Journal of Algebra 321 (2009) 1600–1631

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Standard isotrivial fibrations with $p_g = q = 1$

Francesco Polizzi

Dipartimento di Matematica, Università della Calabria, Via Pietro Bucci, 87036 Arcavacata di Rende (CS), Italy

ARTICLE INFO

Article history:
Received 20 April 2007
Available online 19 January 2009
Communicated by Steven Dale Cutkosky

Keywords:
Surfaces of general type
Isotrivial fibrations
Actions of finite groups

ABSTRACT

A smooth, projective surface S of general type is said to be a *standard isotrivial fibration* if there exists a finite group G acting faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$. If T is smooth then $S = T$ is called a *quasi-bundle*. In this paper we classify the standard isotrivial fibrations with $p_g = q = 1$ which are not quasi-bundles, assuming that all the singularities of T are rational double points. As a by-product, we provide several new examples of minimal surfaces of general type with $p_g = q = 1$ and $K_S^2 = 4, 6$.

0. Introduction

Recently, there has been considerable interest in understanding the geometry of complex projective surfaces with small birational invariants, and in particular of surfaces with $p_g = q = 1$. Any surface S of general type verifies $\chi(\mathcal{O}_S) > 0$, hence $q(S) > 0$ implies $p_g(S) > 0$. It follows that the surfaces of general type with $p_g = q = 1$ are the irregular ones with the lowest geometric genus, hence it would be important to achieve their complete classification. So far, this has been obtained only in the cases $K_S^2 = 2, 3$ (see [Ca81,CaCi91,CaCi93,Pol05,CaPi06]). If S is any surface with $q = 1$, its Albanese map $\alpha : S \to E$ is a fibration over an elliptic curve E; we denote by g_{alb} the genus of the general fiber of α. The universal property of the Albanese morphism implies that α is the unique fibration on S with irrational base. As the title suggests, this paper considers surfaces with $p_g = q = 1$ which are *standard isotrivial fibrations*. This means that there exists a finite group G which acts faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$. If T is smooth then $S = T$ is called a *quasi-bundle* or a surface *isogenous to an unmixed product* (see [Se90,Se96,Ca00]). Quasi-bundles of general type with $p_g = q = 1$ are classified in [Pol08] and [CarPol]. In the present work we consider the case where all the singularities of T are rational double points

E-mail address: polizzi@mat.unical.it.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.10.028

(RDPs). Our classification procedure combines ideas from [Pol08] and combinatorial methods of finite group theory. Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration; then the two projections $\pi_C : C \times F \to C$, $\pi_F : C \times F \to F$ induce two morphisms $\alpha : S \to C/G$, $\beta : S \to F/G$, whose smooth fibers are isomorphic to F and C, respectively. We have $q(S) = g(C/G) + g(F/G)$, then if $q(S) = 1$ we may assume that $E := C/G$ is an elliptic curve and $F/G \cong \mathbb{P}^1$. Consequently, the morphism α is the Albanese fibration of S and $g_{\mathrm{alb}} = g(F)$. If $p_g(S) = q(S) = 1$ and T contains only RDPs, we show that S is a minimal surface (Proposition 3.5) and that $2 \leqslant g(F) \leqslant 4$. Therefore we can use the classification of finite groups acting on Riemann surfaces of low genus given in [Br90,KuKi90,KuKu90, Bre00,Vin00,Ki03]. In particular we obtain $|G| \leqslant 168$ and so the problem can be attacked with the computer algebra program GAP4, whose database includes all groups of order less than 2000, with the exception of 1024 (see [GAP4]). Computer algebra is a powerful tool when dealing with this kind of problems; a recent example is the paper [BaCaGr06], where the MAGMA database of finite groups (identical to the GAP4 database) is exploited in order to achieve the classification of surfaces with $p_g = q = 0$ isogenous to a product. In our case we have tried to minimize the amount of computer calculations, doing everything "by hand" whenever possible and using GAP4 only when working with groups of big order or cumbersome presentation. Nevertheless, the computer's aid has been extremely useful in order to obtain some of the non-generation results of Section 2 and some of the existence results of Section 7. The aim of this paper is to prove the following

Main Theorem. *Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration of general type with $p_g = q = 1$, which is not a quasi-bundle, and assume that T contains only* RDPs. *Then S is a minimal surface, K_S^2 is even and the singularities of T are exactly* $8 - K_S^2$ *nodes. Moreover, the occurrences for K_S^2, $g(F)$, $g(C)$ and G are precisely those listed in the table below.*

K_S^2	$g(F) = g_{\mathrm{alb}}$	$g(C)$	G	`IdSmallGroup`(G)
6	3	10	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$
6	3	13	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32,9)$
6	3	13	$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32,11)$
6	3	19	$G(48,33)$	$G(48,33)$
6	3	19	$\mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2$	$G(48,3)$
6	3	64	$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168,42)$
6	4	3	D_4	$G(8,3)$
6	4	4	A_4	$G(12,3)$
6	4	7	$D_{2,12,7}$	$G(24,10)$
6	4	10	$\mathbb{Z}_3 \times A_4$	$G(36,11)$
6	4	19	$D_4 \ltimes (\mathbb{Z}_3)^2$	$G(72,40)$
6	4	31	S_5	$G(120,34)$
4	2	3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$G(4,2)$
4	2	4	$\mathbb{Z}_6$	$G(6,2)$
4	2	4	S_3	$G(6,1)$
4	2	5	D_4	$G(8,3)$
4	2	7	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$G(12,5)$
4	2	7	D_6	$G(12,4)$
4	2	9	$D_{2,8,3}$	$G(16,8)$
4	2	13	$\mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3)$	$G(24,8)$
4	2	25	$\mathrm{GL}_2(\mathbb{F}_3)$	$G(48,29)$
4	3	3	D_4	$G(8,3)$
4	3	4	A_4	$G(12,3)$
4	3	5	$D_{2,8,5}$	$G(16,6)$
4	3	5	$D_{4,4,-1}$	$G(16,4)$
4	3	7	$\mathbb{Z}_2 \times A_4$	$G(24,13)$
2	2	3	Q_8	$G(8,4)$
2	2	3	D_4	$G(8,3)$

Here `IdSmallGroup`(G) denotes the label of G in the GAP4 database of small groups. For instance, `IdSmallGroup`$(D_4) = G(8,3)$ means that D_4 is the third in the list of groups of order 8. We emphasize that all quasi-bundles with $\chi(\mathcal{O}_S)=1$ verify $K_S^2=8$ (see [Se90], Proposition 3.5), whereas imposing some RDPs allows us to obtain surfaces with lower K_S^2. In particular, as a by-product of our classification, we produce several examples with $p_g=q=1$ and $K_S^2=6$. In the survey paper [BaCaPi06] the minimal surfaces of general type with these invariants are referred as "mysterious." Actually, there was only one example hitherto known, described by C. Rito in [Ri07]. It verifies $g_{\mathrm{alb}}=3$ and is obtained as a double cover of a Kummer surface; the construction makes use of the computer algebra program MAGMA in order to find a branch curve with the right singularities. We note that Rito's surface is not a standard isotrivial fibration, because the reducible fibers of its Albanese pencil contain no *HJ*-strings (cf. Theorem 3.2). Therefore all examples with $p_g=q=1$, $K_S^2=6$ and $g_{\mathrm{alb}}=3$ described in the present paper were previously unknown; in addition, we provide the first examples with $g_{\mathrm{alb}}=4$. Our viewpoint also sheds some new light on surfaces with $p_g=q=1$, $K_S^2=4$ and $g_{\mathrm{alb}}=2,3$. An example with $K_S^2=4$ and $g_{\mathrm{alb}}=2$ was previously given by Catanese [Ca99] as the minimal resolution of a bidouble cover of $\mathbb{P}^2$; examples with $K_S^2=4$ and $g_{\mathrm{alb}}=3$ were constructed by Ishida [Is05] as the minimal resolution of a double cover of the 2-fold symmetric product $E^{(2)}$ of an elliptic curve. Both covers of Catanese and Ishida contain non-rational singularities, whereas in all our examples T has only nodes; it follows that all surfaces with $K_S^2=4$ presented here are new. Finally, we obtain two examples with $K_S^2=2$; they can be also constructed as double covers of $E^{(2)}$ and in both cases we describe the six-nodal branch curve in detail (Proposition 7.9). These two examples belong to the same irreducible component of the moduli space of surfaces of general type with $K_S^2=2$, $\chi(\mathcal{O}_S)=1$, which is in fact irreducible [Ca81]; then it would be desirable to know whether any two surfaces in our list, with the same K_S^2 and g_{alb}, are deformation equivalent. We conjecture that the answer is negative, but this question is at the present not solved. One could obtain some partial information by computing in every case the index of the paracanonical system, which is a topological invariant ([CaCi91], Theorem 1.4; see also [Pol08], Theorem 6.3), but we will not develop this point here.

We shall now explain in more detail the steps of our classification procedure. The crucial fact is that, since G acts on both C and F, the geometry of S is encoded in the geometry of the two G-covers $h: C \to C/G$, $f: F \to F/G$. This allows us to "detopologize" the problem by transforming it into an equivalent problem about the existence of a pair $(\mathcal{V}, \mathcal{W})$ of generating vectors for G of type $(0 \mid m_1, \dots, m_r)$ and $(1 \mid n_1, \dots, n_s)$, respectively (see Section 1 for the definitions). These vectors must satisfy some additional properties in order to obtain a quotient $T=(C \times F)/G$ with only RDPs and whose desingularization S has the desired invariants (Proposition 5.6).

In Section 1 we present some preliminaries and we fix the algebraic set-up. In Proposition 1.3, which is essentially a reformulation of Riemann's existence theorem, we show that a smooth projective curve Y of genus $\mathfrak{g}'$ admits a G-cover $X \to Y$, branched in r points $P_1, \dots, P_r$ with branching numbers $m_1, \dots, m_r$, if and only if G contains a generating vector $\mathcal{V}$ of type $(\mathfrak{g}' \mid m_1, \dots, m_r)$. For every $h \in G$ we give a formula that computes the number of fixed points of h on X in terms of $\mathcal{V}$ (Proposition 1.4).

In Section 2 we collect some non-generation results for finite groups which will be useful in the sequel of the paper. They are obtained either by direct computation or by using the GAP4 database of small groups. For every group we refer to the presentation given in the corresponding table of Appendix A. The reader that finds these results too dry or boring might skip this section for the moment and come back to it when reading Section 7.

In Section 3 we establish the main properties of standard isotrivial fibrations (following [Se96]) and we compute their invariants in the case where T has only RDPs.

In Sections 4 and 5 we show that if S is a standard isotrivial fibration of general type with $p_g=q=1$ and T contains only RDPs, then S is a minimal surface, K_S^2 is even and the singularities of T are exactly $8-K_S^2$ nodes. Furthermore we prove Proposition 5.6, which plays a crucial role in this paper as it provides the translation of our classification problem "from geometry to algebra."

In Section 6 we show our Main Theorem assuming that the group G is abelian; the proof is extended to the non-abelian case in Section 7.

The tables in Appendix A contain the automorphism groups acting on Riemann surfaces of genus 2, 3 and 4 so that the quotient is isomorphic to $\mathbb{P}^1$. In the last two cases we listed only the non-abelian groups. Tables 1–3 are adapted from [Br90, pp. 252, 254, 255], whereas Table 4 is adapted from [Ki03, Theorem 1] and [Vin00]. For every G we give a presentation, the branching data and the `IdSmallGroup`(G).

Finally, in Appendix B we give an example of GAP4 script used during the preparation of this work.

Notations and conventions. All varieties, morphisms, etc. in this article are defined over the field $\mathbb{C}$ of the complex numbers. By "surface" we mean a projective, non-singular surface S, and for such a surface K_S denotes the canonical class, $p_g(S)=h^0(S,K_S)$ is the *geometric genus*, $q(S)=h^1(S,K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S)=1-q(S)+p_g(S)$ is the *Euler characteristic*. If T is a normal surface, a desingularization $\lambda: S\to T$ is said to be *minimal* if λ does not contract any (-1)-curve in S. Such a minimal desingularization always exists and it is determined uniquely by T [BPV84, p. 86]; it is worth pointing out that S is not necessarily a minimal surface (cf. Proposition 3.5).

Throughout the paper we use the following notation for groups:

- $\mathbb{Z}_n$: cyclic group of order n.
- $D_{p,q,r}=\mathbb{Z}_p\ltimes\mathbb{Z}_q=\langle x,y \mid x^p=y^q=1,\ xyx^{-1}=y^r\rangle$: split metacyclic group of order pq, note that $r^p\equiv 1 \bmod q$. The group $D_{2,n,-1}$ is the dihedral group of order $2n$, that will be denoted by D_n.
- S_n, A_n: symmetric, alternating group on n symbols. We write the composition of permutations from the right to the left; for instance, $(13)(12)=(123)$.
- $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{SL}_n(\mathbb{F}_q)$, $\mathrm{PSL}_n(\mathbb{F}_q)$: general linear, special linear and projective special linear group of $n\times n$ matrices over a field with q elements.
- Whenever we give a presentation of a semi-direct product $H\ltimes N$, the first generators represent H and the last generators represent N. The action of H on N is specified by conjugation relations.
- The order of a finite group G is denoted by $|G|$. If H is a subgroup of G, the centralizer of H in G is denoted by $C_G(H)$ and the normalizer of H in G by $N_G(H)$. The conjugacy relation in G is denoted by $\sim_G$.
- The subgroup generated by $x_1,\dots,x_n\in G$ is denoted by $\langle x_1,\dots,x_n\rangle$. The derived subgroup of G is denoted by G'. The center of G is denoted by $Z(G)$. The set of elements of G different from the identity is denoted by $G^\times$.
- If $x\in G$, the order of x is denoted by $o(x)$ and the conjugacy class of x by $\mathrm{Cl}(x)$. If $x,y\in G$, their commutator is defined as $[x,y]=xyx^{-1}y^{-1}$.
- All groups are represented in multiplicative format.

1. Algebraic background

In this section we present some preliminaries and we fix the algebraic set-up. Many of the results that we collect here are standard, so proofs are often omitted. We refer the reader to [Br90, Section 2], [Bre00, Chapter 3], [H71] and [Pol08, Section 1] for more details.

Definition 1.1. Let G be a finite group and let

$$\mathfrak{g}'\geqslant 0,\quad m_r\geqslant m_{r-1}\geqslant\cdots\geqslant m_1\geqslant 2$$

be integers. A *generating vector* for G of type $(\mathfrak{g}'\mid m_1,\dots,m_r)$ is a $(2\mathfrak{g}'+r)$-tuple of elements

$$\mathcal{V}=\{g_1,\dots,g_r;\ h_1,\dots,h_{2\mathfrak{g}'}\}$$

such that the following conditions are satisfied:

- the set $\mathcal{V}$ generates G;
- $o(g_i)=m_i$;
- $g_1g_2\cdots g_r\prod_{i=1}^{\mathfrak{g}'}[h_i,h_{i+\mathfrak{g}'}]=1$.

If such a $\mathcal{V}$ exists, then G is said to be $(\mathfrak{g}'\mid m_1,\dots,m_r)$-generated.

For convenience we make abbreviations such as $(4 \mid 2^3, 3^2)$ for $(4 \mid 2, 2, 2, 3, 3)$ when we write down the type of the generating vector $\mathcal{V}$.

Proposition 1.2. *If an abelian group G is $(\mathfrak{g}' \mid m_1, \dots, m_r)$-generated, then $r \neq 1$.*

Proof. If $r = 1$ and $\mathcal{V} = \{g_1,\ h_1, \dots, h_{2\mathfrak{g}'}\}$ is a generating vector, we have

$$1 = g_1 \prod_{i=1}^{\mathfrak{g}'} [h_i, h_{i+\mathfrak{g}'}] = g_1,$$

a contradiction because $o(g_1) = m_1 \geqslant 2$. $\square$

The following result, which is essentially a reformulation of Riemann's existence theorem, translates the problem of finding Riemann surfaces with automorphisms into the group theoretic problem of finding groups G which contain suitable generating vectors.

Proposition 1.3. *A finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $\mathfrak{g}$ if and only if there exist integers $\mathfrak{g}' \geqslant 0$ and $m_r \geqslant m_{r-1} \geqslant \cdots \geqslant m_1 \geqslant 2$ such that G is $(\mathfrak{g}' \mid m_1, \dots, m_r)$-generated, with generating vector $\mathcal{V} = \{g_1, \dots, g_r;\ h_1, \dots, h_{2\mathfrak{g}'}\}$, and the following Riemann–Hurwitz relation holds*:

$$2\mathfrak{g} - 2 = |G| \left(2\mathfrak{g}' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{1}$$

If this is the case then $\mathfrak{g}'$ is the genus of the quotient Riemann surface $Y := X/G$ and the G-cover $X \to Y$ is branched in r points $P_1, \dots, P_r$ with branching numbers $m_1, \dots, m_r$, respectively. In addition, the subgroups $\langle g_i \rangle$ and their conjugates provide all the non-trivial stabilizers of the action of G on X.

Let G, $\mathcal{V}$ and X be as in Proposition 1.3. For any $h \in G$ set $H := \langle h \rangle$ and define

$$\mathrm{Fix}_X(h) = \mathrm{Fix}_X(H) := \{x \in X \mid hx = x\}.$$

Proposition 1.4. *If $o(h) = m$ then*

$$\left|\mathrm{Fix}_X(h)\right| = \left|N_G(H)\right| \cdot \sum_{\substack{1 \leqslant i \leqslant r \\ m \mid m_i \\ H \sim_G \langle g_i^{m_i/m} \rangle}} \frac{1}{m_i}.$$

Proof. (See [Bre00], Lemma 10.4.) Let x be in $\mathrm{Fix}_X(h)$ and let R_i be a set of coset representatives of $\langle g_i \rangle$ in G. Then

$$\begin{aligned}
\mathrm{Fix}_X(h) &= \biguplus_{1 \leqslant i \leqslant r} \left\{ \sigma x \;\middle|\; \sigma \in R_i,\ H \leqslant \left\langle \sigma g_i \sigma^{-1} \right\rangle \right\} \\
&= \biguplus_{1 \leqslant i \leqslant r} \left\{ \sigma x \mid \sigma \in R_i,\ H = \left\langle \sigma g_i^{m_i/m} \sigma^{-1} \right\rangle \right\}.
\end{aligned}$$

Taking the cardinalities on both sides, we get

$$\begin{aligned}\left|\mathrm{Fix}_X(h)\right| &= \sum_{1\leqslant i\leqslant r}\left|\left\{\sigma x \mid \sigma\in R_i,\ H=\left\langle\sigma g_i^{m_i/m}\sigma^{-1}\right\rangle\right\}\right| \\ &= \sum_{1\leqslant i\leqslant r}\left|\left\{\sigma\in R_i \mid H=\left\langle\sigma g_i^{m_i/m}\sigma^{-1}\right\rangle\right\}\right| \\ &= \sum_{1\leqslant i\leqslant r}\frac{1}{m_i}\left|\left\{\sigma\in G \mid H=\left\langle\sigma g_i^{m_i/m}\sigma^{-1}\right\rangle\right\}\right|,\end{aligned}$$

where the set in the ith summand has cardinality $|N_G(H)|$ if H is G-conjugate to $\langle g_i^{m_i/m}\rangle$, and is empty otherwise. □

Corollary 1.5. *If $o(h)=2$ then*

$$\left|\mathrm{Fix}_X(h)\right| = \frac{|G|}{|\mathrm{Cl}(h)|}\cdot\sum_{\substack{2|m_i\\ H\sim_G\langle g_i^{m_i/2}\rangle}}\frac{1}{m_i}. \tag{2}$$

If $o(h)=2$ and $h\in Z(G)$ then

$$\left|\mathrm{Fix}_X(h)\right| = |G|\cdot\sum_{\{i|h\in\langle g_i\rangle\}}\frac{1}{m_i}. \tag{3}$$

Proof. Since $H\cong\mathbb{Z}_2$ we have $N_G(H)=C_G(H)$, so Proposition 1.4 implies (2). The proof of (3) is now immediate. □

2. Some non-generation results

This section contains some non-generation results for finite groups which will be useful in the sequel of our classification procedure. They are obtained either by direct computation or by using the GAP4 database of small groups. We will first use them in Section 7. For every group we refer to the presentation given in the corresponding table of Appendix A.

Lemma 2.1. *Let G be a non-abelian finite group containing a unique element ℓ of order 2. Then G is not $(1\mid 2^2)$-generated.*

Proof. Assume that G is $(1\mid 2^2)$-generated, with generating vector $\mathcal{V}=\{\ell_1,\ell_2;\ h_1,h_2\}$. Since ℓ is the only element of order 2 in G, it follows $\ell\in Z(G)$ and $\ell_1=\ell_2=\ell$, hence $[h_1,h_2]=1$. Therefore $G=\langle\ell,h_1,h_2\rangle$ would be abelian, a contradiction. □

Proposition 2.2. *Referring to Table 2 of Appendix A, the groups G in cases (2b), (2d), (2h) are not $(1\mid 2^2)$-generated.*

Proof. It is sufficient to show that G satisfies the hypotheses of Lemma 2.1.

- Case (2b). $G=Q_8$. Take $\ell=-1$.
- Case (2d). $G=D_{4,3,-1}$. Take $\ell=x^2$.
- Case (2h). $G=\mathrm{SL}_2(\mathbb{F}_3)$. Take $\ell=\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$. □

Proposition 2.3. *Referring to Table 2 of Appendix A, the groups G in cases (2d), (2e), (2f), (2g), (2h), (2i) are not $(1\mid 2^1)$-generated.*

Proof. We do a case-by-case analysis.

• Case $(2d)$. $G = D_{4,3,-1}$.

Looking at the presentation of G, one checks that $G' = \langle y \rangle \cong \mathbb{Z}_3$. Therefore G contains no commutators of order 2, so it cannot be $(1 \mid 2^1)$-generated.

• Case $(2e)$. $G = D_6$.

We have $G' = \langle y^2 \rangle \cong \mathbb{Z}_3$, so G contains no commutators of order 2.

• Case $(2f)$. $G = D_{2,8,3}$.

We have $G' = \langle y^2 \rangle \cong \mathbb{Z}_4$ and the only commutator of order 2 is y^4. A direct computation shows that if $[h_1, h_2] = y^4$ then either $\langle h_1, h_2 \rangle \cong D_4$ or $\langle h_1, h_2 \rangle \cong Q_8$. In particular $\langle h_1, h_2 \rangle \neq G$, hence G is not $(1 \mid 2^1)$-generated.

• Case $(2g)$. $G = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) = G(24, 8)$.

We have $G' = \langle y, w \rangle \cong \mathbb{Z}_6$ and the only commutator of order 2 is y. If $[h_1, h_2] = y$ then $\langle h_1, h_2 \rangle \cong D_4$, so G is not $(1 \mid 2^1)$-generated.

• Case $(2h)$. $G = \mathrm{SL}_2(\mathbb{F}_3)$.

The group G contains a unique element of order 2, namely $\ell = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. A direct computation shows that $G' \cong Q_8$ and that ℓ can be expressed as a commutator in 24 different ways. Moreover, if $[h_1, h_2] = \ell$ we have $\langle h_1, h_2 \rangle \cong Q_8$, so G is not $(1 \mid 2^1)$-generated.

• Case $(2i)$. $G = \mathrm{GL}_2(\mathbb{F}_3)$.

It is well known that $G' = \mathrm{SL}_2(\mathbb{F}_3)$; then G' contains a unique element of order 2, namely ℓ. Either by direct computation or by using GAP4, one can check that there are 96 different ways to write ℓ as a commutator in G. If $[h_1, h_2] = \ell$ and both h_1 and h_2 belong to $\mathrm{SL}_2(\mathbb{F}_3)$, then $\langle h_1, h_2 \rangle \cong Q_8$; otherwise $\langle h_1, h_2 \rangle \cong D_4$. In both cases $\langle h_1, h_2 \rangle \neq G$, hence G is not $(1 \mid 2^1)$-generated. □

Proposition 2.4. *Referring to Table* 3 *of Appendix A, the groups G in cases* $(3d)$, $(3e)$, $(3i)$, $(3j)$, $(3l)$, $(3n)$, $(3o)$, $(3p)$, $(3q)$, $(3r)$, $(3s)$, $(3t)$, $(3u)$, $(3v)$, $(3w)$ *are not* $(1 \mid 2^1)$*-generated.*

Proof. We have already proven the statement in cases $(3d)$, $(3e)$ and $(3n)$: see Proposition 2.3, cases $(2d)$, $(2e)$ and $(2h)$. Now let us consider the remaining cases.

• Case $(3i)$. $G = \mathbb{Z}_2 \times D_4$.

The group G cannot be generated by two elements, so in particular it cannot be $(1 \mid 2^1)$-generated.

• Case $(3j)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4) = G(16, 13)$.

We have $G' = \langle z^2 \rangle \cong \mathbb{Z}_2$. By direct computation or by using GAP4 (see Appendix B for the corresponding script) we can check that if $[h_1, h_2] = z^2$ then either $\langle h_1, h_2 \rangle \cong D_4$ or $\langle h_1, h_2 \rangle \cong Q_8$, so $\langle h_1, h_2 \rangle \neq G$.

• Case $(3l)$. $G = D_{2,12,5}$.

We have $G' = \langle y^4 \rangle \cong \mathbb{Z}_3$, so G contains no commutators of order 2.

• Cases $(3o)$ and $(3p)$. $G = S_4$.

We have $G' = A_4$. If $[h_1, h_2]$ has order 2 then $\langle h_1, h_2 \rangle \cong D_4$ or $\langle h_1, h_2 \rangle \cong A_4$, so G is not $(1 \mid 2^1)$-generated.

• Case $(3q)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) = G(32, 9)$.

We have $G' = \langle yz^2 \rangle \cong \mathbb{Z}_4$ and the only commutator of order 2 is $(yz^2)^2 = z^4$. If $[h_1, h_2] = z^4$ then $\langle h_1, h_2 \rangle$ has order 8 or 16, hence $\langle h_1, h_2 \rangle \neq G$.

• Case $(3r)$. $G = \mathbb{Z}_2 \ltimes D_{2,8,5} = G(32, 11)$.

We have $G' = \langle yz^2 \rangle \cong \mathbb{Z}_4$ and the only commutator of order 2 is $(yz^2)^2 = z^4$. If $[h_1, h_2] = z^4$ then $\langle h_1, h_2 \rangle$ has order 8 or 16, hence $\langle h_1, h_2 \rangle \neq G$.

• Case $(3s)$. $G = \mathbb{Z}_2 \times S_4$.

If $[h_1, h_2]$ has order 2 then $|\langle h_1, h_2 \rangle| \leqslant 24$, so $\langle h_1, h_2 \rangle \neq G$.

• Case $(3t)$. $G = G(48, 33)$.

We have $G' = \langle t, z, w \rangle \cong Q_8$ and the only commutator of order 2 is t. If $[h_1, h_2] = t$ then $\langle h_1, h_2 \rangle \cong D_4$ or $\langle h_1, h_2 \rangle \cong Q_8$, so G is not $(1 \mid 2^1)$-generated.

• Case $(3u)$. $G = \mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2 = G(48, 3)$.

We have $G' = \langle y, z \rangle \cong (\mathbb{Z}_4)^2$. If $[h_1, h_2]$ has order 2 then $\langle h_1, h_2 \rangle \cong A_4$, so G is not $(1 \mid 2^1)$-generated.

• Case $(3v)$. $G = S_3 \ltimes (\mathbb{Z}_4)^2 = G(96, 64)$.

We have $G' = \langle y, z \rangle$ and $|G'| = 48$. The elements of order 2 in G' are z^2, y^2z^2y, yz^2y^2. If $[h_1, h_2]$ has order 2 then $|\langle h_1, h_2 \rangle| \leqslant 16$, so G is not $(1 \mid 2^1)$-generated.

• Case $(3w)$. $G = \mathrm{PSL}_2(\mathbb{F}_7)$.

Since G is simple we have $G' = G$. If $[h_1, h_2]$ has order 2 then either $\langle h_1, h_2 \rangle \cong D_4$ or $\langle h_1, h_2 \rangle \cong A_4$, so G is not $(1 \mid 2^1)$-generated. $\square$

Proposition 2.5. *Referring to Table* 3 *of Appendix A, the groups G in cases* $(3i)$, $(3j)$, $(3s)$, $(3v)$ *are not* $(1 \mid 4^1)$-*generated.*

Proof. We do a case-by-case analysis.

• Case $(3i)$. $G = \mathbb{Z}_2 \times D_4$.

We have $G' = \langle (1, y^2) \rangle \cong \mathbb{Z}_2$; therefore G contains no commutators of order 4 and so it cannot be $(1 \mid 4^1)$-generated.

• Case $(3j)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4) = G(16, 13)$.

We have $G' = \langle z^2 \rangle \cong \mathbb{Z}_2$, so G contains no commutators of order 4 and we conclude as in the previous case.

• Case $(3s)$. $G = \mathbb{Z}_2 \times S_4$.

We have $G' \cong A_4$, so G contains no commutators of order 4.

• Case $(3v)$. $G = S_3 \ltimes (\mathbb{Z}_4)^2 = G(96, 64)$.

If $[h_1, h_2]$ has order 4 then $|\langle h_1, h_2 \rangle| \leqslant 48$, so G is not $(1 \mid 4^1)$-generated. $\square$

Proposition 2.6. *Referring to Table* 4 *of Appendix A, the groups G in cases* $(4g)$, $(4h)$, $(4i)$, $(4j)$, $(4k)$, $(4l)$, $(4o)$, $(4p)$, $(4q)$, $(4s)$, $(4t)$, $(4u)$, $(4v)$, $(4w)$, $(4y)$, $(4z)$, $(4aa)$, $(4ab)$ *are not* $(1 \mid 2^1)$-*generated.*

Proof. Again a case-by-case analysis.

• Cases $(4g)$ and $(4h)$. $G = D_6$.

See Proposition 2.3, case $(2e)$.

• Case $(4i)$. $G = D_8$.

We have $G' = \langle y^2 \rangle \cong \mathbb{Z}_4$ and the only commutator of order 2 is y^4. If $[h_1, h_2] = y^4$ then $\langle h_1, h_2 \rangle \cong D_4$, hence G is not $(1 \mid 2^1)$-generated.

• Case $(4j)$. $G = G(16, 9)$.

We have $G' = \langle z \rangle \cong \mathbb{Z}_4$ and the only commutator of order 2 is z^2. If $[h_1, h_2] = z^2$ then $\langle h_1, h_2 \rangle \cong Q_8$, hence G is not $(1 \mid 2^1)$-generated.

• Cases $(4k)$ and $(4l)$. $G = \mathbb{Z}_3 \times S_3$.

We have $G' \cong \mathbb{Z}_3$, so G contains no commutators of order 2.

• Case $(4o)$. $G = D_{4,5,-1}$.

We have $G' = \langle y \rangle \cong \mathbb{Z}_5$, so G contains no commutators of order 2.

• Case $(4p)$. $G = D_{4,5,2}$.

We have $G' = \langle y \rangle \cong \mathbb{Z}_5$, so G contains no commutators of order 2.

• Case $(4q)$. $G = S_4$.

See Proposition 2.4, cases $(3o)$ and $(3p)$.

• Case $(4s)$. $G = \mathrm{SL}_2(\mathbb{F}_3)$.

See Proposition 2.3, case $(2h)$.

• Case $(4t)$. $G = D_{2,16,7}$.

We have $G' = \langle y^2 \rangle \cong \mathbb{Z}_8$ and the only commutator of order 2 is y^8. If $[h_1, h_2] = y^8$ then $\langle h_1, h_2 \rangle \cong D_4$ or $\langle h_1, h_2 \rangle \cong Q_8$, hence G is not $(1 \mid 2^1)$-generated.

• Cases $(4u)$ and $(4v)$. $G = (\mathbb{Z}_2)^2 \ltimes (\mathbb{Z}_3)^2 = G(36, 10)$.

We have $G' = \langle z, w \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, so G contains no commutators of order 2.

• Case $(4w)$. $G = \mathbb{Z}_6 \times S_3$. We have $G' \cong \mathbb{Z}_3$, so G contains no commutators of order 2.

• Case $(4y)$. $G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2 = G(36, 9)$.

We have $G' = \langle y, z \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, so G contains no commutators of order 2.

• Case $(4z)$. $G = D_4 \ltimes \mathbb{Z}_5 = G(40, 8)$.

We have $G' = \langle y^2, z\rangle \cong \mathbb{Z}_{10}$ and the only commutator of order 2 is y^2. If $[h_1, h_2] = y^2$ then $\langle h_1, h_2\rangle \cong D_4$, hence G is not $(1 \mid 2^1)$-generated.

• Case (4*aa*). $G = A_5$.

Since G is simple we have $G' = G$. If $[h_1, h_2]$ has order 2 then $\langle h_1, h_2\rangle \cong A_4$, hence G is not $(1 \mid 2^1)$-generated.

• Case (4*ab*). $G = \mathbb{Z}_3 \times S_4$.

We have $G' = A_4$. If $[h_1, h_2]$ has order 2 then $|\langle h_1, h_2\rangle| \leqslant 36$, hence G is not $(1 \mid 2^1)$-generated. □

3. Standard isotrivial fibrations

In this section we establish the basic properties of standard isotrivial fibrations. Definition 3.1 and Theorem 3.2 can be found in [Se96].

From now on, S will always denote a smooth, projective surface of general type.

Definition 3.1. We say that S is a *standard isotrivial fibration* if there exists a finite group G acting faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$. The two maps $\alpha : S \to C/G$, $\beta : S \to F/G$ will be referred as the *natural projections*. If T is smooth then $S = T$ is called a *quasi-bundle*, or a surface *isogenous to an unmixed product*.

The stabilizer $H \subseteq G$ of a point $y \in F$ is a cyclic group ([FK92], p. 106). If H acts freely on C, then T is smooth along the scheme-theoretic fiber of $\sigma : T \to F/G$ over $\bar{y} \in F/G$, and this fiber consists of the curve C/H counted with multiplicity $|H|$. Thus, the smooth fibers of σ are all isomorphic to C. On the contrary, if $x \in C$ is fixed by some non-zero element of H, then T has a cyclic quotient singularity over the point $\overline{(x, y)} \in (C \times F)/G$. In this case, the fiber of $\overline{(x, y)}$ on the minimal desingularization $\lambda : S \to T$ is an *HJ*-string (abbreviation of Hirzebruch–Jung string), that is to say, a connected union of smooth rational curves $Z_1, \ldots, Z_n$ with self-intersection $\leqslant -2$, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i, and $Z_i Z_j = 0$ if $|i - j| \geqslant 2$ ([BPV84], III 5.4). These observations lead to the following statement, which describes the singular fibers that can arise in a standard isotrivial fibration (see [Se96], Theorem 2.1).

Theorem 3.2. *Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration and let us consider the natural projection $\beta : S \to F/G$. Take any point over $\bar{y} \in F/G$ and let Λ denote the fiber of β over $\bar{y}$. Then*

(i) *the reduced structure of Λ is the union of an irreducible curve Y, called the central component of Λ, and either none or at least two mutually disjoint HJ-strings, each meeting Y at one point. These strings are in one-to-one correspondence with the branch points of $C \to C/H$, where $H \subseteq G$ is the stabilizer of y;*
(ii) *the intersection of a string with Y is transversal, and it takes place at only one of the end components of the string;*
(iii) *Y is isomorphic to C/H, and has multiplicity equal to $|H|$ in Λ.*

Evidently, a completely similar statement holds if we consider the natural projection $\alpha : S \to C/G$.

Remark 3.3. The *HJ*-strings arising from the minimal resolution of RDPs are precisely the A_n-cycles.

Theorem 3.2 and Remark 3.3 now imply

Corollary 3.4. *Let us suppose that T has at worst* RDPs, *and let Λ be any fiber of $\beta : S \to F/G$. Then Λ contains either none or at least two A_n-cycles. An analogous statement holds if we consider any fiber Φ of $\alpha : S \to C/G$.*

It is worth pointing out that a standard isotrivial fibration is not necessarily a minimal surface; indeed, the central component of some reducible fiber might be a (-1)-curve. A criterion for minimality is provided by the following

Proposition 3.5. *If T has at worst* RDPs *then both fibrations $\alpha : S \to C/G$ and $\beta : S \to F/G$ are relatively minimal. In addition, if either $g(C/G) > 0$ or $g(F/G) > 0$ then S is a minimal model.*

Proof. Let us suppose that β is not relatively minimal; then there is a singular fiber Λ whose central component Y is a (-1)-curve. Corollary 3.4 implies that Λ contains (at least) two disjoint A_n-cycles Z_1, Z_2 such that $YZ_1 = YZ_2 = 1$. Thus by blowing down Y we obtain a surface S' with two (-1)-curves E_1, E_2 such that $E_1E_2 = 1$, a contradiction because S is of general type (cf. [BPV84], Proposition 4.6, p. 79). The proof for α is similar. The last part of the statement follows at once because a fibration over a curve of strictly positive genus is minimal if and only if it is relatively minimal. $\square$

Now set $\mathfrak{g}'_1 := g(F/G)$ and $\mathfrak{g}'_2 := g(C/G)$. By Proposition 1.3 it follows that there exist

- integers $2 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r$ such that G is $(\mathfrak{g}'_1 \mid m_1, \ldots, m_r)$-generated and
- integers $2 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_s$ such that G is $(\mathfrak{g}'_2 \mid n_1, \ldots, n_s)$-generated.

Proposition 3.6. *If T has at worst* RDPs, *then*

- *m_i divides $2g(C) - 2$ for all $i \in \{1, \ldots, r\}$;*
- *n_j divides $2g(F) - 2$ for all $j \in \{1, \ldots, s\}$.*

Proof. Take any $i \in \{1, \ldots, r\}$. By Theorem 3.2 there exists a fiber Λ of $\beta : S \to F/G$ having the form $\Lambda = Y + Z$, where Y is a component of multiplicity m_i and Z is a (possibly empty) union of (-2)-curves. Setting $Y = m_iY'$ we obtain $K_S\Lambda = m_iK_SY'$; since Λ is algebraically equivalent to C this implies $2g(C) - 2 = m_iK_SY'$. Thus m_i divides $2g(C) - 2$. Clearly, we can prove the second claim in the same way. $\square$

Corollary 3.7. *Assuming that T has at worst* RDPs, *the following holds*:

- *if $g(C) = 2$ then $m_i = 2$ for all $i \in \{1, \ldots, r\}$;*
- *if $g(F) = 2$ then $n_j = 2$ for all $j \in \{1, \ldots, s\}$.*

Corollary 3.8. *Suppose that T has only* RDPs. *If either $g(C) = 2$ or $g(F) = 2$ then T has at worst nodes.*

Proof. If $g(C) = 2$ then by Corollary 3.7 it follows that the non-trivial stabilizers of the action of G on F are isomorphic to $\mathbb{Z}_2$, and this implies that the singularities of T are at worst nodes. If $g(F) = 2$ the argument is the same. $\square$

The invariants of S can be computed using

Proposition 3.9. *Let V be a smooth algebraic surface, and let G be a finite group acting on V with only isolated fixed points. Suppose that the quotient $T := V/G$ has at worst* RDPs, *and let $\lambda : S \to T$ be the minimal desingularization. Let t_n be the number of singular points of type A_n in T. Then we have*

(i) $|G| \cdot K_S^2 = K_V^2$.
(ii) $|G| \cdot e(S) = e(V) + |G| \cdot \sum_n \frac{(n+1)^2-1}{n+1} t_n$.
(iii) $H^0(S, \Omega_S^1) = H^0(V, \Omega_V^1)^G$.

Proof. (i) This is immediate because G acts on V with only isolated fixed points and the singularities of T are at worst RDPs.

(ii) Let $\pi : V \to T$ be the projection, T^o be the smooth locus of T and $V^o := \pi^{-1}(T^o)$; finally set $S^o = \lambda^{-1}(T^o)$. Let $p \in T$ be a singularity of type A_n; since p is covered by $\frac{|G|}{n+1}$ points in V, we obtain

$$e\big(V^o\big) = e(V) - \sum_n \frac{|G|}{n+1} t_n.$$

On the other hand, since G acts on V^o without fixed points, we have $|G| \cdot e(S^o) = e(V^o)$. Finally, notice that S is obtained from S^o by attaching all the A_n-cycles; since every A_n-cycle Z verifies $e(Z) = n+1$, the additivity of the Euler number implies

$$\begin{aligned}|G| \cdot e(S) &= |G| \cdot e\big(S^o\big) + |G| \cdot \sum_n (n+1) t_n \\ &= e(V) + |G| \cdot \sum_n \frac{(n+1)^2 - 1}{n+1} t_n.\end{aligned}$$

(iii) See [Fre71]. $\square$

So we have

Proposition 3.10. *Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration such that T has at worst* RDPs. *Denote by t_n the number of singular points of type A_n in T. Then the invariants of S are*

- $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$;
- $e(S) = \frac{4(g(C)-1)(g(F)-1)}{|G|} + \sum_n \frac{(n+1)^2-1}{n+1} t_n$;
- $q(S) = g(C/G) + g(F/G)$.

In particular this implies (cf. [Se96]):

Corollary 3.11. *The following are equivalent*:

- *$t_n = 0$ for any $n \geqslant 1$;*
- *$K_S^2 = 2e(S)$;*
- *S is a quasi-bundle.*

Remark 3.12. By Corollary 3.4 it follows $\sum_n t_n \neq 1$.

4. The case $\chi(\mathcal{O}_S) = 1$

Proposition 4.1. *Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration, such that T contains at worst* RDPs. *In addition, let us assume $\chi(\mathcal{O}_S) = 1$. Then there are the following possibilities*:

- *$1 \leqslant K_S^2 \leqslant 8$ and T contains $8 - K_S^2$ points of type A_1;*
- *$K_S^2 = 3$ and T contains two points of type A_3;*
- *$K_S^2 = 2$ and T contains one point of type A_1 and two points of type A_3;*
- *$K_S^2 = 1$ and T contains two points of type A_1 and two points of type A_3.*

Proof. If a minimal surface of general type with $\chi(\mathcal{O}_S) = 1$ contains some A_n-cycle then $n \leqslant 10$ [Mi84]. Thus by Proposition 3.10 we have

$$\frac{1}{2}K_S^2+\frac{3}{2}t_1+\frac{8}{3}t_2+\frac{15}{4}t_3+\frac{24}{5}t_4 \\ +\frac{35}{6}t_5+\frac{48}{7}t_6+\frac{63}{8}t_7+\frac{80}{9}t_8+\frac{99}{10}t_9+\frac{120}{11}t_{10}=e(S).$$

Noether formula gives $e(S)=12-K_S^2$, so we obtain

$$41580K_S^2+41580t_1+73920t_2+103950t_3+133056t_4 \\ +161700t_5+190080t_6+218295t_7+246400t_8 \\ +274428t_9+302400t_{10}=332640. \tag{4}$$

We can check by direct computation that the only non-negative integers K_S^2, $t_1,\dots,t_{10}$ which satisfy (4) are

- $1\leqslant K_S^2\leqslant 8$, $t_1=8-K_S^2$;
- $K_S^2=3$, $t_3=2$;
- $K_S^2=2$, $t_1=1$, $t_3=2$;
- $K_S^2=1$, $t_1=2$, $t_3=2$.

This completes the proof. $\square$

Proposition 4.2. *Let $\lambda: S\to T=(C\times F)/G$ be as in Proposition* 4.1. *If S is not a quasi-bundle, then $K_S^2\leqslant 6$.*

Proof. Since S is not a quasi-bundle we have $K_S^2\leqslant 7$. On the other hand, if $K_S^2=7$ then $t_1=1$ and $t_n=0$ for $n\geqslant 2$. But this is impossible by Remark 3.12. $\square$

If $\chi(\mathcal{O}_S)=1$ then Proposition 3.6 can be refined in the following way.

Proposition 4.3. *Let S be as in Proposition* 4.1 *and let us assume $K_S^2=6$ or $K_S^2=5$. Then*

- *m_i divides $g(C)-1$ for all $i\in\{1,\dots,r\}$, except at most one*;
- *n_j divides $g(F)-1$ for all $j\in\{1,\dots,s\}$, except at most one.*

Proof. Suppose $K_S^2=6$ or $K_S^2=5$. Then T contains either 3 or 2 nodes (Proposition 4.1) and by Theorem 3.2 the corresponding (-2)-curves must belong to the same fiber of $\beta: S\to F/G$. It follows that, for all i except one, there is a subgroup H of G, isomorphic to $\mathbb{Z}_{m_i}$, which acts freely on C. Now Riemann–Hurwitz formula applied to $C\to C/H$ gives

$$g(C)-1=m_i\big(g(C/H)-1\big),$$

so m_i divides $g(C)-1$. The second statement can be proven in the same way. $\square$

Set $\mathbf{m}:=(m_1,\dots,m_r)$ and $\mathbf{n}:=(n_1,\dots,n_s)$, where we make the usual abbreviations such as $(2^3,3^2)$.

Proposition 4.4. *Let us assume $\chi(\mathcal{O}_S)=1$ and $K_S^2=6$ or $K_S^2=5$. Then $g(F)=2$ implies $\mathbf{n}=(2^1)$, whereas $g(C)=2$ implies $\mathbf{m}=(2^1)$.*

Proof. If $g(F)=2$ then Corollary 3.7 yields $\mathbf{n}=(2^s)$. On the other hand, if $s\geqslant 2$ then Proposition 4.3 implies that 2 divides $g(F)-1=1$, a contradiction. An analogous proof works in the case $g(C)=2$. $\square$

5. Standard isotrivial fibrations with $p_g = q = 1$. Building data

From now on we suppose that $\lambda : S \to T = (C \times F)/G$ is a standard isotrivial fibration with $p_g = q = 1$, such that T has at worst RDPs. Since $q = 1$, we may assume that $E := C/G$ is an elliptic curve and that $F/G \cong \mathbb{P}^1$, that is $\mathfrak{g}'_1 = 0$ and $\mathfrak{g}'_2 = 1$. Then the natural projection $\alpha : S \to E$ is the Albanese morphism of S and $g_{\mathrm{alb}} = g(F)$. Moreover by Proposition 3.5 it follows that S is a minimal model. Let $\mathcal{V} = \{g_1, \ldots, g_r\}$ be a generating vector for G of type $(0 \mid m_1, \ldots, m_r)$, inducing the G-cover $F \to \mathbb{P}^1$ and let $\mathcal{W} = \{\ell_1, \ldots, \ell_s;\ h_1, h_2\}$ be a generating vector of type $(1 \mid n_1, \ldots, n_s)$ inducing $C \to E$. Then Riemann–Hurwitz formula implies

$$2g(F) - 2 = |G|\left(-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right),$$

$$2g(C) - 2 = |G| \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right). \tag{5}$$

Proposition 5.1. *Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, such that T has at worst* RDPs. *Then*

$$\frac{K_S^2}{4(g(F) - 1)} = \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right). \tag{6}$$

Proof. Using Proposition 3.10 and the second relation in (5) we obtain

$$\frac{|G| \cdot K_S^2}{4(g(F) - 1)} = 2\big(g(C) - 1\big) = |G| \cdot \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right),$$

so the claim follows. □

Proposition 5.2. *The case $p_g = q = 1$, $K_S^2 = 5$ does not occur.*

Proof. If $K_S^2 = 5$ occurs, Proposition 5.1 gives

$$\big(g(F) - 1\big) \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right) = \frac{5}{4}. \tag{7}$$

If $s \geqslant 2$ then $g(F) - 1 \leqslant \frac{5}{4}$, hence $g(F) = 2$. This yields $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) = \frac{5}{4}$, hence $\mathbf{n} = (2^1, 4^1)$, which contradicts Proposition 4.4. Therefore we must have $s = 1$, i.e. $\mathbf{n} = (n^1)$. This implies

$$\frac{5}{4} = \big(g(F) - 1\big)\left(1 - \frac{1}{n}\right) \geqslant \frac{1}{2}\big(g(F) - 1\big),$$

hence $g(F) \leqslant 3$. Using (7), we obtain $(1 - \frac{1}{n}) = \frac{5}{4}$ if $g(F) = 2$ and $(1 - \frac{1}{n}) = \frac{5}{8}$ if $g(F) = 3$; but both cases are impossible, because n must be a positive integer. □

Proposition 5.3. *The case $p_g = q = 1$, $K_S^2 = 3$ does not occur.*

Proof. If $K_S^2 = 3$ then either $g(F) = 3$ or $g(F) = 2$ [CaCi91,CaCi93]. In the former case Proposition 5.1 implies $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) = \frac{3}{8}$, which is impossible. In the latter case we have $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) = \frac{3}{4}$, hence $\mathbf{n} = (4^1)$ which contradicts Corollary 3.7. $\square$

Proposition 5.4. *If $p_g = q = 1$ and $K_S^2 = 2$, then T contains only nodes.*

Proof. If $K_S^2 = 2$ we have $g(F) = 2$ [Ca81,CaCi91,CaCi93]. So the claim follows by Corollary 3.8. $\square$

Summing up and using Proposition 4.1 we obtain

Proposition 5.5. *Let $\lambda : S \to T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, such that T has at worst* RDPs. *Then K_S^2 is even and the only singularities of T are $8 - K_S^2$ nodes.*

Now let us observe that the cyclic subgroups $\langle g_1\rangle, \dots, \langle g_r\rangle$ and their conjugates provide the non-trivial stabilizers of the action of G on F, whereas $\langle \ell_1\rangle, \dots, \langle \ell_s\rangle$ and their conjugates provide the non-trivial stabilizers of the actions of G on C. The singularities of T arise from the points in $C \times F$ with non-trivial stabilizer; since the action of G on $C \times F$ is the diagonal one, it follows that the set $\mathcal{S}$ of all non-trivial stabilizers for the action of G on $C \times F$ is given by

$$\mathcal{S} = \left(\bigcup_{\sigma \in G}\bigcup_{i=1}^{r}\langle \sigma g_i \sigma^{-1}\rangle\right) \cap \left(\bigcup_{\sigma \in G}\bigcup_{j=1}^{s}\langle \sigma \ell_j \sigma^{-1}\rangle\right) \cap G^{\times}. \tag{8}$$

Notice that Proposition 5.5 implies that every element of $\mathcal{S}$ has order 2. Moreover the (reduced) fiber of the covering $C \times F \to T$ over each node has cardinality $\frac{|G|}{2}$, so the number of nodes of T is given by

$$8 - K_S^2 = t_1 = \frac{2}{|G|}\sum_{h \in \mathcal{S}}\left|\mathrm{Fix}_C(h)\right| \cdot \left|\mathrm{Fix}_F(h)\right|.$$

Proposition 3.10 yields

$$K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}, \tag{9}$$

so we can write down the basic equality

$$\big(g(C)-1\big)\big(g(F)-1\big) + \frac{1}{4}\sum_{h \in \mathcal{S}}\left|\mathrm{Fix}_C(h)\right| \cdot \left|\mathrm{Fix}_F(h)\right| = |G|. \tag{10}$$

We call $(G, \mathcal{V}, \mathcal{W})$ the *building data* of S. In fact, we have the following structure result.

Proposition 5.6. *Let G be a finite group which is both $(0 \mid m_1, \dots, m_r)$-generated and $(1 \mid n_1, \dots, n_s)$-generated, with generating vectors $\mathcal{V} = \{g_1, \dots, g_r\}$ and $\mathcal{W} = \{\ell_1, \dots, \ell_s;\ h_1, h_2\}$, respectively. Denote by*

$$f : F \longrightarrow \mathbb{P}^1 = F/G,$$

$$h : C \longrightarrow E = C/G$$

the two G-coverings induced by $\mathcal{V}$ and $\mathcal{W}$ and let $g(F)$, $g(C)$ be the genera of F and C, that are related to $|G|$, $\mathbf{m}$, $\mathbf{n}$ by (5). Finally, define $\mathcal{S}$ as in (8). Assume moreover that

- $g(C) \geqslant 2$, $g(F) \geqslant 2$;
- *every element of* $\mathcal{S}$ *has order* 2;
- *equality* (10) *is satisfied.*

Then the quotient $T := (C \times F)/G$ *contains exactly* $8 - K_T^2$ *nodes and its minimal desingularization* S *is a minimal surface of general type whose invariants are*

$$p_g(S) = q(S) = 1, \qquad K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}.$$

Conversely, every standard isotrivial fibration S*, with* $p_g(S) = q(S) = 1$ *and such that* T *has only* RDPs*, arises in this way.*

Proof. We have already shown that, if $\lambda : S \to T = (C \times F)/G$ is a standard isotrivial fibration with $p_g = q = 1$, such that T has at worst RDPs, then the assumptions above must be satisfied. Vice versa, if all the assumptions are satisfied then the quotient $T = (C \times F)/G$ is a nodal surface with $q(T) = 1$, whose number of nodes is given by

$$\begin{aligned} t_1 &= \frac{2}{|G|} \sum_{h \in \mathcal{S}} \left|\mathrm{Fix}_C(h)\right| \cdot \left|\mathrm{Fix}_F(h)\right| \\ &= \frac{2}{|G|} \cdot 4\big(|G| - \big(g(C)-1\big)\big(g(F)-1\big)\big) \quad \big(\text{using (10)}\big) \\ &= 8 - \frac{8(g(C)-1)(g(F)-1)}{|G|}. \end{aligned}$$

Let S be the minimal desingularization of T; by using Proposition 3.10 and relation (9) we obtain

$$\begin{aligned} e(S) &= \frac{1}{2} K_S^2 + \frac{3}{2} t_1 \\ &= \frac{1}{2} K_S^2 + \frac{3}{2}\big(8 - K_S^2\big) \\ &= 12 - K_S^2. \end{aligned}$$

Thus Noether formula yields $\chi(\mathcal{O}_S) = 1$, that implies $p_g(S) = q(S) = 1$. Again by (9) we have $K_S^2 > 0$, hence S is a surface of general type, which must be minimal by Proposition 3.5. $\square$

Remark 5.7. The surface S is a quasi-bundle if and only if $\mathcal{S} = \emptyset$ (see [Pol08, Proposition 7.2]).

6. Standard isotrivial fibrations with $p_g = q = 1$. The abelian case

The aim of this section is to prove

Theorem 6.1. *Let* $\lambda : S \to T = (C \times F)/G$ *be a standard isotrivial fibration with* $p_g = q = 1$*, which is not a quasi-bundle, such that* T *has only* RDPs*. Assume in addition that the group* G *is abelian. Then* $K_S^2 = 4$*,* $g(F) = 2$ *and there are three cases*:

- $g(C) = 3$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$;
- $g(C) = 4$, $G = \mathbb{Z}_6$;
- $g(C) = 7$, $G = \mathbb{Z}_2 \times \mathbb{Z}_6$.

All possibilities occur.

The proof of Theorem 6.1 will be a consequence of the following results.

Proposition 6.2. *If G is abelian then*

$$K_S^2 = 4, \qquad g(F) = 2, \qquad \mathbf{n} = (2^2).$$

Proof. Since G is abelian, the G-cover $h: C \to E$ is branched in at least two points (Proposition 1.2); thus $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) \geqslant 1$. By using Proposition 5.1 this gives

$$K_S^2 = 4\big(g(F) - 1\big) \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right) \geqslant 4\big(g(F) - 1\big). \tag{11}$$

Since $K_S^2 \leqslant 6$ (Proposition 4.2), we obtain $g(F) = 2$ and so $K_S^2 \geqslant 4$. Thus Proposition 5.5 implies $K_S^2 = 6$ or $K_S^2 = 4$.

- If $K_S^2 = 6$ then $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) = \frac{3}{2}$, that is either $\mathbf{n} = (2^3)$ or $\mathbf{n} = (4^2)$; since $g(F) = 2$, both possibilities contradict Proposition 4.4.
- If $K_S^2 = 4$ then $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) = 1$, hence $\mathbf{n} = (2^2)$. $\square$

Corollary 6.3. *If G is abelian then* $|G|$ *is even and* $|G| \geqslant 4$.

Proof. By Propositions 3.10 and 6.2 we obtain $|G| = 2(g(C) - 1)$, so $|G|$ is even. If $|G| = 2$ then $g(C) = 2$, so S would be a minimal surface of general type with $p_g = q = 1$, $K_S^2 = 4$ and a rational pencil $|C|$ of genus 2 curves; but this contradicts [Xi85, p. 51]. Thus $|G| \geqslant 4$. $\square$

Proposition 6.4. *If* $|G| = 4$ *then the only possibility is*

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \mathbf{m} = (2^5).$$

This case occurs.

Proof. If $|G| = 4$ then Proposition 6.2 and relations (5) imply $g(C) = 3$ and

$$-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) = \frac{1}{2},$$

so there are two possibilities:

- $\mathbf{m} = (2^2, 4^2)$,
- $\mathbf{m} = (2^5)$.

First let us rule out the case $\mathbf{m} = (2^2, 4^2)$. If it occurs, then $G = \mathbb{Z}_4 = \langle x \mid x^4 = 1 \rangle$. Up to automorphisms of G, we may assume

$$g_1 = g_2 = x^2, \qquad g_3 = x, \qquad g_4 = x^3,$$
$$\ell_1 = \ell_2 = x^2.$$

Then $\mathscr{S} = \{x^2\}$ and by using Corollary 1.5 we obtain

$$\big|\mathrm{Fix}_F\big(x^2\big)\big| = 6, \qquad \big|\mathrm{Fix}_C\big(x^2\big)\big| = 4.$$

It follows that equality (10) is not satisfied, so this case does not occur.

It remains to show that the possibility $\mathbf{m} = (2^5)$ actually occurs. In this case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, because $\mathbb{Z}_4$ is not $(0|2^5)$-generated. Our example is the following.

- $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbf{m} = (2^5)$, $g(C) = 3$.

Set $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y \mid x^2 = y^2 = [x, y] = 1\rangle$ and

$$g_1 = x, \qquad g_2 = y, \qquad g_3 = g_4 = g_5 = xy,$$
$$\ell_1 = \ell_2 = x, \qquad h_1 = h_2 = y.$$

We have $\mathcal{S} = \{x\}$ and by using Corollary 1.5 we obtain

$$\left|\mathrm{Fix}_F(x)\right| = 2, \qquad \left|\mathrm{Fix}_C(x)\right| = 4.$$

Equality (10) is satisfied, hence Proposition 5.6 implies that this case occurs. $\square$

Lemma 6.5. *If G is cyclic then $m_1 \geqslant 3$.*

Proof. If G is cyclic then it contains a unique element h of order 2. By Proposition 6.2 we have $\mathbf{n} = (2^2)$, hence $|\mathrm{Fix}_C(h)| = 2 \cdot \frac{|G|}{2} = |G|$. On the other hand, if $m_1 = 2$ then $|\mathrm{Fix}_F(h)| \geqslant \frac{|G|}{2}$. Since $K_S^2 = 4$ we have

$$4 = t_1 = \frac{2}{|G|} \cdot \left|\mathrm{Fix}_C(h)\right| \cdot \left|\mathrm{Fix}_F(h)\right| \geqslant |G|.$$

Thus $G = \mathbb{Z}_4$, which contradicts Proposition 6.4. $\square$

Proposition 6.6. *If G is abelian and $|G| > 4$ there are two possibilities*:

- $G = \mathbb{Z}_6$, $\mathbf{m} = (3, 6^2)$;
- $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbf{m} = (2, 6^2)$.

Both cases occur.

Proof. The abelian group G acts as a group of automorphisms on the genus 2 curve F so that $F/G \cong \mathbb{P}^1$. Let us look at Table 1 of Appendix A. By using Corollary 6.3 and Lemma 6.5 we may rule out cases $(1a)$, $(1b)$, $(1c)$, $(1d)$, $(1e)$, $(1f)$, $(1h)$, $(1i)$. It remains to show that cases $(1g)$ and $(1j)$ occur.

- Case $(1g)$. $G = \mathbb{Z}_6$, $\mathbf{m} = (3, 6^2)$, $g(C) = 4$.

Set $\mathbb{Z}_6 = \langle x \mid x^6 = 1\rangle$ and

$$g_1 = x^4, \qquad g_2 = x, \qquad g_3 = x,$$
$$\ell_1 = \ell_2 = x^3, \qquad h_1 = h_2 = x.$$

Then $\mathcal{S} = \{x^3\}$ and

$$\left|\mathrm{Fix}_F\left(x^3\right)\right| = 2, \qquad \left|\mathrm{Fix}_C\left(x^3\right)\right| = 6.$$

Equality (10) is satisfied, so this case occurs.

- Case $(1j)$. $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbf{m} = (2, 6^2)$, $g(C) = 7$.

Let x, y be obvious generators of G of order 2 and 6, respectively, and set

$$g_1 = x, \qquad g_2 = y^5, \qquad g_3 = xy,$$
$$\ell_1 = \ell_2 = y^3, \qquad h_1 = x, \qquad h_2 = y.$$

Then $\mathscr{S} = \{y^3\}$ and

$$\left|\mathrm{Fix}_F\left(y^3\right)\right| = 2, \qquad \left|\mathrm{Fix}_C\left(y^3\right)\right| = 12.$$

Equality (10) is satisfied, so this case occurs.

The proof of Theorem 6.1 is complete. □

7. Standard isotrivial fibrations with $p_g = q = 1$. The non-abelian case

By Proposition 5.5 we have $K_S^2 = 6$, 4 or 2. We deal with the three cases separately.

7.1. The case $K_S^2 = 6$

Proposition 7.1. *If $K_S^2 = 6$ then we have two possibilities*:

- $g(F) = 3$, $\mathbf{n} = (4^1)$;
- $g(F) = 4$, $\mathbf{n} = (2^1)$.

Proof. Formula (6) in this case gives

$$\frac{3}{2} = \big(g(F) - 1\big) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right). \tag{12}$$

If $s \geqslant 2$ then $3/2 \geqslant g(F) - 1$, hence $g(F) = 2$ which contradicts Proposition 4.4. Then $s = 1$, i.e. $\mathbf{n} = (n^1)$. Using (12) we obtain $3/2 \geqslant 1/2(g(F) - 1)$ which implies $g(F) \leqslant 4$. The case $g(F) = 2$ is impossible, otherwise $1 - 1/n = 3/2$; therefore either $g(F) = 3$ or $g(F) = 4$. Using again (12) we see that we have $\mathbf{n} = (4^1)$ in the former case and $\mathbf{n} = (2^1)$ in the latter one. □

Proposition 7.2. *If $K_S^2 = 6$ and $g(F) = 3$ there are precisely the following cases*:

G	IdSmall Group(G)	$\mathbf{m}$
$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24, 3)$	$(3^2, 6)$
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32, 9)$	$(2, 4, 8)$
$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32, 11)$	$(2, 4, 8)$
$G(48, 33)$	$G(48, 33)$	$(2, 3, 12)$
$\mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2$	$G(48, 3)$	$(3^2, 4)$
$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168, 42)$	$(2, 3, 7)$

Proof. By Proposition 3.10 we have $3 \cdot |G| = 8(g(C) - 1)$, so 8 divides $|G|$. The non-abelian group G acts as a group of automorphisms on the genus 3 curve F so that $F/G \cong \mathbb{P}^1$. In addition, since $\mathbf{n} = (4^1)$, it follows that G must be $(1 \mid 4^1)$-generated. Now let us look at Table 3 of Appendix A; by using Propositions 2.5 and 4.3 we are only left with cases $(3n)$, $(3q)$, $(3r)$, $(3t)$, $(3u)$, $(3w)$.

- Case $(3n)$. $G = \mathrm{SL}_2(\mathbb{F}_3)$, $\mathbf{m} = (3^2, 6)$, $g(C) = 10$.

Set

$$g_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$
$$\ell_1 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, h_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and $\ell = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since $(g_3)^3 = (\ell_1)^2 = \ell$ and $\ell \in Z(G)$ it follows $\mathcal{S} = \mathrm{Cl}(\ell) = \{\ell\}$. By using Corollary 1.5 we obtain

$$\left|\mathrm{Fix}_F(\ell)\right| = 4, \qquad \left|\mathrm{Fix}_C(\ell)\right| = 6$$

so equality (10) is satisfied and this case occurs.

- Case $(3q)$. $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) = G(32, 9)$, $\mathbf{m} = (2, 4, 8)$, $g(C) = 13$.

Set

$$g_1 = x, \qquad g_2 = xz, \qquad g_3 = z^7,$$
$$\ell_1 = yz^6, \qquad h_1 = x, \qquad h_2 = z.$$

Since $(\ell_1)^2 = (g_3)^4 = z^4$ and $z^4 \in Z(G)$, we have $\mathcal{S} = \mathrm{Cl}(z^4) = \{z^4\}$; moreover $z^4 \notin \langle g_1 \rangle$ and $z^4 \notin \langle g_2 \rangle$, so we obtain

$$\left|\mathrm{Fix}_F\left(z^4\right)\right| = 4, \qquad \left|\mathrm{Fix}_C\left(z^4\right)\right| = 8.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(3r)$. $G = \mathbb{Z}_2 \ltimes D_{2,8,5} = G(32, 11)$, $\mathbf{m} = (2, 4, 8)$, $g(C) = 13$.

Set

$$g_1 = x, \qquad g_2 = xz, \qquad g_3 = z^7,$$
$$\ell_1 = yz^6, \qquad h_1 = x, \qquad h_2 = z.$$

Since $(\ell_1)^2 = (g_3)^4 = z^4$ and $z^4 \in Z(G)$, we have $\mathcal{S} = \mathrm{Cl}(z^4) = \{z^4\}$; moreover $z^4 \notin \langle g_1 \rangle$ and $z^4 \notin \langle g_2 \rangle$, so we obtain

$$\left|\mathrm{Fix}_F\left(z^4\right)\right| = 4, \qquad \left|\mathrm{Fix}_C\left(z^4\right)\right| = 8.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(3t)$. $G = G(48, 33)$, $\mathbf{m} = (2, 3, 12)$, $g(C) = 19$.

Set

$$g_1 = xz, \qquad g_2 = zy^2, \qquad g_3 = xy,$$
$$\ell_1 = z, \qquad h_1 = y^2, \qquad h_2 = xy^2z.$$

Since $(\ell_1)^2 = (g_3)^6 = t$ and $t \in Z(G)$ we have $\mathcal{S} = \mathrm{Cl}(t) = \{t\}$; moreover $t \notin \langle g_1 \rangle$, so we obtain

$$\left|\mathrm{Fix}_F(t)\right| = 4, \qquad \left|\mathrm{Fix}_C(t)\right| = 12.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(3u)$. $G = \mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2 = G(48, 3)$, $\mathbf{m} = (3^2, 4)$, $g(C) = 19$.

Set

$$g_1 = x, \qquad g_2 = x^{-1}y^{-1}, \qquad g_3 = y,$$
$$\ell_1 = yz^2, \qquad h_1 = x, \qquad h_2 = xyx.$$

We have $(\ell_1)^2 = (g_3)^2 = y^2$, so $\mathcal{S} = \mathrm{Cl}(y^2)$. One checks that $|C_G(y^2)| = 16$, hence $|\mathcal{S}| = 3$ (in fact, $\mathcal{S} = \{y^2, xy^2x^2, x^2y^2x\}$). For every $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 4, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

• Case $(3w)$. $G = \mathrm{PSL}_2(\mathbb{F}_7)$, $\mathbf{m} = (2, 3, 7)$, $g(C) = 64$.

It is well known that G can be embedded in S_8; in fact $G = \langle (375)(486), (126)(348)\rangle$. Set

$$g_1 = (12)(34)(58)(67), \qquad g_2 = (154)(367), \qquad g_3 = (1247358),$$
$$\ell_1 = (1825)(3647), \qquad h_1 = (2576348), \qquad h_2 = (1673428).$$

The group G contains 21 elements of order 2, which belong to a unique conjugacy class (see [CCPW] or [Bar99]). Therefore $\mathcal{S} = \mathrm{Cl}(g_1) = \mathrm{Cl}((\ell_1)^2)$ and $|\mathcal{S}| = 21$. It follows that for all $h \in \mathcal{S}$ we have

$$\left|\mathrm{Fix}_F(h)\right| = 4, \qquad \left|\mathrm{Fix}_C(h)\right| = 2,$$

so equality (10) is satisfied and this case occurs. Notice that in this example the Albanese fiber F of S is isomorphic to the Klein plane quartic $\{x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0\} \subset \mathbb{P}^2$; in particular it is not hyperelliptic.

This completes the proof of Proposition 7.2. $\square$

Proposition 7.3. *If $K_S^2 = 6$ and $g(F) = 4$ there are precisely the following cases*:

G	`IdSmallGroup(G)`	$\mathbf{m}$
D_4	$G(8,3)$	$(2^4, 4)$
A_4	$G(12,3)$	$(2, 3^3)$
$D_{2,12,7}$	$G(24,10)$	$(2, 6, 12)$
$\mathbb{Z}_3 \times A_4$	$G(36,11)$	$(3^2, 6)$
$D_4 \ltimes (\mathbb{Z}_3)^2$	$G(72,40)$	$(2, 4, 6)$
S_5	$G(120,34)$	$(2, 4, 5)$

Proof. By Proposition 3.10 we have $|G| = 4(g(C) - 1)$, so 4 divides $|G|$. Moreover, since $\mathbf{n} = (2^1)$, the group G must be $(1 \mid 2^1)$-generated. Now let us look at Table 4 of Appendix A; by using Propositions 2.6 and 4.3 we are only left with cases $(4c)$, $(4f)$, $(4r)$, $(4x)$, $(4ac)$, $(4ad)$.

• Case $(4c)$. $G = D_4$, $\mathbf{m} = (2^4, 4)$, $g(C) = 3$.

Set

$$g_1 = x, \qquad g_2 = xy, \qquad g_3 = x, \qquad g_4 = xy^2, \qquad g_5 = y,$$
$$\ell_1 = y^2, \qquad h_1 = y, \qquad h_2 = x.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{y^2\}$ and

$$\left|\mathrm{Fix}_F\left(y^2\right)\right| = 2, \qquad \left|\mathrm{Fix}_C\left(y^2\right)\right| = 4,$$

so equality (10) is satisfied and this case occurs.

• Case $(4f)$. $G = A_4$, $\mathbf{m} = (2, 3^3)$, $g(C) = 4$.

Set

$$g_1 = (12)(34), \qquad g_2 = (134), \qquad g_3 = (134), \qquad g_4 = (123),$$
$$\ell_1 = (12)(34), \qquad h_1 = (123), \qquad h_2 = (124).$$

Then $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{(12)(34), (13)(24), (14)(23)\}$. For all $h \in \mathcal{S}$ we have

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 2,$$

so equality (10) is satisfied and this case occurs.

• Case $(4r)$. $G = D_{2,12,7}$, $\mathbf{m} = (2, 6, 12)$, $g(C) = 7$.

Set

$$g_1 = x, \qquad g_2 = y^5x, \qquad g_3 = y,$$
$$\ell_1 = y^6, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\ell_1 = (g_3)^6$; since $\ell_1 \in Z(G)$ it follows $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{y^6\}$. On the other hand $\ell_1 \notin \langle g_1 \rangle$ and $\ell_1 \notin \langle g_2 \rangle$, so we obtain

$$\left|\mathrm{Fix}_F\left(y^6\right)\right| = 2, \qquad \left|\mathrm{Fix}_C\left(y^6\right)\right| = 12.$$

Thus equality (10) is satisfied and this case occurs.

• Case $(4x)$. $G = \mathbb{Z}_3 \times A_4$, $\mathbf{m} = (3^2, 6)$, $g(C) = 10$.

Set $\mathbb{Z}_3 = \langle z \mid z^3 = 1 \rangle$ and

$$g_1 = \big(z, (123)\big), \qquad g_2 = \big(z, (234)\big), \qquad g_3 = \big(z, (12)(34)\big),$$
$$\ell_1 = \big(1, (12)(34)\big), \qquad h_1 = \big(1, (123)\big), \qquad h_2 = \big(z, (14)(23)\big).$$

Since $\ell_1 = (g_3)^3$ we obtain $\mathcal{S} = \mathrm{Cl}(\ell_1)$ and so $|\mathcal{S}| = 3$. For all $h \in \mathcal{S}$ we have

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 6,$$

so equality (10) is satisfied and this case occurs.

• Case $(4ac)$. $G = D_4 \ltimes (\mathbb{Z}_3)^2 = G(72, 40)$, $\mathbf{m} = (2, 4, 6)$, $g(C) = 19$.

Set

$$g_1 = xzy, \qquad g_2 = y, \qquad g_3 = y^2z^2x,$$
$$\ell_1 = y^2, \qquad h_1 = xy, \qquad h_2 = xz.$$

We have $\ell_1 = (g_2)^2$ and so $\mathcal{S} = \mathrm{Cl}(\ell_1)$; since $|C_G(\ell_1)| = 8$, it follows $|\mathcal{S}| = 9$. Moreover $g_1 \notin \mathrm{Cl}(\ell_1)$ and $(g_3)^3 \notin \mathrm{Cl}(\ell_1)$, hence for all $h \in \mathcal{S}$ we have

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

• Case (4ad). $G = S_5$, $\mathbf{m} = (2, 4, 5)$, $g(C) = 31$.
Set

$$g_1 = (12), \qquad g_2 = (1543), \qquad g_3 = (12345),$$
$$\ell_1 = (14)(35), \qquad h_1 = (145), \qquad h_2 = (1432).$$

We have $\ell_1 = (g_2)^2$, hence $\mathscr{S} = \mathrm{Cl}(\ell_1)$ and $|\mathscr{S}| = 15$. For all $h \in \mathscr{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4,$$

so equality (10) is satisfied and this case occurs.
This completes the proof of Proposition 7.3. □

7.2. The case $K_S^2 = 4$

Proposition 7.4. *If $K_S^2 = 4$ then we have two possibilities*:

- $g(F) = 2$, $\mathbf{n} = (2^2)$;
- $g(F) = 3$, $\mathbf{n} = (2^1)$.

Proof. If $K_S^2 = 4$ then Proposition 5.1 gives

$$\big(g(F) - 1\big)\sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right) = 1.$$

If $s \geqslant 2$ then $g(F) - 1 \leqslant 1$, which implies $g(F) = 2$ and $\mathbf{n} = (2^2)$. So we may assume $s = 1$, i.e. $\mathbf{n} = (n^1)$. In this case we have $\frac{1}{2}(g(F) - 1) \leqslant 1$, then $g(F) \leqslant 3$. On the other hand, $g(F) = 2$ gives $1 - \frac{1}{n} = 1$, a contradiction; therefore $g(F) = 3$ and $\mathbf{n} = (2^1)$. □

In Proposition 6.2 we have proven that if G is abelian then $K_S^2 = 4$ and $g_{\mathrm{alb}} = 2$. However we can also obtain the same invariants with non-abelian G:

Proposition 7.5. *If $K_S^2 = 4$, $g(F) = 2$ and G is not abelian there are precisely the following cases*:

G	IdSmall Group(G)	$\mathbf{m}$
S_3	$G(6,1)$	$(2^2, 3^2)$
D_4	$G(8,3)$	$(2^3, 4)$
D_6	$G(12,4)$	$(2^3, 3)$
$D_{2,8,3}$	$G(16,8)$	$(2, 4, 8)$
$\mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3)$	$G(24,8)$	$(2, 4, 6)$
$\mathrm{GL}_2(\mathbb{F}_3)$	$G(48,29)$	$(2, 3, 8)$

Proof. By Proposition 3.10 we have $|G| = 2(g(C) - 1)$. Moreover, since $\mathbf{n} = (2^2)$, it follows that G is $(1 \mid 2^2)$-generated. Let us look at Table 2 of Appendix A. Using Proposition 2.2 we can rule out cases (2b), (2d) and (2h). Now we check the remaining possibilities.
• Case (2a). $G = S_3$, $\mathbf{m} = (2^2, 3^2)$, $g(C) = 4$.

Set

$$g_1 = (12), \qquad g_2 = (12), \qquad g_3 = (123), \qquad g_4 = (132),$$
$$\ell_1 = \ell_2 = (12), \qquad h_1 = h_2 = (13).$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{(12), (13), (23)\}$ and for every $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 2.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(2c)$. $G = D_4$, $\mathbf{m} = (2^3, 4)$, $g(C) = 5$.

Set

$$g_1 = x, \qquad g_2 = xy, \qquad g_3 = y^2, \qquad g_4 = y,$$
$$\ell_1 = \ell_2 = x, \qquad h_1 = h_2 = y.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{x, xy^2\}$ and for every $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(2e)$. $G = D_6$, $\mathbf{m} = (2^3, 3)$, $g(C) = 7$.

Set

$$g_1 = x, \qquad g_2 = xy, \qquad g_3 = y^3, \qquad g_4 = y^2,$$
$$\ell_1 = xy, \qquad \ell_2 = xy^5, \qquad h_1 = x, \qquad h_2 = y^2.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{xy, xy^3, xy^5\}$ and for every $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(2f)$. $G = D_{2,8,3}$, $\mathbf{m} = (2, 4, 8)$, $g(C) = 9$.

Set

$$g_1 = x, \qquad g_2 = xy^7, \qquad g_3 = y,$$
$$\ell_1 = x, \qquad \ell_2 = xy^6, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{x,\ xy^2, xy^4, xy^6\}$. Moreover $(g_2)^2 = (g_3)^4 = y^4$ and $y^4 \notin \mathcal{S}$, hence for every $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

- Case $(2g)$. $G = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) = G(24, 8)$, $\mathbf{m} = (2, 4, 6)$, $g(C) = 13$.

Set

$$g_1 = x, \qquad g_2 = wxz, \qquad g_3 = zw,$$
$$\ell_1 = \ell_2 = x, \qquad h_1 = z, \qquad h_2 = w.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1)$; since $C_G(\ell_1) = \langle x, y\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, it follows $|\mathcal{S}| = 6$. Moreover $(g_2)^2 \notin \mathrm{Cl}(\ell_1)$ and $(g_3)^3 \notin \mathrm{Cl}(\ell_1)$, so for every $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

• Case $(2i)$. $G = \mathrm{GL}_2(\mathbb{F}_3)$, $\mathbf{m} = (2, 3, 8)$, $g(C) = 25$.

Set

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},$$
$$\ell_1 = \ell_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\ell = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We have $\mathcal{S} = \mathrm{Cl}(\ell_1)$ and $C_G(\ell_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, hence $|\mathcal{S}| = 12$. Moreover $(g_3)^4 = \ell \notin \mathrm{Cl}(\ell_1)$, so for all $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 2, \qquad \left|\mathrm{Fix}_C(h)\right| = 4.$$

Thus equality (10) is satisfied and this case occurs.

This completes the proof of Proposition 7.5. $\square$

Proposition 7.6. *If $K_S^2 = 4$ and $g(F) = 3$ there are precisely the following cases*:

G	`IdSmallGroup(`G`)`	$\mathbf{m}$
D_4	$G(8,3)$	$(2^2, 4^2)$
D_4	$G(8,3)$	(2^5)
A_4	$G(12,3)$	$(2^2, 3^2)$
$D_{2,8,5}$	$G(16,6)$	$(2, 8^2)$
$D_{4,4,-1}$	$G(16,4)$	(4^3)
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(2, 6^2)$

Proof. By Proposition 3.10 we have $|G| = 4(g(C) - 1)$, so 4 divides $|G|$. Moreover, since $\mathbf{n} = (2^1)$, the group G is $(1 \mid 2^1)$-generated. Now let us look at Table 3 of Appendix A; by using Proposition 2.4 we are only left with cases $(3b)$, $(3c)$, $(3f)$, $(3g)$, $(3h)$, $(3m)$.

• Case $(3b)$. $G = D_4$, $\mathbf{m} = (2^2, 4^2)$, $g(C) = 3$.

Set

$$g_1 = x, \qquad g_2 = x, \qquad g_3 = y, \qquad g_4 = y^3,$$
$$\ell_1 = y^2, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{y^2\}$ and

$$\left|\mathrm{Fix}_F\left(y^2\right)\right| = 4, \qquad \left|\mathrm{Fix}_C\left(y^2\right)\right| = 4,$$

so equality (10) is satisfied and this case occurs.

• Case $(3c)$. $G = D_4$, $\mathbf{m} = (2^5)$, $g(C) = 3$.

Set

$$g_1 = y^2, \qquad g_2 = xy, \qquad g_3 = xy^3, \qquad g_4 = x, \qquad g_5 = x,$$
$$\ell_1 = y^2, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{y^2\}$ and

$$\left|\mathrm{Fix}_F\left(y^2\right)\right| = 4, \qquad \left|\mathrm{Fix}_C\left(y^2\right)\right| = 4,$$

so equality (10) is satisfied and this case occurs.

• Case $(3f)$. $G = A_4$, $\mathbf{m} = (2^2, 3^2)$, $g(C) = 4$.

Set

$$g_1 = (12)(34), \qquad g_2 = (12)(34), \qquad g_3 = (123), \qquad g_4 = (132),$$
$$\ell_1 = (12)(34), \qquad h_1 = (123), \qquad h_2 = (124).$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{(12)(34), (13)(24), (14)(23)\}$ and for all $h \in \mathcal{S}$ we obtain

$$\left|\mathrm{Fix}_F(h)\right| = 4, \qquad \left|\mathrm{Fix}_C(h)\right| = 2,$$

so equality (10) is satisfied and this case occurs.

• Case $(3g)$. $G = D_{2,8,5}$, $\mathbf{m} = (2, 8^2)$, $g(C) = 5$.

Set

$$g_1 = x, \qquad g_2 = xy^{-1}, \qquad g_3 = y,$$
$$\ell_1 = y^4, \qquad h_1 = x, \qquad h_2 = y.$$

Since $\ell_1 = (g_2)^4 = (g_3)^4$ and $\ell_1 \in Z(G)$, it follows $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{y^4\}$. We have

$$\left|\mathrm{Fix}_F\left(y^4\right)\right| = 4, \qquad \left|\mathrm{Fix}_C\left(y^4\right)\right| = 8,$$

so equality (10) is satisfied and this case occurs.

• Case $(3h)$. $G = D_{4,4,-1}$, $\mathbf{m} = (4^3)$, $g(C) = 5$.

Set

$$g_1 = x, \qquad g_2 = x^{-1}y^{-1}, \qquad g_3 = y,$$
$$\ell_1 = y^2, \qquad h_1 = x, \qquad h_2 = y.$$

Since $\ell_1 = (g_3)^2$ and $\ell_1 \in Z(G)$ we have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{y^2\}$. Moreover $\ell_1 \notin \langle g_1 \rangle$ and $\ell_2 \notin \langle g_2 \rangle$, so we obtain

$$\left|\mathrm{Fix}_F\left(y^2\right)\right| = 4, \qquad \left|\mathrm{Fix}_C\left(y^2\right)\right| = 8.$$

Thus equality (10) is satisfied and this case occurs.

• Case $(3m)$. $G = \mathbb{Z}_2 \times A_4$, $\mathbf{m} = (2, 6^2)$, $g(C) = 7$.

Let $\mathbb{Z}_2 = \langle z \mid z^2 = 1\rangle$ and set

$$g_1 = \big(1, (12)(34)\big), \qquad g_2 = \big(z, (123)\big), \qquad g_3 = \big(z, (234)\big),$$
$$\ell_1 = \big(1, (12)(34)\big), \qquad h_1 = \big(z, (123)\big), \qquad h_2 = \big(z, (124)\big).$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{(1, (12)(34)), (1, (13)(24)), (1, (14)(23))\}$. For all $h \in \mathcal{S}$ we obtain

$$\big|\mathrm{Fix}_F(h)\big| = 4, \qquad \big|\mathrm{Fix}_C(h)\big| = 4,$$

so equality (10) is satisfied and this case occurs.

This completes the proof of Proposition 7.6. □

7.3. *The case $K_S^2 = 2$*

Lemma 7.7. *If $K_S^2 = 2$ then $\mathbf{n} = (2^1)$.*

Proof. If $K_S^2 = 2$ we have $g(F) = 2$ [Ca81,CaCi91,CaCi93]. Therefore by Proposition 5.1 we obtain $\sum_{j=1}^{s}(1 - \frac{1}{n_j}) = \frac{1}{2}$, that is $\mathbf{n} = (2^1)$. □

Proposition 7.8. *If $K_S^2 = 2$ there are precisely the following possibilities*:

G	`IdSmallGroup(`G`)`	$\mathbf{m}$
Q_8	$G(8,4)$	(4^3)
D_4	$G(8,3)$	$(2^3, 4)$

Proof. Proposition 3.10 yields $|G| = 4(g(C) - 1)$, so 4 divides $|G|$. Since $\mathbf{n} = (2^1)$, G must be $(1 \mid 2^1)$-generated. Now let us look at Table 2 of Appendix A. By using Proposition 2.3 we may rule out cases (2d), (2e), (2f), (2g), (2h) and (2i), so the proof will be complete if we show that cases (2b) and (2c) occur.

• Case (2b). $G = Q_8$, $\mathbf{m} = (4^3)$, $g(C) = 3$.

Set

$$g_1 = j, \qquad g_2 = i, \qquad g_3 = k,$$
$$\ell_1 = -1, \qquad h_1 = i, \qquad h_2 = j.$$

We have $\mathcal{S} = \mathrm{Cl}(\ell_1) = \{-1\}$ and

$$\big|\mathrm{Fix}_F(-1)\big| = 6, \qquad \big|\mathrm{Fix}_C(-1)\big| = 4,$$

so equality (10) is satisfied and this case occurs.

• Case (2c). $G = D_4$, $\mathbf{m} = (2^3, 4)$, $g(C) = 3$.

Set

$$g_1 = xy^2, \qquad g_2 = xy^3, \qquad g_3 = y^2, \qquad g_4 = y,$$
$$\ell_1 = y^2, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\mathcal{S} = \mathrm{Cl}(y^2) = \{y^2\}$ and

$$\left|\mathrm{Fix}_F\left(y^2\right)\right| = 6, \qquad \left|\mathrm{Fix}_C\left(y^2\right)\right| = 4,$$

so equality (10) is satisfied and this case occurs. $\square$

Proposition 7.8 shows that there exist two families of standard isotrivial fibrations with $p_g = q = 1$, $K_S^2 = 2$. The first family, that we denote by $\mathfrak{M}_{D_4}$, has dimension 2 because it depends on the choice of four points on $\mathbb{P}^1$ and one point on E (up to projective equivalence); the second family, that we denote by $\mathfrak{M}_{Q_8}$, has dimension 1 because it depends on the choice of three points on $\mathbb{P}^1$ and one point on E. Now we can provide a geometric description of $\mathfrak{M}_{D_4}$ and $\mathfrak{M}_{Q_8}$; to this purpose, let us recall some facts about surfaces of general type with $p_g = q = 1$, $K_S^2 = 2$ (see [Ca81] and [CaCi91] for further details). Let $(E, \oplus, 0)$ be an elliptic curve E with group law $\oplus$ and identity element 0, and let

$$E^{(2)} = \mathrm{Sym}^2(E) = \{x + y \mid x, y \in E\}$$

be its double symmetric product. Then the Abel–Jacobi map $E^{(2)} \to E$, $x + y \to x \oplus y$ gives to $E^{(2)}$ the structure of a $\mathbb{P}^1$-bundle over E. For any $a \in E$, let us consider the following divisors on $E^{(2)}$:

$$\mathfrak{f}_a := \{x + y \in E \mid x \oplus y = a\};$$
$$\mathfrak{h}_a := \{x + a \mid x \in E\}.$$

In both cases the corresponding algebraic equivalence classes do not depend on a, hence we may denote them by $\mathfrak{f}$ and $\mathfrak{h}$, respectively. We have $\mathrm{NS}(E^{(2)}) = \mathbb{Z}\mathfrak{f} \oplus \mathbb{Z}\mathfrak{h}$. The antibicanonical system $|-2K_{E^{(2)}}| = |4\mathfrak{h}_0 - 2\mathfrak{f}_0|$ is a linear pencil, whose general elements are smooth elliptic curves of the form

$$\mathfrak{b}_a := \left\{x + (x \oplus a) \,\middle|\, x \in E\right\}, \quad a \oplus a \neq 0.$$

If $\div a$ denotes the inverse element of $a \in E$, we have $\mathfrak{b}_a = \mathfrak{b}_{\div a}$. It follows that the singular members of $|-2K_{E^{(2)}}|$ are precisely the three double curves $2\mathfrak{b}_{\xi_1}, 2\mathfrak{b}_{\xi_2}, 2\mathfrak{b}_{\xi_3}$, where the ξ_i are the three 2-torsion points of E different from 0. The $\mathfrak{b}_{\xi_i}$ are three divisors on $E^{(2)}$ which are algebraically but not linearly equivalent to $2\mathfrak{h}_0 - \mathfrak{f}_0$ (in fact, $\mathfrak{b}_{\xi_i} \in |2\mathfrak{h}_0 - \mathfrak{f}_{\xi_i}|$). In [Ca81] it is shown that any surface S of general type with $p_g = q = 1$, $K_S^2 = 2$ is a double cover of $E^{(2)}$ branched along a divisor $\mathfrak{B}$ algebraically equivalent to $6\mathfrak{h} - 2\mathfrak{f}$ and having at worst simple singularities. In particular the Albanese pencil $\{F\}$ of S is the pullback of the ruling $\{\mathfrak{f}\}$ of $E^{(2)}$. Since the group of translations of E acts transitively on the set of linear equivalence classes of divisors algebraically equivalent to $6\mathfrak{h} - 2\mathfrak{f}$, we may assume $\mathfrak{B} \in |6\mathfrak{h}_0 - 2\mathfrak{f}_0|$. Therefore the surfaces in $\mathfrak{M}_{D_4}$ and $\mathfrak{M}_{Q_8}$ must correspond to special curves with six nodes in the linear system $|6\mathfrak{h}_0 - 2\mathfrak{f}_0|$. Indeed we can prove

Proposition 7.9. *Let S be the double cover of $E^{(2)}$ branched along a curve $\mathfrak{B} \in |6\mathfrak{h}_0 - 2\mathfrak{f}_0|$. Then the following holds.*

(i) *If $S \in \mathfrak{M}_{D_4}$ we have*

$$\mathfrak{B} = \mathfrak{B}' + \mathfrak{b}_{\xi_i} + \mathfrak{f}_{\xi_i},$$

where $\mathfrak{B}' \in |-2K_{E^{(2)}}|$ and $i \in \{1, 2, 3\}$.

(ii) *If $S \in \mathfrak{M}_{Q_8}$ we have*

$$\mathfrak{B} = \mathfrak{b}_{\xi_1} + \mathfrak{b}_{\xi_2} + \mathfrak{b}_{\xi_3} + \mathfrak{f}_0.$$

In both cases the isotrivial fibration $|C|$ of S is obtained as the pullback of the antibicanonical pencil $|-2K_{E^{(2)}}|$ of $E^{(2)}$.

Proof. Since $C^2 = 0$ and $CF = 8$, it follows that the image of $|C|$ in $E^{(2)}$ via the double cover $S \to E^{(2)}$ is a linear pencil whose general element $\mathfrak{c}$ verifies $\mathfrak{c}^2 = 0$, $\mathfrak{c}\mathfrak{f} = 4$. This implies $|\mathfrak{c}| = |-2K_{E^{(2)}}|$ [CaCi93, p. 404]. Moreover, since $\mathbf{n} = (2^1)$, exactly one component of $\mathfrak{B}$ is algebraically equivalent to $\mathfrak{f}$. If $S \in \mathfrak{M}_{D_4}$ then $\mathbf{m} = (2^2, 4)$, so exactly one of the curves $\mathfrak{b}_{\xi_i}$ is contained in $\mathfrak{B}$; this implies (i). If $S \in \mathfrak{M}_{Q_8}$ then $\mathbf{m} = (4^3)$, so all the $\mathfrak{b}_{\xi_i}$ are components of $\mathfrak{B}$; this implies (ii). $\square$

Notice that in both cases all the components of $\mathfrak{B}$ not contained in $\{\mathfrak{f}\}$ are invariant under translation in $E^{(2)}$; this explains why the Albanese pencil of S turns out to be isotrivial.

Acknowledgments

The author wishes to thank I. Bauer, A. Broughton, G. Carnovale, F. Catanese, C. Ciliberto, G. Infante, J. Van Bon for useful discussions and suggestions. Part of this research was done during the winter semester of the academic year 2006–2007, when the author was visiting the Department of Mathematics at the Imperial College (London, UK) and was supported by an "Accademia dei Lincei" grant. He is especially grateful to A. Corti for his warm hospitality.

Appendix A

This appendix contains the classification of finite groups of automorphisms acting on Riemann surfaces of genus 2, 3 and 4 so that the quotient is isomorphic to $\mathbb{P}^1$. In the last two cases we listed only the non-abelian groups. Tables 1–3 are adapted from [Br90, pp. 252, 254, 255], whereas Table 4 is adapted from [Ki03, Theorem 1] and [Vin00]. For every G we give a presentation, the vector $\mathbf{m}$ of branching data and the `IdSmallGroup`(G), that is the number of G in the GAP4 database of small groups. The author wishes to thank S.A. Broughton who kindly communicated to him that the group $G(48, 33)$ (Table 3, case $(3t)$) was missing in [Br90].

Table 1
Abelian groups of automorphisms acting with rational quotient on Riemann surfaces of genus 2.

Case	G	`IdSmallGroup`(G)	$\mathbf{m}$
$(1a)$	$\mathbb{Z}_2$	$G(2,1)$	(2^6)
$(1b)$	$\mathbb{Z}_3$	$G(3,1)$	(3^4)
$(1c)$	$\mathbb{Z}_4$	$G(4,1)$	$(2^2,4^2)$
$(1d)$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$G(4,2)$	(2^5)
$(1e)$	$\mathbb{Z}_5$	$G(5,1)$	(5^3)
$(1f)$	$\mathbb{Z}_6$	$G(6,2)$	$(2^2,3^2)$
$(1g)$	$\mathbb{Z}_6$	$G(6,2)$	$(3,6^2)$
$(1h)$	$\mathbb{Z}_8$	$G(8,1)$	$(2,8^2)$
$(1i)$	$\mathbb{Z}_{10}$	$G(10,2)$	$(2,5,10)$
$(1j)$	$\mathbb{Z}_2\times\mathbb{Z}_6$	$G(12,5)$	$(2,6^2)$

Table 2
Non-abelian groups of automorphisms acting with rational quotient on Riemann surfaces of genus 2.

Case	G	`IdSmallGroup`(G)	**m**	Presentation
$(2a)$	S_3	$G(6,1)$	$(2^2,3^2)$	$\langle x,y \mid x=(123),\ y=(12)\rangle$
$(2b)$	Q_8	$G(8,4)$	(4^3)	$\langle i,j,k \mid i^2=j^2=k^2=-1,\ ij=k,\ jk=i, ki=j\rangle$
$(2c)$	D_4	$G(8,3)$	$(2^3,4)$	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(2d)$	$D_{4,3,-1}$	$G(12,1)$	$(3,4^2)$	$\langle x,y \mid x^4=y^3=1,\ xyx^{-1}=y^{-1}\rangle$
$(2e)$	D_6	$G(12,4)$	$(2^3,3)$	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(2f)$	$D_{2,8,3}$	$G(16,8)$	$(2,4,8)$	$\langle x,y \mid x^2=y^8=1,\ xyx^{-1}=y^3\rangle$
$(2g)$	$G=\mathbb{Z}_2\ltimes((\mathbb{Z}_2)^2\times\mathbb{Z}_3)$	$G(24,8)$	$(2,4,6)$	$\langle x,y,z,w \mid x^2=y^2=z^2=w^3=1,$ $[y,z]=[y,w]=[z,w]=1,$ $xyx^{-1}=y,\ xzx^{-1}=zy,\ xwx^{-1}=w^{-1}\rangle$
$(2h)$	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$(3^2,4)$	$\langle x,y \mid x=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\ y=\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)\rangle$
$(2i)$	$\mathrm{GL}_2(\mathbb{F}_3)$	$G(48,29)$	$(2,3,8)$	$\langle x,y \mid x=\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right),\ y=\left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)\rangle$

Table 3
Non-abelian groups of automorphisms acting with rational quotient on Riemann surfaces of genus 3.

Case	G	`IdSmallGroup`(G)	**m**	Presentation
$(3a)$	S_3	$G(6,1)$	$(2^4,3)$	$\langle x,y \mid x=(12),\ y=(123)\rangle$
$(3b)$	D_4	$G(8,3)$	$(2^2,4^2)$	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3c)$	D_4	$G(8,3)$	(2^5)	$\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3d)$	$D_{4,3,-1}$	$G(12,1)$	$(4^2,6)$	$\langle x,y \mid x^4=y^3=1,\ xyx^{-1}=y^{-1}\rangle$
$(3e)$	D_6	$G(12,4)$	$(2^3,6)$	$\langle x,y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
$(3f)$	A_4	$G(12,3)$	$(2^2,3^2)$	$\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(3g)$	$D_{2,8,5}$	$G(16,6)$	$(2,8^2)$	$\langle x,y \mid x^2=y^8=1,\ xyx^{-1}=y^5\rangle$
$(3h)$	$D_{4,4,-1}$	$G(16,4)$	(4^3)	$\langle x,y \mid x^4=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3i)$	$\mathbb{Z}_2\times D_4$	$G(16,11)$	$(2^3,4)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
$(3j)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_4)$	$G(16,13)$	$(2^3,4)$	$\langle x,y,z \mid x^2=y^2=z^4=1,\ [x,z]=[y,z]=1,\ xyx^{-1}=yz^2\rangle$
$(3k)$	$D_{3,7,2}$	$G(21,1)$	$(3^2,7)$	$\langle x,y \mid x^3=y^7=1,\ xyx^{-1}=y^2\rangle$
$(3l)$	$D_{2,12,5}$	$G(24,5)$	$(2,4,12)$	$\langle x,y \mid x^2=y^{12}=1,\ xyx^{-1}=y^5\rangle$
$(3m)$	$\mathbb{Z}_2\times A_4$	$G(24,13)$	$(2,6^2)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x=(12)(34),\ y=(123)\rangle$
$(3n)$	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$(3^2,6)$	$\langle x,y \mid x=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\ y=\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)\rangle$
$(3o)$	S_4	$G(24,12)$	$(3,4^2)$	$\langle x,y \mid x=(1234),\ y=(12)\rangle$
$(3p)$	S_4	$G(24,12)$	$(2^3,3)$	$\langle x,y \mid x=(1234),\ y=(12)\rangle$
$(3q)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_8)$	$G(32,9)$	$(2,4,8)$	$\langle x,y,z \mid x^2=y^2=z^8=1,\ [x,y]=[y,z]=1,\ xzx^{-1}=yz^3\rangle$
$(3r)$	$\mathbb{Z}_2\ltimes D_{2,8,5}$	$G(32,11)$	$(2,4,8)$	$\langle x,y,z \mid x^2=y^2=z^8=1,\ yzy^{-1}=z^5,\ xyx^{-1}=yz^4,\ xzx^{-1}=yz^3\rangle$
$(3s)$	$\mathbb{Z}_2\times S_4$	$G(48,48)$	$(2,4,6)$	$\langle z \mid z^2=1\rangle\times\langle x,y \mid x=(12),\ y=(1234)\rangle$
$(3t)$	$G(48,33)$	$G(48,33)$	$(2,3,12)$	$\langle x,y,z,w,t \mid x^2=z^2=w^2=t,\ y^3=1,$ $t^2=1,\ yzy^{-1}=w,\ ywy^{-1}=zw,\ zwz^{-1}=wt,$ $[x,y]=[x,z]=1\rangle$
$(3u)$	$\mathbb{Z}_3\ltimes(\mathbb{Z}_4)^2$	$G(48,3)$	$(3^2,4)$	$\langle x,y,z \mid x^3=y^4=z^4=1,\ [y,z]=1,\ xyx^{-1}=z,\ xzx^{-1}=(yz)^{-1}\rangle$
$(3v)$	$S_3\ltimes(\mathbb{Z}_4)^2$	$G(96,64)$	$(2,3,8)$	$\langle x,y,z,w \mid x^2=y^3=z^4=w^4=1,$ $[z,w]=1,\ xyx^{-1}=y^{-1},\ xzx^{-1}=w,$ $xwx^{-1}=z,\ yzy^{-1}=w,\ ywy^{-1}=(zw)^{-1}\rangle$
$(3w)$	$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168,42)$	$(2,3,7)$	$\langle x,y \mid x=(375)(486),\ y=(126)(348)\rangle$

Table 4
Non-abelian groups of automorphisms acting with rational quotient on Riemann surfaces of genus 4.

Case	G	IdSmall Group(G)	**m**	Presentation
(4*a*)	S_3	$G(6,1)$	(2^6)	$\langle x, y \mid x=(12),\ y=(123)\rangle$
(4*b*)	S_3	$G(6,1)$	$(2^2,3^3)$	$\langle x, y \mid x=(12),\ y=(123)\rangle$
(4*c*)	D_4	$G(8,3)$	$(2^4,4)$	$\langle x, y \mid x^2=y^4=1,\ xyx^{-1}=y^{-1}\rangle$
(4*d*)	Q_8	$G(8,4)$	$(2,4^3)$	$\langle i, j, k, -1 \mid i^2=j^2=k^2=-1,\ ij=k,\ jk=i,\ ki=j\rangle$
(4*e*)	D_5	$G(10,1)$	$(2^2,5^2)$	$\langle x, y \mid x^2=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
(4*f*)	A_4	$G(12,3)$	$(2,3^3)$	$\langle x, y \mid x=(12)(34),\ y=(123)\rangle$
(4*g*)	D_6	$G(12,4)$	(2^5)	$\langle x, y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
(4*h*)	D_6	$G(12,4)$	$(2^2,3,6)$	$\langle x, y \mid x^2=y^6=1,\ xyx^{-1}=y^{-1}\rangle$
(4*i*)	D_8	$G(16,7)$	$(2^3,8)$	$\langle x, y \mid x^2=y^8=1,\ xyx^{-1}=y^{-1}\rangle$
(4*j*)	$G(16,9)$	$G(16,9)$	$(4^2,8)$	$\langle x, y, z, w \mid x^2=y^2=z^2=w,\ w^2=1,$ $xzx^{-1}=z^{-1},\ yzy^{-1}=z^{-1},\ yxy^{-1}=(xz)^{-1}\rangle$
(4*k*)	$\mathbb{Z}_3\times S_3$	$G(18,3)$	$(2^2,3^2)$	$\langle z \mid z^3=1\rangle\times\langle x, y \mid x=(12),\ y=(123)\rangle$
(4*l*)	$\mathbb{Z}_3\times S_3$	$G(18,3)$	$(3,6^2)$	$\langle z \mid z^3=1\rangle\times\langle x, y \mid x=(12),\ y=(123)\rangle$
(4*m*)	$\mathbb{Z}_2\ltimes(\mathbb{Z}_3)^2$	$G(18,4)$	$(2^2,3^2)$	$\langle x, y, z \mid x^2=y^3=z^3=1,\ xyx^{-1}=y^{-1},\ xzx^{-1}=z^{-1},\ [y,z]=1\rangle$
(4*n*)	$\mathbb{Z}_2\times D_5$	$G(20,4)$	$(2^3,5)$	$\langle z \mid z^2=1\rangle\times\langle x, y \mid x^2=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
(4*o*)	$D_{4,5,-1}$	$G(20,1)$	$(4^2,5)$	$\langle x, y \mid x^4=y^5=1,\ xyx^{-1}=y^{-1}\rangle$
(4*p*)	$D_{4,5,2}$	$G(20,3)$	$(4^2,5)$	$\langle x, y \mid x^4=y^5=1,\ xyx^{-1}=y^2\rangle$
(4*q*)	S_4	$G(24,12)$	$(2^3,4)$	$\langle x, y \mid x=(1234),\ y=(12)\rangle$
(4*r*)	$D_{2,12,7}$	$G(24,10)$	$(2,6,12)$	$\langle x, y \mid x^2=y^{12}=1,\ xyx^{-1}=y^7\rangle$
(4*s*)	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$(3,4,6)$	$\langle x, y \mid x=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\ y=\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right)\rangle$
(4*t*)	$D_{2,16,7}$	$G(32,19)$	$(2,4,16)$	$\langle x, y \mid x^2=y^{16}=1,\ \ xyx^{-1}=y^7\rangle$
(4*u*)	$(\mathbb{Z}_2)^2\ltimes(\mathbb{Z}_3)^2$	$G(36,10)$	$(2^3,3)$	$\langle x, y, z, w \mid x^2=y^2=z^3=w^3=1,$ $yzy^{-1}=z^2,\ xwx^{-1}=w^2,$ $[x,y]=[x,z]=[y,w]=[z,w]=1\rangle$
(4*v*)	$(\mathbb{Z}_2)^2\ltimes(\mathbb{Z}_3)^2$	$G(36,10)$	$(2,6^2)$	$\langle x, y, z, w \mid x^2=y^2=z^3=w^3=1,$ $yzy^{-1}=z^2,\ xwx^{-1}=w^2,$ $[x,y]=[x,z]=[y,w]=[z,w]=1\rangle$
(4*w*)	$\mathbb{Z}_6\times S_3$	$G(36,12)$	$(2,6^2)$	$\langle z \mid z^6=1\rangle\times\langle x, y \mid x=(12), y=(123)\rangle$
(4*x*)	$\mathbb{Z}_3\times A_4$	$G(36,11)$	$(3^2,6)$	$\langle z \mid z^3=1\rangle\times\langle x, y \mid x=(12)(34),\ y=(123)\rangle$
(4*y*)	$\mathbb{Z}_4\ltimes(\mathbb{Z}_3)^2$	$G(36,9)$	$(3,4^2)$	$\langle x, y, z \mid x^4=y^3=z^3=1,\ xyx^{-1}=yz^2,\ xzx^{-1}=y^2z^2,\ [y,z]=1\rangle$
(4*z*)	$D_4\ltimes\mathbb{Z}_5$	$G(40,8)$	$(2,4,10)$	$\langle x, y, z \mid x^2=y^4=z^5=1,\ xyx^{-1}=y^{-1},\ xzx^{-1}=z,\ yzy^{-1}=z^{-1}\rangle$
(4*aa*)	A_5	$G(60,5)$	$(2,5^2)$	$\langle x, y \mid x=(12)(34),\ y=(12345)\rangle$
(4*ab*)	$\mathbb{Z}_3\times S_4$	$G(72,42)$	$(2,3,12)$	$\langle z \mid z^3=1\rangle\times\langle x, y \mid x=(12), y=(1234)\rangle$
(4*ac*)	$D_4\ltimes(\mathbb{Z}_3)^2$	$G(72,40)$	$(2,4,6)$	$\langle x, y, z, w \mid x^2=y^4=z^3=w^3=1,$ $xyx^{-1}=y^{-1},\ xzx^{-1}=w,$ $yzy^{-1}=w,\ ywy^{-1}=z^2,\ [z,w]=1\rangle$
(4*ad*)	S_5	$G(120,34)$	$(2,4,5)$	$\langle x, y \mid x=(12),\ y=(12345)\rangle$

Appendix B

The following is the GAP4 script that we used in order to check that the group $G=\mathbb{Z}_2\ltimes(\mathbb{Z}_2\times\mathbb{Z}_4)=G(16,13)$ is not $(1\mid 2^1)$-generated (see Proposition 2.4, case (3*j*)). In fact, the output shows that if $[h_1,h_2]$ has order 2 then either $\langle h_1,h_2\rangle\cong G(8,3)=D_4$ or $\langle h_1,h_2\rangle\cong G(8,4)=Q_8$. Completely similar scripts can be used in order to check the other results stated in Propositions 2.3–2.6, although in almost all cases it is also possible to carry out the computations "by hand."

```
gap> f:=FreeGroup("x", "y", "z");
<free group on the generators [x,y,z]>
gap> x:=f.1; y:=f.2; z:=f.3;
x
y
z
gap> G:=f/[x^2, y^2, z^4,
Comm(x,z), Comm(y,z), x*y*x^-1*(y*z^2)^-1]; # insert the presentation
of G
```

```
<fp group on the generators [x,y,z]>
gap> x:=G.1; y:=G.2; z:=G.3;
x
y
z
gap> IdSmallGroup(G); # check the IdSmallGroup(G)
[16,13]
gap> for h1 in G do
> for h2 in G do
> H:=Subgroup(G, [h1,h2]);
> if Order(h1*h2*h1^-1*h2^-1)=2 then # check whether [h1,h2] has order 2
> Print(IdSmallGroup(H), " "); # identify the subgroup generated by h1
and h2
> fi; od; od; Print("\n");
[8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3]
[8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3]
[8,3] [8,3] [8,4] [8,3] [8,4] [8,3] [8,4] [8,4]
[8,3] [8,4] [8,4] [8,3] [8,3] [8,4] [8,4] [8,3]
[8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3]
[8,3] [8,4] [8,4] [8,3] [8,3] [8,4] [8,4] [8,3]
[8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3]
[8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3]
[8,3] [8,3] [8,4] [8,3] [8,4] [8,3] [8,4] [8,4]
[8,3] [8,4] [8,4] [8,3] [8,3] [8,4] [8,4] [8,3]
[8,3] [8,4] [8,4] [8,3] [8,3] [8,4] [8,4] [8,3]
[8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3] [8,3]
gap>
```

References

[BaCaGr06] I. Bauer, F. Catanese, F. Grunewald, The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves, e-print math.AG/0610267, 2006, Pure Appl. Math. Q., volume in honour of F. Bogomolov's 60-th birthday, in press.

[BaCaPi06] I. Bauer, F. Catanese, R. Pignatelli, Complex surfaces of general type: Some recent progress, in: Global Methods in Complex Geometry, Springer-Verlag, 2006, pp. 3–58.

[BPV84] W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces, Springer-Verlag, 1984.

[Bar99] R. Barlow, Zero-cycles on Mumford's surface, Math. Proc. Cambridge Philos. Soc. 126 (1999) 505–510.

[Bre00] T. Breuer, Characters and Automorphism Groups of Compact Riemann Surfaces, Cambridge University Press, 2000.

[Br90] S.A. Broughton, Classifying finite group actions on surfaces of low genus, J. Pure Appl. Algebra 69 (1990) 233–270.

[Ca81] F. Catanese, On a class of surfaces of general type, in: Algebraic Surfaces, CIME, Liguori, 1981, pp. 269–284.

[Ca99] F. Catanese, Singular bidouble covers and the construction of interesting algebraic surfaces, in: Contemp. Math., vol. 241, 1999, pp. 97–119.

[Ca00] F. Catanese, Fibred surfaces, varieties isogenous to a product and related moduli spaces, Amer. J. Math. 122 (2000) 1–44.

[CaCi91] F. Catanese, C. Ciliberto, Surfaces with $p_g = q = 1$, Sympos. Math. 32 (1991) 49–79.

[CaCi93] F. Catanese, C. Ciliberto, Symmetric product of elliptic curves and surfaces of general type with $p_g = q = 1$, J. Algebraic Geom. 2 (1993) 389–411.

[CaPi06] F. Catanese, R. Pignatelli, Fibrations of low genus I, Ann. Sci. Ecole Norm. Sup. 39 (6) (2006) 1011–1049.

[CarPol] G. Carnovale, F. Polizzi, The classification of surfaces of general type with $p_g = q = 1$ isogenous to a product, e-print arXiv: 0704.0446, 2007, Adv. Geom., in press.

[CCPW] J.H. Conway, R.T. Curtis, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Oxford University Press, 1985.

[FK92] H.M. Farkas, I. Kra, Riemann Surfaces, 2nd edition, Grad. Texts in Math., vol. 71, Springer-Verlag, 1992.

[Fre71] E. Freitag, Über die Struktur der Funktionenkörper zu hyperabelschen Gruppen I, J. Reine Angew. Math. 247 (1971) 97–117.

[GAP4] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4, http://www.gap-system.org, 2006.

[H71] W.J. Harvey, On the branch loci in Teichmüller space, Trans. Amer. Math. Soc. 153 (1971) 387–399.

[Is05] H. Ishida, Bounds for the relative Euler–Poincaré characteristic of certain hyperelliptic fibrations, Manuscripta Math. 118 (2005) 467–483.

[Ki03] H. Kimura, Classification of automorphism groups, up to topological equivalence, of compact Riemann surfaces of genus 4, J. Algebra 264 (2003) 26–54.

[KuKi90] A. Kuribayashi, H. Kimura, Automorphism groups of compact Riemann surfaces of genus five, J. Algebra 134 (1) (1990) 80–103.

[KuKu90] I. Kuribayashi, A. Kuribayashi, Automorphism groups of compact Riemann surfaces of genera three and four, J. Pure Appl. Algebra 65 (3) (1990) 277–292.

[Mi84] Y. Miyaoka, The maximum number of quotient singularities on surfaces with given numerical invariants, Math. Ann. 268 (1984) 159–171.

[Pol05] F. Polizzi, On surfaces of general type with $p_g = q = 1$, $K_S^2 = 3$, Collect. Math. 56 (2) (2005) 181–234.

[Pol08] F. Polizzi, On surfaces of general type with $p_g = q = 1$ isogenous to a product of curves, Comm. Algebra 36 (6) (2008) 2023–2053.

[Ri07] C. Rito, On surfaces with $p_g = q = 1$ and non-ruled bicanonical involution, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (1) (2007) 81–102.

[Se90] F. Serrano, Fibrations on algebraic surfaces, in: A. Lanteri, M. Palleschi, D.C. Struppa (Eds.), Geometry of Complex Projective Varieties, Cetraro, 1990, Mediterranean Press, 1993, pp. 291–300.

[Se96] F. Serrano, Isotrivial fibred surfaces, Ann. Mat. Pura Appl. CLXXI (1996) 63–81.

[Vin00] C.R. Vinroot, Symmetry and Tiling groups for genus 4 and 5, Rose–Hulman Inst. Technol. Undergrad. Math. J. 1 (1) (2000).

[Xi85] G. Xiao, Surfaces fibrees in courbes de genre deux, Lecture Notes in Math., vol. 1137, 1985.