



A Local Boundedness Result for a Class of Obstacle Problems with Non-Standard Growth Conditions

Mariapia De Rosa¹ · Antonio Giuseppe Grimaldi¹

Received: 15 February 2022 / Accepted: 22 July 2022 / Published online: 16 September 2022
© The Author(s) 2022

Abstract

We prove the local boundedness for solutions to a class of obstacle problems with non-standard growth conditions. The novelty here is that we are able to establish the local boundedness under a sharp bound on the gap between the growth exponents.

Keywords Non-standard growth · Local boundedness · Sharp bound

Mathematics Subject Classification 35J47 · 49N60

1 Introduction

This paper deals with the local boundedness of the solutions $u \in W^{1,p}(\Omega)$ to variational obstacle problems of the type

$$\min \left\{ \int_{\Omega} F(x, z, Dz) dx : z \in \mathcal{K}_{\psi}(\Omega) \right\}. \quad (1.1)$$

Here, Ω is a bounded open set of \mathbb{R}^n , $n \geq 2$, the function $\psi : \Omega \rightarrow [-\infty, +\infty)$, called *obstacle*, belongs to the Sobolev class $W^{1,p}(\Omega)$ and

$$\mathcal{K}_{\psi}(\Omega) := \{z \in u_0 + W_0^{1,p}(\Omega) : z \geq \psi \text{ a.e. in } \Omega\} \quad (1.2)$$

Communicated by Antonino Maugeri.

✉ Antonio Giuseppe Grimaldi
antoniogiuseppe.grimaldi@unina.it
Mariapia De Rosa
mariapia.derosa@unina.it

¹ Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli “Federico II”, Via Cintia, 80126 Napoli, Italy

is the class of admissible functions, where $u_0 \in W^{1,p}(\Omega)$ is a fixed boundary datum. To avoid trivialities, in what follows, we shall assume that $\mathcal{K}_\psi(\Omega)$ is not empty. We assume that the energy density $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$(s, \xi) \mapsto F(x, s, \xi) \text{ is convex,} \tag{1.3}$$

$$c_1|\xi|^p \leq F(x, s, \xi) \leq c_2(1 + |s|^\gamma + |\xi|^q), \tag{1.4}$$

for almost all $x \in \Omega$ and all $s \in \mathbb{R}, \xi \in \mathbb{R}^n$, where $1 < p \leq q, 0 \leq \gamma$ and $c_1, c_2 > 0$ are fixed constants. We recall that $u \in \mathcal{K}_\psi(\Omega)$ is a solution to (1.1) if and only if $F(x, u, Du) \in L^1_{loc}(\Omega)$ and the minimality condition

$$\int_{\text{supp}(u-\varphi)} F(x, u, Du)dx \leq \int_{\text{supp}(u-\varphi)} F(x, \varphi, D\varphi)dx \tag{1.5}$$

is satisfied for all $\varphi \in \mathcal{K}_\psi(\Omega)$.

The obstacle problem has been first considered in the works by Stampacchia [19], related to the capacity of a set in potential theory, and Fichera [10], who solved the so-called *Signorini problem* in elastostatics.

The study of the local boundedness for minimizers of integral functionals is a classic topic in Partial Differential Equations and Calculus of Variations starting from the classical result by De Giorgi [13] and often is the first step in the analysis of the regularity properties of the solutions.

In the last years, there has been an intense research activity concerning the local boundedness of minimizers of unconstrained problems, in case of energy densities satisfying non-standard growth conditions. In [5, 6, 8], the authors established the local boundedness of minimizers of integral functionals satisfying the anisotropic growth conditions

$$c_1 \sum_{i=1}^n |\xi_i|^{p_i} \leq F(x, \xi) \leq c_2(1 + |\xi|^q).$$

In this case, local boundedness is proven under the condition $q < \bar{p}^*$, where $\frac{1}{\bar{p}} = \sum_{i=1}^n \frac{1}{p_i}$ and $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$. Recently, in [7], the previous boundedness result has been obtained also in the borderline case $q = \bar{p}^*$. Moreover, the functionals, there considered, allows the dependence on u . Other related boundedness results that deserved to be quoted are [2, 9, 15, 16, 18].

The analogous study for solutions of obstacle problems under non-standard growth has been exploited in [3], where the local boundedness has been established assuming that

$$q < p_n^* = \frac{np}{n-p} \tag{1.6}$$

provided the obstacle is locally bounded. We also point out a particular case of p, q -growth condition considered in the paper by Chlebicka and De Filippis [4]. There the

authors proved the local boundedness for solutions to double-phase obstacle problems, where the model functional is given by

$$\mathcal{F}(w, \Omega) := \int_{\Omega} [|Dw|^p + a(x)|Dw|^q]dx,$$

for a non-negative function $a \in C^{0,\alpha}$, with $\alpha \in (0, 1]$. For this kind of problems, the local boundedness of the solutions is proven assuming that the obstacle, beside the boundedness, belongs to the Sobolev space $W^{1,K(\cdot)}$, with $K(t) = t^p + a(x)t^q$, and the exponents p, q, α satisfy the relations

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n} \quad \text{and} \quad 1 < p < q \leq n. \tag{1.7}$$

A restriction on the closeness between the growth exponents cannot be avoided, since thanks to the well-known Marcellini’s counterexample [17], we know that if

$$q > \frac{(n - 1)p}{n - 1 - p} = p_{n-1}^* \quad 1 < p < n - 1,$$

minimizers of functionals with p, q -growth can be unbounded even in the unconstrained setting. In a very recent paper [14], Hirsch and Schäffner proved that the sharp bound (1.8) is sufficient to the local boundedness for unconstrained minimizers.

The aim of this paper is to prove that (1.8) is sufficient also to establish the local boundedness of solutions to the obstacle problem. Here, we improve the previous local boundedness result for obstacle problems contained in [3] in different directions: We admit the dependence of the integrand F not only on x and Du , but on u too; the bound (1.8) is less restrictive than (1.6) and it is essentially sharp for local boundedness, since, in view of a counterexample by Franchi, Serapioni and Serra Cassano [11], the conclusion of Theorem 1.1 is false if condition (1.8) is replaced by

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{n - 1} + \varepsilon$$

for any $\varepsilon > 0$, already for unconstrained minimizers. Furthermore, our result shows that solutions to double-phase obstacle problems are locally bounded under weaker assumptions on the data with respect to the one considered in [4] (see Remark 4.1).

More precisely, we are to going to prove the following

Theorem 1.1 *Let $u \in W^{1,p}(\Omega)$ be a solution to (1.1) under assumptions (1.3) and (1.4), for exponents $1 < p \leq q, 0 \leq \gamma$ verifying*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n - 1} \tag{1.8}$$

and

$$\gamma \leq p_n^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n \\ \text{any finite exponent in } (q, \infty) & \text{if } p \geq n \end{cases} \tag{1.9}$$

If $\psi \in L^\infty_{loc}(\Omega)$, then $u \in L^\infty_{loc}(\Omega)$ and the following estimate

$$\sup_{B_{R_0/2}} |u| \leq C(\sup_{B_{R_0}} |\psi| + \|u\|_{W^{1,p}(B_{R_0})})^\pi,$$

holds for every ball $B_{R_0} \Subset \Omega$, for $\pi := \pi(n, p, q)$ and with $C := C(n, p, q, R_0)$.

Observe that, due to the local nature of our regularity results, we are not requiring further properties on the boundary datum u_0 in (1.2).

The proof of Theorem 1.1 is achieved following the strategy first proposed in [3], i.e., using the well-known De Giorgi method that consists in deriving a suitable Caccioppoli inequality on the superlevel sets of the solution to (1.1). In order to do so, one has to use test functions obtained truncating the solution. Here, the difficulties come from the set of admissible test functions that must belong to the admissible class $\mathcal{K}_\psi(\Omega)$ and this is where the local boundedness of the obstacle ψ comes into play. We also remark that the crucial tool in order to achieve the result under the sharp bound on the gap between the exponents is the Sobolev inequality on the spheres as done in [1, 14].

The paper is organized as follows: In Sect. 2, we introduce some notations and collect some results that will be needed in the sequel; in Sect. 3, we derive a Caccioppoli inequality for the minimizer of (1.1); Sect. 4 is devoted to the proof of Theorem 1.1.

2 Notations and Preliminary Results

In this paper, we will denote by C or c a general positive constant that may vary from line to line. Relevant dependencies on parameters will be highlighted using parentheses. With the symbol $B_r(x)$, we will denote the ball with center x and radius r . We shall omit the dependence on the center and on the radius when they are clear from the context. Moreover, we shall denote by $S_r = \{x \in \mathbb{R}^n : |x| = r\}$ the sphere of radius r on \mathbb{R}^n .

The following is a well-known iteration lemma (see [13, Lemma 6.1] for the proof).

Lemma 2.1 *Let $\Phi : [\rho, R] \rightarrow \mathbb{R}$ be a bounded non-negative function. Assume that for all $\rho \leq t < s \leq R$, it holds*

$$\Phi(t) \leq \theta\Phi(s) + A + \frac{B}{(s-t)^2} + \frac{C}{(s-t)^\gamma}$$

where $\theta \in (0, 1)$, $A, B, C \geq 0$ and $\gamma > 0$ are constants. Then, there exists a constant $c = c(\theta, \gamma)$ such that

$$\Phi(\rho) \leq c \left(A + \frac{B}{(R-\rho)^2} + \frac{C}{(R-\rho)^\gamma} \right).$$

A key ingredient in the proof of Theorem 1.1 is the following lemma, that can be found in [14, Lemma 2.1].

Lemma 2.2 *Let $n \geq 2$. For any $0 < \rho < \sigma < \infty$, $v \in L^1(B_\sigma)$ and $s > 1$, we set*

$$I(\rho, \sigma, v) := \inf \left\{ \int_{B_\sigma} |v||D\eta|^s dx : \eta \in C_0^1(B_\sigma), \eta \geq 0, \eta = 1 \text{ in } B_\rho \right\}.$$

Then, for every $\delta \in (0, 1]$

$$I(\rho, \sigma, v) \leq (\sigma - \rho)^{s-1+\frac{1}{\delta}} \left(\int_\rho^\sigma \left(\int_{S_r} |v| d\mathcal{H}^{n-1} \right)^\delta dr \right)^{\frac{1}{\delta}}.$$

Next Lemma, whose proof can be found in [13, Lemma 7.1], allows us to iterate the Caccioppoli type estimate and it is crucial to establish the local boundedness result.

Lemma 2.3 *Let $\alpha > 0$ and let (J_i) be a sequence of real positive numbers, such that*

$$J_{i+1} \leq A\lambda^i J_i^{1+\alpha},$$

with $A > 0$ and $\lambda > 1$. If $J_0 \leq A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}$, then

$$J_i \leq \lambda^{-\frac{i}{\alpha}} J_0 \quad \text{and} \quad \lim_{i \rightarrow +\infty} J_i = 0.$$

We conclude this subsection recalling the Sobolev inequality on spheres (see, e.g., [12, Chapter 16]).

Lemma 2.4 *Let $v \in W^{1,m}(S_1, d\mathcal{H}^{n-1})$ with $m \in [1, n - 1)$. Then, there exists $c = c(n, m)$ such that*

$$\left(\int_{S_1} |v|^{m^*} d\mathcal{H}^{n-1} \right)^{\frac{1}{m^*}} \leq c \left(\int_{S_1} (|Dv|^m + |v|^m) d\mathcal{H}^{n-1} \right)^{\frac{1}{m}},$$

where $\frac{1}{m^} = \frac{1}{m} - \frac{1}{n-1}$.*

3 Caccioppoli Inequality

If $u \in W^{1,p}(\Omega)$, $k \in \mathbb{R}$ and $B_R \subset \Omega$ is a ball, we set

$$A_{k,R} := \{x \in B_R : u(x) > k\}.$$

The main result of this section is the following Caccioppoli inequality.

Theorem 3.1 *Let $u \in W^{1,p}(\Omega)$ be a solution to (1.1) under assumptions (1.3) and (1.4), for exponents $1 < p \leq q$ verifying (1.8) and $0 \leq \gamma$. Assume that $\psi \in L_{loc}^\infty(\Omega)$. Then, the following inequality*

$$\int_{B_\rho} |D(u - k)_+|^p dx \leq \frac{C}{(R - \rho)^\mu} \|(u - k)_+\|_{W^{1,p}(B_R)}^q |A_{k,R}|^{q(\frac{1}{q^*} - \frac{1}{p})} + Ck^\gamma |A_{k,R}| \tag{3.1}$$

holds for all balls $B_\rho \subset B_R \subset B_{R_0} \Subset \Omega$ and for every $k \geq \max\{\sup_{B_{R_0}} \psi, 1\}$, where

$$\frac{1}{q^*} := \min\left\{\frac{1}{q} + \frac{1}{n-1}, 1\right\}, \tag{3.2}$$

$\mu := q - 1 + \frac{q}{q^*}$ and with $C := C(n, q, R_0)$.

Proof Let us fix $B_{R_0}(x_0) \Subset \Omega$. Let $\frac{R_0}{2} \leq \rho \leq s < t \leq R \leq R_0$ and let $\eta \in C_0^\infty(B_t)$ be a cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ in B_s , $|D\eta| \leq \frac{2}{t-s}$. By virtue of the assumption $\psi \in L_{loc}^\infty(\Omega)$, we may fix $k \geq \max\{\sup_{B_{R_0}} \psi, 1\}$ and define $u_k := (u - k)_+ = \max\{u - k, 0\}$. Note that, $\varphi = u - \eta^\sigma u_k$, with $\sigma > q$, belongs to $\mathcal{K}_\psi(\Omega)$. Indeed,

$$\varphi = \begin{cases} u \geq \psi & \text{if } u \leq k \\ u - \eta^\sigma(u - k) = (1 - \eta^\sigma)(u - k) + k \geq k \geq \psi & \text{if } u \geq k \end{cases} \tag{3.3}$$

By the minimality of u and the convexity assumption (1.3), we get

$$\begin{aligned} \int_{A_{k,s}} F(x, u, Du) dx &\leq \int_{A_{k,t}} F(x, u + \varphi, Du + D\varphi) dx \\ &= \int_{A_{k,t}} F\left(x, (1 - \eta^\sigma)u + \eta^\sigma k, (1 - \eta^\sigma)Du - \sigma\eta^\sigma \frac{D\eta}{\eta}(u - k)\right) dx \\ &\leq \int_{A_{k,t}} \left[(1 - \eta^\sigma)F(x, u, Du) + \eta^\sigma F\left(x, k, -\sigma \frac{D\eta}{\eta}(u - k)\right) \right] dx. \end{aligned}$$

Taking into account that $\eta = 1$ on B_s , $\sigma > q$, $t \leq R$ and the growth assumption (1.4), we obtain

$$\begin{aligned} &\int_{A_{k,s}} F(x, u, Du) dx \\ &\leq \int_{A_{k,t} \setminus A_{k,s}} (1 - \eta^\sigma)F(x, u, Du) dx + c_2 \int_{A_{k,t}} \eta^\sigma \left(1 + k^\gamma + \sigma^q \frac{|D\eta|^q}{\eta^q} (u - k)^q\right) dx \\ &\leq \int_{A_{k,t} \setminus A_{k,s}} F(x, u, Du) dx + Ck^\gamma |A_{k,R}| + C \int_{A_{k,t}} |D\eta|^q (u - k)^q dx. \end{aligned} \tag{3.4}$$

Exploiting Lemma 2.4 after the use of Hölder’s inequality in case $q \in (1, \frac{n-1}{n-2})$, we infer

$$\left(\int_{S_1} |v|^q d\mathcal{H}^{n-1}\right)^{\frac{1}{q}} \leq c(n, q) \left(\int_{S_1} (|Dv|^{q^*} + |v|^{q^*}) d\mathcal{H}^{n-1}\right)^{\frac{1}{q^*}}, \tag{3.5}$$

for every $v \in W^{1,q_*}(S_1, d\mathcal{H}^{n-1})$, where $q_* \geq 1$ is given by

$$\frac{1}{q_*} := \min\left\{\frac{1}{q} + \frac{1}{n-1}, 1\right\}.$$

We set

$$A(s, t) := \{\eta \in C_0^\infty(B_t) : \eta = 1 \text{ in } B_s, \quad 0 \leq \eta \leq 1\}.$$

Combining (3.5) applied to $v(y) = u_k^q(x_0 + ry)$ with $r \in (s, t)$ and Lemma 2.2 with $\delta = \frac{q_*}{q}$, we get

$$\begin{aligned} & \inf_{A(s,t)} \int_{B_t} |D\eta|^q u_k^q dx \\ & \leq (t-s)^{-(q-1+\frac{q}{q_*})} \left(\int_s^t r^{(n-1)\frac{q_*}{q}} \left(\int_{S_1} u_k^q(x_0 + ry) d\mathcal{H}^{n-1}(y) \right)^{\frac{q_*}{q}} dr \right)^{\frac{q}{q_*}} \\ & \leq c(n, q)(t-s)^{-(q-1+\frac{q}{q_*})} \left(\int_s^t r^{(n-1)\frac{q_*}{q}} \int_{S_1} (r^{q_*} |Du_k(x_0 + ry)|^{q_*} \right. \\ & \quad \left. + u_k^{q_*}(x_0 + ry)) d\mathcal{H}^{n-1}(y) dr \right)^{\frac{q}{q_*}} \\ & = c(n, q)(t-s)^{-(q-1+\frac{q}{q_*})} \left(\int_s^t r^{(n-1)q_* (\frac{1}{q} + \frac{1}{n-1} - \frac{q_*}{q})} \int_{S_r} (|Du_k(x_0 + y)|^{q_*} \right. \\ & \quad \left. + r^{-q_*} u_k^{q_*}(x_0 + y)) d\mathcal{H}^{n-1}(y) dr \right)^{\frac{q}{q_*}}. \end{aligned}$$

Using the fact that $\frac{R_0}{2} \leq s < t \leq R_0$, we deduce

$$\begin{aligned} & \inf_{A(s,t)} \int_{B_t} |D\eta|^q u_k^q dx \\ & \leq C(n, q, R_0)(t-s)^{-(q-1+\frac{q}{q_*})} \left(\int_s^t \int_{S_r} (|Du_k(x_0 + y)|^{q_*} + u_k^{q_*}(x_0 + y)) d\mathcal{H}^{n-1}(y) dr \right)^{\frac{q}{q_*}}. \end{aligned} \tag{3.6}$$

Now, notice that inequality (1.8) implies $q_* \leq p$. Thus, using Hölder’s inequality in estimate (3.6), we obtain

$$\begin{aligned}
 \inf_{\mathcal{A}(s,t)} \int_{B_t} |D\eta|^q u_k^q dx &\leq \frac{C(n, q, R_0)}{(t-s)^\mu} \|u_k\|_{W^{1,q^*}(B_t \setminus B_s)}^q \\
 &\leq \frac{C(n, q, R_0)}{(t-s)^\mu} \|u_k\|_{W^{1,p}(B_t \setminus B_s)}^q |A_{k,t}|^{q(\frac{1}{q^*} - \frac{1}{p})} \\
 &\leq \frac{C(n, q, R_0)}{(t-s)^\mu} \|u_k\|_{W^{1,p}(B_R)}^q |A_{k,R}|^{q(\frac{1}{q^*} - \frac{1}{p})}, \tag{3.7}
 \end{aligned}$$

where we denote $\mu = q - 1 + \frac{q}{q^*}$.

Since estimate (3.4) holds for every $\eta \in \mathcal{A}(s, t)$, by (3.7), we get

$$\begin{aligned}
 &\int_{A_{k,s}} F(x, u, Du) dx \\
 &\leq \int_{A_{k,t} \setminus A_{k,s}} F(x, u, Du) dx + Ck^\gamma |A_{k,R}| \\
 &\quad + \frac{C(n, q, R_0)}{(t-s)^\mu} \|u_k\|_{W^{1,p}(B_R)}^q |A_{k,R}|^{q(\frac{1}{q^*} - \frac{1}{p})}.
 \end{aligned}$$

Adding the integral $\int_{A_{k,s}} F(x, u, Du) dx$ to both sides of the previous estimate, by Lemma 2.1 we get

$$\begin{aligned}
 &\int_{A_{k,\rho}} F(x, u, Du) dx \\
 &\leq C(n, q, R_0) \left\{ k^\gamma |A_{k,R}| + \frac{1}{(R-\rho)^\mu} \|u_k\|_{W^{1,p}(B_R)}^q |A_{k,R}|^{q(\frac{1}{q^*} - \frac{1}{p})} \right\}. \tag{3.8}
 \end{aligned}$$

Eventually, inequality (3.8) and the growth condition at (1.4) yield

$$\begin{aligned}
 &\int_{B_\rho} |D(u-k)_+|^p dx \\
 &\leq C(n, q, R_0) \left\{ k^\gamma |A_{k,R}| + \frac{1}{(R-\rho)^\mu} \|u_k\|_{W^{1,p}(B_R)}^q |A_{k,R}|^{q(\frac{1}{q^*} - \frac{1}{p})} \right\},
 \end{aligned}$$

i.e., the conclusion. □

4 Proof of the Main Result

Before giving the proof of Theorem 1.1, we need to introduce some notations.

For a fixed ball $B_{R_0} \Subset \Omega$, define two sequences by setting

$$\rho_i := \frac{R_0}{2} \left(1 + \frac{1}{2^i} \right),$$

$$k_i := 2d \left(1 - \frac{1}{2^{i+1}} \right),$$

where $d \geq \max\{\sup_{B_{R_0}} \psi, 1\}$ will be determined later. Note that,

$$\frac{R_0}{2} \leq \rho_i \leq R_0, \quad \rho_i \searrow \frac{R_0}{2}$$

and

$$d \leq k_i \leq 2d, \quad k_i \nearrow 2d.$$

The use of estimate (3.1) on the concentric balls $B_{\rho_{i+1}} \subset B_{\rho_i}$ translates into

$$\begin{aligned} & \|D(u - k_{i+1})_+\|_{L^p(B_{\rho_{i+1}})}^p \\ & \leq \frac{C}{(\rho_i - \rho_{i+1})^\mu} \|(u - k_{i+1})_+\|_{W^{1,p}(B_{\rho_i})}^q |A_{k_{i+1},\rho_i}|^{q(\frac{1}{q^*} - \frac{1}{p})} + Ck_{i+1}^\gamma |A_{k_{i+1},\rho_i}|, \end{aligned} \tag{4.1}$$

for every $i \in \mathbb{N}$.

Moreover, define the sequence (J_i) setting

$$J_i := \|(u - k_i)_+\|_{W^{1,p}(B_{\rho_i})}^p.$$

We begin proving an inequality that will be crucial for the proof of Theorem 1.1.

Lemma 4.1 *Let $u \in W^{1,p}(\Omega)$ be a solution to (1.1) under assumptions (1.3) and (1.4), for exponents $1 < p \leq q$, $0 \leq \gamma$ verifying (1.8), (1.9), respectively. Then, there exists a constant $\tilde{C} > 0$ such that for every $i \in \mathbb{N}$*

$$J_{i+1} \leq \frac{\tilde{C}}{d^\theta} \lambda^i J_i^{1+\alpha},$$

for some $\lambda > 1$ and $\alpha > 0$, where $\frac{1}{d^\theta} := \max \left\{ \frac{1}{d^{p_n^* - p}}, \frac{1}{d^{q(\frac{1}{q^*} - \frac{1}{p})p_n^*}}, \frac{1}{d^{p_n^* - \gamma}} \right\}$.

Proof We observe that $k_{i+1} - k_i < u - k_i$ on A_{k_{i+1},ρ_i} for every $i \in \mathbb{N}$. By Sobolev inequality, we get

$$\begin{aligned} |A_{k_{i+1},\rho_i}| & \leq \int_{A_{k_{i+1},\rho_i}} \left(\frac{u - k_i}{k_{i+1} - k_i} \right)^{p_n^*} dx \leq \frac{\|(u - k_i)_+\|_{L^{p_n^*}(B_{\rho_i})}^{p_n^*}}{(k_{i+1} - k_i)^{p_n^*}} \\ & \leq C(n, p) \frac{J_i^{\frac{p_n^*}{p}}}{(k_{i+1} - k_i)^{p_n^*}}, \end{aligned} \tag{4.2}$$

where p_n^* was introduced in (1.9).

Furthermore, we have

$$\begin{aligned}
 & \| (u - k_{i+1})_+ \|_{W^{1,p}(B_{\rho_i})}^p \\
 &= \int_{B_{\rho_i}} (u - k_{i+1})_+^p dx + \int_{B_{\rho_i}} |D(u - k_{i+1})_+|^p dx \\
 &= \int_{A_{k_{i+1}, \rho_i}} (u - k_{i+1})^p dx + \int_{A_{k_{i+1}, \rho_i}} |D(u - k_{i+1})|^p dx \\
 &\leq \int_{A_{k_i, \rho_i}} (u - k_i)^p dx + \int_{A_{k_i, \rho_i}} |D(u - k_i)|^p dx = J_i.
 \end{aligned} \tag{4.3}$$

Inserting estimates (4.2) and (4.3) in (4.1), we obtain

$$\|D(u - k_{i+1})_+\|_{L^p(B_{\rho_{i+1}})}^p \leq \frac{C}{(\rho_i - \rho_{i+1})^\mu} \frac{J_i^{\frac{q}{p}(1+p_n^*(\frac{1}{q^*}-\frac{1}{p}))}}{(k_{i+1} - k_i)^{q(\frac{1}{q^*}-\frac{1}{p})p_n^*}} + C \frac{k_{i+1}^\gamma J_i^{\frac{p_n^*}{p}}}{(k_{i+1} - k_i)^{p_n^*}}. \tag{4.4}$$

By using Hölder’s and Sobolev inequalities and estimate (4.2), it follows

$$\| (u - k_{i+1})_+ \|_{L^p(B_{\rho_{i+1}})}^p \leq \| (u - k_i)_+ \|_{L^{p_n^*}(B_{\rho_i})}^p |A_{k_{i+1}, \rho_i}|^{1-\frac{p}{p_n^*}} \leq C \frac{J_i^{\frac{p_n^*}{p}}}{(k_{i+1} - k_i)^{p_n^*-p}}. \tag{4.5}$$

Summing (4.4) and (4.5), we obtain

$$J_{i+1} \leq C \left[\frac{J_i^{\frac{p_n^*}{p}}}{(k_{i+1} - k_i)^{p_n^*-p}} + \frac{1}{(\rho_i - \rho_{i+1})^\mu} \frac{J_i^{\frac{q}{p}(1+p_n^*(\frac{1}{q^*}-\frac{1}{p}))}}{(k_{i+1} - k_i)^{q(\frac{1}{q^*}-\frac{1}{p})p_n^*}} + \frac{k_{i+1}^\gamma J_i^{\frac{p_n^*}{p}}}{(k_{i+1} - k_i)^{p_n^*}} \right], \tag{4.6}$$

for a constant $C := C(n, p, q, R_0)$.

Recalling the definition of k_i, ρ_i and using the fact that $k_i \leq 2d$, we can write inequality (4.6) as follows

$$\begin{aligned}
 & J_{i+1} \\
 &\leq C \left[\frac{J_i^{\frac{p_n^*}{p}}}{d^{p_n^*-p}} 2^{(i+2)(p_n^*-p)} + \frac{2^{(i+2)(\mu+q(\frac{1}{q^*}-\frac{1}{p})p_n^*)}}{R_0^\mu} \frac{J_i^{\frac{q}{p}(1+p_n^*(\frac{1}{q^*}-\frac{1}{p}))}}{d^{q(\frac{1}{q^*}-\frac{1}{p})p_n^*}} + \frac{2^{(i+2)p_n^*}}{d^{p_n^*-\gamma}} J_i^{\frac{p_n^*}{p}} \right] \\
 &\leq \frac{C}{R_0^\mu d^\theta} 2^{i\tau} (J_i^{\frac{p_n^*}{p}} + J_i^{\frac{q}{p}(1+p_n^*(\frac{1}{q^*}-\frac{1}{p}))} + J_i^{\frac{p_n^*}{p}}),
 \end{aligned} \tag{4.7}$$

where

$$\frac{1}{d^\theta} := \max \left\{ \frac{1}{d^{p_n^* - p}}, \frac{1}{d^{q(\frac{1}{q_*} - \frac{1}{p})p_n^*}}, \frac{1}{d^{p_n^* - \gamma}} \right\},$$

$$\tau := \max \left\{ p_n^* - p, \mu + q \left(\frac{1}{q_*} - \frac{1}{p} \right) p_n^*, p_n^* \right\}.$$

Notice that by the definition of J_i and d , we easily derive

$$J_i \leq \| (u - \sup_{B_{R_0}} \psi)_+ \|_{W^{1,p}(B_{R_0})}^p \quad \forall i \in \mathbb{N}.$$

Setting

$$\sigma := \min \left\{ \frac{p_n^*}{p}, \frac{q}{p} \left(1 + p_n^* \left(\frac{1}{q_*} - \frac{1}{p} \right) \right) \right\} > 1 \tag{4.8}$$

and

$$\beta := \max \left\{ \frac{p_n^*}{p} - \sigma, \frac{q}{p} \left(1 + p_n^* \left(\frac{1}{q_*} - \frac{1}{p} \right) \right) - \sigma \right\} \geq 0,$$

estimate (4.7) can be written as

$$J_{i+1} \leq \frac{C}{R_0^\mu d^\theta} 2^{i\tau} (1 + \| (u - \sup_{B_{R_0}} \psi)_+ \|_{W^{1,p}(B_{R_0})}^p)^\beta J_i^\sigma = \frac{\tilde{C}}{d^\theta} 2^{i\tau} J_i^\sigma.$$

□

Now, we are in position to give the proof of our main result.

Proof of Theorem 1.1 Notice that $u \geq \psi$ and $\psi \in L^\infty_{\text{loc}}(\Omega)$ imply that u is bounded from below. Hence, we only need to show that u is bounded from above.

In order to show that the minimizer u is bounded in the ball $B_{R_0/2}$, we prove that the sequence (J_i) satisfies the assumptions of Lemma 2.3. Indeed, Lemma 2.3 yields

$$\lim_{i \rightarrow +\infty} J_i = 0. \tag{4.9}$$

On the other hand, we have

$$\lim_{i \rightarrow +\infty} J_i = \| (u - 2d)_+ \|_{W^{1,p}(B_{R_0/2})}^p. \tag{4.10}$$

Thus, (4.9) and (4.10) allow us to conclude that

$$\sup_{B_{R_0/2}} u \leq 2d.$$

Denote

$$\varepsilon := \frac{1}{q_*} - \frac{1}{p}.$$

If $\gamma < p_n^*$ and $\varepsilon > 0$, by Lemma 4.1, the sequence (J_i) satisfies assumptions of Lemma 2.3 with

$$A = \frac{\tilde{C}}{d^\theta}, \quad \lambda = 2^\tau \quad \text{and} \quad \alpha = \sigma - 1,$$

where σ was defined in (4.8).

We also have that

$$J_0 \leq A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}} \quad \text{with} \quad d \geq \tilde{C}^{\frac{1}{\theta}} \lambda^{\frac{1}{\alpha\theta}} \|u\|_{W^{1,p}(B_{R_0})}^{\frac{\alpha}{\theta}}.$$

Therefore, (4.9) holds.

If $\gamma < p_n^*$ and $\varepsilon = 0$, Lemma 4.1 implies that the sequence (J_i) satisfies Lemma 2.3 with

$$A = \tilde{C}, \quad \lambda = 2^\tau \quad \text{and} \quad \alpha = \sigma - 1.$$

Therefore, by Lemma 2.3, we have that (4.9) holds if

$$J_0 \leq A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}. \tag{4.11}$$

By definition, $J_0 = \|(u - d)_+\|_{W^{1,p}(B_{R_0})}^p$. We choose $d > 0$ large enough so that (4.11) holds; this is possible since $u \in W^{1,p}(B_{R_0})$ and

$$J_0 = \|(u - d)_+\|_{W^{1,p}(A_d, R_0)}^p \leq \|u\|_{W^{1,p}(A_d, R_0)}^p \longrightarrow 0 \quad \text{as } d \rightarrow +\infty.$$

With this choice of d , we get (4.9).

Eventually, the case $\gamma = p_n^*$ and $\varepsilon \geq 0$ can be treated as the previous one.

Remark 4.1 Let us consider functionals F satisfying the double-side bound

$$\nu H(x, z) \leq F(x, s, z) \leq LH(x, z), \tag{4.12}$$

for constants $0 \leq \nu \leq L$, where

$$H(x, z) := |z|^p + a(x)|z|^q, \tag{4.13}$$

with $1 < p < q$ and $0 \leq a(x) \in L^\infty(\Omega)$. Then, under the closeness condition (1.8) on the exponents p, q , solutions to (1.1) with F verifying (4.12) are locally bounded provided the obstacle is locally bounded. The arguments are essentially the same of

Theorem 1.1. Indeed, an analogous version of Theorem 3.1 and Lemma 4.1 can be proved similarly with exponent $\gamma = 0$.

Let $u \in W^{1,p}(\Omega)$ be a solution to (1.1) with energy density satisfying the growth condition (4.12). Using the same notation introduced in the proof of Theorem 3.1, we have

$$\begin{aligned} \int_{A_{k,s}} F(x, u, Du)dx &\leq \int_{A_{k,t}} F(x, u + \varphi, Du + D\varphi)dx \\ &\leq L \int_{A_{k,t}} H(x, Du + D\varphi)dx \\ &= L \int_{A_{k,t}} H\left(x, (1 - \eta^\sigma)Du - \sigma\eta^\sigma \frac{D\eta}{\eta}(u - k)\right)dx. \end{aligned} \tag{4.14}$$

By the very definition of $H(x, z)$, we have that for a.e. $x \in \Omega$ and every $z_1, z_2 \in \mathbb{R}^n$

$$H(x, z_1 + z_2) \leq 2^q (H(x, z_1) + H(x, z_2)). \tag{4.15}$$

Using (4.15), the fact that $\eta \leq 1$ and the boundedness of function a , we estimate inequality (4.14) as follows

$$\begin{aligned} \int_{A_{k,s}} F(x, u, Du)dx &\leq C(q, L) \int_{A_{k,t}} H(x, (1 - \eta^\sigma)Du)dx \\ &\quad + C(q, L) \int_{A_{k,t}} H\left(x, -\sigma\eta^\sigma \frac{D\eta}{\eta}(u - k)\right)dx \\ &\leq C(q, L) \int_{A_{k,t}} (1 - \eta^\sigma)^p H(x, Du)dx \\ &\quad + C(q, L, \|a\|_\infty) \int_{A_{k,t}} (1 + |D\eta|^q (u - k)^q)dx. \end{aligned}$$

Taking into account that $\eta = 1$ on B_s , the growth assumption (4.12) and $t \leq R$, we obtain

$$\begin{aligned} \int_{A_{k,s}} H(x, Du)dx &\leq C \int_{A_{k,s}} F(x, u, Du)dx \\ &\leq C \int_{A_{k,t} \setminus A_{k,s}} H(x, Du)dx + C|A_{k,R}| + C \int_{A_{k,t}} |D\eta|^q (u - k)^q dx. \end{aligned}$$

Recalling the definition of $H(x, Du)$, the previous estimate yields the Caccioppoli inequality (3.1) with $\gamma = 0$.

It is worth noting that the local boundedness of the solutions to double-phase obstacle problems with integrand F satisfying (4.12) is proven under weaker assumptions on the data with respect to the one assumed in [4]. Indeed, the gap (1.8) is better than

(1.7), we do not require any regularity on the function $a(x)$ and the obstacle is only assumed to be bounded.

5 Conclusions

The main result of this paper states that solutions to a class of obstacle problems, satisfying p, q -growth conditions under the sharp relation

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1},$$

are locally bounded.

Let us briefly describe the contents of the paper. After recalling some notation and preliminary results in Sect. 2, we focus on deriving the intermediate steps that will put us in the position to prove our main result. In particular, in Sect. 3, we show that the solution to the obstacle problem satisfies a suitable Caccioppoli inequality (see Theorem 3.1). In Sect. 4, we give the proof of Lemma 4.1 that allows us to iterate the Caccioppoli type estimate, so that we are eventually able to prove Theorem 1.1.

Acknowledgements The authors wish to thank the anonymous referees for useful comments which improve the initial version of the paper.

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Bella, P., Schäffner, M.: Local boundedness and harnack inequality for solutions of linear nonuniformly elliptic equations. *Comm. Pure Appl. Math.* **74**, 0453–0477 (2021)
2. Carozza, M., Giannetti, F., Leonetti, F., di Napoli, A.P.: Pointwise bounds for minimizers of some anisotropic functionals. *Nonlinear Anal.* **177**, 254–269 (2018)
3. Caselli, M., Eleuteri, M., di Napoli, A.P.: Regularity results for a class of obstacle problems with p, q -growth conditions. *ESAIM: Control, Opt. Calculus Var.* **27**, 19 (2021)
4. Chlebicka, I., De Filippis, C.: Removable sets in non-uniformly elliptic problems. *Annali di Mat.* **199**, 619–649 (2020)
5. Cupini, G., Marcellini, P., Mascolo, E.: Regularity under sharp anisotropic general growth conditions. *Discrete Contin. Dyn. Syst. Ser. B* **11**, 66–86 (2009)
6. Cupini, G., Marcellini, P., Mascolo, E.: Local boundedness of solutions to quasilinear elliptic systems. *Manuscripta Math.* **137**(3), 287–315 (2012)
7. Cupini, G., Marcellini, P., Mascolo, E.: Local boundedness of minimizers with limit growth conditions. *J. Optim. Theory Appl.* **166**, 1–22 (2015)

8. Cupini, G., Marcellini, P., Mascolo, E.: Regularity of minimizers under limit growth conditions. *Nonlinear Anal.: Theory, Methods Appl.* **153**, 294–310 (2017)
9. Dall’Aglia, A., Mascolo, E., Papi, G.: Local boundedness for minima of functionals with nonstandard growth conditions. *Rendiconti di Mat.* **18**, 305–326 (1998)
10. Fichera, G.: Problemi elastostatici con vincoli unilaterali: il problema di signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. Ia* **7**(8), 91–140 (1967)
11. Franchi, B., Serapioni, R., Cassano, F.S.: Irregular solutions of linear degenerate elliptic equations. *Potential Anal.* **9**(3), 201–216 (1998)
12. Gilbarg, D., Trudinger, N.: *Elliptic partial differential equations of second order*. Springer, Berlin (1998)
13. Giusti, E.: *Direct Methods in the Calculus of Variations*. World Scientific, Singapore (2003)
14. Hirsch, J., Schäffner, M.: Growth conditions and regularity, an optimal local boundedness result. *Commun. Contemp. Math.* **23**(3), 2050029 (2021)
15. Leonetti, F., Petricca, P.V.: Regularity for vector valued minimizers of some anisotropic integral functionals. *J. Inequal. Pure Appl. Math.* **7**, Art. 88 (2006)
16. Leonetti, F., Petricca, P.V.: Regularity for minimizers of integrals with nonstandard growth. *Nonlinear Anal.: Theory, Methods Appl.* **129**, 258–264 (2015)
17. Marcellini, P.: Regularity and existence of solutions of elliptic equations with p, q -growth conditions. *J. Differ. Equ.* **90**(1), 1–30 (1991)
18. MoscarIELLO, G., Nania, L.: Hölder continuity of minimizers of functionals with non standard growth conditions. *Ricerche mat.* **30**, 259–273 (1991)
19. Stampacchia, G.: Formes bilineaires coercitives sur les ensembles convexes. *C. R. Math. Acad. Sci. Paris* **258**, 4413–4416 (1964)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.