly*, 27(2), 274–283. <https://doi.org/10.1016/j.ecresq.2011.10.003>
- Neisworth, J. T., & Bagnato, S. J. (2004). The mismeasure of young children: The authentic assessment alternative. *Infants & Young Children*, 17(3), 198–212.
- Reikerås, E. (2016). Central skills in toddlers' and pre-schoolers' mathematical development, observed in play and everyday activities. *Nordic Studies in Mathematics Education*, 21(4), 57–77. <https://doi.org/10.7146/nomad.v21i4.148742>
- Reikerås, E., Løge, I. K., & Knivsberg, A.-M. (2012). The mathematical competencies of toddlers expressed in their play and daily life activities in Norwegian kindergartens. *International Journal of Early Childhood*, 44(1), 91–114. <https://doi.org/10.1007/s13158-011-0050-x>
- Rinvold, R. A., & Skorpen, L. B. (2024). Play responsive assessment of children's mathematical proficiencies in Norwegian kindergarten. In H. Palmér, C. Björklund, E. Reikerås, & J. Elofsson (Eds.), *Teaching mathematics as to be meaningful—Foregrounding play and children's perspectives: Results from the POEM5 conference, 2022* (pp. 185–196). Springer. https://doi.org/10.1007/978-3-031-37663-4_14
- Steinweg, A. S. (2006). *Lerndokumentation Mathematik*. Senatsverwaltung für Bildung, Wissenschaft und Forschung.
- Steinweg, A. S. (2015). *Lerndokumentation Mathematik*. Senatsverwaltung für Bildung, Wissenschaft und Forschung.
- Wynn, K. (1990). Children's understanding of counting. *Cognition*, 36(2), 155–193. [https://doi.org/10.1016/0010-0277\(90\)90003-3](https://doi.org/10.1016/0010-0277(90)90003-3)
- Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, 24(2), 220–251. [https://doi.org/10.1016/0010-0285\(92\)90008-P](https://doi.org/10.1016/0010-0285(92)90008-P)

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Pre-Service Teachers' and Children's Conceptual Understanding of Quadrilaterals



Mark Sprenger

Introduction

Conceptual understanding is a basis for mathematical knowledge and for applying mathematical relationships (Hischer, 2021). Many studies show that children often have difficulties developing an adequate conceptual understanding for different geometric shapes like “quadrilateral”, “rectangle”, “square” or “triangle” (Bartolini Bussi & Baccaglioni-Frank, 2015; Bruns et al., 2021; Clements et al., 1999; Maier, 2019; Maier & Benz, 2020; Tsamir et al., 2014; Unterhauser, 2020). Results indicate that children still have difficulties in understanding these terms in secondary school (Fuys et al., 1988; Heinze, 2002). In order to adequately address these difficulties in school, it is important that teachers themselves have a solid conceptual understanding of geometrical shapes. However, studies also indicate that this is not always the case (Fujita & Jones, 2007; Gokbulut & Sen, 2019; Molitoris Miller, 2018; Pagiling & Nur’aini, 2022). Some studies on teachers’ conceptual understanding, especially those on quadrilaterals, often have small sample sizes (Gokbulut & Sen, 2019; Molitoris Miller, 2018; Pagiling & Nur’aini, 2022). In studies about quadrilaterals, it is mainly only aspects to do with definitions which are in focus. While the difficulties faced by primary school children as well as pre-service teachers due to a lack of conceptual understanding have been noted, what is not known is if they face the same difficulties. Consequently, the present study investigates primary school children’s as well as pre-service teachers’ conceptual understanding of quadrilaterals.

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Theoretical and Empirical Background

The discussion of the term “concept” has a long tradition in the history of science and philosophy (Hischer, 2021; Ruwisch & Weigand, 2023). It is therefore important to first clarify what a concept is, especially in regard to geometrical shapes. Concepts can be defined in a variety of ways. For example, a square can be defined by its “sides”, “lengths” and “angles”, or by its diagonal attributes or based on the definition of a rectangle, as a more general class of objects. A mathematical definition of a concept consists of condensed information, containing only the essential attributes. A concept refers not only to a specific object, but to a class of objects that share common attributes (Franke & Reinhold, 2016; Meyer & Tiedemann, 2017).

In order to describe a concept Vinner and Hershkowitz (1980) related the term “concept definition” to the term “concept image” for geometric concepts. A formal concept definition is one, that is accepted by the mathematical community, whereas a personal concept definition is formed by the individual and can evolve over time and under varying circumstances (Tsamir et al., 2014). Concept image refers to the entire cognitive structure associated with a concept, encompassing all mental images as well as the related properties and processes (Tall & Vinner, 1981), which include not only verbal, visual, and auditory associations, but potentially also other sensory connections (Vinner, 2011). Concept images vary from person to person and from one culture to another. In contrast, the concept definition refers to a linguistic description used to specify the features of a concept precisely (Tall & Vinner, 1981).

The understanding of a concept can be acquired through various ways. Children mainly build concepts “by actively dealing with objects in connection with language” (Franke & Reinhold, 2016, p. 118). Within cognitive psychology, there are two different views to describe the processes of categorization and development of conceptual understanding (Maier & Benz, 2020; Tsamir et al., 2014). The first is the classical view (Klausmeier & Sipple, 1980; Medin & Smith, 1981, 1984) in which all examples of a concept share a set of defining features. “The features of a new stimulus would then be judged against the features of a known category in order to determine if it is or is not an example of that category” (Tsamir et al., 2014, p. 498). The second is the prototypical view (Clarke, 2004; Medin & Smith, 1984; Szagun, 2011) that focus on ideal examples, called prototypes. Prototypes are generally acquired first, often with other characteristic features, than just the defining features. When categorizing examples or non-examples, the prototypes serve as a basis for comparison. The classical view focusses on the concept definition whereas the prototypical view focusses on, the individual mental representation, that one forms of a shape (concept image):

A concept name when seen or when heard is a stimulus to our memory. Something is evoked by the concept name in our memory. Usually it is not the concept definition, even in the case the concept does have a definition. It is what we call ‘concept image’ (Vinner, 1991, p. 68).

The concept image can reflect only a few prototypical examples and does not necessarily encompass the full range of examples that do not fulfill the requirements of the concept definition (Franke & Reinhold, 2016). Examples of the concept image that are not in line with the definition are considered to be non-examples (Levenson et al., 2011). The concept definition in combination with a broad concept image, of relevant examples, can play a fundamental role in the development of the conceptual understanding (Maier & Benz, 2020).

Selected Studies of Different Age

Franke and Reinhold (2016) differentiate how the concept definition and the concept image can be distinguished through: (1) knowing definitions and properties of the concept (2) knowledge of the hierarchical structure (3) capture of the range of the concept, e.g. distinguishing examples from non-examples, drawing own examples, building or designing own examples of shapes (concept image) and (4) being familiar with the applications of the concept (Maier & Benz, 2020).

Studies focusing on the concept definition reveal that the conflict between the perceptual experience and the theoretical needs of a mathematical definition persists in older children (Bartolini Bussi & Baccaglini-Frank, 2015; Clarke, 2004; Medin & Smith, 1984; Szagun, 2011). Therefore Bartolini Bussi and Baccaglini-Frank (2015) pointed out that the “process of harmonization of figural and conceptual components must be fostered through specific interventions, because research shows that it does not occur spontaneously, and attachment to prototypes is not overcome with age alone” (p. 392).

Studies focusing not only on young children's concept definition but also on their concept image are summarized by Bruns et al. (2021). With regard to the quadrilateral, they note that studies with primary school children show that the explanations are often based on the number of angles or sides. In some cases, children's explanations and their identifications do not match. For example, children can insist that a quadrilateral must have four angles, but still would not choose a rectangle as an example of a quadrilateral. Children mainly identify squares as examples of quadrilaterals, although a small percentage also identify rectangles. Overall, typical (rectangles, square) rather than atypical representatives (quadrilaterals with concave sides) are correctly identified.

Non-examples (e.g. figures without straight sides), on the other hand, tend to be incorrectly perceived as examples of the term quadrilateral (Bruns et al., 2021). Regarding quadrilateral, non-defining attributes influence their identification. In particular, primary school children often have difficulty identifying “obtuse-angled” and “concave quadrilaterals”. If a figure is not shown with a closed line segment, it is usually not correctly recognized as a representative. When square-like figures have rounded angles, they are often identified as representative. This is less common for figures with curved sides. In general, children's correct identification of the quadrilateral concept increases in higher grades (Bruns et al., 2021).

With reference to Currie and Pegg (1998), de Villiers (1994), Erez and Yerushalmy (2006), Monaghan (2000) and Pickreign (2007), Fujita and Jones (2007) summarise that a “number of international studies have shown that many learners have problems with a hierarchical classification of quadrilaterals and the related issue of defining such shapes” (p 4). Therefore it can be concluded that children of all ages have difficulties with the concept of quadrilaterals. This raises the question of how those, who teach the children recognize examples of the concept.

Pre-service teachers (N = 44) definitions of various quadrilaterals were examined by Molitoris Miller (2018) at a large Mid-Atlantic state university. Molitoris Miller (2018) described with pre-service elementary teachers (PSTs) teacher students who had not yet studied geometry or definitions on the college level. She showed that the majority of PSTs were most likely to align their statements with definitions that included necessary attributes, but which were not sufficient or minimal attributes. This means that they knew the necessary attributes to describe essential characteristics of a concept, but were unable to use it for a definition. For a definition, all essential characteristics must be included, but only these. Similarly, it was found that the PSTs’ definitions were equally likely to reflect a hierarchical or partial relationship among of shape types. The PSTs were most comfortable with squares, followed by “parallelograms”, then “rectangles”, “trapezoids”, “rhombuses”, and finally “kites”.

Understandings of the concepts “triangle”, “circle”, and “cylinder” of early-years teachers (N = 18; N = 45; depending on which parts of the questionnaire could be answered) were examined by Tsamir et al. (2014). The teachers were asked to define each concept and then identify various examples and non-examples. Tsamir et al. (2014) also examined whether the teachers had used correct and precise mathematical language and whether they had referred to defining and non-defining attributes. The results showed that teachers were generally able to identify examples and non-examples of triangles and to define triangles. They were able to identify examples and non-examples of circles, although they had difficulties with the task of defining circles. They also had difficulties identifying examples and non-examples of cylinders and defining cylinders.

Teachers’ conceptual understanding, particularly about quadrilaterals, was also investigated in two qualitative studies, one with four primary school teachers (Gokbulut & Sen, 2019) and one with four secondary school teachers (Pagiling & Nur’aini, 2022). The study of Gokbulut and Sen (2019), conducted in Turkey, showed that the four primary teachers gave mainly standard definitions for prototype quadrilaterals like squares and rectangles. Important attributes, such as “parallelism” or the inclusion of right angles were often omitted, or incorrectly assigned. This indicated that these teachers had limited knowledge of the definitions. The study of Pagiling and Nur’aini (2022), conducted in Indonesia, investigated the concept definition and the classification of quadrilaterals of four secondary school teachers. One teacher demonstrated a comprehensive hierarchical conceptual understanding of quadrilaterals by recognising the relationships between different types, for example, that a square was a special case of a rectangle and a rhombus and also a parallelogram. Two teachers’ conceptual understanding could be described as

partial prototype understanding. They were able to recognise some relationships between quadrilaterals, for example that a rhombus is a parallelogram, but had difficulty recognising similar connections for squares and rectangles. One teacher referred solely to prototypical representations of quadrilaterals.

All of these studies indicate that some (preservice) teachers are likely to lack proficient conceptual understanding of geometric shapes, in formulating precise mathematical definitions, which include the necessary and sufficient attributes. Instead, standard definitions or prototypes can dominate their conceptual understanding. The ability to classify geometric figures hierarchically (e.g. squares as special cases of rectangles) can be limited in many teachers. In all of the studies, teachers' knowledge was strongly characterised by prototypes, which led to incorrect classifications and a limited understanding of less familiar shapes. Teachers showed better conceptual understanding for familiar shapes, such as squares and triangles, while less familiar shapes, such as trapezoids, or rhombuses were less well defined.

Although these research results indicate that primary school children and pre-service teachers can have difficulties in understanding concepts, the extent to which the same difficulties exist in both groups has not been evaluated. Previous studies are all based on rather small samples and mainly focused on concept definition. To address the conceptual image and, above all, to compare primary school childrens' and pre-service teachers' conceptual image, a study addressing the following research questions was conducted.

RQ1: To what extent can primary school children identify and distinguish between examples and non-examples of quadrilaterals?

RQ2: To what extent can pre-service teachers identify and distinguish between examples and non-examples of quadrilaterals?

RQ3: Is there a difference between primary school childrens' and pre-service teachers' responses?

Design of the Study

Before describing the sample and the process of the study, it is important to highlight a particular feature that arises from the German language. This characteristic may have an impact on the results. The term "Viereck" is predominantly used in the German language to refer to a quadrilateral and commonly used in everyday language. The term Viereck is derived from the components "Vier" (meaning four) and "Eck" (meaning corner), thus, emphasizing the angles of the shape. In English-speaking contexts, the term is often referred to as quadrilateral with a focus on sides. In contrast to the German-speaking context the term quadrilateral is not used in everyday English.

In order to answer the research questions, a quasi-longitudinal study was undertaken. To conduct the study in a manner that is efficient in terms of research

economy, an opportunity sample was employed (Döring & Bortz, 2016). The sample comprises 178 primary school children and 128 pre-service teachers, and, thus, in total 306 participants. The former sample comprised nine primary school classes as follows: grade 1: $n = 42$, grade 2: $n = 42$, grade3: $n = 59$, grade 4: $n = 35$. The latter sample comprised pre-service teachers studying primary school education at the same university. Most pre-service teachers were in their second year, $n = 111$, in their third year were, $n = 10$, and $n = 7$ were in their fourth year. The surveys were administered by the respective primary school teachers, as well as the respective lecturers teaching the pre-service teachers. Data was collected from the primary school children using a paper-and-pencil test and from the pre-service teachers using an online questionnaire.

To assess primary school children's and pre-service teachers' conceptual image, a task consisting of 18 items was used for operationalization. The selection of the task was mainly based on the studies of Aslan and Arnas (2007), Bruns et al. (2021), Clements et al. (1999), Fujita and Jones (2007) and Tsamir et al. (2014). Participants were presented with various quadrilaterals and non-quadrilaterals and asked to circle (paper-and-pencil test) or select (online questionnaire) all quadrilaterals. The selection of the different items (see Fig. 1) was based on the studies of Aslan and Arnas (2007), Bruns et al. (2021) and Clements et al. (1999). The selected figures were varied in their position, their size and number of angles.

This systematic variation was used to find out to what extent primary school children and pre-service teachers recognized quadrilaterals independently of typical prototypes (Tsamir et al., 2014).

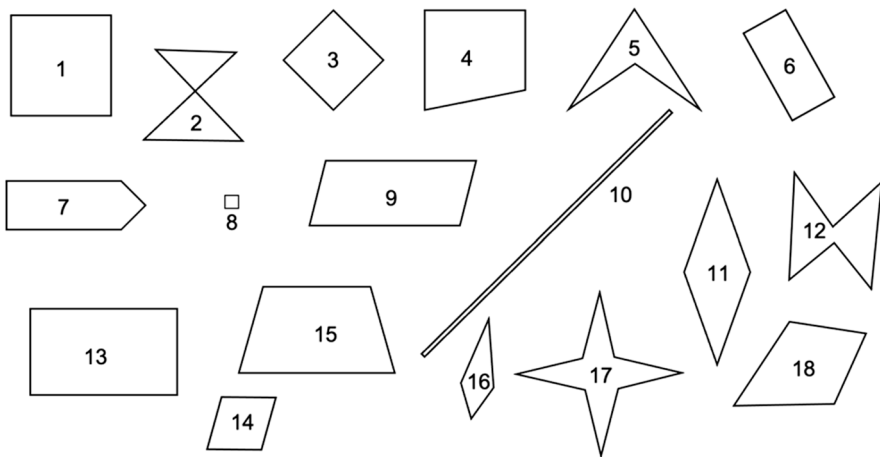


Fig. 1 Figures for the task “Selecting the representatives of the concept quadrilateral”. Note: 1 prototypical square, 2 overlapping square resp. two triangles connected at the angle, 3 square at the angle, 4 trapezoid at the angle, 5 concave quadrilateral, 6 rectangle at the angle, 7 pentagon, 8 small square, 9 parallelogram, 10 narrow rectangle, 11 rhombus at the angle, 12 concave hexagon, 13 prototypical rectangle, 14 small rhombus, 15 trapezoid, 16 kite, 17 octagon, 18 irregular quadrilateral

Results

To address research question 1 and 2, the results provide information on identifying examples and non-examples of quadrilaterals for the different groups (grade 1 to 4 and the pre-service teachers) (see Table 1). It can be seen that in grade 1 ($n = 42$), 100% of the primary school children answered the “prototypical square” (1) and “pentagon” (in German “Fünfeck”—which consists of “five” and “corner” (7) items correctly. Over 75% of the primary school children gave correct answers to the items “trapezoid at the angle” (4), “small square” (8), “concave hexagon” (12), “prototypical rectangle” (13), “small rhombus” (14), and “octagon” (17). Fifty percent or more also gave correct answers for the items “square at the angle” (3), “rectangle at the angle” (6), “parallelogram” (9), “trapezoid” (15), and “irregular quadrilateral” (18). Fewer than 50% of participants identified correctly the “concave quadrilateral” (5), “narrow rectangle” (10), “rhombus at the angle” (11) and kite (16), with the item concave quadrilateral (5) having the lowest number of participants providing a correct answer with 11.9%.

In grade 2 ($n = 42$), the prototypical square (1) has the highest recognition rate at 100%. Correct responses for the trapezoid at the angle (4) and pentagon (7) items were just over 95%. In addition to these items, over 75% of the primary school children correctly identified the square on the angle (3), small square (8), concave

Table 1 Solution frequencies at the grade and item levels, in %

Item	Grade / group				
	Grade 1	Grade 2	Grade 3	Grade 4	Pre-service teachers
	$n = 42$	$n = 42$	$n = 59$	$n = 35$	$n = 128$
1 prototypical square	100.0	100.0	98.3	100.0	98.4
3 square at the angle	69.0	88.1	91.5	82.9	98.4
4 trapezoid at the angle	85.7	95.2	91.5	54.3	91.4
5 concave quadrilateral	11.9	21.4	15.3	17.1	46.9
6 rectangle at the angle	61.9	69.0	81.4	42.9	98.4
7 pentagon	100.0	95.2	98.3	97.1	97.7
8 small square	81.0	90.5	86.4	74.3	96.1
9 parallelogram	69.0	71.4	76.3	34.3	93.0
10 narrow rectangle	23.8	45.2	52.5	28.6	93.0
11 rhombus at the angle	23.8	45.2	57.6	28.6	90.6
12 concave hexagon	88.1	83.3	79.7	85.7	95.3
13 prototypical rectangle	76.2	69.0	84.7	45.7	97.7
14 small rhombus	85.7	95.2	93.2	71.4	93.8
15 trapezoid	64.3	69.0	81.4	31.4	91.4
16 kite	23.8	40.5	45.8	25.7	81.3
17 octagon	85.7	76.2	69.5	94.3	97.7
18 irregular quadrilateral	64.3	85.7	88.1	42.9	89.8

Note: The item 2 overlapping square resp. two triangles connected at the angle was not included in the results, as it does not allow a clear statement to be made in its current form

hexagon (12), octagon (17), and irregular quadrilateral (18) items. There were also over 50% correct responses for the items rectangle at the angle (6), parallelogram (9), prototypical rectangle (13), and trapezoid (15). The items concave quadrilateral (5), narrow rectangle (10), rhombus at the angle (11), and kite (16) had less than 50% correct answers. The concave quadrilateral (5) received the lowest number of correct answers with 21.4%.

In grade 3 ($n = 59$), almost all primary school children (98.3%) chose the correct answer for prototypical square (1) and pentagon (7). There were also more than 75% correct responses for the items square on the angle, trapezoid at the angle (4), rectangle at the angle (6), small square (8), parallelogram (9), concave hexagon (12), prototypical rectangle (13), small rhombus (14), trapezoid (15), and irregular quadrilateral (18). The primary school children also answered more than 50% correctly for the items narrow rectangle (10), rhombus at the angle (11), kite (16), and small rhombus (14). The correct answers for the items concave quadrilateral (5) and kite (16) were below 50%. The concave quadrilateral (5) had the lowest number of correct answers with 15.3%.

In grade 4 ($n = 35$), 100% of the primary school children gave the correct answer for the prototypical square (1) item. Over 75% of the primary school children gave the correct answer for the square at the angle (3), pentagon (7), concave hexagon (12), and octagon (17) items. At least 50% correct answers were also given for the trapezoid at the angle (4), small square (8), and small rhombus (14) items. Answers were lower for the items concave quadrilateral (5), parallelogram (9), narrow rectangle (10), rhombus at the angle (11), prototypical rectangle (13), trapezoid (15), kite (16), and irregular quadrilateral (18). At the end of the scale is the concave quadrilateral (5) with 17.1%.

In the group of pre-service teachers ($n = 128$), the correct answers for the items prototypical square (1), square at the angle (3), and rectangle at the angle (6) were close to 98%. Furthermore, there were more than 75% correct responses for the items trapezoid at the angle (4), pentagon (7), small square (8), parallelogram (9), narrow rectangle (10), rhombus at the angle (11), concave hexagon (12), prototypical rectangle (13), small rhombus (14), trapezoid (15), kite (16), octagon (17), and irregular quadrilateral (18). The correct answer for the concave quadrilateral (5) was 46.9%.

Research question 3 required a comparison of solution frequencies between the groups (grade 1 to 4 and the pre-service teachers). This comparison reveals notable differences for certain items. For instance, the solution frequencies for items such as prototypical square (1), pentagon (7) or small rhombus (14) show only minor differences, whereas the disparities are more pronounced for items like concave quadrilateral (5) or narrow rectangle (10).

To better classify the data presented so far, the respective frequencies of correct answers within the groups (grade 1 to 4 and the pre-service-teachers) are considered (see Table 2). This shows that in grade 1 and 2, 4.8% of the primary school children were able to rate all items correctly. In grade 3 it was 5.1%, in grade 4 11.4%, and among the pre-service teachers 40.6%. These results put the data in Table 2 into perspective. For example, although 21.4% of the primary school children in grade 2

Table 2 Solution frequencies of the individual grade levels or groups, in %

Solution frequencies	Grade / group				
	Grade 1	Grade 2	Grade 3	Grade 4	Pre-service teachers
	n = 42	n = 42	n = 59	n = 35	n = 128
0 incorrect identifications	4.8	2.4	5.1	11.4	40.6
1 incorrect identifications	2.4	14.3	8.5	5.7	34.4
2 incorrect identifications	7.1	9.5	16.9	5.7	7.0
3 incorrect identifications	9.5	21.4	23.7	5.7	8.6
4 incorrect identifications	9.5	11.9	8.5	0.0	1.6
5 incorrect identifications	19.0	7.1	8.5	2.9	0.8
6 incorrect identifications	7.1	4.8	14.3	2.4	0.0
7 incorrect identifications	7.1	4.8	8.5	0.0	0.8
8 incorrect identifications	11.9	9.5	3.4	14.3	3.9
9 incorrect identifications	11.9	4.8	1.7	5.7	0.0
10 incorrect identifications	2.4	7.1	1.7	17.1	0.8
11 incorrect identifications	0.0	2.4	1.7	11.4	0.8
12 incorrect identifications	4.8	0.0	1.7	8.6	0.0
13 incorrect identifications	2.4	0.0	0.0	8.6	0.0
14 incorrect identifications	0.0	0.0	0.0	0.0	0.0
15 incorrect identifications	0.0	0.0	0.0	0.0	0.8
16 incorrect identifications	0.0	0.0	0.0	0.0	0.0
17 incorrect identifications	0.0	0.0	0.0	0.0	0.0

recognized the concave quadrilateral (5) as such (see Table 1), all but 4.2% of the primary school children were unable to correctly identify all of the quadrilaterals. Five or more incorrect identifications were made by 66.7% in grade 1, 38.7% in grade 2, 37.3% in grade 3, 71.4% in grade 4 and 7.8% of pre-service teachers.

In relation to research question 3, a difference between the groups (grade 1 to 4 and the pre-service teachers) becomes evident. Overall, the pre-service teachers provide fewer incorrect solutions compared to the primary school children. The analysis of the results in relation to the items shows that almost all primary school children and pre-service teachers gave correct answers to the prototypical square (1) item. There was also a high frequency of correct answers for the pentagon (7) in all groups. The other items showed a mixed picture in some cases, with stronger outliers in grade 4 in particular. For example, for the trapezoid at the angle (4) item, the correct answers of primary school children in grades 1, 2, and 3 and pre-service teachers ranged from 85.7% to 98.4%, while a value of 54.3% was achieved in grade 4. It is interesting to analyze the items for which the primary school children's solution frequencies were around or below 50%. These are for example the items narrow rectangle (10), rhombus at the angle (11) and kite (16). The kite (16) item shows relatively low frequencies of correct answers in each grade or group. At the bottom of the range is the concave quadrilateral (5), for which less than 50% of each group (grade 1 to 4 and the pre-service teachers), including the pre-service teachers, indicated that it was a quadrilateral.

Looking at the data of the non-representatives for the concept quadrilateral, it can be seen that there are no special results for the items pentagon (7), concave hexagon (12) and octagon (17). They show similar solution frequencies as many representative-items of the quadrilateral.

The analysis of the solution frequencies of the items across the different groups provides insights into the development of conceptual understanding. The data indicates that the percentage of participants who could correctly identify quadrilaterals increases as they get older.

A central finding of the study is the clear difference between prototypical and non-prototypical geometric shapes. While the prototypical square (1) is recognized almost consistently across all groups (between 98.3% and 100%), the recognition rates for non-prototypical quadrilaterals such as the kite (16) (between 23.8 and 81.3), concave quadrilateral (5) (between 11.9% and 46.9%) are significantly lower. A closer look at the individual group levels also reveals specific features. In the fourth grade, comparatively low solution frequencies can be observed for some geometric figures. For example, the parallelogram (9) achieved 69.0% and 93.0% in grades 1 to 3 and among pre-service teachers. In the fourth grade, the item was recognised as a quadrilateral by just 34.3% of the children.

Interpretation and Discussion

The participants' responses indicate that there are differences between concept images of prototypical and non-prototypical geometric shapes (Clarke, 2004; Szagun, 2011). The prototypical square is recognized consistently across nearly all age groups (grades 1 to 4 and pre-service teachers, ranging from 98.3% to 100%), whereas recognition rates for non-prototypical quadrilaterals, such as the concave quadrilateral, are much lower (ranging from 11.9% to 46.9%) These findings suggest that learners become familiar with prototypical shapes early on, while non-prototypical geometric figures are integrated into their cognitive schemas more gradually. This observation corroborates previous research on the conceptual knowledge of primary school children (Bartolini Bussi & Baccaglini-Frank, 2015; Bruns et al., 2021; Clements et al., 1999; Maier, 2019; Maier & Benz, 2020; Tsamir et al., 2014; Unterhauser, 2020).

A more detailed examination of the individual age groups reveals trends. In the fourth grade, there is a notable absence of correct responses for certain geometric figures, such as the parallelogram (9). This could indicate that children at this level have specific difficulties in recognizing certain geometric features. We can only hypothesize about potential causes. It is possible that these decreases in performance are attributable to instructional factors, such as a deficiency in visual and practical exercises during classroom activities. Another possible explanation could be that the teacher did not have the necessary conceptual knowledge to effectively support the students' learning. However, considering that the children are in the fourth grade, it is reasonable to assume that they have already been taught by

different teachers. Nevertheless, the small sample size prevents drawing reliable conclusions. To obtain more substantiated insights, additional data collection would be required.

The analysis of solution frequencies across different age groups in this study indicate a potential development of the conceptual image among primary school children and pre-service teachers. The results indicate that the correct identification of quadrilaterals and non-squares increases during schooling. This tendency was also shown in the study by Bruns et al. (2021). However, the present study suggests that this development is not completed even after primary school, and that the ability to distinguish and classify shapes continues to improve beyond primary school.

The results indicate that conceptual understanding can develop from first grade through adulthood (with the exception of fourth grade). Nevertheless, the conceptual understanding among pre-service teachers remains highly variable and insufficiently developed, shown by the diverse responses. A possible explanation for the development could be that enhanced networking and structuring of learners' knowledge (Stern, 2002), along with a more systematic study of geometric definitions and relationships, contribute to this development. The fact that the development of geometric understanding does not occur to the expected extent may also be due to teachers lacking the knowledge. If teachers lack the necessary knowledge, students will not have the chance to acquire it, and this will have an impact on their knowledge when they become preservice teachers. Thereofre, there are consequences for teacher education.

The present findings have pedagogical implications. Specifically, mathematics learning opportunities should not focus exclusively on prototypical shapes, but should incorporate non-prototypical geometric figures. It seems necessary to facilitate the recognition of non-prototypical shapes, potentially through the use of enactive activities (Bartolini Bussi & Baccaglini-Frank, 2015; Franke & Reinhold, 2016). Such approaches could help foster a deeper and more flexible conceptual understanding.

The present study is not representative. Neither the sample size nor the selection of the sample (opportunity sample) allows for generalizable conclusions. However, for the examined sample, it can be summarized that the understanding of concept image develops differently over the different groups. While simple and prototypical forms are recognized with a high degree of reliability at an early stage, the identification of non-prototypical figures may not be achieved even by some adults. The results of this study underscore the necessity of providing ongoing learning opportunities in geometry to support a comprehensive and sustainable development of conceptual understanding, particularly concerning the concept image. Future research could focus on teaching experiments and intervention studies to identify which learning opportunities are most supportive. Studies that can be based on a representative sample could also be useful.

References

- Aslan, D., & Arnas, Y. A. (2007). Three- to six-year-old children's recognition of geometric shapes. *International Journal of Early Years Education*, 15(1), 83–104. <https://doi.org/10.1080/09669760601106646>
- Bartolini Bussi, M. G., & Baccaglini-Frank, A. (2015). Geometry in early years: Sowing seeds for a mathematical definition of squares and rectangles. *ZDM—Mathematics Education*, 47(3), 391–405. <https://doi.org/10.1007/s11858-014-0636-5>
- Bruns, J., Unterhauser, E., & Gasteiger, H. (2021). Geometrisches Begriffsverständnis in der Grundschule am Beispiel der Begriffe Viereck, Rechteck und Quadrat [geometrical concept understanding in elementary school using the example of quadrangle, rectangle, and square]. *Journal für Mathematik-Didaktik*, 42, 581–623. <https://doi.org/10.1007/s13138-021-00185-4>
- Clarke, B. A. (2004). A shape is not defined by its shape: Developing young children's geometric understanding. *Australian Research in Early Childhood Education*, 11(2), 110–127.
- Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, 30(2), 192–212. <https://doi.org/10.2307/749610>
- Currie, P., & Pegg, J. (1998). Investigating students' understanding of the relationships among quadrilaterals. In C. Kanes (Ed.), *Teaching mathematics in new times: Proceedings of the annual conference of the mathematics education research group of Australia* (pp. 177–184).
- de Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics*, 14, 11–18.
- Döring, N., & Bortz, J. (2016). *Forschungsmethoden und Evaluation in den Sozial- und Humanwissenschaften* [Research methods and evaluation in the social and human sciences] (5th ed.). Springer.
- Erez, M. M., & Yerushalmy, M. (2006). “If you can turn a rectangle into a square, you can turn a square into a rectangle ...” young students experience the dragging tool. *International Journal of Computers for Mathematical Learning*, 11(3), 271–299. <https://doi.org/10.1007/s10758-006-9106-7>
- Franke, M., & Reinhold, S. (2016). *Didaktik der Geometrie in der Grundschule* [Didactics of geometry in elementary school] (3rd ed.). Springer Spektrum.
- Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education*, 9(1), 3–20. <https://doi.org/10.1080/14794800008520167>
- Fuys, D., Geddes, D., & Tischler, R. (1988). *The van Hiele model of thinking in geometry among adolescents*. National Council of Teachers of Mathematics.
- Gokbulut, Y., & Sen, M. (2019). Primary teachers' knowledge of quadrilaterals. *Higher Education Studies*, 9(2), 1–9. <https://doi.org/10.5539/hes.v9n2p1>
- Heinze, A. (2002). «...aber ein Quadrat ist kein Rechteck»—Schülerschwierigkeiten beim Verwenden einfacher geometrischer Begriffe in Jahrgang 8 [“... but a square is not a rectangle”—Students' difficulties in using simple geometric terms in grade 8]. *Zentralblatt für Didaktik der Mathematik*, 34(2), 51–55. <https://doi.org/10.1007/BF02655704>
- Hischer, H. (2021). *Grundlegende Begriffe der Mathematik: Entstehung und Entwicklung: Struktur—Funktion—Zahl* [Fundamental concepts of mathematics: Origin and development: Structure—function—number]. Springer Berlin Heidelberg. <https://doi.org/10.1007/978-3-662-62233-9>
- Klausmeier, H. J., & Sipple, T. S. (1980). *Learning and teaching concepts*. Academic Press.
- Levenson, E., Tirosh, D., & Tsamir, P. (2011). *Preschool geometry: Theory, research, and practical perspectives*. Sense Publishers.
- Maier, A. (2019). *Geometrisches Begriffsverständnis von 4—bis 6-jährigen Kindern—In England und Deutschland* [Geometrical concept understanding in 4- to 6-year-old children—In England and Germany]. Waxmann.

- Maier, A., & Benz, C. (2020). "A triangle is like a tent": Children's conception of geometric shapes. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years* (pp. 229–248). Springer International Publishing. https://doi.org/10.1007/978-3-030-34776-5_14
- Medin, D. L., & Smith, E. E. (1981). Strategies and classification learning. *Journal of Experimental Psychology: Human Learning and Memory*, 7(4), 241–253. <https://doi.org/10.1037/0278-7393.7.4.241>
- Medin, D. L., & Smith, E. E. (1984). Concepts and concept formation. *Annual Review of Psychology*, 35(1), 113–138. <https://doi.org/10.1146/annurev.ps.35.020184.000553>
- Meyer, M., & Tiedemann, K. (2017). *Sprache im Fach Mathematik* [Language in the subject of mathematics]. Springer.
- Molitoris Miller, S. (2018). An analysis of the form and content of quadrilateral definitions composed by novice pre-service teachers. *The Journal of Mathematical Behavior*, 50, 142–154. <https://doi.org/10.1016/j.jmathb.2018.02.006>
- Monaghan, F. (2000). What difference does it make? Children's views of the differences between some quadrilaterals. *Educational Studies in Mathematics*, 42(2), 179–196. <https://doi.org/10.1023/A:1004175020394>
- Pagiling, S. L., & Nur'aini, K. D. (2022). Specialized content knowledge lower secondary school teachers on quadrilaterals. *PYTHAGORAS Jurnal Pendidikan Matematika*, 17(1), 333–345. <https://doi.org/10.21831/pythagoras.v17i1.42446>
- Pickreign, J. (2007). Rectangles and rhombi: How well do preservice teachers know them? *IUMPST: The Journal, 1* (Content Knowledge). [www.k-12prep.math.ttu.edu].
- Ruwisch, S., & Weigand, H.-G. (2023). Begriffe bilden [forming concepts]. In R. Bruder, A. Büchter, H. Gasteiger, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* [Handbook of Mathematics Education] (pp. 281–311). Springer. https://doi.org/10.1007/978-3-662-66604-3_9
- Stern, E. (2002). Wie abstrakt lernt das Grundschulkind? Neuere Ergebnisse der entwicklungspsychologischen Forschung [How abstract does a primary school child learn? Recent results from developmental psychological research]. In H. Petillon (Ed.), *Individuelles und soziales Lernen in der Grundschule—Kindperspektive und pädagogische Konzepte* [Individual and social learning in primary school—Child perspective and pedagogical concepts] (pp. 27–42). Leske + Budrich.
- Szagan, G. (2011). *Sprachentwicklung beim Kind: Ein Lehrbuch* [Language development in children: A textbook] (4th ed., corresponds to 3rd, updated ed. 2010). Beltz.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151–169. <https://doi.org/10.1007/BF00305619>
- Tsamir, P., Tirosh, D., Levenson, E., Barkai, R., & Tabach, M. (2014). Early-years teachers' concept images and concept definitions: Triangles, circles, and cylinders. *ZDM*, 47(3), 497–509. <https://doi.org/10.1007/s11858-014-0641-8>
- Unterhauser, E. (2020). *Geometrisches Begriffsverständnis in der frühen Bildung: Eine Interviewstudie zu den Begriffen Vier- und Dreieck bei Kindergartenkindern* [Geometric Conceptual Understanding in Early Childhood Education: An Interview Study on the Concepts of Quadrilateral and Triangle among Kindergarten Children]. Springer. <https://doi.org/10.1007/978-3-658-28590-6>
- Vinner, S. (1991). The Role of Definitions in the Teaching and Learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65–81). Springer. https://doi.org/10.1007/0-306-47203-1_5
- Vinner, S. (2011). The role of examples in the learning of mathematics and in everyday thought processes. *ZDM*, 43(2), 247–256. <https://doi.org/10.1007/s11858-010-0304-3>
- Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. *Proceedings of the 4th conference of the International Group for the Psychology of Mathematics Education*, 177–184.

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Sandwiches: Problem Solving on Combinatorics in Early Mathematics Education



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Introduction

In research on early mathematics education, probably the most prevalent topic is children's learning of numbers and counting. Numbers and counting are part of number sense but are also crucial in mathematical topics such as algebra, trigonometry, and combinatorics. However, early mathematics education research seldom focuses on those mathematical topics. The data in this paper are from a Swedish educational design research study, based on work described by McKenney and Reeves (2012) which explored and developed a problem-solving and problem-posing approach in with 6-year-olds. In the study, different mathematical topics have been explored in different design cycles. In the design cycle presented in this paper, the students were to solve a combinatorics problem-solving task.

In Sweden, 6-year-olds are enrolled in preschool class, a year of schooling between preschool and school. The curriculum highlights the need for children to gain the ability to engage in problem-solving, with problem solving being considered both content and purpose. Combinatorics is not mentioned in the curriculum of the preschool class, but both international (English, 1991, 2005) and national (Palmér & van Bommel, 2018; van Bommel & Palmér, 2021) studies show that combinatorics is suitable as content in mathematics education for young students. The central question in combinatorics is: in how many different ways can something be done? Previous Swedish studies by Palmér and van Bommel showed that

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6-year-olds could make sense of the topic and that a problem-solving approach helped students to understand the concept of permutations. Permutation is the term used to express the possible ways a set or number of items can be ordered or arranged. In these studies, students' choice of representations when working on combinatorics (Palmér & van Bommel, 2018, 2024), digital tools (van Bommel & Palmér, 2021), and digital animations (Ebbelind et al., 2023) as part of the lessons have been explored.

Wathne and Carlsen (2022) focus on 8–9 years old students' multimodal mathematical reasoning, including documentation, spoken dialogue and gestures, when solving a combinatorics problem about toppings of two types of bread. As a result of their study, they emphasized the need to pay “more attention to combinatorial problems in the mathematics teaching in the first years of schooling as this is a rich mathematical structure” (Wathne & Carlsen, 2022, p. 15). They point out the importance of further studies investigating the way the combinations are displayed and emphasize the importance of including younger students in the studies. This study contributes to this field of research, by responding to Wathne and Carlsen's (2022) call. In this study, our research question was:

- In what way do the students depict combinations when solving the problem-solving task on combinatorics?

Combinatorics

Combinatorics is the field in mathematics that includes both permutations and combinations. These terms date far back in history. “Permutations” was used as long ago as 1000 BCE. Around 400 BCE, both terms were used regarding an attempt to find the number of different syllables possible in the Greek language (i.e. *The first attempt on record to solve a difficult problem in permutations and combinations*, Xenocrates of Chalcedon). Eventually, this resulted in a specific field in mathematics concerned with counting the number of ways it is possible to combine “things” (often referred to as objects or items). The counting can be done using the multiplication principle, which means that if there are m possible outcomes for a first stage and n possible outcomes for a second stage, the total procedure can be carried out in $m \times n$ ways. In the task in this paper, the first stage would be a choice between two types of bread ($m = 2$), then the possible number of outcomes in the second stage would be three (three different topics, $n = 3$), resulting in 2×3 combinations. (Using $m \times n$ is however not how young children most often would calculate the number of combinations, see next section.)

In combinatorial tasks, it is important to distinguish between problem situations where the order of the items is relevant. An example where the order is relevant is the problem where three bears can be arranged on the sofa in six unique ways (van Bommel & Palmér, 2021). For the bears on the sofa, the task is to find the number of ways to arrange a set of objects (all three bears) in a specific order, known as

permutations. However, in other tasks such as the sandwich task used in Wathne and Carlsen's (2022) study, the order is irrelevant. The task is to find the number of ways one can select a subset of objects (such as one slice of bread and one topping at a time) from a more extensive set, known as combinations. The approach and solution strategy in combinatorial problems are affected by such differences in combinatorial tasks. In this paper, we will focus on a type of combinatorial task where the order is not important, where repetition is allowed, and where the counting principle of combining is central.

Combinatorics in Early Ages

A literature search on combinatorics in early mathematics (up to age 8) provides only a limited but relevant research base. English (1991, 2005) was one of the first to study young children and combinatorics, emphasizing that the biggest challenge for younger children when working with combinatorics was to systematically represent combinations. Young children's work with combinatorics tends to be non-systematic or random, where the same permutation may appear as a solution several times without the students noticing. In her 1991 study, English identified five strategies that young children use when working with combinatorial tasks: i) random selection of objects—with duplicates, ii) trial and error with random item selection—with rejection of duplicates, iii) emerging pattern for the choice of objects—with rejection of duplicates, iv) consistent and complete cyclical item selection—with rejection of duplicates and v) "odometer pattern" in item selection. Students who used the two first strategies had no system for how they tried out new permutations. Sometimes, they checked each new permutation by comparing it with the previous ones and thus detected duplicates. However, it was common for solutions based on these strategies to contain many duplicates. The emerging pattern (iii) for the choice of objects implies that the students started to work systematically by keeping one element constant and varying the others, but the system was not followed throughout the solution process. When students used a consistent and complete strategy (iv) or an odometer pattern (v), they exhausted all possible permutations because other objects were held constant, so no duplicates occurred. English found that the younger children in her studies (4.5–6 years) used only the less effective strategies, while the older children (7–10 years) also used more systematic strategies. A similar result emerged in a study by Outhred (1996), where students in grades 1–4 were to combine ice cream scoops of different flavors. In that study, the frequency of correct solutions correlated with the degree of systematization in students' documentation of permutations, where systematization was more common for older students. However, it is noted that it is most important that young students are encouraged to use strategies and representations they are acquainted with as a starting point (Maher & Yankelewitz, 2010; Wathne & Carlsen, 2022). Finding all possible combinations requires systematization. For three bears on a sofa this entails matching one seat with one bear and combining this fixed position with the remaining two bears and

their seating, then repeating the process by fixing another bear in the same seat. The elementary ideas of combinatorics “support children’s development of beginning probability ideas and problem-solving skills” (English, 2005, p. 122) and underpin other areas in mathematics such as counting (Lockwood, 2013).

Palmér and van Bommel have studied 6-year-olds working on a combinatorial task (three bears on a sofa). Their results of these studies indicated, for example, that students who chose to draw bears made fewer duplicates than students who, instead of bears, drew dots or lines in corresponding colors (Palmér & van Bommel, 2018). This result appeared somewhat surprising because dots and lines are a more abstract form of representation than bears, which in studies of cardinality is often connected with students’ progress in abstract thinking (Hughes, 1986). A follow-up interview study (Palmér & van Bommel, 2024) showed that the connection between representations and systematization was even more complex, as some students expressed that they were not able to draw bears and therefore chose to draw lines or dots. This indicated that the students did not always choose the representation they wanted, which might affect the strategy they used and thus also the presence of duplicates in their solutions. To draw students’ attention to systematization, a digital application of the task was developed where the program made the students aware of duplicates. Students who explored this application before they worked on the task with paper and pencil were compared with a control group working only with paper and pencil. The results showed that the work with the application alerted the students to duplicates when working with paper and pencil later, and their solutions contained significantly fewer duplicates than the control group’s (van Bommel & Palmér, 2021). Based on the positive results, another digital element in the form of animations was also explored, which, in a similar way, had positive outcomes on students’ learning of combinatorics (Ebbelind et al., 2023).

Method

This intervention was conducted in line with educational design research (Anderson & Shattuck, 2012; McKenney & Reeves, 2012) and thus aims to provide both practical and theoretical results of relevance for research and educational practices. In line with this approach, the study employs an iterative design in designing, implementing, and evaluating education to develop theories and methods that improve teaching and research (Anderson & Shattuck, 2012). In addition to local design principles, connected to the specific mathematical content of the task in this study, global design principles were derived from previous phases of the study. The global design principles (DP1-DP4) have previously been described in van Bommel and Palmér (2016) and drew upon criteria used by Lesh and Zawojewski (2007) and Way (2003). The local design principles (DP5-DP6) are adjusted to the specific mathematical content used in this task.

- DP1 the task should not require being able to read or write
- DP2 the mathematical idea in the task is understandable but not yet grasped by the children
- DP3 the task should be possible to solve by using different strategies without limitations
- DP4 the mathematical concepts introduced in the task have to be defined
- DP5 the task should touch upon intuitive notions of combinatorics
- DP6 the task should focus on combinations rather than permutations

The cyclic design of the study enabled us to consider experiences from previous cycles regarding the design of the task and the instruction given in class. The empirical material presented in this paper is from one design cycle from this intervention, involving 155 students from eight classes from four schools. The five teachers who worked in these classes had completed a 3-year university course in preschool teacher education and thus had foundational knowledge in education and mathematics. Further, these teachers had participated in several of the previous design cycles and, hence, were familiar with the aim of the study and the problem-solving task on combinatorics. The students were familiar with problem solving, but had not worked with combinatorics problems previously. The students were divided into pairs, but depending on the class and students, the students worked individually, in pairs, or in groups of three. It was up to the teachers to decide if some students needed to work individually, and sometimes, the number of students did not allow for an even divide of students in pairs.

The study adheres to Swedish ethical guidelines for research, including the requirement that caregivers provide signed informed consent (Swedish Research Council, 2017). Only the problem-solving solutions documented by the students whose caregivers gave permission were collected—90 documented solutions in total (from individual students, pairs and groups).

The Problem-Solving Task and the Generated Data

The teachers started the lesson by describing the task verbally (DP1), explaining that there were two types of bread (light, dark) and three kinds of toppings (cheese, salami, liver pate). They posed the question: How many different sandwiches can you make with two types of bread and three types of toppings? (DP2, DP5) Each slice of bread may only have one topping (DP4, DP6). Then, the students were given a paper illustrating the bread and the toppings (Fig. 1) (DP1).

Students documented their solutions on paper and were offered colored pens (DP3). The lesson ended with a whole-class discussion, during which the task and the different solutions were discussed (DP3). After this, the students' documented solutions were gathered and given to the researchers. In addition, the teachers took notes after the students had finished documenting their solutions. For instance, notes were taken for clarification if it was unclear what the drawing represented.

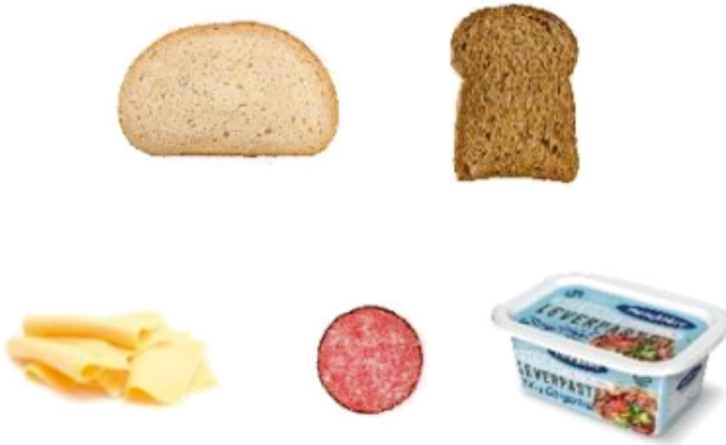


Fig. 1 Worksheet: two types of bread, three toppings

Notes were also taken regarding other relevant information, for instance, when students argued for their solutions or if they expressed a plan for finding the combinations.

Framework for Analysis

Within combinatorics, different combinations need to be distinguished from each other in order to obtain the number of possible combinations. The combinations can be noted and represented in different ways, often in line with the different solution strategies used. Further, English (1991) pointed out that the level of systematization can differ. Thus, three aspects were focused on in the analysis: the number of combinations in the students' drawings, the ways the combinations were depicted, and the level of systematization in the solution.

We distinguished between solutions that showed all six combinations and those that did not. When not all combinations were found, we categorized the solutions with only two or three combinations if the students had argued that "there are no more slices of bread" or "there are no more toppings to put on the bread". Through the teachers' notes, we were also able to identify if there were duplicate combinations in students' final answers. Some students had duplicate combinations but noticed these and did not include them in their total count of combinations; these duplicates were then not considered as such in our analysis.

We applied an inductive approach to how the combinations were depicted. Each documentation was analyzed, and the ways of depicting the combinations were noted. This resulted in approximately 30 different descriptions, which we then grouped into the following overarching themes.

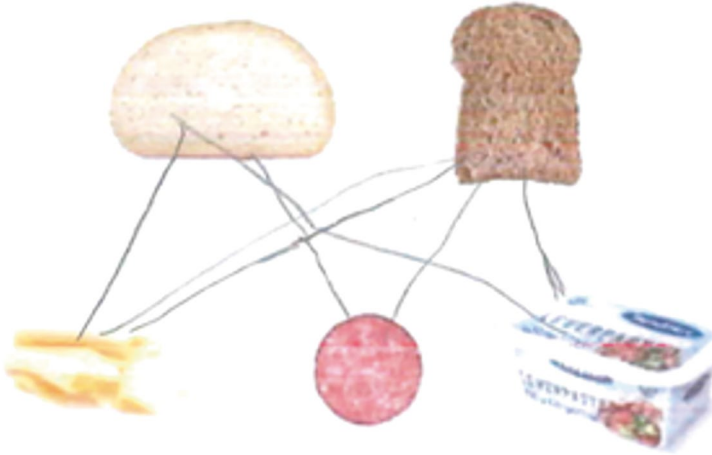


Fig. 2 Lines depicting combinations

- Lines depicting combinations (example Fig. 2)
- Drawings depicting combinations
- Coding system depicting combinations (example Fig. 5)
- Mixture

As for the Drawings, two cases were distinguished: bread and topping side by side (example Fig. 3) and whole sandwich (example Fig. 4).

Thus, these themes were not predetermined but were based on the description of each solution. Some students used a mixture of the above themes, which were categorized separately as Mixture.

Finally, the level of systematization in the documentation was used, based on English (1991) for categorization of systematic (iv, v) and non-systematic strategies (i, ii, iii). For instance, the documented solutions in Figs. 2 and 4 were classified as systematic, whereas the documentation in Fig. 3 was not. The teachers' notes were important, they described the processes. The teachers did not guide the students' solution process but took notes when the students had finished working on the task.

Results

Just over half of the documented solutions (57 out of 90, Table 1) showed all combinations. The way the combinations were depicted differed, but the majority used lines or drawings of whole sandwiches. Very few showed a coding system to distinguish between the different combinations within one picture. Apart from the example in Fig. 5, one other documentation showed digits (using the digits 1 and 2) to represent the two different slices of bread; this was done together with drawing lines and whole sandwiches and was therefore categorized as a "mixture".

Fig. 3 Drawings depicting combinations: bread and topping side by side



Fig. 4 Drawings depicting combinations: whole sandwich

In 33 of the documented solutions, not all combinations were found (Table 1). In 18 of these, the students argued that there were no more slices of bread (4) or no more toppings to use (14), resulting in four solutions with two combinations, and 14 solutions with three combinations (Table 1). When arguing that there are no more pieces of bread, only two toppings were used. The argument that there were no more toppings resulted in drawing a third slice of bread (Fig. 6).

Table 1 Depiction of combinations related to number of combinations found

		All combinations		Not all combinations			Total
		Unique	Duplicates	2: no more slices of bread	3: no more toppings	2-5	
						Unique	Duplicates
Combinations depicted by							
Lines		13	1	2	8	2	26
Drawing	Bread & topping	1		1			2
	Whole sandwich	23	1		3	7	35
Coding system						1	1
Mixture		17	1	1	3	4	26
Total		54	3	4	14	14	90
		57		33			

**Fig. 5** Coding system depicting combinations

For the remaining 15 who did not show all combinations, the students were asked if there could be any more, but they did not find any.

Very few of the documented problem-solving solutions included duplicates as part of their final answer (three out of 90, Table 1). Some students drew duplicates, but when counting the total number of (different) combinations, these were detected, and the students adjusted their final answer.

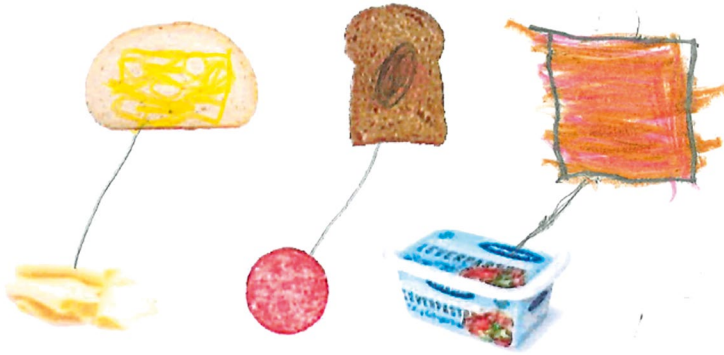


Fig. 6 An extra slice of bread is drawn to match the third topping

Fig. 7 Lines representing the number of combinations complemented with drawings of the bread



In 52 solutions lines were used to depict a combination; 26 only used lines, while the remaining 26 (categorized as a mixture) complemented the lines with drawings of bread and toppings, or drawings of whole sandwiches. This was done in different ways, with some students using the original pictures on the worksheet (Fig. 2) while others drew separate items for each combination (see Fig. 5). Figure 7 shows an example where the bread was drawn but not the toppings.

Regarding the systematization of the combinations, solutions in which lines were used to show what was being combined with what were often categorized as

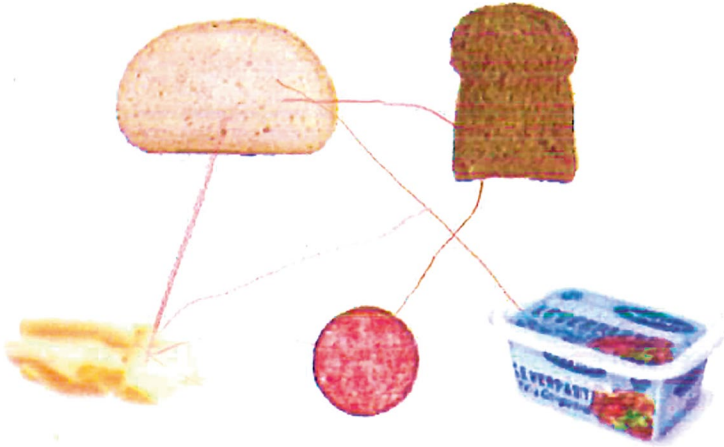


Fig. 8 Non-systematic use of lines

systematic (22 of 26). Figure 3 shows an example of a solution where lines are used in a systematic way, whereas Fig. 8 shows a non-systematic use of lines.

Among the drawings, only half could be considered systematic. The same goes for the documented solutions where a mixture of depictions were used. Figure 7 shows a systematic use of drawings and lines (mixture). Figure 9 shows an example where the students have used a mixture of drawing and lines in a non-systematic way as the salami is connected to three slices of bread.

The few duplicates appeared in solutions with non-systematic problem-solving strategies, such as trial and error (ii, English, 1991) or random selection of objects (i, English, 1991). However, far from all non-systematic strategies led to duplicates. Figures 9 and 10 shows examples of non-systematic, emerging strategies, (iii, English, 1991) with a duplicate (light bread with salami in both cases).

In some cases, the students' detected duplicates themselves and crossed these out (Fig. 11) or marked these in other ways and did not count the duplicate combination in their final total.

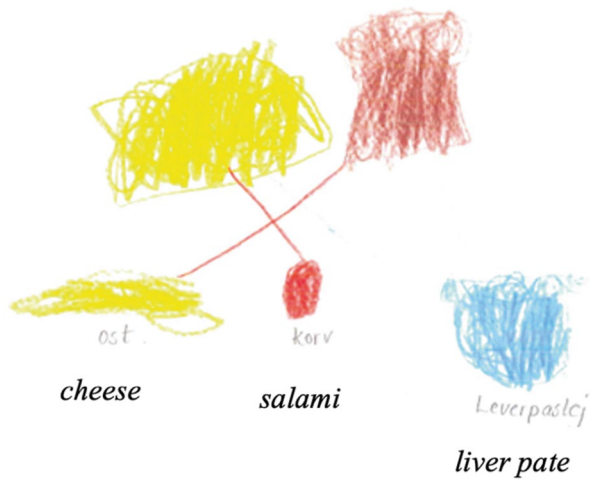
Students who argued for two or three combinations could be considered to have a systematic strategy (28 documented solutions). However, with only two or three possible combinations, it is debatable whether this was in fact systematic. In Fig. 12, we can see such a solution, and the line positions indicate neither a non-systematic strategy nor a systematic problem-solving strategy.

The ones who found two to five solutions differed in systematization, with most applying a non-systematic strategy (10 out of 15) and the remaining five using emerging systematic strategies (iii, English, 1991), as can be seen in Fig. 9.

Fig. 11 Duplicates detected by the students and crossed out



Fig. 12 Solution with two combinations (Notes by teacher)



strategy to find and represent combinations, such as Figs. 2, 4, and 7. The results in this study indicate that young students are able to enact a systematic strategy and, maybe more importantly, that this systematic strategy, in most cases, leads to a result where they find all solutions.

English (2005) found that the younger children in her studies (4.5–6 years) used less effective strategies, while the older children (7–10 years) used more systematic

strategies. The task in this study was previously used with 8–9-year-old students (Wathne & Carlsen, 2022). The 6-year-old students in our study showed a frequent use of systematic strategies. However, unlike Wathne and Carlsen (2022), we did not include the students' use of gestures in the analysis. This means that we cannot know if these systematic strategies helped the students in understanding each other's mathematical arguments in the same way that gestures helped the students in Wathne and Carlsen's (2022) study. It is of interest to further investigate the interaction between students and see how systematic strategies can help young students in understanding each other's combinatorial arguments. Also, the results of this study need to be seen in the context of Swedish preschool class, where the teachers were qualified in both education and mathematics which may differ from other contexts. Further, even though combinatorics was a new mathematical content for the students, problem solving was not, which may have positively influenced their work on the task.

Systematizing may also be connected to a student's interpretation of a task. In previous studies by Palmér and van Bommel (2018), many students interpreted combinatorics as requiring every item to be changed in order for a combination to be considered unique. In this study, quite a few documented solutions (18 of 90) were connected to an explicit argumentation about using a slice of bread or a specific topping only once. It is interesting to note, however, that the argument for three combinations sometimes included drawings of an extra slice of bread, making their argumentation inconsistent (Fig. 6). In previous studies by Palmér and van Bommel (2018) and van Bommel and Palmér (2021), few students found all solutions, and duplicates were seldom noticed. The task provided in this study resulted in very few duplicates. In the end, only four solutions included duplicates as a combination (Fig. 10), and 57 out of 90 documented solutions showed all the possible combinations. It might be that the different types of combinatorial tasks influenced this outcome. Therefore, more research is needed about whether the order aspect in combinatorial tasks is crucial or not. Further, the task design might influence students' choice of strategy as pointed out in one of our previous studies (Palmér & van Bommel, 2024). A next version of the task will therefore include pictures of a whole bread, a whole salami and a whole cheese instead of single slices of bread, salami and cheese to see if this influences the students' arguments 'there is no more bread' and 'there is no more topping'.

Finally, we make a general comment relevant to problem-solving lessons. The students differed in how they worked on problem-solving tasks, and this study provides an example of the variety of strategies a teacher is likely to meet in a classroom. This variety is a strength, as it enables the students to use different representations during classroom practice. However, it is still essential to develop students' strategies which lead to enhanced understanding. Thus, it is important to promote the idea of systematic search and how to systematize the search for combinations. We conclude, in line with English (2005), Maher and Yankelewitz (2010), and Wathne and Carlsen (2022), by saying that young students need to be able to use

and build upon their personal choice of representations when solving combinatorial problem-solving tasks and to make assumptions, think systematically, and experience the satisfaction of generalization within mathematics.

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References

- Anderson, T., & Shattuck, J. (2012). Design-based research: A decade of progress in education research? *Educational Researcher*, 41(1), 16–25. <https://doi.org/10.3102/0013189X11428813>
- Ebbelind, A., Palmér, H., Danielsson, K., Patron, E., & Wernholm, M. (2023). Förskoloklasselever utforskar kombinatorik genom digitala animeringar. [Preschool students explore combinatorics through digital animations.]. *Utbildning och Lärande*, 17(4), 87–104. <https://doi.org/10.58714/ul.v17i4.18265>
- English, L. D. (1991). Young children's combinatoric strategies. *Educational Studies in Mathematics*, 22(5), 451–474. <https://doi.org/10.1007/BF00367908>
- English, L. D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 121–141). Springer. <https://doi.org/10.1007/b105829>
- Hughes, M. (1986). *Children and number: Difficulties in learning mathematics*. Blackwell Publishing.
- Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Information Age.
- Lockwood, E. (2013). A model of students' combinatorial thinking. *The Journal of Mathematical Behavior*, 32(2), 251–265. <https://doi.org/10.1016/j.jmathb.2013.02.008>
- Maher, C. A., & Yankelewitz, D. (2010). Representations as tools for building arguments. In C. A. Maher, A. B. Powell, & E. B. Uptegrove (Eds.), *Combinatorics and reasoning: Representing, justifying and building isomorphisms* (pp. 17–26). Springer. https://doi.org/10.1007/978-0-387-98132-1_3
- McKenney, S., & Reeves, T. (2012). *Conducting educational design research*. Routledge. <https://doi.org/10.4324/9780203818183>
- Outhred, L. (1996). Children's drawing of multiplicative structures: Cartesian product and area. In J. T. Mulligan & M. C. Mitchelmore (Eds.), *Children's number learning* (pp. 185–202). Australian Association of Mathematics Teachers.
- Palmér, H., & van Bommel, J. (2018). The role of and connection between systematization and representation when young children work on a combinatorial task. *European Early Childhood Education Research Journal*, 26(4), 562–573. <https://doi.org/10.1080/1350293X.2018.1487141>
- Palmér, H., & van Bommel, J. (2024). Young students' choice of representation when solving a problem-solving task on combinatorics. In H. Palmér, C. Björklund, E. Reikerås, & J. Elofsson (Eds.), *Teaching mathematics as to be meaningful—Foregrounding play and children's perspectives. Results from the POEM5 conference* (pp. 119–129). Springer. https://doi.org/10.1007/978-3-031-37663-4_9
- Swedish Research Council. (2017). *God forskningssed [Good Custom in Research]*.
- van Bommel, J., & Palmér, H. (2016). Young children exploring probability – with focus on their documentations. *NOMAD Nordic Studies in Mathematics Education*, 21(4), 95–114. <https://doi.org/10.7146/nomad.v21i4.148744>

- van Bommel, J., & Palmér, H. (2021). Enhancing young children's understanding of a combinatorial task by using a duo of digital and physical artefacts. *Early Years*, 41(2–3), 218–231. <https://doi.org/10.1080/09575146.2018.1501553>
- Wathne, U., & Carlsen, M. (2022). Third grade students' multimodal mathematical reasoning when collaboratively solving combinatorial problems in small groups. *Mathematical Thinking and Learning*, 24(3), 221–240. <https://doi.org/10.1080/10986065.2022.2099611>
- Way, J. (2003). *The development of children's notions of probability [Doctoral dissertation]*. University of Western Sydney.

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Similarities and Differences in Norwegian and Swedish Early Childhood Preservice Teachers' Responses to the Use of Digital Technologies for Supporting Mathematical Explorations



Mona Karbaschi Vee, Dorota Lembrér, and Tamsin Meaney

Introduction

In this paper, we investigate early childhood preservice teachers' views on using digital technologies with young children when exploring mathematical ideas in Norway and Sweden. These two Nordic countries have similar historical understandings of early childhood education and care (ECEC) (see for example Einarsdottir et al., 2015). However, differences are appearing as a result of specific influences on these societies (Ringsmose, 2017). Traditionally, children attending ECEC in the Nordic countries are expected to learn through play, supported by teachers focusing on children's interests. However, societal influences on learning and teaching in relationship to ECEC policy (see for example Fosse et al., 2018; Hildén et al., 2023) suggest that there may be differences in how the integration of digital technologies into children's explorations of mathematics is viewed in the different countries. In a study of ECEC preservice teachers' views of integrating digital technologies in eight countries, Madsen et al. (2023) found similarities and differences and recommended that further research should be undertaken to investigate how specific educational situations might affect their understandings.

In regard to integrating digital technologies into mathematics learning experiences, there has been much focus on the digital competency of teachers, including ECEC teachers, and how they develop the digital competency of the children with whom they work. In Nordic curricula, definitions of the term "digital competence"

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are complex, as they often include skills, literacies and *bildung* (Erstad et al., 2021). Erstad et al. (2021) highlighted three different ways that digital competence has been defined and how it is expressed in curriculum documents. We have summarised these as:

1. New literacy studies: The exploration of literacy within social contexts, both within and beyond formal educational settings, has shaped the direction of research on how children make meaning about digital technologies.
2. Media cultures: “Another influential theoretical position goes back to the ‘cultural studies’ tradition of how cultures and sub-cultures within our societies are formed and develop” (p. 78). This perspective has influenced research on how children make meaning and use digital technologies.
3. Learning sciences: A perspective on how children learn. Erstad et al. (2021) claim that there is a tension between “what research could document about the ways people learn and the ways schools and curricula were organised” (p. 79).

These three perspectives can be found in Nordic curricula, which often highlight the need for digital competency in children’s futures, relating it to societal issues while taking a critical approach to their integration (Godhe, 2019). The Swedish curriculum (The Swedish National Agency of Education, 2018) states that children should have the opportunity to develop their digital skills and a critical and responsible attitude towards digital technology. ECEC teachers should also be “able to use digital tools in a way that stimulates development and learning” (The Swedish National Agency of Education, 2018, p. 16). Similarly, the Norwegian curriculum, known as the Framework Plan (Ministry of Education and Research, 2017), states, “the digital practices in kindergarten shall encourage the children, be creative and learn” (p. 44). It also directly references play in that teachers should “enable the children to explore, play, learn and create using digital forms of expression” (p. 45). This is in alignment with views about how young children’s learn, such as Amrar et al. (2024), who stated, “children’s cognitive structure will build upon the pretend play and mathematical explorations they initiated, as well as the reflexive interactions guided by the teacher” (p. 36).

In regard to the learning area of mathematics in the Framework Plan, there is an emphasis on explorations to help children understand nature and society as well as stimulate their motivation for problem-solving (Ministry of Education and Research, 2017). The Swedish curriculum similarly states “preschool should give children the opportunity to use mathematics to investigate and describe their environment and solve everyday problems” (The Swedish National Agency of Education, 2018, p. 10).

Nevertheless, the use of digital technologies in ECEC is debated in the wider society. According to Pesonen and Valkonen (2023), ECEC is often justified by the need for future workers to have digital skills. They investigated Finnish ECEC staff’s views about using digital technology to consider how their professional experiences were changing (Pesonen & Valkonen, 2023). They found that the traditional child-centred approach to pedagogy was being replaced by the agenda embedded into digital apps or other tools (Pesonen & Valkonen, 2023). This

seemed to occur without ECEC staff being completely aware of how their views were changing. However, Pesonen and Valkonen (2023) found that some Finnish ECEC teachers remained critical about technology, questioning whether children really learnt from instructional games or whether the children simply tried to win the game, without making sense of the learning tasks. In Norway, Christiansen (2022) highlighted how in the broader society young children's use of digital technologies is explicitly questioned. In Sweden Sandberg et al. (2024) raised concerns about how screen time can affect children's physical and psychological development. However, in Sweden, the concern that children spend a long time on digital technologies, especially in the home environment, has been challenged, with a recognition that there is a need to respect parents' management of their children's engagement with digital technologies, rather than criticising their choices.

These different societal expectations about the value of digital technologies are likely to have an impact on ECEC preservice teachers' views about integrating digital technologies into their work with young children, particularly to do with mathematics (e.g. Vee, 2023). Although Vee (2023) suggested that working with children could affect preservice teachers' understandings about the use of digital tools, there has been little research on this. It may also be the case that working in ECEC does not always provide preservice teachers with a more nuanced and extended view about how to integrate digital technologies into their work. For example, it may be that preservice teachers' uncertainty about their professionalism could lead them to rely more on the "easy solutions" offered by commercial digital technologies (Pesonen & Valkonen, 2023). Nevertheless, investigating the impact of experiences in ECEC on preservice teachers' understandings seems valuable. Given the limited time available for developing digital competencies in teacher education (Vee & Meaney, 2022), knowledge about the potential impact of practicum and different societal discussions about digital technologies on these views could inform the development of ECEC teacher education courses.

In this paper, we investigate preservice teachers' views about how digital technologies would support children's explorations with mathematical ideas, after they had participated in practicum. By asking preservice teachers in Sweden and Norway, it would be possible to see whether different societal discussions may have affected their views. The research question is: After completing practicum, do Swedish and Norwegian ECEC preservice teachers value the use of digital technologies for mathematical explorations in similar ways?

Methodological Approach

The data were collected using an online survey from a sample of ECEC preservice teachers in Norway and Sweden. Online surveys allow for data collection within a limited period of time (Braun et al., 2021) and so were useful for our study. After they had completed at least one practicum, 125 ECEC preservice

teachers from the three campuses of the Western Norway University of Applied Sciences and 135 preservice teachers at Malmö University in Sweden completed the survey. The Norwegian participants were in the first year of ECEC teacher education and participated in a compulsory mathematics education course. The Swedish preservice teachers were in their third, fourth or fifth semesters of their education and had completed a 10-week course in early years mathematics. The collection times reflected when the mathematics education courses were completed in the teacher education programmes. We acknowledge that the extra coursework of the Swedish preservice teachers may have affected the results. However, given we were interested in the responses of preservice teachers who had completed a practicum and engaged in a mathematics education course, it was not possible to have them complete the surveys at the same point in their teacher education.

The survey questions were based on what had been highlighted as important issues in previous research on mathematical play, creativity and problem-solving (e.g. Christiansen & Meaney, 2020; Fosse et al., 2020), as well as the use of digital technologies from curriculum documents and societal debates about using digital technologies (e.g. Christiansen, 2022; Sandberg et al., 2024). The survey mostly consisted of statements requiring Likert scale responses. Preservice teachers responded to statements about using apps and robots to provide mathematical learning opportunities and play, using digital technologies to support children's creativity and problem-solving, and whether children were too young to use digital technologies for learning mathematics and whether its use should be limited. For the Likert scale statements, the preservice teachers could choose between "disagree fully", "disagree partly", "undecided", "agree partly", and "agree fully" and "don't know". Two of the questions (questions 1 and 10) provided multiple choice responses. The first asked the preservice teachers to specify which kinds of digital technologies they had seen being used with mathematics during practicum. The second was about the pedagogical approaches they considered valuable when using digital technologies. There were also opportunities to write comments in relationship to the Likert scale statements, but only 19 preservice teachers from Swedish and 10 from Norwegian provided comments.

To determine whether the results show significant differences between the Swedish and Norwegian responses, we used Fisher's exact test. Fisher's exact test is used to analyse association between two categorical variables in a contingency table and is useful when the sample size is small. We used Fisher's exact test to test if the proportions of responses in each category were significantly different from the null hypothesis, which was that there was no difference between Swedish and Norwegian preservice teachers' responses. We considered that a difference between responses for Swedish and Norwegian ECEC preservice teachers was significant if the P-value was less than 0.05. We compared Norwegian and Swedish preservice teachers' answers in each category. If there was a significant difference, then we tested to see if the difference remained when the two categories of agree (agree fully and agree partly) and disagree (disagree fully and disagree partly) were combined.

Results

There were very few significant differences between the Swedish and Norwegian ECEC teachers’ responses, as indicated by the Fisher’s exact tests, in relationship to questions about the connections between mathematical learning opportunities and digital technologies and the general questions about digital technologies in ECEC.

To understand, whether practicum may have influenced the preservice teachers’ views, we began by identifying which digital technologies were seen being used in mathematical work with children. The results are presented in Fig. 1. Figure 1 shows that almost 60% of Norwegian preservice teachers and more than 35% of Swedish preservice teachers had not seen any digital technologies being used in ECEC to support mathematical learning. The Fisher’s exact test shows that some differences were significant, with a higher percentage of Swedish preservice teachers having seen tablets and digital games being used. There is no information about how the tablets were being used, but perhaps digital games may have been seen as a common way to support children to engage with mathematics. However, the differences in percentages between tablets and digital games, especially for Swedish preservice teachers suggest that perhaps they mostly saw tablets being used in other ways, than children engaging in digital games on them. In a study by Demeshkant et al. (2023) in which they surveyed ECEC preservice teachers in eight countries, the term digital games was not used, but instead pedagogical apps. It may be that the preservice teachers in our study considered digital apps to be for entertainment and not for developing mathematical understandings. The preservice teachers also had the possibility to describe other digital technologies that they had seen being used. Very few responded, except a small number who noted the use of robots.

The results from Fig. 1 indicate that the majority of Norwegian preservice teachers and over a third of the Swedish preservice teachers were likely to respond to the other Likert scale statements without having seen digital tools being used to support children’s mathematical understandings, in ECEC. Thus, their opinions could be based on expectations in the society, including the curriculum (Madsen et al., 2023). In our data, the responses suggest a generally positive valuing of digital technologies for mathematics learning, although with some concerns appearing in regard to the general questions.

Figures 2 and 3 indicate that both Swedish and Norwegian preservice teachers were mostly positive about using digital technologies. Fisher’s exact test showed no

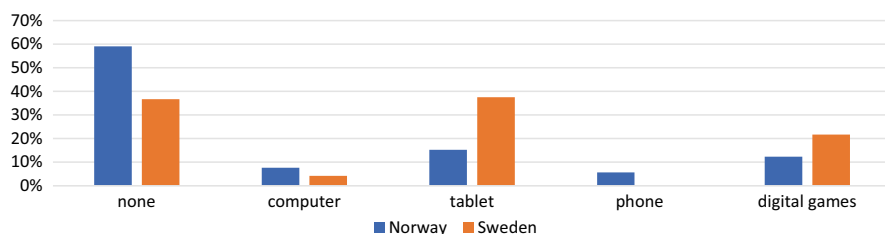


Fig. 1 Responses to the question “What kind of digital technologies have you seen being used with children in ECEC with focus on mathematics?”

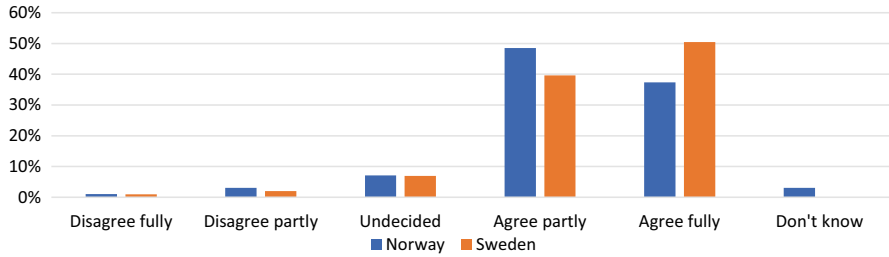


Fig. 2 Responses to the statement “Digital technologies can be used to promote mathematics learning”

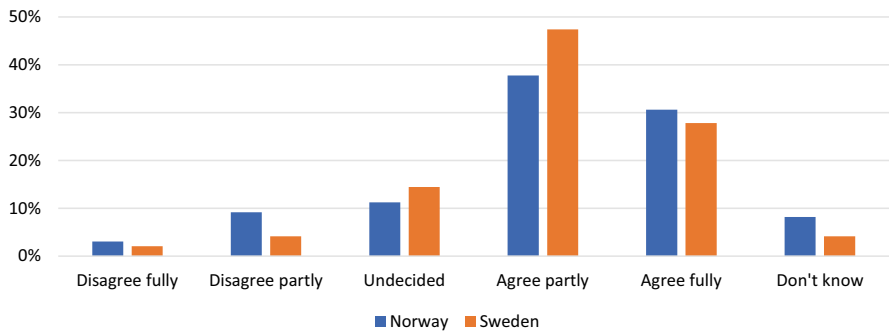


Fig. 3 Responses to the statement, “It is fun to teach children mathematics by using digital technologies”

significant difference between the two groups, suggesting that different societal debates may not be particularly influential. Figure 2 showed that the preservice teachers viewed digital technologies as positively contributing to mathematics learning while Fig. 3 suggests that the preservice teachers expected it to be fun, to teach mathematics through using digital technologies.

Similar positive views were found in the responses to questions which focused on the preservice teachers’ expectations about using specific digital technologies, those of apps and robots, to provide mathematical learning opportunities and play to children. The results from these questions are shown in Figs. 4, 5 and 6.

Figure 4 compares the preservice teachers’ responses to the use of apps and of robots for supporting children’s learning of mathematics. There was a significant difference according to Fisher’s exact test between the responses of Norwegian preservice teachers and Swedish preservice teachers who “did not know” if robots could provide learning opportunities. These results align with Demeshkant et al.’s (2023) general results which indicated that Norwegian preservice teachers were less positive in their future work in ECEC about using toys and other coding devices, which we take to mean robots, than they were about using tablets, which they labelled as iPads in their survey. Almost 80% of Swedish preservice teachers agreed partly or fully with the statement that robots could provide mathematical learning

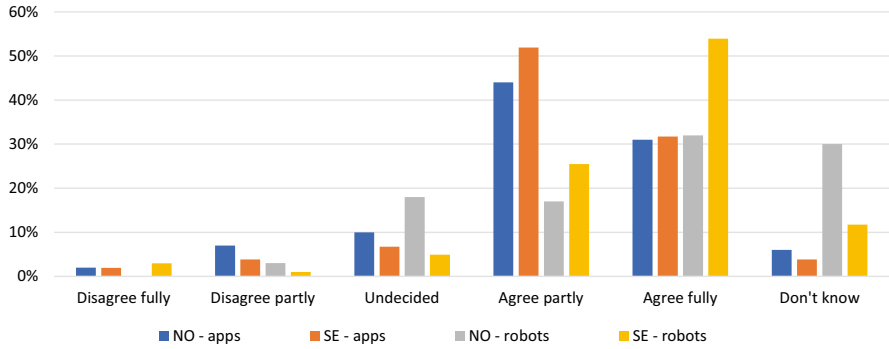


Fig. 4 Responses to the statements “Apps/robots provide opportunities for mathematical learning”

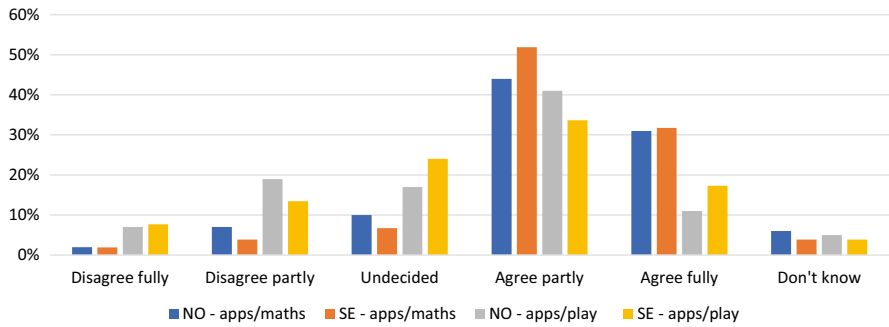


Fig. 5 Responses to the statements “Apps provide opportunities for mathematical learning/play”

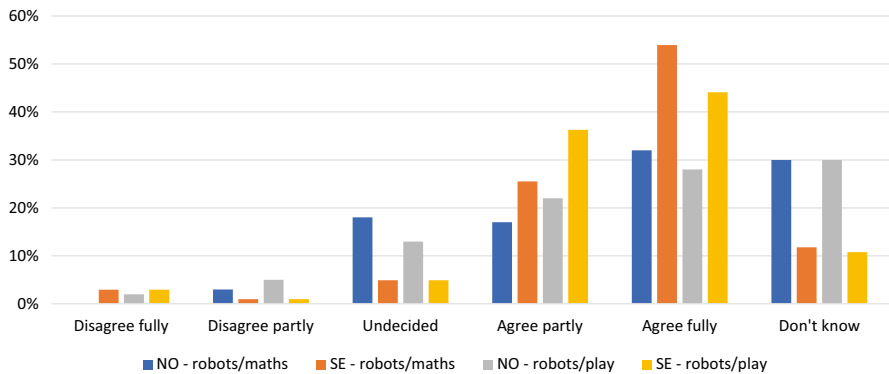


Fig. 6 Responses to the statements “Robots provide opportunities for mathematical learning/play”

opportunities. This was not statistically different to the Norwegian preservice teachers' results, even though most preservice teachers in both groups had not seen robots being used on practicum.

When considering if apps were useful for mathematics learning and for children's play (Fig. 5), both sets of preservice teachers were generally positive, with no significant differences between the Norwegian and Swedish preservice teachers' responses. Although some ECEC teachers in the study of Pesonen and Valkonen (2023) highlighted concerns about how apps support the meaningfulness of learning, most of the Norwegian and Swedish preservice teachers did not seem to share those concerns when responding to these statements.

Nevertheless, the results suggest that fewer preservice teachers valued how apps could contribute to children's play than for supporting mathematical learning opportunities. If apps are not considered to contribute to play, including mathematical play, it is perhaps not surprising that fewer preservice teachers saw digital games as providing mathematical learning opportunities than when using tablets (Fig. 1).

In regard to ECEC mathematics teacher education, it may be that research, such as that of Christiansen (2022), needs to be shared with preservice teachers to provide examples of how digital games can support children to pose and solve their own problems, with mathematical understandings. Otherwise, it may be that preservice teachers will continue to consider digital technologies as not being compatible with children's play, including their mathematical play.

There was a significant difference between the Norwegian and Swedish preservice teachers' responses when considering whether robots could provide opportunities for mathematical learning and for play (Fig. 6). Both groups of preservice teachers were generally positive about robots being used for mathematical learning and for play. When the proportion of responses to partly agree results are combined with fully agreed, the Swedish and Norwegian preservice teachers were similarly positive about using robots, although a higher percentage of Swedish teachers fully agreed compared to the Norwegian preservice teachers. This was particularly the case in relationship to the statement about the value of robots to children's play. These results may be related to societal discussions about children's use of screen time (Sandberg et al., 2024) with robots being considered digital technologies which do not raise the same concerns. However, this explanation does not explain the differences between preservice teachers' views about whether apps and robots provide mathematics learning, where there is little difference between Figs. 5 and 6.

In Fig. 6, a large proportion of preservice teachers responded that they were undecided or did not know about the usefulness of robots. A higher percentage of Norwegian preservice teachers than Swedish preservice teachers admitted that they "did not know" if robots could provide opportunities for mathematics and for play. As well, a higher percentage of Norwegian preservice teachers were undecided about whether robots could provide opportunities for mathematics and for play. However, a smaller percentage of Norwegian preservice teachers were undecided about play than they were about mathematics learning, than Swedish preservice teachers, although the difference was not significant. This could be because few preservice teachers noted that they had seen robots in ECEC.

The generally positive responses about using digital technologies in ECEC suggests that most preservice teachers aligned themselves with the curricular documents. This can be seen in a comment about using digital technologies in ECEC from a Swedish preservice teacher, PT 137, who wrote in the comment section:

What does the Swedish preschool curriculum say about digital technologies? How do we interpret this in connection with the new recommendations? [...] Then it is my responsibility as a future preschool teacher to see how digital technology is used for educational purposes.

In this statement, it can be seen that the curriculum is a driving force, but there is uncertainty about how to achieve it, with this preservice teachers taking this responsibility on as their own. For some preservice teachers, robots, like other digital technologies did not need to be used in ECEC as there were many other ways for children to engage in mathematics. Another Swedish preservice teacher, PT 126, stated:

There are probably teachers who are passionate about mathematics, and then they can probably work well with beebots [a kind of robot], too, but there is so much you can do with maths that is not digital, so I don't see the need for that. Beebots have become just another learning goal that most teachers have poor knowledge of; it just takes a lot of time that we could have for something better.

Therefore, if uncertainties such as those expressed by these two preservice teachers are to be overcome, teacher education programmes need to provide opportunities to show how digital technologies can be used in ways that match their interpretations of the curriculum as well as their philosophies about what should happen in ECEC. Thus, although most preservice teachers expressed general agreement with using digital technologies for learning mathematics there remained some concerns, as there were with some Finnish ECEC teachers (Pesonen & Valkonen, 2023).

As well as generally being positive about how robots could contribute to play, the majority of both Swedish and Norwegian preservice teachers responded positively to statements about children having opportunities to be creative and use problem solving when using digital technologies (Fig. 7). However, the Swedish preservice

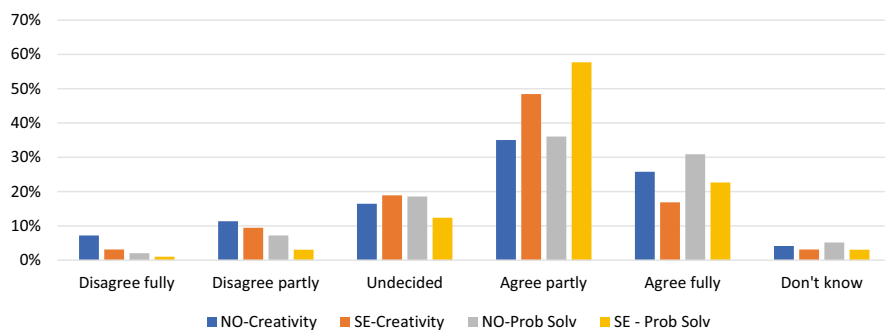


Fig. 7 Responses to the statements “Digital technologies support children’s creativity/problem solving”

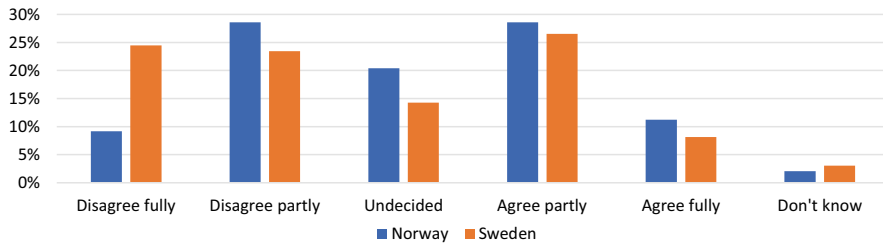


Fig. 8 Responses to the statement “Children were too young to use digital technologies for learning mathematics”

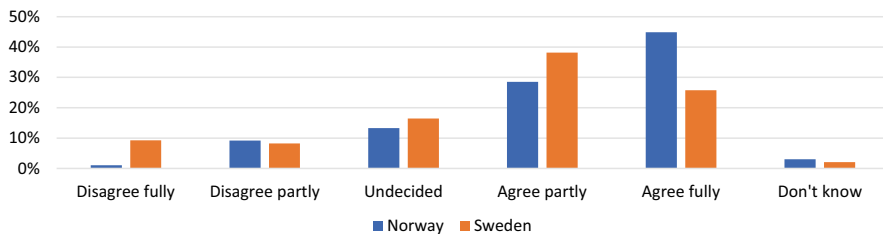


Fig. 9 Responses to the statement “Digital technologies should not be used too often for the learning of mathematics”

teachers were more positive than the Norwegian preservice teachers. The Fisher’s exact test shows a significant difference to the category “Agree partly”. The P–value for problem solving the $P = 0.0019$, and for creativity $P = 0.0414$, with a higher proportion of Swedish preservice teachers agreeing to this statement than Norwegian preservice teachers.

Although the previous results highlighted that the preservice teachers were generally positive about the use of digital technologies, the results in Figs. 8 and 9, show that both groups had concerns, particularly the Norwegian preservice teachers.

In Fig. 8, there is no clear trend about whether the preservice teachers agreed or disagreed with the statement about children using digital technologies in ECEC to learn mathematics. However, the difference between the proportion of responses from the Swedish and Norwegian preservice teachers for the category “disagree fully” was significant, according to Fisher’s exact test. However, when the responses of “disagree fully” were combined with those of “disagree partly” and compared between the two groups of preservice teachers, the difference was no longer significant. The uncertainties shown in the responses in Fig. 8 to this statement suggest that both groups of preservice teachers remained sceptical.

Similar lack of clarity appeared in preservice teachers’ responses to the statement, “using digital technologies should be limited and not used too often for mathematics learning” (Fig. 9), with most of the responses agreeing partly or fully. Although the survey question does not make clear how much is “too often”, perhaps the preservice teachers reconciled their understandings that digital technologies should be used, by restricting how much they were to be used. Given that many preservice teachers did

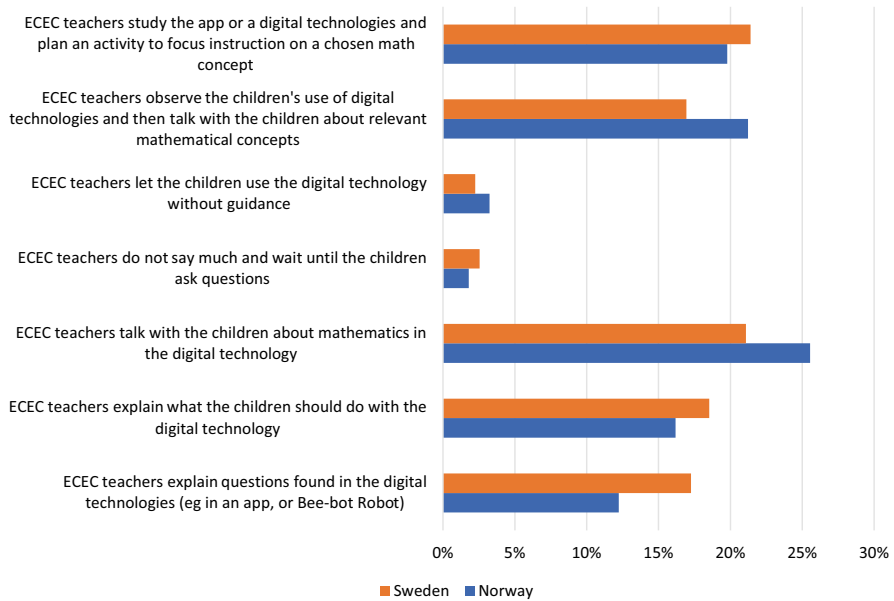


Fig. 10 Responses to the question “How would you describe what ECEC teachers should do when using digital technologies to work with mathematical concepts?”

not see them being used, while on practicum, these results suggest that limiting the use of digital tools would both fulfill curricular requirements while also matching how they should engage children with mathematics in ECEC.

The survey also included a question about what the preservice teachers considered to be their role when children use digital technologies to engage with mathematics (Fig. 10). They could choose a maximum of four options from a range of multiple choices. Both the Swedish and Norwegian preservice teachers most often choose options about the role of the ECEC teacher being to talk with the children about mathematics when using digital technologies (options 2 and 5). Many of the preservice teachers also chose option 1 about planning activities with digital technologies, suggesting they saw it as having an active role in engaging children, which is somewhat different from what has been found in previous research. In research by Vee and Meaney (2022) in Norway, the preservice teachers, who had not yet been on practicum, considered that their main role was to explain aspects of the digital games that children would not be able to understand themselves, while their role in actively developing children’s mathematical understanding was not apparent. Yet, in the survey results, the option of explaining aspects of digital technologies that the children did not understand was the fourth most popular choice among Norwegian and the third most popular choice among Swedish preservice teachers. It may be that experiences on practicum provided the preservice teachers with a better understanding about how to interact with children, rather than passively watch children’s engagement with digital tool.

Some of the comments to the Likert statements provided insights into why the preservice teachers might not be consistent in how they responded to the statements. It was very common that the comments provided insights into the role that the preservice teachers should have in ECEC and what they imagined children should be doing. In the 10 responses from Norwegian preservice teachers to the question about digital technologies in ECEC, of the five written by preservice teachers who had not seen digital technologies in ECEC, three gave negative responses about the use of digital technologies in ECEC. For example, PT 21 wrote:

There are so many other options for promoting learning and play in ECEC. Children are also most likely to be exposed to digital technologies at home.

When the preservice teachers did not consider digital technologies should be used in ECEC, they often mentioned that children could gain these experiences at home, when they got to school or in their future work life. This can be seen in a reflection from a Swedish preservice teacher, PT 75, who wrote:

Digital technologies can be an aid, but we don't have to teach the children in ECEC. They have a lifetime to learn how a computer/tablet works. Our jobs are to give the children the opportunity to creativity, knowledge, language, thinking, patience, turn order, etc. As teachers, we can work without digital technologies.

The preservice teachers' comments also suggested that digital technologies should be used in a limited way to achieve a specific purpose, indicating that they did not think it had a main role in ECEC. For example, a Norwegian preservice teacher, PT 5 wrote, "Digital tools should be used, but one must consider whether it is always the best tool or whether something more concrete is more beneficial (e.g. writing with pencil and paper)". Another Norwegian preservice teacher, PT 8 wrote:

There are many ways to use and learn mathematics in ECEC that do not include digital technologies. In contrast, digital technologies are a large part of today's society and have many good tools that can be used concretely against the areas you want to work towards.

These comments are in alignment with the three digital competencies identified by Erstad et al. (2021) in that they consider both the role of digital technologies in present and future societies, how children are expected to learn and how children make sense of digital technologies as part of their learning when compared to other possibilities. At least for these preservice teachers and potentially others, they did have appropriate professional digital competencies, which could be further developed in teacher education courses. This would help to contribute to preservice teachers making appropriate decisions about what digital technologies could be used in what circumstances to support children's engagement with mathematical ideas.

In many of the comments, the use of digital technologies was often contrasted with expectations about what should happen in ECEC. One of the Norwegian preservice teachers (PT 7) who was not in favour of using digital technologies in ECEC wrote:

I think it is very early to start with screens and that it encourages inactivity. Rather, I think there should be a greater focus on physical activity and social learning up to school age.

The concerns about screen time in relationship to physical activity and socialising is very present in Norwegian societal discussions and were often drawn upon when critiquing the use of digital technologies. In comments such as this, the comparison seemed to be implicitly based on the understanding that play and learning in ECEC were the primary aims for children, but which did not require digital technologies to be achieved. For example, Norwegian preservice teacher PT 32 wrote, “can be included to a certain extent but should be limited with more focus on learning through physical activity and play than is unfortunately seen in some kindergartens”.

The preservice teachers also suggested that digital technologies should be used with a specific purpose. For example, Swedish preservice teacher PT 114 wrote:

[digital technology] should be used with a clear purpose. Is it possible to do something similar without digital technology? How does digital technology foster teaching? Always think through why we choose, e.g., an iPad or something similar.

Another Swedish preservice teacher (PT 17) wrote, “I believe it should be used if there is a purpose. The ECEC teacher needs to be incredibly well prepared and know what they are talking about”. In the comments about the usefulness of digital technologies in ECEC, many Swedish preservice teachers suggested there was a need for skills in using digital technologies in pedagogical practices. One of them, PT 13, wrote, “teachers should gain increased competence when it comes to being able to use digital technologies so that they can feel safe using it in teaching”. Comments such as these reinforce the need for teacher education to build on the digital competency skills that preservice teachers already show by providing specific examples about how digital technologies can support children’s engagement with mathematics, through play and creativity, which would fit more closely with what they consider to be the purpose of ECEC.

Conclusion

In this study, we investigated how Norwegian and Swedish ECEC preservice teachers responded to a set of survey questions about the use of digital technologies for supporting young children’s engagement with mathematics. Although differences were noted in societal discussions about the use of digital technologies in the two countries, including in the curricula (Ministry of Education and Research, 2017; The Swedish National Agency of Education, 2018), on the whole there were few differences in the responses between the two groups. The Swedish and Norwegian preservice teachers had similar views about learning mathematics when using digital technologies in ECEC (Figs. 2 and 3). They also seem to be in agreement about how apps and robots could provide opportunities for mathematics and play (Figs. 5 and 6). However, other statements about reducing opportunities for young children to engage with digital technologies (Figs. 8 and 9) showed less certainty about how often children should use them to learn mathematics in ECEC.

Many of the comments that the preservice teachers added to the survey reflected their concerns about using digital technologies in ECEC, not just with mathematics but more generally. Although some preservice teachers noted a lack of digital competency and, therefore, the pedagogical knowledge needed to integrate them successfully, many expressed comments which seemed to show awareness of the complexity of using digital technologies in ECEC to support children's engagement with mathematics. These could be related to the definitions of digital competency summarised by Erstad et al. (2021) and, thus, could provide a basis for further teacher education courses.

Given that many preservice teachers were not seeing digital tools being used in ECEC, there is a need for teacher education to provide examples of how these tools could be used to support mathematics learning through play and creativity. Such examples could counteract some of the societal views that using digital technologies is connected to excessive screen time and a lack of physical activity and socialising. Although the preservice teachers in both countries showed awareness of curricular requirements, they seemed to need explicit support in their teacher education courses about how to implement those curricular requirements.

Further research is also needed about what might contribute to the similarities found in the responses of the two groups of preservice teachers, given that there are different societal debates about the use of digital technologies in Sweden and Norway. It may be that follow-up interviews with preservice teachers after they have been on practicum could provide further insights.

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References

- Amrar, L., Clerc-Georgy, A., & Dorier, J.-L. (2024). Preschool teacher practices during a collective play-related thinking: Dialectic between pretend play and mathematics. In H. Palmér, C. Björklund, E. Reikerås, & J. Elofsson (Eds.), *Teaching mathematics as to be meaningful—Foregrounding play and children's perspectives: Results from the POEM5 conference, 2022* (pp. 27–38). Springer International Publishing. https://doi.org/10.1007/978-3-031-37663-4_3
- Braun, V., Clarke, V., Boulton, E., Davey, L., & McEvoy, C. (2021). The online survey as a qualitative research tool. *International Journal of Social Research Methodology*, 24(6), 641–654. <https://doi.org/10.1080/13645579.2020.1805550>
- Christiansen, S. F. (2022). Multilingual children's mathematical engagement with apps: What can be learned from multilingual children's mathematical and playful participation when interacting with two different apps? *Conjunctions. Transdisciplinary Journal of Cultural Participation*, 9(2), 1–12. <https://tidsskrift.dk/tcp/article/view/145664/188830>
- Christiansen, S. F., & Meaney, T. (2020). Cultural meetings? Curricula, digital apps and mathematics education. *Journal of Mathematics and Culture*, 14(2), 71–90. <https://journalofmathematicsandculture.wordpress.com/wp-content/uploads/2020/12/christiansen-and-meaney-final-paper.pdf>

- Demeshkant, N., Madsen, S. S., Janeš, A., Klančar, A., Brito, R., Konca, S., Krasin, S., Saure, H. I., O'Connor, J., & Jwaifell, M. (2023). What digital tools teachers are ready to use in kindergarten—international comparative study with early childhood pre-service teachers. In J. L. Shih, A. Kashiwara, W. Chen, & H. Ogata (Eds.), *Proceedings of the 31st international conference on computers in education* (Vol. 1, pp. 855–860). Asia-Pacific Society for Computers in Education. <https://doi.org/10.58459/icce.2023.1103>
- Einarsdóttir, J., Puroola, A.-M., Johansson, E. M., Broström, S., & Emilson, A. (2015). Democracy, caring and competence: Values perspectives in ECEC curricula in the Nordic countries. *International Journal of Early Years Education*, 23(1), 97–114. <https://doi.org/10.1080/009669760.2014.970521>
- Erstad, O., Kjällander, S., & Järvelä, S. (2021). Facing the challenges of 'digital competence'. *Nordic Journal of Digital Literacy*, 16(2), 77–87. <https://doi.org/10.18261/issn.1891-943x-2021-02-04>
- Fosse, T., Lange, T., Hope Lossius, M., & Meaney, T. (2018). Mathematics as the Trojan horse in Norwegian early childhood policy? *Research in Mathematics Education*, 20(2), 166–182. <https://doi.org/10.1080/14794802.2018.1473162>
- Fosse, T., Lange, T., & Meaney, T. (2020). Kindergarten teachers' stories about young children's problem posing and problem solving. In M. Carlsen, I. Erfjord, & S. Hundeland (Eds.), *Mathematics education in the early years. Results from the POEM4 conference, 2018* (pp. 351–368). Springer International Publishing. https://doi.org/10.1007/978-3-030-34776-5_21
- Godhe, A. (2019). Digital literacies or digital competence: Conceptualizations in Nordic curricula. *Media and Communication*, 7(2), 25–35. <https://doi.org/10.17645/mac.v7i2.1888>
- Hildén, E., Löfdahl Hultman, A., & Ribaeus, K. (2023). Teaching as a new mission: Swedish preschool teachers' collegial discussions. *Early Years*, 43(1), 1–14. <https://doi.org/10.1080/009575146.2021.1880374>
- Madsen, S. S., O'Connor, J., Janeš, A., Klančar, A., Brito, R., Demeshkant, N., Konca, A. S., Krasin, S., Saure, H. I., & Gjerdal, B. (2023). International perspectives on the dynamics of pre-service early childhood teachers' digital competences. *Educational Sciences*, 13(7), 633. <https://doi.org/10.3390/educsci13070633>
- Ministry of Education and Research. (2017). *Framework plan for the content and tasks of kindergartens*. Norwegian Directorate for Education and Training. <https://www.udir.no/contentassets/7c4387bb50314f33b828789ed767329e/framework-plan-for-kindergartens%2D%2Drammeplan-engelsk-pdf.pdf>
- Pesonen, J., & Valkonen, S. (2023). Governing education, governing early childhood education and care practitioners' profession? *Contemporary Issues in Early Childhood*, 0(0), 1–14. <https://doi.org/10.1177/14639491231172206>
- Ringsmose, C. (2017). Global education reform movement: Challenge to Nordic childhood. *Global Education Review*, 4(2), 92–102. <http://ger.mercy.edu/index.php/ger/article/view/327/270>
- Sandberg, H., Sjöberg, U., & Sundin, E. (2024). *Små barns digitala vardagsliv: Barndom, föräldraskap och modern familjeliv [young children's digital everyday life: Childhood, parenthood and modern family life]*. Department of Communication and Media, Lund University, Sweden.
- The Swedish National Agency of Education. (2018). *Läroplan för förskolan Lpfö 2018. [Curriculum for the preschool Lpfö18]*. Fritzes.
- Vee, M. K. (2023). How do beginning preservice kindergarten teachers' evaluations of mathematical apps provide insights into their understandings about bildung (danning) for young children? *Nordisk Barnehageforskning*, 20(2), 51–69. <https://doi.org/10.23865/nbf.v20.268>
- Vee, M. K., & Meaney, T. (2022). Kindergarten preservice teachers evaluating mathematical apps. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pålssdóttir, P. Portaankorva-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future: Proceedings of Norma 20, the ninth Nordic conference on mathematics education* (Vol. 20, pp. 249–256). Swedish Society for Research in Mathematics Education.

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